Characterization of Controllability based on Continuity of Closed-loop Eigenvectors: Application to Controller-Driven Sampling Stabilization

Hernan Haimovich and Esteban N. Osella

***Abstract*—This paper presents a novel characterization of controllability for linear time-invariant finite-dimensional systems. This characterization relates eigenvalue controllability with the continuity of the map that assigns to each closed-loop eigenvalue the smallest subspace containing the set of corresponding closed-loop eigenvectors. Application of the given characterization is illustrated on a specific case of controller-driven sampling stabilization, where the sampled system is interpreted as a discrete-time switched system and stability under arbitrary switching is ensured via simultaneous triangularization (Lie-algebraic solvability).**

***Index Terms*—Controllability, eigenvector assignment, subspace-valued maps, continuity, gap metric.**

I. Introduction

The concept of controllability for linear time-invariant (LTI) systems with finite-dimensional state space (i.e. systems identified by constant matrices A and B) was introduced by Kalman in the late 50s and published in 1960 [1]. Since then, several characterizations of controllability appeared, based on invariant subspaces [1], rank of the controllability matrix [2], positive definiteness of the controllability Grammian [2], pole assignability [3], eigenvalue controllability [4].

The standard definition of controllability involves the ability to reach every state from arbitrary initial states in finite time by application of a suitable input. Hautus [4] introduced the notion of eigenvalue controllability and showed that controllability as per the standard definition is equivalent to requiring that every eigenvalue of the A matrix be controllable. Hautus also established the equivalence between stabilizability and the requirement that every *unstable* eigenvalue of A be controllable [5]. Another well-known characterization of controllability states that a system is controllable if and only if the closed-loop eigenvalues can be placed arbitrarily by suitable choice of linear feedback [3]. If an eigenvalue of A is not controllable, then such a quantity will be a closed-loop eigenvalue for every choice of linear feedback and hence no linear feedback can alter its position.

This paper introduces a characterization of eigenvalue controllability that is based on properties of closed-loop **eigenvectors**. It will be shown that an eigenvalue λ of A is controllable if and only if a specific map is continuous at λ. Such a map assigns to each complex number λ (closed-loop eigenvalue), the smallest subspace, i.e. the subspace of lowest dimension, containing the closed-loop **eigenvectors** corresponding to the closed-loop eigenvalue λ.

Properties of closed-loop eigenvectors are of importance in feedback control design for switched linear systems under arbitrary switching [6], [7], [8], [9], in the case when Lie-algebraic solvability is involved [10]. Indeed, the motivation for the characterization given in this paper arises in feedback control design for switched linear systems based on Lie-algebraic solvability [11], [12], [13]. Solvability of a Lie algebra of matrices is equivalent to the simultaneous triangularizability of the matrices. The latter property implies that the matrices involved share a common eigenvector [14]. When a switched control system has no control inputs, the property of Lie-algebraic solvability is known to have little applicability because it is satisfied by a very limited class of matrices and because it lacks robustness. However, the situation can be radically different when control inputs are present [15], to the point where the existence of stabilizing feedback matrices that achieve Lie-algebraic solvability becomes a generic property, i.e. a property valid for almost every set of system parameters [16].

The remainder of this paper is as follows. Section II provides brief motivation, as well as concepts and previous results required. Some numerical examples and our main results are given in Section III. Section IV illustrates application of our main results to controller-driven sampling stabilization. Conclusions are drawn in Section V. A preliminary version of the current results was presented in [17].

H. Haimovich and E.N. Osella are with CIFASIS-CONICET and Depto. de Control, Esc. de Ing. Electrónica, FCEIA, Universidad Nacional de Rosario, Ocampo y Esmeralda, 2000 Rosario, Argentina. {haimovich,osella}@cifasis-conicet.gov.ar

Work partially supported by ANPCyT grant PICT 2013-0852, Argentina.

II. Preliminaries

This section lays the foundation for the presentation of our main results in Section III. Section II-A motivates the development. Section II-B briefly describes the well-known concepts of controllability and stabilizability, and related results. The required linear algebra concepts are addressed in Section II-C, and Section II-D deals with closed-loop eigenvectors.

A. Motivation

Consider a continuous- or discrete-time LTI system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \text{or} \quad x_{t+1} = Ax_t + Bu_t \tag{1}$$

where the dimension of x is n and that of u is m. Let $\boldsymbol{\sigma}(A)$ denote the spectrum of A, i.e. the set of its eigenvalues, and let $\lambda \notin \boldsymbol{\sigma}(A)$ so that $\lambda\mathrm{I} - A$ is invertible. Consider a vector of the form

$$v = (\lambda\mathrm{I} - A)^{-1}Bu \tag{2}$$

for some nonzero u. If the columns of B are linearly independent (l.i.), then $v \neq 0$ and it follows that the vector v is a feedback-assignable eigenvector corresponding to the closed-loop eigenvalue λ, since a matrix K satisfying $Kv = u$ always exists and then

$$[\lambda\mathrm{I} - (A + BK)]v = 0.$$

Next, consider a sequence of (possible, closed-loop) eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k \notin \boldsymbol{\sigma}(A)$ for all k and $\lim_{k\to\infty} \lambda_k = \lambda \in \boldsymbol{\sigma}(A)$. The sequence $\{v_k\}_{k=1}^{\infty}$ given by

$$v_k = (\lambda_k\mathrm{I} - A)^{-1}Bu_k \tag{3}$$

for some u_k is thus a sequence of feedback-assignable eigenvectors corresponding to each of the possible closed-loop eigenvalues λ_k. The following questions are of interest. Can the sequence $\{v_k\}_{k=1}^{\infty}$ be made convergent by appropriate choice of u_k? What would the resulting sequence converge to? The results in this paper will show that $\lambda = \lim_{k\to\infty} \lambda_k$ is a controllable eigenvalue of A if and only if every feedback-assignable eigenvector v corresponding to λ is the limit of a sequence of the form (3).

B. Controllability and Stabilizability

Consider a system of the form (1), with state and input spaces of dimensions n and m, respectively. Results will be given for the general complex case $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$, $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{n\times m}$. However, the results hold for the real case only by replacing $\mathbb{C}$ by $\mathbb{R}$. Matrix B is assumed to have full column rank and hence $m \leq n$.

The results to be given will hold for both continuous- and discrete-time systems, the only difference being that "stable eigenvalue" is to be understood as a complex number with negative real part in the continuous-time case and one with magnitude less than 1 in the discrete-time case. In addition, "stable matrix" refers to a matrix all of whose eigenvalues are stable.

The following well-known definitions and results are given for the sake of clarity. These were developed by Kalman, Wonham, and Hautus (among others) as referenced in Section I, and can be found in almost every standard text on linear systems, such as [18].

Definition 1 (Controllability): The system (1) is controllable if for every final time t_f, initial state x_o and final state x_f, there exists an input evolution u for which $x(0) = x_o$ and $x(t_f) = x_f$. The pair (A, B) is controllable if $\operatorname{rank}[B,\ AB,\ \ldots,\ A^{n-1}B] = n$. An eigenvalue λ of A is (A, B)-controllable (just "controllable" if no confusion arises) if $\operatorname{rank}[\lambda\mathrm{I} - A,\ B] = n$.

Theorem 1: The following are equivalent:

(a) The system (1) is controllable.
(b) The pair (A, B) is controllable.
(c) Every eigenvalue of A is (A, B)-controllable.
(d) For every set $\Lambda \subset \mathbb{C}$ of n not necessarily distinct numbers, there exists K such that $\boldsymbol{\sigma}(A + BK) = \Lambda$.

Definition 2 (Stabilizability): The system (1), or equivalently the pair (A, B), is stabilizable if there exists a matrix K such that $A + BK$ is stable.

Theorem 2: The following are equivalent:

(a) The system (1), or equivalently the pair (A, B), is stabilizable.
(b) Every unstable eigenvalue of A is (A, B)-controllable.

Comparison of item (d) of Theorem 1 with Definition 2 shows a clear relationship between controllability and stabilizability.

C. Subspaces, Linear Maps and Projections

The standard facts about subspaces and linear maps in this subsection can be consulted, for example, in Chapter 0 of [19].

The kernel of a linear map $M : \mathbb{C}^n \to \mathbb{C}^p$ is denoted $\ker M$, its image (range) is $\operatorname{img} M$. If $\mathcal{S}$ is a subspace of $\mathbb{C}^p$, then the subspace $\{x \in \mathbb{C}^n : Mx \subset \mathcal{S}\}$ is denoted $M^{-1}\mathcal{S}$. The latter notation is read as the pre-image (or inverse image) of $\mathcal{S}$ under the linear map M and *does not* imply that M is invertible. The dimension of a subspace $\mathcal{S}$ is denoted by $\dim(\mathcal{S})$. As is well-known, linear maps $M : \mathbb{C}^n \to \mathbb{C}^p$ can be represented by matrices in $\mathbb{C}^{p\times n}$ with a one-to-one correspondence when bases of $\mathbb{C}^n$ and $\mathbb{C}^p$ are fixed.

The concepts in the following two definitions and their properties can be consulted, for example, in [20].

Definition 3: A linear map $P : \mathbb{C}^n \to \mathbb{C}^n$ is called a *projector* if $P^2 = P$. A projector is *orthogonal* if in addition $P^* = P$, where P^* denotes the adjoint of P (conjugate transpose matrix). The projector P is said to be on $\mathcal{S}$ along $\mathcal{T}$ if $\operatorname{img} P = \mathcal{S}$ and $\ker P = \mathcal{T}$.

If P is an orthogonal projector with $\operatorname{img} P = \mathcal{S}$, then P is a projector on $\mathcal{S}$ along $\mathcal{S}^\perp$, where $\mathcal{S}^\perp$ denotes the orthogonal complement of $\mathcal{S}$. Given a subspace $\mathcal{S}$ of $\mathbb{C}^n$, orthogonal projectors on $\mathcal{S}$ are unique, and hence one may refer to *the* orthogonal projector on $\mathcal{S}$.

The set of subspaces of $\mathbb{C}^n$ will be denoted $\mathbb{G}(\mathbb{C}^n)$. Consequently, the expression "$\mathcal{S}$ is a subspace of $\mathbb{C}^n$" can be written as "$\mathcal{S} \in \mathbb{G}(\mathbb{C}^n)$". In the sequel, a notion of distance between subspaces will be required. This notion is given by the gap metric.

Definition 4 (Gap): Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathbb{C}^n$. The *gap* between $\mathcal{S}$ and $\mathcal{T}$ is defined as

$$\boldsymbol{\theta}(\mathcal{S}, \mathcal{T}) = \|P_\mathcal{S} - P_\mathcal{T}\|,$$

where $P_\mathcal{S}$ and $P_\mathcal{T}$ are the orthogonal projectors on $\mathcal{S}$ and $\mathcal{T}$, respectively, and the norm above is the induced 2-norm (Euclidean).

D. Closed-loop Eigenvectors

Given $\lambda \in \mathbb{C}$, the vectors that become corresponding closed-loop eigenvectors for some choice of feedback matrix $K \in \mathbb{C}^{m\times n}$ are the nonzero vectors contained in the set

$$\tilde{\mathcal{V}}(\lambda) := \{v \in \mathbb{C}^n : [\lambda\mathrm{I} - A^{\mathrm{CL}}]v = 0,\ \text{ for some } K \in \mathbb{C}^{m\times n}\},$$

with $A^{\mathrm{CL}} = A + BK$. Note that the set $\tilde{\mathcal{V}}(\lambda)$ is a subspace of $\mathbb{C}^n$, i.e. $\tilde{\mathcal{V}}(\lambda) \in \mathbb{G}(\mathbb{C}^n)$, and hence $\tilde{\mathcal{V}}(\lambda)$ can be referred to as the smallest subspace containing the set of closed-loop eigenvectors corresponding to λ. The only difference between $\tilde{\mathcal{V}}(\lambda)$ and the aforementioned set of eigenvectors is that $0 \in \tilde{\mathcal{V}}(\lambda)$ is not an eigenvector.

Claim 1: The set $\tilde{\mathcal{V}}(\lambda)$ coincides with

$$\mathcal{V}(\lambda) := \{v \in \mathbb{C}^n : (\lambda\mathrm{I} - A)v = Bw,\ \text{ for some } w \in \mathbb{C}^m\}. \tag{4}$$

Proof: Let $v \in \tilde{\mathcal{V}}(\lambda)$. Taking $w := Kv$ shows that $v \in \mathcal{V}(\lambda)$. Hence $\tilde{\mathcal{V}}(\lambda) \subset \mathcal{V}(\lambda)$. Next, take $v \in \mathcal{V}(\lambda)$ and select K such that $Kv = w$. This shows that $v \in \tilde{\mathcal{V}}(\lambda)$ and hence $\mathcal{V}(\lambda) \subset \tilde{\mathcal{V}}(\lambda)$. Consequently, $\tilde{\mathcal{V}}(\lambda) = \mathcal{V}(\lambda)$. ■

The set $\mathcal{V}(\lambda)$ can be written as $\{v \in \mathbb{C}^n : (\lambda\mathrm{I} - A)v \in \operatorname{img} B\}$ and hence

$$\mathcal{V}(\lambda) = (\lambda\mathrm{I} - A)^{-1}\mathcal{B} \quad \text{with} \quad \mathcal{B} := \operatorname{img} B, \tag{5}$$

where $(\lambda\mathrm{I} - A)^{-1}\mathcal{B}$ denotes the pre-image of $\mathcal{B}$ under the linear map $\lambda\mathrm{I} - A$ and does not imply that $\lambda\mathrm{I} - A$ is invertible.

The following facts are straightforward.

Lemma 1: Let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$. Then,

a) If $Av = \lambda v$, then $v \in \mathcal{V}(\lambda)$.
b) If $\lambda \in \boldsymbol{\sigma}(A)$, there may exist $v \in \mathcal{V}(\lambda)$ for which $Av \neq \lambda v$.
c) If $\lambda \notin \boldsymbol{\sigma}(A)$, then $\lambda\mathrm{I} - A$ is invertible and $\dim(\mathcal{V}(\lambda)) = \dim(\mathcal{B}) = m$.

Lemma 1a) states that the eigenvectors of A corresponding to λ, i.e. the open-loop eigenvectors, are contained in $\mathcal{V}(\lambda)$. Note that the condition in Lemma 1a) can be equivalently written as $\ker(\lambda\mathrm{I} - A) \subset \mathcal{V}(\lambda)$. According to Lemma 1b), $\mathcal{V}(\lambda)$ need not coincide with $\ker(\lambda\mathrm{I} - A)$. The first equality in Lemma 1c) follows from (5) and Lemma 3b). The fact that $\dim(\mathcal{B}) = m$ follows from $B \in \mathbb{C}^{n\times m}$ and the assumption that B has full column rank.

III. Controllability and Closed-loop Eigenvectors

This section contains the main results of the paper. Section III-A states our main result, namely Theorem 3, and explains some of the concepts and interpretations involved, as well as some of the consequences. Section III-B provides numerical examples that serve both as illustration and as motivation for our results. The proof of Theorem 3 is given in Section III-C. Section III-D explains the relationship between Theorem 3 and the questions posed in Section II-A.

A. Continuity of $\mathcal{V} : \mathbb{C} \to \mathbb{G}(\mathbb{C}^n)$

In the sequel, $\mathcal{V}(\lambda)$, as defined in (5), will be regarded as a map between the complex numbers and the set of subspaces of $\mathbb{C}^n$, i.e. $\mathcal{V} : \mathbb{C} \to \mathbb{G}(\mathbb{C}^n)$. This map assigns, to each complex number λ, the subspace $\mathcal{V}(\lambda)$. Our main result involves the continuity of the subspace-valued map $\mathcal{V}$. Continuity in this context is defined as follows, where $\boldsymbol{\theta}(\cdot,\cdot)$ denotes the gap as per Definition 4.

Definition 5: A map $\mathcal{V} : \mathbb{C} \to \mathbb{G}(\mathbb{C}^n)$ is continuous at $\lambda \in \mathbb{C}$ if for every convergent sequence $\{\lambda_k\}_{k=1}^\infty$ in $\mathbb{C}$ where $\lim_{k\to\infty} \lambda_k = \lambda$, $\lim_{k\to\infty} \boldsymbol{\theta}(\mathcal{V}(\lambda_k), \mathcal{V}(\lambda)) = 0$.

Our main result is stated as follows.

Theorem 3: The map $\mathcal{V} : \mathbb{C} \to \mathbb{G}(\mathbb{C}^n)$ as defined by (5) is continuous at λ if and only if $[\lambda\mathrm{I} - A,\ B]$ has full rank.

The proof of Theorem 3 is given in Section III-C.

The following straightforward consequences of Theorem 3, given below as Corollary 1 and Corollary 2, provide alternative characterizations of controllability and stabilizability.

Corollary 1: The map $\mathcal{V}$ is continuous in $\mathbb{C}$ if and only if (A, B) is a controllable pair.

Proof: ($\Rightarrow$) $\mathcal{V}$ continuous in $\mathbb{C}$ implies $\mathcal{V}$ continuous at $\lambda \in \boldsymbol{\sigma}(A)$. By Theorem 3 then $[\lambda\mathrm{I} - A,\ B]$ has full rank at every $\lambda \in \boldsymbol{\sigma}(A)$. From Theorem 1 and Definition 1, then (A, B) is controllable.

($\Leftarrow$) From Theorem 1 and Definition 1, (A, B) controllable implies that $[\lambda\mathrm{I} - A,\ B]$ has full rank at every $\lambda \in \boldsymbol{\sigma}(A)$. Then, $[\lambda\mathrm{I} - A,\ B]$ has full rank at every $\lambda \in \mathbb{C}$ and application of Theorem 3 shows that $\mathcal{V}$ is continuous at every $\lambda \in \mathbb{C}$. ■

Corollary 2: The map $\mathcal{V}$ is continuous at every unstable eigenvalue of A if and only if (A, B) is stabilizable.

Proof: The proof is analogous to that of Corollary 1 but employs Theorem 2 instead of Theorem 1. ■

In the next subsection, Theorem 3 is illustrated by means of numerical examples.

B. Numerical Examples

1) Uncontrollable System.: Consider the following second-order, single-input, uncontrollable system, identified by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6}$$

The eigenvalues of A are $\boldsymbol{\sigma}(A) = \{0, 1\}$, where 0 is an uncontrollable eigenvalue and 1 is a controllable one. Computation of the sets $\mathcal{V}(0)$ and $\mathcal{V}(1)$ according to (4) yields

$$\mathcal{V}(0) = \mathrm{Span}\{v^u, v^c\} \qquad \mathcal{V}(1) = \mathrm{Span}\{v^c\}, \tag{7}$$

where $v^u = [1\ 0]'$ and $v^c = [0\ 1]'$ are eigenvectors of A (i.e. open-loop eigenvectors) corresponding to the uncontrollable and controllable eigenvalues, respectively. Whenever $\lambda \notin \boldsymbol{\sigma}(A)$ and since $(\lambda I - A)^{-1}$ is invertible [cf. Lemma 3d) below], $\mathcal{V}(\lambda)$ can be straightforwardly computed from (5) as

$$\begin{aligned} \mathcal{V}(\lambda) &= \mathrm{img}[(\lambda\mathrm{I} - A)^{-1} B] = \mathrm{img} \begin{bmatrix} 0 \\ \frac{1}{\lambda - 1} \end{bmatrix} \\ &= \mathrm{Span}\{v^c\}, \quad \text{for} \quad \lambda \notin \boldsymbol{\sigma}(A). \end{aligned} \tag{8}$$

From (7) and (8) it follows that $\mathcal{V}(\lambda)$ is constant (hence continuous) and one-dimensional at every λ not equal to the uncontrollable eigenvalue, and that its dimension "jumps" at the uncontrollable eigenvalue $\lambda = 0$. Also, from (7) and (8) it follows that $\lim_{\lambda \to 0} \mathcal{V}(\lambda) = \mathrm{Span}\{v^c\} \subset \mathcal{V}(0)$.

2) Controllable system.: As a second example, consider the controllable two-input system identified by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of A are $\boldsymbol{\sigma}(A) = \{-1, -2, 1\}$. By direct computation, one obtains

$$\begin{aligned} \mathcal{V}(-1) &= \mathrm{Span}\{e_1, e_3\}, \\ \mathcal{V}(-2) &= \mathrm{Span}\{e_2, e_3\}, \\ \mathcal{V}(1) &= \mathrm{Span}\{3e_1 + 2e_2, e_3\}, \end{aligned} \tag{9}$$

where $e_1 = [1\ 0\ 0]'$, $e_2 = [0\ 1\ 0]'$ and $e_3 = [0\ 0\ 1]'$. For each $\lambda \in \boldsymbol{\sigma}(A)$, $\mathcal{V}(\lambda)$ contains the eigenvector corresponding to λ, but in this (multi-input) case, note that $\mathcal{V}(\lambda)$ contains other vectors as well [recall Lemma 1a) and b)]. If $\lambda \notin \boldsymbol{\sigma}(A)$, then

$$\begin{aligned} \mathcal{V}(\lambda) &= \mathrm{img}[(\lambda\mathrm{I} - A)^{-1} B] = \mathrm{img} \begin{bmatrix} \frac{1}{\lambda+1} & 0 \\ \frac{1}{\lambda+2} & 0 \\ 0 & \frac{1}{\lambda-1} \end{bmatrix} \\ &= \mathrm{img}\, U_r(\lambda), \quad \text{with } U_r(\lambda) = \begin{bmatrix} (\lambda+2) & 0 \\ (\lambda+1) & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathrm{Span}\{(\lambda+2)e_1 + (\lambda+1)e_2,\ e_3\}. \end{aligned}$$

Note that $\dim(\mathcal{V}(\lambda)) = 2$ for all $\lambda \in \mathbb{C}$, even if $\lambda \in \boldsymbol{\sigma}(A)$. Let $U_r(\lambda)^*$ denote the conjugate transpose of $U_r(\lambda)$. For $\lambda \notin \boldsymbol{\sigma}(A)$, it follows that

$$\begin{aligned} P(\lambda) &:= U_r(\lambda)\big[U_r(\lambda)^* U_r(\lambda)\big]^{-1} U_r(\lambda)^* \\ &= \begin{bmatrix} \frac{|\lambda+2|^2}{|\lambda+2|^2 + |\lambda+1|^2} & \frac{(\lambda+2)\overline{(\lambda+1)}}{|\lambda+2|^2 + |\lambda+1|^2} & 0 \\ \frac{\overline{(\lambda+2)}(\lambda+1)}{|\lambda+2|^2 + |\lambda+1|^2} & \frac{|\lambda+1|^2}{|\lambda+2|^2 + |\lambda+1|^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

is the orthogonal projector onto $\mathcal{V}(\lambda)$, and hence

$$\mathcal{V}(\lambda) = \mathrm{img}\, P(\lambda) \quad \text{whenever } \lambda \notin \boldsymbol{\sigma}(A). \tag{10}$$

One may compute

$$P(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P(-2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$P(1) = \begin{bmatrix} 9/13 & 6/13 & 0 \\ 6/13 & 4/13 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $\mathrm{img}\, P(-1) = \mathrm{Span}\{e_1, e_3\}$, $\mathrm{img}\, P(-2) = \mathrm{Span}\{e_2, e_3\}$, and $\mathrm{img}\, P(1) = \mathrm{Span}\{3e_1 + 2e_2, e_3\}$. These equations jointly with (9–10) show that $\mathcal{V}(\lambda) = \mathrm{img}\, P(\lambda)$ for all $\lambda \in \mathbb{C}$, even when $\lambda \in \boldsymbol{\sigma}(A)$. Therefore, P is the orthogonal projector onto $\mathcal{V}(\lambda)$ for all $\lambda \in \mathbb{C}$. Since P is continuous at every $\lambda \in \mathbb{C}$, then $\mathcal{V}$ is continuous in $\mathbb{C}$, as follows from direct application of Definitions 4 and 5.

C. Proof of Theorem 3

First, some preliminary concepts required are stated. Recall that a sequence is Cauchy if the distance between any two elements in the sequence tends to zero. If the elements of the sequence are subspaces, then distance is measured by the gap metric, and hence a sequence $\{\mathcal{S}_k\}_{k=1}^{\infty}$ of subspaces of $\mathbb{C}^n$ is Cauchy if $\lim_{k,\ell \to \infty} \boldsymbol{\theta}(\mathcal{S}_k, \mathcal{S}_\ell) = 0$. In this context, $\{\mathcal{S}_k\}_{k=1}^{\infty}$ is convergent in $\mathbb{G}(\mathbb{C}^n)$ if there exists $\mathcal{S} \in \mathbb{G}(\mathbb{C}^n)$ such that $\lim_{k \to \infty} \boldsymbol{\theta}(\mathcal{S}_k, \mathcal{S}) = 0$.

Theorem 4: (Adapted from Theorem 13.4.1 of [20]) The metric space $\mathbb{G}(\mathbb{C}^n)$, given by $\mathbb{G}(\mathbb{C}^n)$ endowed with the gap metric, is compact, and, therefore, complete (as a metric space).

Completeness of $\mathbb{G}(\mathbb{C}^n)$ means that every Cauchy sequence of elements in $\mathbb{G}(\mathbb{C}^n)$ is convergent in $\mathbb{G}(\mathbb{C}^n)$. The following properties of the gap metric will be employed (see Chapter 13 of [20]).

Lemma 2: Let $\mathcal{S}, \mathcal{T} \in \mathbb{G}(\mathbb{C}^n)$. Then,

a) $\boldsymbol{\theta}(\mathcal{S}, \mathcal{T}) \leq 1$.
b) $\boldsymbol{\theta}(\mathcal{S}, \mathcal{T}) < 1$ only if $\dim(\mathcal{S}) = \dim(\mathcal{T})$.

The proof of Theorem 3 requires the following facts about subspaces.

Lemma 3: Let $\mathcal{R} \in \mathbb{G}(\mathbb{C}^p)$, $\mathcal{S}, \mathcal{T} \in \mathbb{G}(\mathbb{C}^n)$, let $M : \mathbb{C}^n \to \mathbb{C}^p$ be a linear map, let $N_1 \in \mathbb{C}^{n \times p}$ and $N_2 \in \mathbb{C}^{n \times q}$.

a) $\dim(\mathcal{S} \cap \mathcal{T}) = \dim(\mathcal{S}) + \dim(\mathcal{T}) - \dim(\mathcal{S} + \mathcal{T})$.
b) $\dim(M^{-1}\mathcal{R}) = \dim(\mathcal{R} \cap \mathrm{img}\, M) + \dim(\ker M)$.
c) If $p = n$, then $\dim(\ker M) + \dim(\mathrm{img}\, M) = n$.
d) If $p = n$ and M is invertible, then $M^{-1}\,\mathrm{img}\, N_2 = \mathrm{img}(M^{-1} N_2)$.
e) $\mathrm{rank}[N_1, N_2] = \dim(\mathrm{img}\, N_1 + \mathrm{img}\, N_2)$.

Next, the proof of Theorem 3 is provided. The necessity part of the proof follows from a dimensionality argument, as was illustrated in the first example in Section III-B. The sufficiency part of the proof is more involved and requires the use of orthogonal projectors, as illustrated in the second example in Section III-B.

Proof of Theorem 3: From (5) and Lemma 3b), it follows that

$$\dim(\mathcal{V}(\lambda)) = \dim(\mathcal{B} \cap \operatorname{img}(\lambda \mathrm{I} - A)) + \dim(\ker(\lambda \mathrm{I} - A)).$$

Using Lemma 3a), also

$$\dim(\mathcal{B} \cap \operatorname{img}(\lambda \mathrm{I} - A)) = \dim(\mathcal{B}) + \dim(\operatorname{img}(\lambda \mathrm{I} - A)) - \dim(\mathcal{B} + \operatorname{img}(\lambda \mathrm{I} - A)).$$

Combining the above two equations, recalling that $\dim(\mathcal{B}) = m$, and applying Lemma 3c) yields

$$\dim(\mathcal{V}(\lambda)) = m + n - \dim(\mathcal{B} + \operatorname{img}(\lambda \mathrm{I} - A)). \tag{11}$$

Necessity ($\mathcal{V}$ continuous at $\lambda \implies [\lambda \mathrm{I} - A, B]$ full rank).

If $\lambda \notin \boldsymbol{\sigma}(A)$, then $[\lambda \mathrm{I} - A, B]$ has full rank. Consider next the case $\lambda \in \boldsymbol{\sigma}(A)$. Continuity of $\mathcal{V}$ at λ implies that the dimension of $\mathcal{V}(\tilde{\lambda})$ is constant for all $\tilde{\lambda}$ sufficiently close to λ (see Definition 5 and Lemma 2). Consider $\tilde{\lambda}$ sufficiently close to λ and such that $\tilde{\lambda} \notin \boldsymbol{\sigma}(A)$. By Lemma 1c), $\dim(\mathcal{V}(\tilde{\lambda})) = m$, and hence $\dim(\mathcal{V}(\lambda)) = m$. According to (11), then $\dim(\mathcal{B} + \operatorname{img}(\lambda \mathrm{I} - A)) = n$ and by Lemma 3e), then $\operatorname{rank}[\lambda \mathrm{I} - A, B] = n$.

Sufficiency ($\mathcal{V}$ continuous at $\lambda \impliedby [\lambda \mathrm{I} - A, B]$ full rank).

For every $\lambda \notin \boldsymbol{\sigma}(A)$, then

$$\mathcal{V}(\lambda) = (\lambda \mathrm{I} - A)^{-1}\mathcal{B} = \frac{\operatorname{Adj}(\lambda \mathrm{I} - A)}{\det(\lambda \mathrm{I} - A)}\mathcal{B},$$

where $\operatorname{Adj}(\cdot)$ denotes the adjugate matrix, i.e. the transpose of the cofactor matrix. Since $\det(\lambda \mathrm{I} - A)$ is a scalar and $\mathcal{V}(\lambda)$ is a subspace,

$$\mathcal{V}(\lambda) = \operatorname{Adj}(\lambda \mathrm{I} - A)\mathcal{B} \quad \text{whenever} \quad \lambda \notin \boldsymbol{\sigma}(A).$$

For $\lambda \in \mathbb{C}$, consider the matrix $M(\lambda) := \operatorname{Adj}(\lambda \mathrm{I} - A)B$, whence

$$\mathcal{V}(\lambda) = \operatorname{img} M(\lambda) \quad \text{whenever} \quad \lambda \notin \boldsymbol{\sigma}(A). \tag{12}$$

Since the entries of M are polynomials in the variable λ with coefficients in $\mathbb{C}$, then M can be written in Smith normal form as $M = UST$, where $U(\lambda) \in \mathbb{C}^{n \times n}$, $S(\lambda) \in \mathbb{C}^{n \times m}$ and $T(\lambda) \in \mathbb{C}^{m \times m}$, U and T are invertible and their inverses also are polynomial matrices, and $S(\lambda) = \operatorname{diag}(p_1(\lambda), \ldots, p_m(\lambda)) \in \mathbb{C}^{n \times m}$ where p_i are polynomials with the property that p_i divides p_{i+1}. Let S_r be the matrix formed by the first m rows of S, hence $S_r(\lambda) = \operatorname{diag}(p_1(\lambda), \ldots, p_m(\lambda)) \in \mathbb{C}^{m \times m}$, and U_r the matrix formed by the first m columns of U, so that

$$M(\lambda) = U_r(\lambda)N(\lambda), \text{ with } N(\lambda) = S_r(\lambda)T(\lambda) \in \mathbb{C}^{m \times m}$$

Since U_r consists in the first m columns of the matrix U, then the m columns of $U_r(\lambda)$ are l.i. for every $\lambda \in \mathbb{C}$ and $\operatorname{rank} M(\lambda)$ coincides with $\operatorname{rank} N(\lambda)$. By (12) and Lemma 1c), then $\operatorname{rank} N(\lambda) = m$ whenever $\lambda \notin \boldsymbol{\sigma}(A)$. Therefore, $N(\lambda)$ is invertible and

$$\operatorname{img} M(\lambda) = \operatorname{img} U_r(\lambda) \quad \text{if} \quad \lambda \notin \boldsymbol{\sigma}(A). \tag{13}$$

Since $U_r(\lambda)$ has l.i. columns $\forall \lambda \in \mathbb{C}$, then $U_r(\lambda)^* U_r(\lambda)$ is invertible $\forall \lambda \in \mathbb{C}$ and one may define $P := U_r(U_r^* U_r)^{-1} U_r^*$. Note that $P^2 = P$, $P^* = P$, and $\operatorname{img} P = \operatorname{img} U_r$. Therefore,

$$\mathcal{V}(\lambda) = \operatorname{img} P(\lambda) \quad \text{whenever} \quad \lambda \notin \boldsymbol{\sigma}(A), \tag{14}$$

P is an orthogonal projector (for every λ), and is onto the range of M whenever $\lambda \notin \boldsymbol{\sigma}(A)$. Note that since the rank of U_r is constant $\forall \lambda \in \mathbb{C}$, then P is continuous at every $\lambda \in \mathbb{C}$, even if $\lambda \in \boldsymbol{\sigma}(A)$.

Consider a convergent sequence $\{\lambda_k\}_{k=1}^{\infty}$ in $\mathbb{C}$ so that $\lambda_k \notin \boldsymbol{\sigma}(A)$ for all k and $\lim_{k \to \infty} \lambda_k = \lambda$. By (14), then $\mathcal{V}(\lambda_k) = \operatorname{img} P(\lambda_k)$ for all k. By Definition 4, then

$$\boldsymbol{\theta}(\mathcal{V}(\lambda_j), \mathcal{V}(\lambda_k)) = \|P(\lambda_j) - P(\lambda_k)\| \tag{15}$$

for all j, k. Since P is continuous, the sequence $\{P(\lambda_k)\}_{k=1}^{\infty}$ is Cauchy and by (15) so is the sequence $\{\mathcal{V}(\lambda_k)\}_{k=1}^{\infty}$. By Theorem 4, the latter sequence is convergent. Define

$$\mathcal{V}_o := \lim_{k \to \infty} \mathcal{V}(\lambda_k) = \operatorname{img} P(\lambda)$$

Next, $\mathcal{V}_o \subset \mathcal{V}(\lambda)$ will be established. Let $v \in \mathcal{V}_o$ and take $v_k \in \mathcal{V}(\lambda_k)$ so that $\lim_{k \to \infty} v_k = v$ (this is possible by Theorem 5 in Section III-D). For each k,

$$(\lambda_k \mathrm{I} - A)v_k \in \mathcal{B}.$$

Since $\lambda_k \to \lambda$ and $v_k \to v$ as $k \to \infty$, and since $\mathcal{B}$ is closed (it is a subspace of $\mathbb{C}^n$), it follows that

$$\lim_{k \to \infty} (\lambda_k \mathrm{I} - A)v_k = (\lambda \mathrm{I} - A)v \in \mathcal{B}.$$

Thus, $v \in \mathcal{V}(\lambda)$. Since $v \in \mathcal{V}_o$ is arbitrary, it follows that $\mathcal{V}_o \subset \mathcal{V}(\lambda)$.

Since $\dim(\mathcal{V}(\lambda_k)) = m$ and $\mathcal{V}(\lambda_k) \to \mathcal{V}_o$, then $\dim(\mathcal{V}_o) = m$. By (11) and since $\operatorname{rank}[\lambda \mathrm{I} - A,\ B] = \dim(\mathcal{B} + \operatorname{img}(\lambda \mathrm{I} - A)) = n$, then $\dim(V(\lambda)) = m$. Therefore, $\mathcal{V}_o = \mathcal{V}(\lambda) = \operatorname{img} P(\lambda)$. It has thus been established that if $\operatorname{rank}[\lambda \mathrm{I} - A,\ B] = n$, then $\mathcal{V}(\lambda) = \operatorname{img} P(\lambda)$. Continuity of $\mathcal{V}$ then follows by continuity of P. ■

D. Continuity and Sequences of Eigenvectors

The following theorem will be employed to provide a link between continuity of the map $\mathcal{V}$, as characterized by Theorem 3, and sequences of feedback-assignable eigenvectors.

Theorem 5: (Adapted from Theorem 13.4.2 of [20]) Let $\{\mathcal{S}_k\}_{k=1}^{\infty}$ be a sequence of m-dimensional subspaces in $\mathbb{G}(\mathbb{C}^n)$, such that $\lim_{k \to \infty} \boldsymbol{\theta}(\mathcal{S}_k, \mathcal{S}) = 0$ for some subspace $\mathcal{S} \in \mathbb{G}(\mathbb{C}^n)$. Then $\mathcal{S}$ consists of exactly those vectors $x \in \mathbb{C}^n$ for which there exists a sequence of vectors $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{C}^n$ such that $x_k \in \mathcal{S}_k$ for all $k \geq 1$ and $\lim_{k \to \infty} x_k = x$.

As in Section II-A, consider a sequence $\{\lambda_k\}_{k=1}^{\infty}$, with $\lambda_k \notin \boldsymbol{\sigma}(A)$ for all k and $\lim_{k \to \infty} \lambda_k = \lambda \in \boldsymbol{\sigma}(A)$. The sequence of vectors $\{v_k\}_{k=1}^{\infty}$ where each v_k satisfies (3) for some u_k is such that $v_k \in \mathcal{V}(\lambda_k)$ for all k. Conversely, every vector $v_k \in \mathcal{V}(\lambda_k)$ can be written as in (3) for some u_k.

If λ is an (A, B)-controllable eigenvalue of A, then the map $\mathcal{V}$ is continuous at λ by Theorem 3. According to Definition 5, $\lim_{k \to \infty} \boldsymbol{\theta}(\mathcal{V}(\lambda_k), \mathcal{V}(\lambda)) = 0$, and by Theorem 5, then every $v \in \mathcal{V}(\lambda)$ satisfies $v = \lim_{k \to \infty} v_k$ where $v_k \in \mathcal{V}(\lambda_k)$. In other words, if λ is (A, B)-controllable, then u_k can be chosen so that the vectors v_k in (3) form a sequence that converges to any vector in $\mathcal{V}(\lambda)$. According to Lemma 1a), $\ker(\lambda \mathrm{I} - A) \subset \mathcal{V}(\lambda)$. As a consequence each eigenvector of A which corresponds to a controllable eigenvalue λ is the limit of a sequence of feedback-assignable eigenvectors corresponding to eigenvalues approaching λ.

If λ is not (A, B)-controllable, then the proof of Theorem 3 shows that $\lim_{k \to \infty} \mathcal{V}(\lambda_k) \subset \mathcal{V}(\lambda)$. Hence in this case u_k also can be chosen to make the vectors v_k in (3) form a converging sequence, but the resulting sequence will converge only to some of the vectors in $\mathcal{V}(\lambda)$ and, by Theorem 5, some of the vectors in $\mathcal{V}(\lambda)$ cannot be the limit of any sequence formed by $v_k \in \mathcal{V}(\lambda_k)$ (as was illustrated for $\lambda = 0$ in the first example in Section III-B).

The following Lemma shows that a feedback-assignable eigenvector of a single-input non-scalar controllable system cannot be contained in $\operatorname{img} B$. This property is employed in the next section.

Lemma 4: Consider a controllable system of the form (1), with $n \geq 2$, $m = 1$, and let $v \neq 0$, $v \in \mathcal{V}(\lambda)$ for some $\lambda \in \mathbb{C}$. Then $v \notin \operatorname{img} B$.

Proof: Suppose that $v = Bw$ for some nonzero $w \in \mathbb{C}$. Then, $(\lambda\mathrm{I} - A)Bw = Bu$ for some $u \in \mathbb{C}$ and hence $B(u - \lambda w) + ABw = 0$. Therefore the vectors B and $AB \in \mathbb{C}^n$, with $n \geq 2$, are linearly dependent. This contradicts the fact that (A, B) is controllable. ∎

IV. Controller-driven Sampling Stabilization

We next show how the characterization derived in the previous section can be applied to a simple example of controller-driven sampling stabilization.

A. Problem Statement

Consider the controller-driven sampling setting of [13], where a continuous-time LTI system, described by

$$\dot{x} = A^c x + B^c u, \tag{16}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ may be sampled at varying rates, with the constraint that the possible sampling periods are taken from the finite set $\mathcal{H} := \{h_1, h_2, \ldots, h_N\}$. We assume that every $h \in \mathcal{H}$ is non-pathological [21]. Let t_k denote the sampling instants and define $x_k := x(t_k)$ and $u_k := u(t_k)$. The state evolution at the sampling instants can be obtained from (16) as

$$x_{k+1} = A_{i(k)} x_k + B_{i(k)} u_k, \tag{17}$$

where $i(k) \in \underline{N} := \{1, 2, \ldots, N\}$ and

$$A_j = e^{A^c h_j}, \qquad B_j = \int_0^{h_j} e^{A^c t} dt B^c, \text{ for all } j \in \underline{N}. \tag{18}$$

Note that (17) can be interpreted as a discrete-time switched linear system. The controller-driven sampling scheme allows the controller, at sampling instant t_k, to select the next sampling instant, namely t_{k+1}, arbitrarily within the constraint that $t_{k+1} - t_k \in \mathcal{H}$. Based on such sampling instant knowkedge, at time t_k the controller can apply feedback control of the form $u_k = K_{i(k)} x_k$, so that

$$x_{k+1} = A^{\mathrm{CL}}_{i(k)} x_k, \quad \text{with } A^{\mathrm{CL}}_j = A_j + B_j K_j \text{ for all } j \in \underline{N}. \tag{19}$$

The problem addressed is to design the feedback matrices K_j for $j \in \underline{N}$, so that the discrete-time switched linear closed-loop system (19) is stable irrespective of the way in which the controller selects the sampling periods. In switched systems terminology, we aim at designing K_j for all $j \in \underline{N}$ so that system (19) is stable *under arbitrary switching*, i.e. irrespective of the *switching signal* $i : \mathbb{Z}_+ \to \underline{N}$. More specifically, we aim at providing conditions that ensure the existence of K_j for all $j \in \underline{N}$ so that the closed-loop matrices A^{CL}_j are individually stable and admit simultaneuos triangularization (i.e. generate a solvable Lie algebra). The latter properties (individual stability + simultaneous triangularization) are known to ensure stability under arbitrary switching [10].

We have previously addressed this more specific problem in [13], where the conditions that we provided for the existence of suitable K_j required the continuous-time matrix A^c to be nonsingular. The characterization derived in the previous sections will allow us to provide sufficient conditions without assuming nonsingularity of A^c.

B. Lie-algebraic Solvability-based Control Design

We first provide two results, namely Lemmas 5 and 6, which are analogous to corresponding results in [13] but are formulated employing the current notation. We define the sets of continuous- and discrete-time feedback-assignable eigenvectors:

$$\mathcal{V}^c(\lambda) := (\lambda\mathrm{I} - A^c)^{-1} \operatorname{img} B^c, \quad \mathcal{V}^d_j(\lambda) := (\lambda\mathrm{I} - A_j)^{-1} \operatorname{img} B_j.$$

The following lemma is a slight generalization of Lemma 1 of [13]. This result shows that the set of continuous-time feedback-assignable eigenvectors corresponding to the eigenvalue 0 is contained in the set of discrete-time feedback-assignable eigenvectors corresponding to the eigenvalue 1 *irrespective of the sampling period*. In addition, these two sets coincide if the sampling period is non-pathological.

Lemma 5: Let $A^c \in \mathbb{R}^{n\times n}$, $B^c \in \mathbb{R}^{n\times m}$, $h_j \in \mathbb{R}$, and consider the matrices A_j and B_j as in (18). Then, $\mathcal{V}^c(0) \subseteq \mathcal{V}^d_j(1)$, with equality if, in addition, h_j is non-pathological.

Proof: Let $v \in \mathcal{V}^c(0)$. Then, there exists $K \in \mathbb{R}^{m\times n}$ such that

$$-(A^c + B^c K)v = 0. \tag{20}$$

Left-multiply the above equation by $\int_0^{h_j} e^{A^c t} dt$ to obtain

$$-\left(\int_0^{h_j} e^{A^c t} dt A^c + \int_0^{h_j} e^{A^c t} dt B^c K\right) v = 0.$$

Solving the first integral above and using (18) yields

$$[\mathrm{I} - (A_j + B_j K)]v = 0. \tag{21}$$

This shows that $v \in \mathcal{V}^d_j(1)$. Next, let $v \in \mathcal{V}^d_j(1)$. Then, (21) must hold for some $K \in \mathbb{R}^{m\times n}$. If h_j is non-pahological, then $\int_0^{h_j} e^{A^c t} dt$ is invertible and we may left-multiply (21) by the inverse to reverse the above steps and reach (20), showing that $v \in \mathcal{V}^c(0)$. ∎

The following is the equivalent to Lemma 2 of [13] using the current notation. This result gives a sufficient condition for the existence of feedback matrices that stabilize the discrete-time switched system (17) by achieving simultaneous triangularization.

Lemma 6: Consider matrices $A_j \in \mathbb{R}^{n\times n}$ and $B_j \in \mathbb{R}^{n\times n-1}$ with $\operatorname{rank} B_j = n-1$ for all $j \in \underline{N}$. Suppose that for scalars $\lambda_j \in \mathbb{R}$ satisfying $|\lambda_j| < 1$, we have

$$\bigcap_{j\in\underline{N}} \mathcal{V}^d_j(\lambda_j) \not\subset \operatorname{img} B_k \tag{22}$$

for every $k \in \underline{N}$. Then, there exist $T \in \mathbb{R}^{n\times n}$ invertible and $K_j \in \mathbb{R}^{n-1\times n}$ such that $\boldsymbol{\rho}(A_j + B_j K_j) < 1$ (spectral radius less than 1) and $T^{-1}(A_j + B_j K_j)T$ is upper triangular for all $j \in \underline{N}$.

Proof: By (22) and since 0 is a subset of every subspace, then $\bigcap_{j\in\underline{N}} \mathcal{V}^d_j(\lambda_j) \neq 0$ and there exists a nonzero $v \in \bigcap_{j\in\underline{N}} \mathcal{V}^d_j(\lambda_j)$ satisfying $v \notin \operatorname{img} B_k$ for every $k \in \underline{N}$. Therefore, there exist $F_j \in \mathbb{R}^{n-1\times n}$ so that $(A_j + B_j F_j)v = \lambda_j v$. Let $\bar{v} = v/\|v\|$ and select $U \in \mathbb{R}^{n\times n-1}$ such that $U'U = I$ and $U'\bar{v} = 0$. The matrix $T \doteq [\bar{v}|U]$ satsifies $T'T = I$. We have $\operatorname{rank}(U'B_j) = \operatorname{rank} B_j = n-1$. Then, the matrix $U'B_j$ has a right-inverse and hence there exists $G_j \in \mathbb{R}^{n-1\times n-1}$ such that $M_j \doteq U'(A_j + B_j F_j)U + U'B_j G_j$ is upper triangular and $\boldsymbol{\rho}(M_j) < 1$. Let $K_j = F_j + G_j U'$. It follows that $T^{-1}(A_j + B_j K_j)T = \begin{bmatrix} \lambda_j & \bar{v}'(A_j + B_j K_j)U \\ 0 & M_j \end{bmatrix}$ is stable and upper triangular. ∎

With the sole aim of illustrating an application of the continuity concept developed, we next generalize Theorem 2 of [13] by avoiding the assumption that A^c is nonsingular. Therefore, and to avoid more lengthy derivations, we will concentrate on the case $n = 2$, $m = 1$. Consider then the case $A^c \in \mathbb{R}^{2\times 2}$ and $B^c \in \mathbb{R}^{2\times 1}$. For every $\lambda \in \mathbb{R} \setminus \boldsymbol{\sigma}(A_j)$, define the following quantities

$$\tilde{v}_j(\lambda) := (\lambda\mathrm{I} - A_j)^{-1} B_j, \qquad v_j(\lambda) := \frac{\tilde{v}_j(\lambda)}{\|\tilde{v}_j(\lambda)\|}.$$

Note that $v_j(\lambda) \in \mathcal{V}^d_j(\lambda)$. The function $v_j : \mathbb{R} \setminus \boldsymbol{\sigma}(A_j) \to \mathbb{R}^2$ is continuous and satisfies $\|v_j(\lambda)\| = 1$ at every $\lambda \in \mathbb{R} \setminus \boldsymbol{\sigma}(A_j)$. In addition, due to the fact that the components of $\tilde{v}_j$ are rational functions of λ and considering only real λ, the following left limit must exist, even if $1 \in \boldsymbol{\sigma}(A_j)$:

$$v_j^- := \lim_{\lambda\to 1^-} v_j(\lambda). \tag{23}$$

We are now ready to formulate the aforementioned generalization.

Proposition 1: Let $A^c \in \mathbb{R}^{2\times 2}$, $B^c \in \mathbb{R}^{2\times 1}$, and let the pair (A^c, B^c) be controllable. Then, there exists $\bar{v} \in \mathcal{V}^c(0)$ such that

$$\bar{v} = s_j v_j^-, \quad s_j \in \{1, -1\}, \quad \text{for all } j \in \underline{N}, \tag{24}$$

with v_j^- as in (23). Let $p \in \mathbb{R}^2$ be nonzero and satisfy $p'v = 0$ for all $v \in \mathcal{V}^c(0)$. Then, $\psi_j(\lambda) := p' s_j v_j(\lambda) \neq 0$ for all $j \in \underline{N}$ and all $\lambda \in \mathbb{R} \setminus (\boldsymbol{\sigma}(A_j) \cup \{1\})$. Let $\bar{\lambda}_j \in \mathbb{R} \cup \{-\infty\}$ denote the greatest real eigenvalue[1] of A_j that is less than 1 and define $\bar{\lambda} := \max_{j\in\underline{N}} \bar{\lambda}_j$. Then each of the quantities $\psi_j(\lambda)$ has constant sign for $\bar{\lambda} < \lambda < 1$. If all the $\psi_j(\lambda)$ have the same sign for all $j \in \underline{N}$ (and all $\bar{\lambda} < \lambda < 1$), then there exist $T \in \mathbb{R}^{2\times 2}$ invertible and $K_j \in \mathbb{R}^{1\times 2}$ such that $\boldsymbol{\rho}(A_j + B_j K_j) < 1$ and $T^{-1}(A_j + B_j K_j)T$ is upper triangular for all $j \in \underline{N}$.

Proof: Since (A^c, B^c) is controllable and the sampling periods in $\mathcal{H}$ are non-pathological, then (A_j, B_j) are controllable. By Corollary 1, then $\mathcal{V}_j^d(\lambda)$ is continuous at $\lambda = 1$. Since the left limit (23) exists and $v_j(\lambda) \in \mathcal{V}_j^d(\lambda)$, then $v_j^- \in \mathcal{V}_j^d(1)$ by Theorem 5, and also $-v_j^- \in \mathcal{V}_j^d(1)$. By Lemma 5 we have $\mathcal{V}^c(0) = \mathcal{V}_j^d(1)$ for all $j \in \underline{N}$ because each h_j is non-pathological. Hence $v_j^- \in \mathcal{V}^c(0)$ for all $j \in \underline{N}$. Since $\dim(\mathcal{V}^c(0)) = m = 1$ and $\|v_j^-\| = 1$, then either $v_j^- = v_k^-$ or $v_j^- = -v_k^-$ whenever $j \neq k$. This establishes (24).

For a contradiction, suppose that $\psi_j(\lambda) = p' s_j v_j(\lambda) = 0$ for some $j \in \underline{N}$ and some $\lambda \in \mathbb{R} \setminus (\boldsymbol{\sigma}(A_j) \cup \{1\})$. Note that $\|s_j v_j(\lambda)\| = 1$ and hence $s_j v_j(\lambda) \neq 0$. Since $p \perp s_j v_j(\lambda)$ and $p \perp \mathcal{V}^c(0)$ with $p, v_j(\lambda) \in \mathbb{R}^2$, then $s_j v_j(\lambda) \in \mathcal{V}^c(0) = \mathcal{V}_j^d(1)$. Since also $s_j v_j(\lambda) \in \mathcal{V}_j^d(\lambda)$, we have $s_j v_j(\lambda) \in \mathcal{V}_j^d(1) \cap \mathcal{V}_j^d(\lambda)$. Recalling (24), this means that also $\bar{v} \in \mathcal{V}_j^d(1) \cap \mathcal{V}_j^d(\lambda)$ and hence

$$(\mathrm{I} - A_j)\bar{v} = B_j u \quad \text{and} \quad (\lambda \mathrm{I} - A_j)\bar{v} = B_j w, \tag{25}$$

for some $u, w \in \mathbb{R}$. Subtracting the above equations, we reach $(1 - \lambda)\bar{v} = B_j(u - w)$, which implies that $\bar{v} \in \operatorname{img} B_j$ because $\lambda \neq 1$. According to Lemma 4, this contradicts the fact that (A_j, B_j) is controllable. We have thus established that $\psi_j(\lambda) \neq 0$ for all $j \in \underline{N}$ and all $\lambda \in \mathbb{R} \setminus (\boldsymbol{\sigma}(A_j) \cup \{1\})$.

By definition of $\bar{\lambda}$, then $\psi_j(\lambda) \neq 0$ for all $j \in \underline{N}$ and all $\bar{\lambda} < \lambda < 1$. Also by definition, then $\psi_j(\lambda)$ is continuous for all $\bar{\lambda} < \lambda < 1$. Continuity and being not equal to 0 then imply that the sign of $\psi_j(\lambda)$ cannot change.

Every $s_j v_j(\lambda)$ is continuous, and $\psi_j(\lambda) = p' s_j v_j(\lambda)$ is nonzero and their signs coincide for all $\bar{\lambda} < \lambda < 1$. Also $\|s_j v_j(\lambda)\| = 1$, $\lim_{\lambda\to 1^-} s_j v_j(\lambda) = \bar{v}$ for all $j \in \underline{N}$, and $p'\bar{v} = 0$. Then, a simple geometrical diagram in $\mathbb{R}^2$ shows that there exist λ_j satisfying $\max\{-1, \bar{\lambda}\} < \lambda_j < 1$ so that $s_j v_j(\lambda_j) = s_k v_k(\lambda_k)$ for all $j, k \in \underline{N}$. By Lemma 4, $s_j v_j(\lambda_j) \notin \operatorname{img} B_j$. Also $\mathcal{V}_j^d(\lambda_j) = \operatorname{img}[s_j v_j(\lambda_j)]$ because $\dim(\mathcal{V}_j^d(\lambda_j)) = 1$. Then $\bigcap_{j\in\underline{N}} \mathcal{V}_j^d(\lambda_j) \not\subseteq \operatorname{img} B_k$ is established. The result follows by application of Lemma 6. ■

Proposition 1 generalizes Theorem 2 of [13] by avoiding the assumption that the continuous-time matrix A^c is non-singular. If A^c is non-singular, then $0 \notin \boldsymbol{\sigma}(A^c)$ and hence $1 \notin \boldsymbol{\sigma}(A_j)$ (provided h_j is non-pathological) so that the function $v_j(\lambda)$ is well-defined and differentiable at $\lambda = 1$. The proof of Proposition 1 is based on the continuity of $\mathcal{V}_j^d(\lambda)$ at $\lambda = 1$, and the fact that $\mathcal{V}_j^d(1) = \mathcal{V}^c(0)$ for all $j \in \underline{N}$ whenever the sampling periods are non-pathological (cf. Lemma 5). This proof avoids the use of the derivative of $v_j(\lambda)$ with respect to λ, as was the case in Theorem 2 of [13]. A reformulation of Proposition 1 might be possible by means of the concept of differentiability of subspaces [22], although this goes beyond the scope of this note.

[1] If A_j has no real eigenvalues less than 1, then $\bar{\lambda}_j := -\infty$.

V. Conclusions

We have provided a novel characterization of eigenvalue controllability for LTI finite-dimensional systems. This characterization relates eigenvalue controllability with the continuity of the map that assigns to each closed-loop eigenvalue the smallest subspace containing the set of corresponding eigenvectors. Continuity of this map is related to the convergence of sequences of feedback-assignable eigenvectors. Application of this concept was illustrated in an example of feedback stabilization in a controller-driven sampling setting, based on simultaneous triangularization (i.e. Lie-algebraic solvability).

References

[1] R. Kalman, "On the general theory of control systems," *Automatic and Remote Control, [Proc. 1st Int'l Congress, International Federation of Automatic Control (IFAC), Moscow, 1960]*, vol. 1, pp. 481–492, 1961.

[2] ——, "Contributions to the theory of optimal control," *Bol. Soc. Matem. Mexicana (Ser. 2)*, vol. 5, pp. 102–119, 1960.

[3] W. Wonham, "On pole assignment in multi-input controllable linear systems," *IEEE Trans. on Automatic Control*, vol. 12, no. 6, pp. 660–665, 1967.

[4] M. Hautus, "Controllability and observability conditions of linear autonomous systems," *Nederl. Akad. Wetensch., Proc. Ser. A 72*, pp. 443–448, 1969.

[5] ——, "Stabilization, controllability and observability of linear autonomous systems," *Indagationes Mathematicae*, vol. 73, pp. 448–455, 1970.

[6] D. Liberzon, *Switching in systems and control*. Boston, MA: Birkhauser, 2003.

[7] O. Karabacak and N. Şengör, "A dwell time approach to the stability of switched linear systems based on the distance between eigenvector sets," *Int. J. Systems Science*, vol. 40, no. 8, pp. 845–853, 2009.

[8] H. Lin and P. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Trans. on Automatic Control*, vol. 54, no. 2, pp. 308–322, 2009.

[9] Y. Kouhi and N. Bajcinca, "Nonsmooth control design for stabilizing switched linear systems by left eigenstructure assignment," in *18th IFAC World Congress, Milano, Italy*, 2011, pp. 380–385.

[10] D. Liberzon, J. Hespanha, and A. Morse, "Stability of switched systems: a Lie-algebraic condition," *Systems and Control Letters*, vol. 37, pp. 117–122, 1999.

[11] H. Haimovich and J. Braslavsky, "Feedback stabilization of switched systems via iterative approximate eigenvector assignment," in *Proc. 49th IEEE Conf. on Decision and Control, Atlanta, GA, USA*, 2010, pp. 1269–1274.

[12] H. Haimovich, J. Braslavsky, and F. Felicioni, "Feedback stabilisation of switching discrete-time systems via Lie-algebraic techniques," *IEEE Trans. on Automatic Control*, vol. 56, no. 5, pp. 1129–1135, 2011.

[13] H. Haimovich and E. Osella, "On controller-driven varying-sampling-rate stabilization via Lie-algebraic solvability," *Nonlinear Analysis: Hybrid Systems*, vol. 7, no. 1, pp. 28–38, 2013.

[14] K. Erdmann and M. Wildon, *Introduction to Lie algebras*. Springer-Verlag London, 2006.

[15] A. Agrachev, Y. Baryshnikov, and D. Liberzon, "On robust Lie-algebraic stability conditions for switched linear systems," *Systems and Control Letters*, vol. 61, pp. 347–353, 2012.

[16] H. Haimovich and J. Braslavsky, "Sufficient conditions for generic feedback stabilisability of switching systems via Lie-algebraic solvability," *IEEE Trans. on Automatic Control*, vol. 58, no. 3, pp. 814–820, 2013.

[17] H. Haimovich, "Characterization of controllability based on continuity of closed-loop eigenvectors," in *Congr. Argentino de Control Automático (AADECA), Buenos Aires, Argentina*, 2014.

[18] C.-T. Chen, *Linear system theory and design*, 3rd ed. Oxford University Press, 1999.

[19] W. M. Wonham, *Linear multivariable control: a geometric approach*, 3rd ed. New York: Springer-Verlag, 1985.

[20] I. Gohberg, P. Lancaster, and L. Rodman, *Invariant subspaces of matrices with applications*. John Wiley & Sons, Inc. New York, 1986.

[21] T. Chen and B. Francis, *Optimal sampled-data control systems*. Springer-Verlag, 1995.

[22] J. Ferrer, M. García, and F. Puerta, "Differentiable families of subspaces," *Linear Algebra and its Applications*, vol. 199, suppl. 1, pp. 229–252, 1994.