J. Math. Anal. Appl. 430 (2015) 417–427

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuous and localized Riesz bases for L^2 spaces defined by Muckenhoupt weights

Hugo Aimar [a], Wilfredo A. Ramos [b,c,*]

[a] Instituto de Matemática Aplicada del Litoral, IMAL (UNL-CONICET), CCT CONICET Santa Fe, Predio "Dr. Alberto Cassano", Colectora Ruta Nac. 168, Paraje El Pozo, 3000 Santa Fe, Argentina
[b] Instituto de Matemática Aplicada del Litoral, IMAL (UNL-CONICET), Argentina
[c] Departamento de Matemática, Facultad de Ciencias Exactas Naturales y Agrimensura, Universidad Nacional del Nordeste, Argentina

ARTICLE INFO

Article history:
Received 14 January 2015
Available online 6 May 2015
Submitted by R.H. Torres

Keywords:
Riesz bases
Haar wavelets
Basis perturbations
Muckenhoupt weights
Cotlar's Lemma

ABSTRACT

Let w be an A_∞-Muckenhoupt weight in $\mathbb{R}$. Let $L^2(wdx)$ denote the space of square integrable real functions with the measure $w(x)dx$ and the weighted scalar product $\langle f, g\rangle_w = \int_{\mathbb{R}} fg\ wdx$. By regularization of an unbalanced Haar system in $L^2(wdx)$ we construct absolutely continuous Riesz bases with supports as close to the dyadic intervals as desired. Also the Riesz bounds can be chosen as close to 1 as desired. The main tool used in the proof is Cotlar's Lemma.

1. Introduction and statement of the main result

A sequence $\{f_k, k \in \mathbb{Z}\}$ in a Hilbert space H is said to be a Bessel sequence with bound B if the inequality

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B\,\|f\|_H^2$$

holds for every $f \in H$. If $\{f_k, k \in \mathbb{Z}\}$ is a Bessel sequence with bound B and $\{e_k, k \in \mathbb{Z}\}$ is an orthonormal basis for the separable Hilbert space H, then the operator T on H defined by

$$Tf := \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle e_k$$

is bounded on H with bound $\sqrt{B}$. Conversely if T is bounded on H, then $\{f_k,\ k \in \mathbb{Z}\}$ is a Bessel sequence with bound $\|T\|^2$.

* Corresponding author.
E-mail addresses: haimar@santafe-conicet.gov.ar (H. Aimar), wramos@santafe-conicet.gov.ar (W.A. Ramos).

http://dx.doi.org/10.1016/j.jmaa.2015.05.003

When $\{f_k,\ k \in \mathbb{Z}\}$ itself is an orthonormal basis and $e_k = f_k$, T is the identity. Of particular interest is the case of $H = L^2$ when the Bessel system and the orthonormal basis are built on scaling and translations of the underlying space. In such cases the operator T has a natural decomposition as $T = \sum_{j\in\mathbb{Z}} T_j$. Sometimes the orthonormal basis can be chosen in such a way that the T_j's become almost orthogonal in the sense of Cotlar. We aim to use Cotlar's Lemma to produce smooth and localized Riesz bases for $L^2(\mathbb{R}, wdx)$ when w is a Muckenhoupt weight.

To introduce the problem let us start by some simple illustrations. Let ψ be a Daubechies compactly supported wavelet in $\mathbb{R}$. Assume that supp$\psi \subset [-N, N]$. The system $\{\tilde{\psi}^j_k(x) = 2^{\frac{j}{2}}\psi(2^j x^3 - k) :\ j, k \in \mathbb{Z}\}$ is a compactly supported orthonormal basis for $L^2(\mathbb{R}, 3x^2dx)$. More generally if $w(x)$ is a non-negative locally integrable function in $\mathbb{R}$ and $W(x) = \int_0^x w(y)dy$, then the system $\overline{\psi}^j_k(x) = 2^{\frac{j}{2}}\psi(2^j W(x) - k)$ is an orthonormal basis for $L^2(wdx)$. In fact, changing variables

$$\int_{\mathbb{R}} \overline{\psi}^j_k(x)\overline{\psi}^l_m(x)w(x)dx = 2^{\frac{l+j}{2}} \int_{\mathbb{R}} \psi(2^j W(x) - k)\psi(2^l W(x) - m)w(x)dx$$
$$= \int_{\mathbb{R}} \psi^j_k(z)\psi^l_m(z)dz$$

and we have the orthonormality of the system $\{\overline{\psi}^j_k : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ in $L^2(\mathbb{R}, wdx)$. As it is easy to verify in the case of $w(x) = 3x^2$, for j fixed the length of the supports of $\overline{\psi}^j_k$ tend to zero as $|k| \to +\infty$. On the other hand for $k = 0$ the scaling parameter is $2^{-\frac{1}{3}}$.

Notice also that if w is bounded above and below by positive constants the sequence $\overline{\psi}^j_k$ is an orthonormal basis for $L^2(wdx)$ with a metric control on the sizes of the supports provided by the scale.

A Riesz basis in $L^2(wdx)$ is a Schauder basis $\{f_k\}$ such that there exist two constants A and B called the Riesz bounds of $\{f_k\}$ for which

$$A\sum |c_k|^2 \le \left\|\sum c_k f_k\right\|^2_{L^2(wdx)} \le B\sum |c_k|^2$$

for every $\{c_k\}$ in $l^2(\mathbb{R})$, the space of square summable sequences of real numbers. In this note we aim to give sufficient conditions on a weight w defined on $\mathbb{R}$ more general than $0 < c_1 \le w(x) \le c_2 < \infty$, in order to construct, for every $\delta > 0$, a system $\Psi = \{\psi_I(x),\ I \in \mathcal{D}\}$ ($\mathcal{D}$ are the dyadic intervals in $\mathbb{R}$) with the following properties,

(i) Ψ is a Riesz basis for $L^2(wdx)$ with bounds $(1 - \delta)$ and $(1 + \delta)$,
(ii) each ψ^j_k is absolutely continuous,
(iii) for each I, ψ_I is supported on a neighborhood I^ϵ of I such that

$$0 < \frac{|I^\epsilon|}{|I|} - 1 < \delta.$$

As we have shown in the above example with $w(x) = 3x^2$, we have that $\{\overline{\psi}^j_k\}$ satisfies (i) and (ii) but not (iii).

An orthonormal basis in $L^2(\mathbb{R}, wdx)$ satisfying (iii) but not (ii) when w is locally integrable is the following unbalanced version of the Haar system (see [12]). Let $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}^j$ be the family of standard dyadic intervals in $\mathbb{R}$. Each I in $\mathcal{D}^j$ takes the form $I = [k2^{-j},\ (k + 1)2^{-j})$ for same integer k. For $I \in \mathcal{D}^j$ we have that $|I| = 2^{-j}$. We shall frequently use a_I and b_I to denote the left and right points of I respectively, for each $I \in \mathcal{D}$, define

$$h_I^w(x) = \frac{1}{\sqrt{w(I)}} \left\{ \sqrt{\frac{w(I_r)}{w(I_l)}} \chi_{I_l}(x) - \sqrt{\frac{w(I_l)}{w(I_r)}} \chi_{I_r}(x) \right\} \tag{1.1}$$

where $w(E) = \int_E w\ dx$, I_l is the left half of I and I_r is its right half. Notice that with the above notation h_I^w is the standard Haar basis h_I for $L^2(\mathbb{R})$ when $w = 1$.

The real numbers with the usual distance and measure $d\mu = wdx$ with w a Muckenhoupt weight, is a space of homogeneous type. Some constructions of wavelet type bases on spaces of homogeneous type are contained in [2] and [3]. Those in [2] are not regular and those in [3] are not compactly supported.

In this note we prove that the A_∞ Muckenhoupt condition on a weight w is sufficient for building a Riesz basis in $L^2(wdx)$ satisfying (i), (ii), and (iii).

Aside from Cotlar's Lemma, other fundamental tools we shall use are the basic properties of Muckenhoupt weights and a result due to Favier and Zalik [8] on small Bessel perturbations of Riesz bases.

In [10] N. Govil and R. Zalik gave a spline based regularization method of the Haar system to produce a regular and compactly supported Riesz basis with bounds as close to one as desired and supported on small neighborhoods of the dyadic intervals. In [1] the same type of result is obtained via regularizing by convolution. In both cases the main tool is contained in Theorem 5 in [8].

Let $1 < p < \infty$. A locally integrable nonnegative function w defined on $\mathbb{R}$ is said to be an A_p Muckenhoupt weight if there exists $C > 0$ such that

$$\left(\int_J wdx\right) \left(\int_J w^{-\frac{1}{p-1}} dx\right)^{p-1} \leq C\,|J|^p,$$

for every interval J. The class A_∞ is defined by $A_\infty = \cup_{1<p<\infty} A_p$.

The typical nontrivial examples of A_∞ weights are the powers of the distance to a fixed point. In particular $w(x) = |x|^\alpha$ belongs to A_∞ for every $\alpha > -1$. For the general theory of Muckenhoupt weights, introduced by B. Muckenhoupt in [11], see the book [9].

A simple and well known result for A_∞ weights that implies the doubling condition for the measure $w(x)dx$, due to B. Muckenhoupt, is the inequality

$$\left(\frac{|E|}{|J|}\right)^p \leq C \frac{w(E)}{w(J)} \tag{1.2}$$

which holds for some constant C and every measurable subset E of any interval J, provided that $w \in A_p$. From (1.2) it follows easily that the function $W(x) = \int_0^x w(y)dy$ defines a one to one and onto change of variables on $\mathbb{R}$ with Jacobian w. Set W^{-1} to denote the inverse function of W.

In order to produce a regularization of the system h_I^w given by (1.1) we first use the change of variables defined by W^{-1} to obtain another orthonormal basis $\{H_I^w\}$ in the spaces L^2 with respect to the translation invariant measure dx. Next we regularize by convolution with a smooth and compactly supported function φ the functions H_I^w to produce a Riesz basis for $L^2(\mathbb{R}, dx)$ which we shall denote by $\{H_I^{w,\epsilon}\}$. Finally in order to obtain the desired regularization $h_I^{w,\epsilon}$ of $\{h_I^w\}$ we go back to $L^2(\mathbb{R}, wdx)$ by reversing the change of variables induced by W^{-1}. Since the regularizing function φ can be assumed to be as smooth as desired, the regularity of each $h_I^{w,\epsilon}$ is only limited by the regularity of $W(x)$ which is at least locally absolutely continuous. Let us precisely define the three families $\{H_I^w\}$, $\{H_I^{w,\epsilon}\}$ and $\{h_I^{w,\epsilon}\}$.

For each $I \in \mathcal{D}$ set $H_I^w = h_I^w \circ W^{-1}$. Notice that

$$H_I^w(x) = \frac{1}{\sqrt{|I'|}} \left\{ \sqrt{\frac{|I'_r|}{|I'_l|}} \chi_{I'_l}(x) - \sqrt{\frac{|I'_l|}{|I'_r|}} \chi_{I'_r}(x) \right\} \tag{1.3}$$

where $I' = \{W(y),\ y \in I\}$. Now take a function φ to be C^∞, nonnegative, non-increasing to the right of 0, even and supported in $(-1,1)$ with $\int_{\mathbb{R}} \varphi = 1$. With the standard notation $\varphi_t(x) = \frac{1}{t}\varphi(\frac{x}{t})$, $t > 0$, define

$$H_I^{w,\epsilon}(x) = \left(\varphi_{\epsilon w(I)} * H_I^w\right)(x). \tag{1.4}$$

Finally, set $h_I^{w,\epsilon}(x) = \left(H_I^{w,\epsilon} \circ W\right)(x)$ for ϵ positive small enough.

The main result in this note is contained in the following statement.

Theorem 1.1. *Let w be a weight in $A_\infty(\mathbb{R})$. Then there exists $\epsilon_0 > 0$ depending only on w such that*

a) for each positive $\epsilon < \epsilon_0$, the system $\{h_I^{w,\epsilon},\ I \in \mathcal{D}\}$ is a Riesz basis for $L^2(wdx)$ of absolutely continuous functions,
b) the Riesz bounds of $\{h_I^{w,\epsilon},\ I \in \mathcal{D}\}$ can be taken as close to one as desired by taking ϵ small enough,
c) for each dyadic interval $I = [a_I, b_I]$ the support of $h_I^{w,\epsilon}$ is an interval $I^\epsilon = [a_I^\epsilon, b_I^\epsilon]$ with $a_I^\epsilon \nearrow a_I$, $b_I^\epsilon \searrow b_I$ when $\epsilon \to 0$ and for some constant C, $0 < \frac{|I^\epsilon|}{|I|} - 1 < C\epsilon^{\frac{1}{p}}$ if $w \in A_p$.

Let us point out that the regularity of each $h_I^{w,\epsilon}$ can be better than absolute continuity if w is smooth. In particular, when $w \equiv 1$ the functions $h_I^{w,\epsilon}$ are C^∞. In other words we get a basis for $L^2(dx)$ with full regularity and small supports. To get simultaneously these two properties we have to pay loosing orthogonality.

In Section 2 we give the basic result used in Section 3 in order to prove Theorem 1.1.

2. Preliminaries and basic results

In this section we introduce three basic results from functional and harmonic analysis which we shall use in Section 3 to prove Theorem 1.1. We shall refer to them as Coifman–Fefferman inequality, Cotlar's Lemma and Favier–Zalik stability, respectively.

Aside from (1.2) another important property of A_∞ weights that we shall use in the proof Theorem 1.1 is contained in the next statement which is proved as Theorem 2.9 on p. 401 in [9] and originally proved in [5].

Coifman–Fefferman. *If $w \in A_p$, $1 < p < \infty$ then there exist positive and finite constants C, γ such that the inequality*

$$\frac{w(E)}{w(J)} \le C\left(\frac{|E|}{|J|}\right)^\gamma \tag{2.1}$$

holds for every interval J and every measurable subset E of J.

The original proof of Cotlar's Lemma is contained in [6]. For more easily available proofs see [7] or [12].

Cotlar's Lemma. *Let $\{T_i:\ i \in \mathbb{Z}\}$ be a sequence of bounded operators in a Hilbert space H. Assume that they are almost orthogonal in the sense that there exists a sequence $s : \mathbb{Z} \to (0,\infty)$ with $\sum_{k\in\mathbb{Z}} \sqrt{s(k)} = A < \infty$ such that*

$$\|T_i^* T_j\| + \left\|T_i T_j^*\right\| \le s(i-j)$$

for every $i, j \in \mathbb{Z}$. Then

$$\left\| \sum_{i=-N}^{N} T_i \right\| \leq A$$

for every positive integer N.

The third result, due to S. Favier and R. Zalik, deals with the perturbation of Riesz bases and is contained in Theorem 5 of [8]. A basis $\{f_n\}$ for a Hilbert space H is said to be a Riesz basis with bounds A and B if and only if the inequalities

$$A\|f\|^2 \leq \sum |\langle f_n, f\rangle|^2 \leq B\|f\|^2$$

hold for every $f \in H$ (see, for example, Theorem 6.1.1 in [4]).

Favier–Zalik stability. *Let $\{f_n\}$ be a Riesz basis for a Hilbert space $\mathcal{H}$ with bounds A and B. Let $\{g_n\}$ be a sequence in $\mathcal{H}$ such that $\{f_n - g_n\}$ is a Bessel sequence with bound $M < A$. Then $\{g_n\}$ is a Riesz basis with bound $[1 - (\frac{M}{A})^{\frac{1}{2}}]^2 A$ and $[1 - (\frac{M}{B})^{\frac{1}{2}}]^2 B$.*

The next lemma is a consequence of (1.2). It will be crucial in the proof of Theorem 1.1.

Lemma 2.1. *Let w be a weight in A_p. For a given dyadic interval I, set a_I, b_I and c_I to denote the left endpoint of I, the right endpoint of I and the center of I respectively. As before I_l and I_r denote the left and right halves of I. Then*

a) with C the constant in (1.2) and $\epsilon < (\frac{1}{2})^p \frac{1}{2C}$ we have that $2\epsilon w(I) < w(I_l)$ and $2\epsilon w(I) < w(I_r)$;
b) with C as above and $\epsilon < \frac{1}{C}\frac{1}{3^p}$ we also have that $\sum_{I\in\mathcal{D}^j} \chi_{W^\epsilon(I)}(x) \leq 2$ for every $j \in \mathbb{Z}$, where $W^\epsilon(I)$ is the $\epsilon w(I)$ neighborhood of the interval $W(I)$, in other words $W^\epsilon(I) = (W(a_I) - \epsilon w(I), W(b_I) + \epsilon w(I))$.

Proof. *a*) Using (1.2) with $J = I$, $E = I_l$ we obtain

$$\frac{w(I_l)}{w(I)} \geq \frac{1}{C}\left(\frac{|I_l|}{|I|}\right)^p = \frac{1}{C2^p} > 2\epsilon.$$

The same inequality is true for I_r instead of I_l.

b) Let us consider I, K and J three consecutive intervals in $\mathcal{D}^j$ with $b_I = a_K$ and $b_K = a_J$. Let M be the interval obtained as the union of I, J and K. From (1.2) we see that

$$\epsilon < \frac{1}{C}\frac{1}{3^p} = \frac{1}{C}\left(\frac{|K|}{|M|}\right)^p \leq \frac{w(K)}{w(M)}$$

Hence $\epsilon(w(I)+w(J)) \leq \epsilon w(M) < w(K) = W(a_J) - W(b_I)$, so that $W(b_I) + \epsilon w(I) < W(a_J) - \epsilon w(J)$. Then, no point $x \in \mathbb{R}$ can belong to more than two of the intervals W_I^ϵ. □

3. Proof of Theorem 1.1

Throughout this section w is a weight in $A_p(\mathbb{R})$ for some $1 < p < \infty$. We shall use the standard inner product notation $\langle\cdot,\cdot\rangle$ for the scalar product in $L^2(dx)$. We shall write $\langle\cdot,\cdot\rangle_w$ to denote the inner product in $L^2(wdx)$.

Notice first that $\{h_I^w : I \in \mathcal{D}\}$ defined in (1.1) is an orthonormal basis for $L^2(\mathbb{R}, wdx)$. For $j \in \mathbb{Z}$, set

$$\mathcal{V}_j = \{f \in L^2(wdx) : f \text{ is constant on each } I \in \mathcal{D}^j\},$$

and observe that $\bigcup_{j\in\mathbb{Z}}\mathcal{V}_j$ is dense in $L^2(wdx)$. By (2.1) wdx is doubling and hence $\int_{\mathbb{R}} w=\infty$. Thus, we have $\bigcap_{j\in\mathbb{Z}}\mathcal{V}_j=\{0\}$. For $I\in\mathcal{D}$ fixed, the two dimensional vector space of those functions f defined on I which are constant on each half I_l and I_r of I has $\{\frac{\chi_I}{\sqrt{w(I)}},\ h_I^w\}$ as an orthonormal basis with the $L^2(wdx)$ inner product. For $j\in\mathbb{Z}$, we define $\mathcal{W}_j$ as the $L^2(wdx)$ orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. In other words, as usual, $\mathcal{V}_{j+1}=\mathcal{V}_j\oplus\mathcal{W}_j$.

From the above mentioned properties of the multiresolution $\{\mathcal{V}_j:\ j\in\mathbb{Z}\}$ we see that

$$L^2(wdx)=\bigoplus_{j\in\mathbb{Z}}\mathcal{W}_j.$$

Since, for $j\in\mathbb{Z}$ fixed, the family $\{h_I^w:\ I\in\mathcal{D}^j\}$ is an orthonormal basis of $\mathcal{W}_j$ we get that $\{h_I^w:\ I\in\mathcal{D}\}$ is an orthonormal basis for $L^2(wdx)$.

Given a set $E\subset\mathbb{R}$ we shall write E' to denote the image of E by W. In other words $E'=\{W(x),\ x\in E\}$. We write $\mathcal{D}'=\bigcup_{j\in\mathbb{Z}}\mathcal{D}'_j$ to denote the family of all the images I' of intervals $I\in\mathcal{D}$ through W, here $\mathcal{D}$ denote the family of all dyadic intervals in $\mathbb{R}$ defined above. Notice that $|I'|=w(I)$.

For each $I\in\mathcal{D}$ we shall use H_I^w to denote the composition $h_I^w\circ W^{-1}$. It is easy to see that $H_I^w(x)=\frac{1}{\sqrt{|I'|}}\left\{\sqrt{\frac{|I'_r|}{|I'_l|}}\chi_{I'_l}(x)-\sqrt{\frac{|I'_l|}{|I'_r|}}\chi_{I'_r}(x)\right\}$ and that $\{H_I^w,\ I\in\mathcal{D}\}$ is an orthonormal basis of $L^2(\mathbb{R},dx)$. In fact, for $f\in L^2(dx)$ we have $\langle f,H_I^w\rangle=\langle f\circ W,h_I^w\rangle_w$ for every $I\in\mathcal{D}$. Moreover

$$\sum_{I\in\mathcal{D}}|\langle f,H_I^w\rangle|^2=\sum_{I\in\mathcal{D}}|\langle f\circ W,h_I^w\rangle_w|^2=\|f\circ W\|^2_{L^2(wdx)}=\|f\|^2_{L^2(dx)}.$$

Next we regularize by convolution the function H_I^w for $I\in\mathcal{D}$ in order to get $H_I^{w,\epsilon}$, defined by $H_I^{w,\epsilon}=\varphi_{\epsilon w(I)}*H_I^w$. Here $I\in\mathcal{D}$, φ is as described in the introduction, and ϵ is as in Lemma 2.1.

We prove $a)$ in Theorem 1.1 by applying the Favier–Zalik stability result. We shall estimate the Bessel bound in $L^2(dx)$ for the difference $b_I^\epsilon=H_I^w-H_I^{w,\epsilon}$ between the basic element H_I^w and its regularization $H_I^{w,\epsilon}$.

We use the strategy described in the introduction, taking as $\{f_k\}$ the sequence $\{b_I^\epsilon\}$ and as the orthonormal basis $\{e_k\}$ the sequence H_I^w. Precisely, define

$$T_\epsilon f=\sum_{I\in\mathcal{D}}\langle f,b_I^\epsilon\rangle H_I^w$$

and $T_jf=\sum_{J\in\mathcal{D}^j}\langle f,b_J^\epsilon\rangle H_J^w$, thus $T_\epsilon=\sum_j T_j$. To prove that $\{b_I^\epsilon:\ I\in\mathcal{D}\}$ is a Bessel sequence with small bound, we apply Cotlar's Lemma to the sequence $\{T_j\}$ of operators in $L^2(\mathbb{R})$. We begin by estimating $\|T_i^*T_j\|$ and $\|T_iT_j^*\|$ where T_j^* is the adjoint of T_j,

$$T_j^*f=\sum_{J\in\mathcal{D}^j}\langle f,H_J^w\rangle b_J^\epsilon.$$

Since the family $\{H_I^w,I\in\mathcal{D}\}$ is orthonormal, for $i\neq j$ we have $T_i^*T_jf=\sum_{J\in\mathcal{D}^j,I\in\mathcal{D}^i}\langle f,b_J^\epsilon\rangle\langle H_J^w,H_I^w\rangle b_I^\epsilon=0$. On the other hand, for $i=j$, $\|T_j^*T_j\|=\|T_j\|^2$ and $\|T_jf\|_2^2=\sum_{J\in\mathcal{D}^j}|\langle f,b_J^\epsilon\rangle|^2$.

Since H_J^w is piecewise constant, for ϵ small enough the support of b_J^ϵ splits into three intervals, each of them centered at the images through W of the two endpoints a_J, b_J of J and of its center c_J. All of them have the same length $2\epsilon w(J)$. Precisely, with $S_J^\epsilon=\operatorname{supp}\,b_J^\epsilon$ we have that $S_J^\epsilon=\bigcup_{m=1}^3S_J^{\epsilon,m}$, where $S_J^{\epsilon,1}=(W(a_J)-w(J)\epsilon,\ W(a_J)+w(J)\epsilon)$, $S_J^{\epsilon,2}=(W(c_J)-w(J)\epsilon,\ W(c_J)+w(J)\epsilon)$ and $S_J^{\epsilon,3}=(W(b_J)-w(J)\epsilon,\ W(b_J)+w(J)\epsilon)$.

$$|\langle f, b_J^\epsilon\rangle|^2 \leq \left(\int_{S_J^\epsilon} |f|^2\right)\left(\int |b_J^\epsilon|^2\right).$$

In order to estimate $\int |b_I^\epsilon|^2$, let us first notice that $|b_I^\epsilon| \leq |H_I^w| + |H_I^{w,\epsilon}| \leq 2|H_I^w| \leq \frac{2}{\sqrt{w(I)}} \max\{\sqrt{\frac{w(I_r)}{w(I_l)}}, \sqrt{\frac{w(I_l)}{w(I_r)}}\}$, which is bounded by a constant C, depending only on w, times $w(I)^{-\frac{1}{2}}$. Then $\int |b_I^\epsilon|^2 \leq \frac{C^2}{w(I)} |S_I^\epsilon| = 6C^2\epsilon$.

Then, from $b)$ in Lemma 2.1 we have

$$\|T_j f\|_2^2 \leq 6C^2\epsilon \sum_{J\in\mathcal{D}^j} \int_{S_J^\epsilon} |f|^2 \leq 6C^2\epsilon \sum_{J\in\mathcal{D}^j} \int_{W^\epsilon(J)} |f|^2$$

$$\leq 6C^2\epsilon \int_{\mathbb{R}} \left(\sum_{J\in\mathcal{D}^j} \chi_{W^\epsilon(J)}\right) |f|^2 \leq 12C^2\epsilon \|f\|_2^2.$$

Hence $\|T_j^* T_j\| = \|T_j\|^2 \leq 12C^2\epsilon$, and since $\|T_i^* T_j\| = 0$ for $i \neq j$, any $s(k)$ with $s(0) \leq 12C^2\epsilon$ and $s(k) \geq 0$ for $k \neq 0$ is admissible for the estimate $\|T_i^* T_j\| \leq s(i-j)$ required by Cotlar's Lemma.

The behavior of the sequence $\|T_i T_j^*\|$ is more subtle since $T_i T_j^* f = \sum_{I\in\mathcal{D}^i}\sum_{J\in\mathcal{D}^j} \langle f, H_J^w\rangle\langle b_J^\epsilon, b_I^\epsilon\rangle H_I^w$, and now the functions b_J^ϵ are not orthogonal. In this case the Lipschitz smoothness of each b_J^ϵ away from its points of discontinuity, and its mean vanishing properties will play essential roles. These two properties are made precise in the following claims, which we proof later.

Claim 1. *For each $I \in \mathcal{D}$ with $I = [a, b)$ centered at c_I, on each one of the segments $\sigma_1 = (-\infty, W(a))$, $\sigma_2 = (W(a), W(c_I))$, $\sigma_3 = (W(c_I), W(b))$ and $\sigma_4 = (W(b), \infty)$ the function b_I^ϵ is Lipschitz with norm bounded by a constant times $(\epsilon w(I))^{-\frac{3}{2}}$.*

Claim 2. *On each one of the three connected components $S_I^{\epsilon,m}$ of its support we have $\int_{S_I^{\epsilon,m}} b_I^\epsilon = 0$, $m = 1, 2, 3$.*

Let us assume Claims 1 and 2 and continue the proof.

To estimate $\|T_i T_j^*\|$, observe that, since $\{H_I^w,\ I \in \mathcal{D}\}$ is an orthonormal basis, we have

$$\|T_i T_j^* f\|_2^2 = \sum_{I\in\mathcal{D}^i}\left(\sum_{J\in\mathcal{D}^j} \langle f, H_J^w\rangle\langle b_I^\epsilon, b_J^\epsilon\rangle\right)^2. \tag{3.1}$$

Assume first that $j > i$. For a fixed $I \in \mathcal{D}^i$, we consider the partition of $\mathcal{D}^j$ provided by the three sets, $\mathcal{A}(I) = \{J \in \mathcal{D}^j :\ S_J^\epsilon \cap S_I^\epsilon = \emptyset\}$; $\mathcal{B}(I) = \{J \in \mathcal{D}^j \setminus \mathcal{A}(I)\ : b_I^\epsilon$ is continuous and not identically zero on $S_J^\epsilon\}$ and $\mathcal{C}(I) = \mathcal{D}^j \setminus (\mathcal{A}(I) \cup \mathcal{B}(I))$. Since for $J \in \mathcal{A}(I)$ we have that $\langle b_I^\epsilon, b_J^\epsilon\rangle = 0$, then

$$\|T_i T_j^* f\|_2^2 = \sum_{I\in\mathcal{D}^i}\left(\sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)} \langle f, H_J^w\rangle\langle b_I^\epsilon, b_J^\epsilon\rangle\right)^2$$

$$\leq \sum_{I\in\mathcal{D}^i}\left(\sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)} |\langle f, H_J^w\rangle|^2\right)\left(\sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2\right)$$

$$= \sum_{I\in\mathcal{D}^i}\left(\sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)} |\langle f, H_J^w\rangle|^2\right)\left(\sum_{J\in\mathcal{C}(I)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2\right)$$

$$+ \sum_{I\in\mathcal{D}^i}\left(\sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)}|\langle f,H_J^w\rangle|^2\right)\left(\sum_{J\in\mathcal{B}(I)}|\langle b_I^\epsilon,b_J^\epsilon\rangle|^2\right)$$
$$= I_1 + I_2.$$

In order to estimate I_1 notice that $\mathcal{C}(I)$ has at most six elements. On the other hand, from (2.1)

$$|\langle b_I^\epsilon,b_J^\epsilon\rangle| \le \int_{S_J^\epsilon}|b_I^\epsilon(x)|\,|b_J^\epsilon(x)|\,dx$$
$$\le C\frac{\epsilon w(J)}{(w(I)w(J))^{\frac{1}{2}}} \le C\epsilon\frac{1}{2^{(j-i)\frac{\gamma}{2}}},$$

hence

$$I_1 \le C\epsilon^2 2^{-\gamma(j-i)}\sum_{I\in\mathcal{D}^i}\sum_{j\in\mathcal{B}(I)\cup\mathcal{C}(I)}|\langle f,H_J^w\rangle|^2$$
$$\le C\epsilon^2 2^{-\gamma(j-i)}\sum_{J\in\mathcal{D}^j}|\langle f,H_J^w\rangle|^2\,\sharp\{I\in\mathcal{D}^i:\ J\notin\mathcal{A}(I)\} \le C\epsilon^2 2^{-\gamma(j-i)}\|f\|_2^2,$$

which has again the desired form to apply Cotlar's Lemma with $s(j-i) = C\epsilon 2^{-\frac{\gamma}{2}(j-i)}$.

For a given interval I, set $\tilde{I}$ to denote the concentric with I and twice its length. Since for $J\in\mathcal{B}(I)$ the function b_I^ϵ is Lipschitz on the support of b_J^ϵ, if x_J^m is the center of the m-th connected component of the support of b_J^ϵ, from Claims 2 and 1 and applying again (2.1) we get

$$\sum_{J\in\mathcal{B}(I)}|\langle b_I^\epsilon,b_J^\epsilon\rangle|^2 = \sum_{J\in\mathcal{B}(I)}\left|\sum_{m=1}^{3}\int_{S_J^{\epsilon,m}}b_J^\epsilon(x)\,(b_I^\epsilon(x)-b_I^\epsilon(x_J^m))\,dx\right|^2$$
$$\le \sum_{J\in\mathcal{B}(I)}\frac{C}{(\epsilon w(I))^3}\left(\sum_{m=1}^{3}\int_{S_J^{\epsilon,m}}|b_J^\epsilon(x)|\,|x-x_J^m|\,dx\right)^2$$
$$\le C\sum_{I\in\mathcal{B}(I)}\frac{1}{\epsilon^3 w(I)^3}|S_J^\epsilon|^2\frac{1}{w(J)}\epsilon^2 w(J)^2$$
$$\le C\epsilon\sum_{J\in\mathcal{B}(I)}\left(\frac{w(J)}{w(I)}\right)^2\frac{w(J)}{w(I)}$$
$$\le C\epsilon\sum_{J\in\mathcal{B}(I)}\left(\frac{|J|}{|I|}\right)^{2\gamma}\frac{1}{w(I)}\int_J w(x)dx$$
$$\le C\epsilon\left(\frac{1}{2}\right)^{2(j-i)\gamma}\frac{1}{w(I)}\int_{\mathbb{R}}\sum_{J\in\mathcal{B}(I)}\chi_J(x)w(x)dx$$
$$\le C\epsilon\left(\frac{1}{2}\right)^{2\gamma(j-i)}\frac{w(\tilde{I})}{w(I)}$$
$$\le C\epsilon\left(\frac{1}{2}\right)^{2\gamma(j-i)}$$

So that, for $j > i$

$$\sum_{J\in\mathcal{B}(I)}|\langle b_I^\epsilon,b_J^\epsilon\rangle|^2 \le C\epsilon 2^{-2(j-i)\gamma} \tag{3.2}$$

hence

$$I_2 \leq C\epsilon 2^{-2(j-i)\gamma} \sum_{I\in\mathcal{D}^i} \sum_{J\in\mathcal{B}(I)\cup\mathcal{C}(I)} |\langle f, H_J^w\rangle|^2 \leq C\epsilon 2^{-2(j-i)\gamma} \|f\|_2^2,$$

finally

$$\left\|T_i T_j^* f\right\|_2^2 \leq I_1 + I_2 \leq C\epsilon 2^{-\gamma(j-i)} \|f\|_2^2.$$

Hence, for $j > i$ taking $s(j-i) = C\epsilon^{\frac{1}{2}} 2^{-\frac{\gamma}{2}(j-i)}$ we have a good sequence in order to use Cotlar's Lemma.

For $i \geq j$, with the above notation for $J \in \mathcal{D}^j$ given, we have the three classes $\mathcal{A}(J)$, $\mathcal{B}(J)$ and $\mathcal{C}(J)$,

$$\begin{aligned}
\left\|T_i T_j^* f\right\|_2^2 &\leq C \sum_{I\in\mathcal{D}^i} \left(\sum_{\{J\in\mathcal{D}^j /\ S_I^\epsilon \cap S_J^\epsilon \neq \emptyset\}} |\langle f, H_J^w\rangle|^2 |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \right) \\
&\leq C \sum_{J\in\mathcal{D}^j} |\langle f, H_J^w\rangle|^2 \left(\sum_{I\in\mathcal{C}(J)\cup\mathcal{B}(J)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \right) \\
&\leq C \sum_{J\in\mathcal{D}^j} |\langle f, H_J^w\rangle|^2 \left(\sum_{I\in\mathcal{C}(J)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \right) + C \sum_{J\in\mathcal{D}^j} |\langle f, H_J^w\rangle|^2 \left(\sum_{I\in\mathcal{B}(J)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \right).
\end{aligned}$$

For the first term, notice that if $I \in \mathcal{C}(J)$, we obtain from (2.1) as before

$$\begin{aligned}
|\langle b_I^\epsilon, b_J^\epsilon\rangle| &\leq \int_{S_I^\epsilon} |b_J^\epsilon(x)|\, |b_I^\epsilon(x)|\, dx \\
&\leq C \frac{\epsilon w(I)}{w(J)^{\frac{1}{2}} w(I)^{\frac{1}{2}}} \leq C\epsilon 2^{-(i-j)\frac{\gamma}{2}},
\end{aligned}$$

since the number of elements in $\mathcal{C}(J)$ is bounded we get that

$$\sum_{J\in\mathcal{D}^j} |\langle f, H_J^w\rangle|^2 \left(\sum_{I\in\mathcal{C}(J)} |\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \right) \leq C\epsilon^2 2^{-\gamma(i-j)} \|f\|_2^2.$$

For the second term observe that if $I \in \mathcal{B}(J)$ and y_I^m is the center of the interval $S_I^{\epsilon,m}$, since the integral of b_I^ϵ vanishes on each connected component $S_I^{\epsilon,m}$, we have

$$|\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 \leq \left(\left| \sum_{m=1}^{3} \int_{S_I^{\epsilon,m}} b_I^\epsilon(y)\, (b_J^\epsilon(y) - b_J^\epsilon(y_I^m))\, dy \right| \right)^2,$$

then, from Claim 1,

$$\begin{aligned}
|\langle b_I^\epsilon, b_J^\epsilon\rangle|^2 &\leq \left(\frac{C}{\epsilon^{\frac{3}{2}} w(J)^{\frac{3}{2}}} \sum_{m=1}^{3} \int_{S_I^{\epsilon,m}} |b_I^\epsilon(y)|\, |y - y_I^m|\, dy \right)^2 \\
&\leq \left(\frac{3C\epsilon w(I)\, |S_I^\epsilon|}{\epsilon^{\frac{3}{2}} w(J)^{\frac{3}{2}} w(I)^{\frac{1}{2}}} \right)^2 \leq C\epsilon \left(\frac{w(I)}{w(J)} \right)^3.
\end{aligned}$$

Hence

$$\left\|T_iT_j^*f\right\|_2^2 \leq C\epsilon^2 2^{-(i-j)\gamma}\left\|f\right\|_2^2 + C\epsilon 2^{-(i-j)2\gamma}\sum_{J\in\mathcal{D}^j}\left|\langle f, H_J^w\rangle\right|^2\left(\frac{1}{w(J)}\sum_{I\in\mathcal{B}(J)}w(I)\right)$$
$$\leq C\epsilon^2 2^{-(i-j)\gamma}\left\|f\right\|_2^2 + C\epsilon 2^{-(i-j)2\gamma}\left\|f\right\|_2^2.$$

Then $\left\|T_iT_j^*\right\| \leq C\epsilon^{\frac{1}{2}}2^{-\frac{\gamma}{2}(i-j)}$, for $i \geq j$.

So far we have the hypotheses of Cotlar's Lemma for the sequence $\{T_j\}$ with $s(k) = C\epsilon^{\frac{1}{2}}2^{-\frac{\gamma}{2}|k|}$, $k \in \mathbb{Z}$. Then $\|T_\epsilon\| \leq C\epsilon^{\frac{1}{4}}$, $0 < \epsilon < \epsilon_0 = \min\{\frac{2-p}{2C}, \frac{3-p}{C}\}$ where C is the constant in (1.2). Now from the Favier–Zalik stability lemma, we get that $\{H_I^{w,\epsilon} : I \in \mathcal{D}\}$ is a Riesz basis for $L^2(\mathbb{R}, dx)$ with bounds $\left(1-\sqrt{C\epsilon^{\frac{1}{4}}}\right)^2$ and $\left(1+\sqrt{C\epsilon^{\frac{1}{4}}}\right)^2$. Since $h_I^{w,\epsilon} = H_I^{w,\epsilon}\circ W$ and for $f \in L^2(wdx)$ we have the identity

$$\sum_{I\in\mathcal{D}}\langle f, h_I^{w,\epsilon}\rangle_w^2 = \sum_{I\in\mathcal{D}}\langle f\circ W^{-1}, H_I^{w,\epsilon}\rangle^2$$

we immediately see that $\{h_I^{w,\epsilon} : I \in \mathcal{D}\}$ is a Riesz basis for $L^2(\mathbb{R}, wdx)$ with bounds $\left(1\pm\sqrt{C\epsilon^{\frac{1}{4}}}\right)^2$. This proves a).

The absolute continuity of each $h_I^{w,\epsilon}$ follows from the regularity of $H_I^{w,\epsilon}$ and the absolute continuity of W. Part b) in the statement of Theorem 1.1 follows directly from the Riesz bounds for $\{h_I^{w,\epsilon} : I \in \mathcal{D}\}$ obtained before.

Let us prove c). With a_I and b_I the left and right endpoint of I we have that the support of $h_I^{w,\epsilon}$ is the interval $I_\epsilon = [W^{-1}(W(a_I) - \epsilon w(I)),\ W^{-1}(W(b_I) + \epsilon w(I))] = [a_I^\epsilon, b_I^\epsilon]$ containing I. Notice that since $W(a_I) - W(a_I^\epsilon) = \epsilon w(I)$ and $W(b_I^\epsilon) - W(b_I) = \epsilon w(I)$, from the continuity of W^{-1} it follows that $a_I^\epsilon \to a_I$ and $b_I^\epsilon \to b_I$ when $\epsilon \to 0$. A more quantitative estimate of the rate of approximation can be obtained using again (1.2). In fact, set I^* to denote the interval concentric with I with three times its length. Let J be the interval $[a_I^\epsilon, a_I]$, then from (1.2)

$$\frac{a_I - a_I^\epsilon}{3|I|} = \frac{|J|}{|I^*|} \leq C\left(\frac{w(J)}{w(I^*)}\right)^{\frac{1}{p}} = C\left(\frac{\epsilon w(I)}{w(I^*)}\right)^{\frac{1}{p}} \leq C\epsilon^{\frac{1}{p}}.$$

In a similar way $\frac{b_I^\epsilon - b_I}{|I|} \leq C\epsilon^{\frac{1}{p}}$. Hence $\frac{|I_\epsilon|}{|I|} = 1 + \frac{a_I - a_I^\epsilon}{|I|} + \frac{b_I^\epsilon - b_I}{|I|}$ and $0 < \frac{|I_\epsilon|}{|I|} - 1 < C\epsilon^{\frac{1}{p}}$ where C depends on the A_p constant of w. Notice that the rate of approximation is better as p tends to 1.

Let us finally prove Claims 1 and 2.

Proof of Claim 1. Since for $x, y \in \sigma_i$, $i = 1, \ldots, 4$ we have that $H_I^w(x) = H_I^w(y)$, then

$$\left|b_I^\epsilon(x) - b_I^\epsilon(y)\right| = \left|H_I^{w,\epsilon} * \varphi_{\epsilon w(I)}(x) - H_I^{w,\epsilon} * \varphi_{\epsilon w(I)}(y)\right|$$
$$= \left|\int_{\mathbb{R}}\frac{H_I^w(z)}{\epsilon w(I)}\left(\varphi\left(\frac{x-z}{\epsilon w(I)}\right) - \varphi\left(\frac{y-z}{\epsilon w(I)}\right)\right)dz\right|.$$

Since φ is smooth, applying the mean value theorem we get that

$$\left|b_I^\epsilon(x) - b_I^\epsilon(y)\right| \leq \frac{\|\varphi'\|_\infty}{\epsilon^2 w(I)^2}|x-y|\int_{\{|x-z|\leq\epsilon w(I)\}\cup\{|y-z|\leq\epsilon w(I)\}}\left|H_I^w(z)\right|dz$$
$$\leq c\frac{\|\varphi'\|_\infty}{(\epsilon w(I))^{\frac{3}{2}}}|x-y|$$

as desired. $\square$

Proof of Claim 2. It is easy to see that $\int b_I^\epsilon \, dx = 0$. In fact, we can see from (1.3)

$$\sqrt{|I'|}\int_{I'} H_I^w(x)dx = \frac{\sqrt{|I'_r|}}{\sqrt{|I'_l|}}\int_{I'}\chi_{I'_l}(x)dx - \frac{\sqrt{|I'_l|}}{\sqrt{|I'_r|}}\int_{I'}\chi_{I'_r}(x)dx$$
$$= \frac{\sqrt{|I'_r|}}{\sqrt{|I'_l|}}|I'_l| - \frac{\sqrt{|I'_l|}}{\sqrt{|I'_r|}}|I'_r| = 0.$$

On the other hand, since $\int \varphi(z)dz = 1$, we also have that $\int H_I^{w,\epsilon}dx = 0$.

Notice that, after normalization, $\int_{S_I^{\epsilon,1}} b_I^\epsilon dx = 0$ since $\int_{-\delta}^{\delta}[\chi_{(0,\infty)}(x) - \left(\chi_{(0,\infty)} * \varphi_\delta\right)(x)]dx = 0$ for $\delta > 0$. Since a similar argument proves that $\int_{S_I^{\epsilon,3}} b_I^\epsilon dx = 0$ and $\int b_I^\epsilon = 0$, we also have $\int_{S_I^{\epsilon,2}} b_I^\epsilon dx = 0$. □

References

[1] H.A. Aimar, A.L. Bernardis, O.P. Gorosito, Perturbations of the Haar wavelet by convolution, Proc. Amer. Math. Soc. 129 (12) (2001) 3619–3621.

[2] H. Aimar, A. Bernardis, B. Iaffei, Multiresolution approximations and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type, J. Approx. Theory 148 (1) (2007) 12–34.

[3] P. Auscher, T. Hytönen, Orthonormal bases of regular wavelets in spaces of homogeneous type, Appl. Comput. Harmon. Anal. 34 (2) (2013) 266–296.

[4] O. Christensen, An Introduction to Frames and Riesz Bases, Appl. Numer. Harmon. Anal., Birkhäuser, 2003.

[5] R.R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, Studia Math. 51 (1974) 241–250.

[6] M. Cotlar, A combinatorial inequality and its applications to L^2-spaces, Rev. Mat. Cuyana 1 (1956) 41–55, 1955.

[7] M. De Guzmán, Real Variable Methods in Fourier Analysis, North-Holl. Math. Stud., vol. 46, North-Holland Publishing Co., Amsterdam, 1981. Notas de Matemática [Mathematical Notes], 75.

[8] S.J. Favier, R.A. Zalik, On the stability of frames and Riesz bases, Appl. Comput. Harmon. Anal. 2 (2) (1995) 160–173.

[9] J. García-Cuerva, J.L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holl. Math. Stud., vol. 116, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[10] N.K. Govil, R.A. Zalik, Perturbations of the Haar wavelet, Proc. Amer. Math. Soc. 125 (11) (1997) 3363–3370.

[11] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc. 165 (1972) 207–226.

[12] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton Math. Ser., vol. 43, Princeton University Press, Princeton, NJ, 1993, with the assistance of Timothy S. Murphy. Monographs in Harmonic Analysis, III.