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A procedure to solve a singular stochastic optimal control problem

Laura S. Aragone[1] and Elina M. Mancinelli[2]

[1]CONICET, Pellegrini 250 - 2000 Rosario,Argentina. - laura@fceia.unr.edu.ar
Supported by grant CONICET N∘ 5379, Argentina.

[2]CONICET and Dpto. de Matemática FCFMN, UNSL. Ejército de los Andes 950 (5700), Argentina. - emancin@unsl.edu.ar

Abstract. In this work we deal with a singular stochastic optimal control problem. We present a theoretical iterative method which converges to the analytical solution and we also present a discretization procedure to obtain an approximated solution. We establish the convergence of the discrete solution to the value function and give an example of application with the numerical results.

Keywords. singular controls, Hamilton-Jacobi-Bellman equation, viscosity solution, iterative constructive method, numerical solution

AMS (MOS) subject classification: 49L20, 49L25, 49M25, 93E20

1 Introduction

In this paper we consider a stochastic control problem where the state is governed by the following stochastic differential equation

$$x_t = x + \int_s^t b(\theta, x_\theta, u_\theta)d\theta + \int_s^t \sigma(\theta, x_\theta, u_\theta)dB_\theta + \int_s^t g(\theta)dv_\theta. \tag{1}$$

We denote with $(\Xi, F, F_t, \mathbf{P})$ the probability framework, where F_t is an increasing set of σ-algebras defined on Ξ, $F = \bigcup_t F_t$ and $\mathbf{P}$ is a probability measure defined on the elements of F. b, σ, g are deterministic functions and $(B_t, t \geq 0)$ is a d-dimensional Brownian motion, x is the initial position of the system at time s, and the controls are $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where

$$\mathcal{U} = \{u : [0,T] \to U \subset \mathbb{R}^c : u \text{ non anticipative w.r.t. } F_t\} \ ,$$

$$\mathcal{V} = \left\{v : [0,T] \to \mathbb{R}_+^k : \forall p = 1, ..., k, \ v_p \text{ non decr., non anticip. w.r.t.} F_t\right\} \ .$$

The expected cost for each pair of controls has the form

$$J(s, x, u, v) = E\left\{\int_s^T f(t, x_t, u_t)dt + \int_s^T c(t)dv_t\right\},$$

where $f : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$, $c : [0,T] \to \mathbb{R}_+^k$ are given. We define the cost function W as

$$W(t,x) = \inf\{J(t,x,u,v) : u \in \mathcal{U}, v \in \mathcal{V}\}\,.$$

We suppose that the cost associated to apply the singular control v is positive, i.e. $c^i(\cdot) > 0$, $i = 1, ..., k$. For this kind of problem we can see [8] and the bibliography cited therein.

As it is well known, for classical stochastic control problems, the optimal cost function satisfies the dynamical programming principle and under appropriated conditions on the data, it also satisfies a second order non linear partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation ([6], [7], [10]).

This property is also valid for singular stochastic controls, where the HJB equations is a second order variational inequality given by

$$\min\left\{\inf_{u\in U}(\widetilde{\mathcal{L}}W + f), g^*\nabla W + c\right\} = 0\,. \tag{2}$$

In [9] a probabilistic analysis of this problem is developed and it is established that the cost function is the unique solution in the viscosity sense of equation (2).

In this paper we present a theoretical constructive procedure to find the function W, a numerical approximation procedure for the solution of equation (2) and we prove the convergence of the discrete solution to the value function. We present a numerical example.

In [1], [11], Teo (et all) had treated a stochastic optimal control problems without singular controls described by Itô differential equations. There the optimality conditions of the controls are used instead of the dynamical programming principle. The dynamical programming equation allows us to present an algorithm easier to implement.

2 Description of the problem

We use the following notation

- $\mathbb{R}_+^n = \{y \in \mathbb{R}^n : y^i \geq 0,\ i = 1, ..., n\}$.
- $T > 0$ fixed horizon
- $\Sigma = [0,T] \times \mathbb{R}^d$.

Remark 1 *Throughout this paper, C represents a constant, not always the same in each case, that depends on the problem data.*

2.1 Properties of the optimal cost function

We will assume the following additional hypotheses

- c is a Lipschitz continuous function,
- f is a bounded function
- $g = (g^{ij})$ $d \times k$ is a constant matrix
- b, σ are continuous with respect to (t, x, u)
- f, b and σ satisfy for $0 \leq s, t \leq T$, $x, y \in \mathbb{R}^d$, $u \in U$ the following conditions

$$\left| \begin{array}{l} |f(t,x,u) - f(s,y,u)| \leq C(|t-s| + \|x-y\|), \\ \|b(t,x,u) - b(s,y,u)\| \leq C(|t-s| + \|x-y\|), \\ \|\sigma(t,x,u) - \sigma(s,y,u)\| \leq C(|t-s| + \|x-y\|). \end{array} \right.$$

Under these assumptions the optimal cost function W satisfies the properties stated in the following theorems (the proofs can be seen in [9]).

Theorem 1 *The optimal cost function W is uniformly continuous in Σ and there exists a constant $C \geq 0$ such that $\forall\, 0 \leq s, t \leq T$, $x, y \in \mathbb{R}^d$ the following inequality holds*

$$|W(t,x) - W(s,y)| \leq C(|t-s|^{\frac{1}{2}} + \|x-y\|).$$

Theorem 2 *a) Let $(t,x) \in \Sigma$, then*

$$W(t,x) \leq W(t, x+gh) + c(t) \cdot h$$

for each $h \in \mathbb{R}^n_+$. Moreover, if the equality holds for some $h = (h^i) \in \mathbb{R}^d_+$ then the same equality holds when we replace h by $\bar{h} = \bar{\lambda} h \in \mathbb{R}^d_+$, with $0 \leq \bar{\lambda} \leq 1$.
b) We define for $0 \leq t \leq T$,

$$A_t = \{x : W(t,x) < W(t, x + g \cdot \lambda) + c(t) \cdot \lambda, \quad \forall \lambda \in \mathbb{R}^d_+, \lambda \neq 0\}.$$

Then the process x_t, associated to any optimal policy, is continuous when it is in A_t.

2.2 HJB equation

Dynamical programming principle

For this problem the dynamical programming principle can be stated in the following way

$$W(s,x) = \min_{u,v} E\left\{ \int_s^t f(\theta, x_\theta, u_\theta) d\theta + \int_0^t c(\theta) dv_\theta + W(t, x_t) \right\}. \qquad (3)$$

Based on this result, in [9] it is proved that the optimal cost function W satisfies in Σ the following HJB equation in the viscosity sense:

$$\min\left\{\inf_{u\in U}(\widetilde{\mathcal{L}}W+f)(t,x,u)\,,\ (g^*\nabla_x W(t,x))^i+c^i(t), i=1,2,...,k\right\}=0\,, \tag{4}$$

with final condition given by

$$W(T,x)=0, \quad \forall x\in\mathbb{R}^d\,, \tag{5}$$

where

$$\widetilde{\mathcal{L}}=\frac{\partial}{\partial t}+\sum_i b_i\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}$$

and $a=\sigma\sigma^*$. For the sake of simplicity we write (4) as

$$\min\left\{\inf_{u\in U}(\widetilde{\mathcal{L}}W+f), g^*\nabla W+c\right\}=0\,. \tag{6}$$

The viscosity solution is defined in such a way that allows us to work in the frame of discontinuous functions.

Definition 1 *w is a* **subsolution** *of equation (4) in the viscosity sense if: w is an upper semicontinuous function, satisfies the boundary condition (5) and verifies that $\forall\phi\in C^{1,2}(\Sigma)$, such that (t_0,x_0) is a local maximum of $w-\phi$ in the interior of Σ, then*

$$\min\left\{\inf_{u\in U}(\widetilde{\mathcal{L}}\phi+f), g^*\nabla\phi+c\right\}(t_0,x_0)\geq 0\,. \tag{7}$$

z is a **supersolution** *of (4) in the viscosity sense if: z is a lower semicontinuous function, satisfies the boundary condition (5) and verifies that $\forall\phi\in C^{1,2}(\Sigma)$, such that (t_0,x_0) is a local minimum of $z-\phi$ in the interior of Σ, then*

$$\min\left\{\inf_{u\in U}(\widetilde{\mathcal{L}}\phi+f), g^*\nabla\phi+c\right\}(t_0,x_0)\leq 0\,. \tag{8}$$

v is a **viscosity solution** *of equation (4) iff it is subsolution and supersolution.*

Theorem 3 *There exists $W\in\mathcal{C}(\Sigma)$ such that it is the unique solution of (6) in the viscosity sense with boundary conditions given by (5), where*

$$\mathcal{C}(\Sigma)=\{W: W\in C(\Sigma),\ bounded,\ |W(t,x)-W(t,y)|\leq L\,\|x-y\|\,,\ L\geq 0\}\,.$$

The proof can be seen in [9]). Applying the concept of subsolution and supersolution in the viscosity sense the following comparison principle holds (see [2]).

Theorem 4 *For every w subsolution of (6) and z supersolution of (6) it is verified that $w\leq z$ $\forall(t,x)\in\Sigma$.*

3 Solution by an iterative method

We find the solution of (2) by using a constructive iterative method. We define a sequence of functions, each of them solution of a stopping time control problem. This sequence converges to a viscosity solution of problem (6). From this fact and the property of unicity of viscosity solution we conclude that the sequence converges to the solution of the original problem.

3.1 An auxiliary stopping time problem

We denote by $\mathcal{L}w = \inf_{u \in U} (\widetilde{\mathcal{L}}w + f)$ and we consider the stopping time problem with obstacle Mw given by

$$\left| \begin{array}{l} \min \{\mathcal{L}w, Mw - w\} = 0\,, \\ \\ w(T,x) = 0\,, \end{array} \right. \tag{9}$$

where

$$Mw(t,x) = \inf_{v \in R^k_+} \{w(t, x + g \cdot v) + c \cdot v\}\,.$$

Remark 2 *For $(t,x) \in \Sigma$ such that $w(t,x) < Mw(t,x)$, it is satisfied that $\mathcal{L}w(t,x) = 0$ in the viscosity sense and for (t,x) such that $w(t,x) = Mw(t,x)$, it is verified also in the viscosity sense that $\mathcal{L}w(t,x) \geq 0$.*

Let us denoted by S the operator such that the function Sw associates to each function w, where Sw is the solution of the stopping time problem with obstacle Mw.

Remark 3 *The optimal cost function W is the fixed point of the operator S.*

Properties of operator S

1. If w is a uniformly continuous function over Σ, then Sw is also a uniform continuous function over Σ.
2. Operator S is monotone: if $w \leq \widehat{w}$ then $Sw \leq S\widehat{w}\,$.
3. There exists $K > 0$ such that

$$S(-K) \geq -K. \tag{10}$$

Remark 4 *The techniques introduced in [4] and [5] can be used to prove these properties, we omit the complete proof for the sake of brevity.*

3.2 The iterative construction method

Procedure description

- Step 0: $\nu = 0,\ w^0 = S(+\infty)$ (i.e. w^0 verifies (5) and $\mathcal{L}w^0 = 0$)
- Step 1: Compute Mw^ν
- Step 2: Compute $w^{\nu+1} = S(w^\nu)$ where $S(w^\nu)$ is the solution of the stopping time problem with obstacle Mw^ν, then it is verified that

$$\min\left\{\mathcal{L}w^{\nu+1}, Mw^\nu - w^{\nu+1}\right\} = 0$$

$$w^{\nu+1}(T,x) = 0.$$

- Step 3: $\nu = \nu + 1$, and go to step 1.

Iterative procedure convergence

The procedure generates a sequence of function with the following properties.

Proposition 1 *The sequence of functions generated by the constructive procedure is non increasing and convergent*

1. $w^{\nu+1} \leq w^\nu,\ \forall \nu \in \mathbb{N}$.
2. $w^\nu \geq W,\ \forall \nu \in \mathbb{N}$.
3. $\underline{w} = \lim\limits_{\nu\to\infty} w^\nu$ is a subsolution of equation (9) in the viscosity sense

Proof:
1. As $0 \leq w^0 \leq +\infty$, by the monotony of operator S we have

$$w^1 = S(w^0) \leq S(+\infty) = w^0.$$

Again by the monotony of operator S we obtain by induction that $w^{\nu+1} \leq w^\nu$.

2. As $W \leq +\infty$, it follows from the monotony of operator S that $\forall \nu$ it is verified

$$w^\nu \geq W. \tag{11}$$

3. Let us prove that $\underline{w} = \lim\limits_{\nu\to\infty} w^\nu$ is a viscosity subsolution of equation (6).

Considering the properties of operator S, we get that the sequence $\{w^\nu\}$ is non increasing and converges point-wisely to a function that we will denote by $\underline{w}$. Moreover as the functions w^ν are continuous, we conclude that $\underline{w}$ is an upper semi-continuous function. By virtue of (9), we have that $\forall x \in \mathbb{R}^d$

$$\underline{w}(T,x) = 0\,.$$

Let $\phi \in C^{1,2}(\Sigma)$ and let us consider $(t_0,x_0) \in \Sigma\circ$, a point where the function $\underline{w} - \phi$ has a strict maximum. We can suppose without loss of generality that $(\underline{w} - \phi)(t_0,x_0) = 0$. We should prove

$$\min\{\mathcal{L}\phi(t_0,x_0), M\underline{w}(t_0,x_0) - \underline{w}(t_0,x_0)\} \geq 0.$$

We know that for each ν the function w^ν is a viscosity solution of the stopping time problem with obstacle $M(w^{\nu-1})$ given by

$$\min\left\{\mathcal{L}w^\nu, Mw^{\nu-1} - w^\nu\right\} = 0 \tag{12}$$

and then w^ν is a viscosity subsolution of equation (12).

Let B be a compact neighbourhood of the point (t_0,x_0) such that $B \subset \Sigma$ and let (t_ν,x_ν) be points such that for each ν the function $(w^\nu - \phi)$ has a local maximum in B at the point (t_ν,x_ν).
We first prove that the sequence (t_ν,x_ν) converges to (t_0,x_0). We assume that the sequence (t_ν,x_ν) does not converge to (t_0,x_0) and we will arrive to an absurd. As the sequence $\{(t_\nu,x_\nu)\}$ is included in a compact set then there exist a sub-sequence which converges to a point that we denote $(\bar{t},\bar{x})$ (for the sake of notation simplicity we will continue on denoting (t_ν,x_ν) such sub-sequence).
We analyze the following difference

$$\begin{aligned}\underline{w} - \phi)(\bar{t},\bar{x}) - (\underline{w} - \phi)(t,x) = \quad & (\underline{w} - \phi)(\bar{t},\bar{x}) - (w^\nu - \phi)(t_\nu,x_\nu) \\ & +(w^\nu - \phi)(t_\nu,x_\nu) - (w^\nu - \phi)(t,x) \\ & +(w^\nu - \phi)(t,x) - (\underline{w} - \phi)(t,x)\,.\end{aligned}$$

Taking limit for $\nu \to \infty$ and considering the continuity of functions w^ν and ϕ, the upper semi-continuity of function $\underline{w}$ and that $\forall (t,x) \in B$ and $\forall \nu$ the following inequality holds

$$(w^\nu - \phi)(t_\nu,x_\nu) - (w^\nu - \phi)(t,x) \geq 0\,,$$

then

$$\lim_{\nu\to\infty} (\underline{w} - \phi)(\bar{t},\bar{x}) - (w^\nu - \phi)(t_\nu,x_\nu) \geq 0$$

$$\lim_{\nu\to\infty} (w^\nu - \phi)(t_\nu,x_\nu) - (w^\nu - \phi)(t,x) \geq 0$$

$$\lim_{\nu\to\infty} (w^\nu - \phi)(t,x) - (\underline{w} - \phi)(t,x) = 0\,.$$

From here we obtain that $\forall (t,x) \in B$ it is verified that

$$(\underline{w} - \phi)(\bar{t},\bar{x}) - (\underline{w} - \phi)(t,x) \geq 0,$$

in particular for (t_0, x_0). This fact is impossible because (t_0, x_0) is a point of strict maximum. Therefore we can conclude that

$$(t_\nu, x_\nu) \to (t_0, x_0)\,.$$

From the definition of viscosity subsolution and (12) we get that $\forall \nu$

$$\mathcal{L}\phi(t_\nu, x_\nu) \geq 0\,.$$

Taking into account the regularity of the function ϕ and its derivative and the convergence of (t_ν, x_ν) to (t_0, x_0) we obtain

$$\mathcal{L}\phi(t_0, x_0) \geq 0\,. \tag{13}$$

From the other side, considering the definition of $\underline{w}$ and the fact that $\forall \nu$ w^ν is a viscosity sub-solution of (12), it is verified that $\forall (t, x)$

$$\underline{w}(t, x) \leq w^{\nu+1}(t, x) \leq M w^\nu(t, x)$$

and so

$$\underline{w}(t, x) \leq M w^\nu(t, x)\,.$$

Now, by definition of operator M we get $\forall v \in R^k_+$

$$\underline{w}(t, x) \leq M w^\nu(t, x) \leq w^\nu(t, x + g \cdot v) + c\, v\,.$$

Taking limit for $\nu \to \infty$, we obtain $\forall v \in R^k_+$

$$\underline{w}(t, x) \leq \underline{w}(t, x + g \cdot v) + c\, v\,,$$

and then the following inequality holds

$$\underline{w}(t, x) \leq M \underline{w}(t, x)\,. \tag{14}$$

Finally, considering both inequalities (13) and (14) for $(t, x) = (t_0, x_0)$ we obtain

$$\min \{\mathcal{L}\phi(t_0, x_0), M\underline{w}(t_0, x_0) - \underline{w}(t_0, x_0)\} \geq 0\,,$$

as we wanted to prove.

□

Corollary 1 *Function $\underline{w}$ verifies $\underline{w} = W$.*

Proof: As $\underline{w}$ is a viscosity subsolution of equation (9), by the comparison principle (see [2]), we have that $\underline{w} \leq W$ and taking limit in (11) we get that

$$\underline{w} = W.$$

□

4 The discrete problem

4.1 Fully discrete solution

4.1.1 Elements of the discrete problem

We consider a partition of interval $[0,T]$, of norm h, i.e. the discrete times are given by $E_h = \{nh, n = 0,\ldots,\mu\}$, $\mu = [T\,/\,h]$.

We identify the space discretization with the parameter k, which also indicates the discretization size.

Remark 5 *For the sake of simplicity we consider here that the set U is a finite set.*

Domain $\mathbb{R}^d$ approximation

We consider a family of quasi–uniform triangulations of $\mathbb{R}^d$, which is denoted by $\left\{\mathcal{S}^k\right\}_k$ and verifies:

• For all k, $\mathcal{S}^k$ is a denumerable collection of closed simplices $\left\{S_j^k\right\}_j$ such that $\bigcup_j S_j^k = \mathbb{R}^d$.

• If $S_j^k \in \mathcal{S}^k$, $S_p^k \in \mathcal{S}^k$, $S_j^k \neq S_p^k$, we have

– $(S_j^k)^\circ \bigcap (S_p^k)^\circ = \emptyset$.

– Either $S_j^k \bigcap S_p^k = \emptyset$ or S_j^k and S_p^k have in common a whole $(m-r)$ – edge, $r = 1,\ldots,m$

• $\max_j \left(\mathrm{diam}(S_j^k)\right) = k$.

• $\exists\ \chi_1 > 0$ and $\exists\ \chi_2 > 0$ independent on the discretization, such that, denoting by d_j the diameter of the simplex S_j^k, it is verified

– the simplex S_j^k has a sphere of radius r_j in its interior and it results $r_j \geq \chi_1 d_j$,

– for any simplex S_j^k, $k \leq \chi_2 d_j$.

Let $E_k = \left\{x^j\,;\, j \in \mathbb{N}\right\}$ be the vertices of this triangulation, arbitrarily arranged. Every $x \in \mathbb{R}^d$ is a convex combination of the vertices x^i of the simplex to which x belongs. Hence, $\forall\ u \in U$, $\forall\ n = 0,\ldots,\mu$, $\forall\ x^i \in E_k$, there exists a matrix with components $\gamma_j(nh, x^i, u)$, such that

$$\left| \begin{array}{ll} \gamma_j(nh, x^i, u) \geq 0, & \forall\, j \in \mathbb{N}, \\ \gamma_j(nh, x^i, u) > 0, & \text{for at most (d + 1) values of } j\,, \\ \sum_j \gamma_j(nh, x^i, u) = 1, & \\ x^i + h\, g(nh, x^i, u) = \sum_j \gamma_j(nh, x^i, u) x^j. & \end{array} \right. \tag{15}$$

Remark 6 *We denote by* $\Sigma_{hk} = E_h \times E_k$ *.*

Discretization of HJB equation

We analyze the discretization for the case where the operator $\widetilde{\mathcal{L}}$ has the form

$$\widetilde{\mathcal{L}} = \frac{\partial}{\partial t} + \sum_i b_i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{d} a_{ii} \frac{\partial^2}{\partial x_i^2}.$$

We consider the following scheme of discretization: $\forall j$

$$\left| \begin{array}{l} w_k^h(nh, x_j) = \min \left(\widehat{w}_k^h(nh, x_j), \widehat{\widehat{w}}_k^h(nh, x_j) \right), \\ w_k^h(T, x_j) = 0, \end{array} \right.$$

with

$$\widehat{w}_k^h(nh, x_j) = \frac{1}{2d} \sum_{0 \neq i = -d}^{d} w_k^h \left((n+1)h, x_j + hb(nh, x_j, u) + e_i \sqrt{dh} \right) + hf(nh, x_j, u),$$

where

$$e_i = (e_i^j)_{j=1}^d \quad \textit{with} \quad e_i^j = \begin{cases} \operatorname{sign}(i) & j = |i| \\ 0 & j \neq |i| \end{cases}$$

and

$$\widehat{\widehat{w}}_k^h(nh, x_j) = \min_{i=1,\dots,k} \left(w_k^h((n+1)h, x_j + g \cdot v_i \xi(h)) + c(nh) v_i \xi(h) \right)\,,$$

where

$$v_i = (v_i^j)_{j=1}^k \quad \textit{with} \quad v_i^j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

and $\xi(h)$ is a positive function which verifies

$$\lim_{h \to 0} \xi(h) = 0 \quad \textit{and} \quad \lim_{h \to 0} \frac{\xi(h)}{h} = +\infty\,. \tag{16}$$

Remark 7 *¿From definition of w_k^h we have that it is unique and that it can be computed recursively.*

Discretization consistency

The discrete function $\widehat{w}_k^h(nh, x_j)$ defined below corresponds to the discretization of operator $\widetilde{\mathcal{L}}$. Let us see that the scheme is consistent.

If $\Phi \in C^{1,2}(\Sigma)$, by Taylor expansion, and doing simple but lengthy calculations, we have that

$$\lim_{h\to 0} \frac{1}{h}\left(-\Phi(t,x) + \frac{1}{2d}\sum_{0\neq i=-d}^{d} \Phi\left(t+h, x+hb(t,x,u)+e_i\sqrt{dh}\right)\right) = \\ = \left(\frac{\partial \Phi}{\partial t} + \nabla\Phi.b + \frac{1}{2}\Delta\Phi\right)(t,x)\,.$$

For the impulsive part, if $\Phi \in C^{1,2}(\Sigma)$, by Taylor expansion, doing again simple calculations and having in mind (16), we get

$$\lim_{h\to 0} \frac{1}{\xi(h)}\left(-\Phi(t,x) + \Phi(t+h, x+g\cdot v_i\xi(h))\right) = (g^*\nabla\Phi)^i(t,x)\,.$$

4.2 Convergence of the numerical procedure

As the sequence $\{w_k^h\}$ defined is equibounded by $M_f T$, we can define the function $\underline{w}$, $\overline{w}$ as follows

$$\underline{w}(t,x) = \\ \lim_{\varepsilon, k_0 \to 0} \inf\{w_k^h(t+s, x+y) : |s|\le\varepsilon,\ \|y\|\le\varepsilon,\ k\le k_0,\ (t+s,x+y)\in\Sigma_{hk}\} \tag{17}$$

and

$$\overline{w}(t,x) = \\ \lim_{\varepsilon, k_0 \to 0} \sup\{w_k^h(t+s, x+y) : |s|\le\varepsilon,\ \|y\|\le\varepsilon,\ k\le k_0,\ (t+s,x+y)\in\Sigma_{hk}\}. \tag{18}$$

We are going to show that $\overline{w}$ is a subsolution of HJB equation and $\underline{w}$ is a supersolution. Then, by Theorem (4), we have that $\overline{w} \le \underline{w}$. Moreover, from (17) and (18) we get $\overline{w} \ge \underline{w}$ and in consequence $\overline{w} = \underline{w}$. This implies that there exists w^* such that it is the limit of the sequence w_k^h. As HJB equation has a unique solution, we conclude that $w^* = W$.

Remark 8 *We have employed the definition of w_k^h over the whole set $E_h \times \mathbb{R}^d$, this definition was made by linear interpolation in the space of linear finite elements.*

$\overline{w}$ is subsolution in the viscosity sense of HJB

We must prove that

- $\overline{w}$ is upper semicontinuous
- $\overline{w}(T,x) = 0 \qquad \forall x$
- $\overline{w}$ is subsolution in the viscosity sense of (7)

The first two properties are obvious, let us see the third one. Let K be such that $|w_k^h(t,x)| \le K \ \forall h > 0,\ k > 0,\ (t,x) \in \Sigma$. We use as the tangent function to $\overline{w}$ functions which verify $\Phi(s,y) \to \infty$ for $||y|| \to \infty$ or $(t,y) \to \partial\Sigma$.

Let (t,x) be a global strict maximum of $(\overline{w} - \Phi)$, w.l.g. we can suppose that $(\overline{w} - \Phi)(t,x) = 0$.

From definition of $\bar{w}$, there exists a sequence ε_ν such that $\varepsilon_\nu \to 0$ and if we define

$$\phi_\nu(t,x) = \sup\{w_k^h(s,y) : h \le \varepsilon_\nu,\ k \le \varepsilon_\nu, (s,y) \in \Sigma_{hk}, |t-s| \le \varepsilon_\nu, ||x-y|| \le \varepsilon_\nu\}$$

we have

$$\phi_\nu(t,x) - \frac{1}{\nu} \le \overline{w}(t,x) \le \phi_\nu(t,x)\,.$$

Then, there are $h_\nu \le \varepsilon_\nu, k_\nu \le \varepsilon_\nu, (t_\nu, x_\nu) \in \Sigma_{h_\nu k_\nu}, |t - t_\nu| \le \varepsilon_\nu, ||x - x_\nu|| \le \varepsilon_\nu$ such that

$$w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) \le \phi_\nu(t,x) \le \frac{1}{\nu^2} + w_{k_\nu}^{h_\nu}(t_\nu, x_\nu)$$

and, in consequence

$$w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) - \frac{1}{\nu} \le \overline{w}(t,x) \le \frac{1}{\nu^2} + w_{k_\nu}^{h_\nu}(t_\nu, x_\nu)\,,$$

then

$$\lim_{\nu\to\infty} w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) = \overline{w}(t,x)\,. \tag{19}$$

Let us see that there are $(\hat{t}_\nu, \widehat{x}_\nu) \in \Sigma_{h_\nu k_\nu}$ such that it is a global maximum of $w_{k_\nu}^{h_\nu} - \Phi$ in $\Sigma_{h_\nu k_\nu}$ and such that $(\hat{t}_\nu, \widehat{x}_\nu) \to (t,x)$. Moreover, it results $w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) \to \overline{w}(t,x)$.

Let (τ_p, q_p) be a maximizing sequence of $w_{k_\nu}^{h_\nu} - \Phi$, then

$$\begin{aligned} w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) - \Phi(t_\nu, x_\nu) &\le \lim_{p\to\infty} w_{k_\nu}^{h_\nu}(\tau_p, q_p) - \Phi(t_p, q_p) \\ &= \sup_{(\tau,q)\in\Sigma_{h_\nu k_\nu}} (w_{k_\nu}^{h_\nu}(\tau,q) - \Phi(\tau,q)) \\ &\le K - \inf_{(\tau,q)\in\Sigma_{h_\nu k_\nu}} \Phi(\tau,q) \\ &= K + \sup_{(\tau,q)\in\Sigma_{h_\nu k_\nu}} (-\Phi(\tau,q)) \le 2K \end{aligned} \tag{20}$$

because

$$\overline{w}(s,y) - \Phi(s,y) \leq \overline{w}(t,x) - \Phi(t,x) = 0\,,\ \forall (s,y) \in \Sigma\,, \tag{21}$$

where

$$-\Phi(s,y) \leq K \Rightarrow \sup(-\Phi(s,y)) \leq K\,.$$

The sequence maximizing q_p is bounded, because if we suppose that $||q_p||$ is not bounded, then there is a subsequence $q_{p'}$ such that $||q_{p'}|| \to \infty$, and so

$$\lim_{p' \to \infty} w_{k_\nu}^{h_\nu}(\tau_{p'}, q_{p'}) - \Phi(t_{p'}, q_{p'}) = -\infty\,,$$

which contradicts (20). Then

$$||q_p|| \leq M\,.$$

As net E_{k_ν} is finite in bounded neighbourhoods, we have

$$\begin{aligned} \tau_{p'} &= \hat{t}_\nu \ \forall p' \geq p(\nu) \\ q_{p'} &= \widehat{x}_\nu \ \forall p' \geq p(\nu) \end{aligned}$$

and $(\hat{t}_\nu, \widehat{x}_\nu)$ is the supremum of $w_{k_\nu}^{h_\nu} - \Phi$ in $\Sigma_{h_\nu k_\nu}$.
Then

$$w_{k_\nu}^{h_\nu}(s,y) - \Phi(s,y) \leq w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu) \ \ \forall (s,y) \in \Sigma_{h_\nu k_\nu}$$

and is consequence

$$w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) - \Phi(t_\nu, x_\nu) \leq w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu) \qquad \forall \nu\,.$$

Now, by taking limit $\nu \to \infty$ and considering (19), we can conclude that

$$\overline{w}(t,x) - \Phi(t,x) \leq \varliminf_{\nu \to \infty} w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu)\,. \tag{22}$$

Following similar arguments to those ones employed for the sequence (τ_p, q_p), it is easy to prove that $||\widehat{x}_\nu|| \leq M$ for all ν. Then, we consider a subsequence of $(\hat{t}_\nu, \widehat{x}_\nu)$ which converges to $(\check{t}, \check{x})$. From the definition of $\overline{w}$ we have

$$\varliminf_{\nu \to \infty} w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) \leq \overline{w}(\check{t}, \check{x})$$

and in consequence

$$\overline{w}(t,x) - \Phi(t,x) \leq \overline{w}(\check{t}, \check{x}) - \Phi(\check{t}, \check{x})\,.$$

As (t,x) is a strict global maximum, then $\check{t} = t$ and $\check{x} = x$, and we have that the complete sequence $(\widehat{t}_\nu, \widehat{x}_\nu)$ converges to (t,x) when $\nu \to \infty$.
From (22), we have

$$\overline{w}(t,x) \leq \varliminf_{\nu \to \infty} w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu)$$

and by $\overline{w}$ definition (18)

$$\overline{w}(t,x) \geq \overline{\lim_{\nu\to\infty}}\, w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu),$$

we conclude that

$$\lim_{\nu\to\infty} w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu) = \overline{w}(t,x)\,.$$

Then

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu) \geq w_{k_\nu}^{h_\nu}(s_\nu, y_\nu) - \Phi(s_\nu, y_\nu), \quad \forall (s_\nu, y_\nu) \in \Sigma_{h_\nu k_\nu}\,. \tag{23}$$

By definition of $w_{k_\nu}^{h_\nu}$

$$w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu) \leq \widehat{w}(\widehat{t}_\nu, \widehat{x}_\nu) \tag{24}$$

and

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) \leq \widehat{\widehat{w}}(\widehat{t}_\nu, \widehat{x}_\nu)\,. \tag{25}$$

Considering (24) we have $\forall u \in U$

$$\begin{aligned} w(\widehat{t}_\nu, \widehat{x}_\nu) \leq{}& h_\nu f(\widehat{t}_\nu, \widehat{x}_\nu, u) + \\ &+ \sum_{0\neq i=-d}^{d} \frac{1}{2d} w_{k_\nu}^{h_\nu}\left((n_\nu+1)h_\nu, \widehat{x}_\nu + h_\nu b(\widehat{t}_\nu, \widehat{x}_\nu, u) + e_i\sqrt{dh_\nu}\right)\,. \end{aligned} \tag{26}$$

It is possible to characterize all point in Σ as a linear combination of the nodes of the discretization used for Σ. Let us denote by η_j^i the barycentric coordinates of $(\widehat{x}_\nu^+)^i = \widehat{x}_\nu + h_\nu b(\widehat{t}_\nu, \widehat{x}_\nu, u) + e_i\sqrt{dh_\nu}$ with respect to nodes $(\widehat{x}_\nu^+)_j^i \in E_{k_\nu}$ of the simplices $(\widehat{x}_\nu^+)^i$ to which it belongs. Then we have

$$w_{k_\nu}^{h_\nu}((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)^i) = \sum_j \eta_j^i w_{k_\nu}^{h_\nu}((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)_j^i)$$

and

$$\Phi((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)^i) = \sum_j \eta_j^i \Phi((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)_j^i) + O(k_\nu^2)\,.$$

From these inequalities and considering (23) we obtain for each $i = -d, ..., d$

$$\begin{aligned} & w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu) \geq \\ & \quad \sum_j \eta_j^i w_{k_\nu}^{h_\nu}((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)_j^i) - \sum_j \eta_j^i \Phi((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)_j^i) + O(k_\nu^2)\,. \end{aligned}$$

Moreover, from (24) we have

$$w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu) \leq h_\nu f(\widehat{t}_\nu, \widehat{x}_\nu, u) + \frac{1}{2d} \sum_{0\neq i=-d}^{d} \sum_j \eta_j^i w_{k_\nu}^{h_\nu}((n_\nu+1)h_\nu, (\widehat{x}_\nu^+)_j^i)\,.$$

By applying the Taylor' expansion of order two to function Φ (which belongs to $C^{1,2}(\Sigma)$), and taking limit for $\nu \to \infty$ we arrive to

$$(\widetilde{\mathcal{L}}\Phi + f)(t, x, u) \geq 0\,,$$

and as u is arbitrary, we get

$$\min_{u \in U} (\widetilde{\mathcal{L}}\Phi + f)(t, x, u) \geq 0\,.$$

On the other hand, considering (25)

$$w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu) \leq \min_{i=1,\ldots,n} \left(w_{k_\nu}^{h_\nu}((t_\nu + 1)h_\nu, x_\nu + gv^i \xi(h_\nu)) + cv^i \xi(h_\nu) \right)$$

and following a similar development, it is easy to prove that $\forall i = 1, ..., n$,

$$(g^* \nabla \Phi + c)^i \geq 0\,.$$

Then, $\overline{w}$ is a subsolution.

$\underline{w}$ is a supersolution in the viscosity sense of HJB

We must prove that

- $\underline{w}$ is lower semicontinuous
- $\underline{w}(T, x) = 0 \qquad \forall x$
- $\underline{w}$ is a supersolution in the viscosity sense of (8)

The first two properties are immediate, let us see the third one, i.e.

$$\min \left\{ \inf_{u \in U} (\widetilde{\mathcal{L}}\underline{w} + f), g^* \nabla \underline{w} + c \right\} \leq 0\,.$$

We consider $|w_k^h(t, x)| \leq K \ \forall h > 0,\ k > 0,\ (t, x) \in \Sigma$. We suppose also that the functions Φ which are tangent to the function $\underline{w}$ verify $\Phi(s, y) \to \infty$ as $||y|| \to \infty$ or $(t, y) \to \partial\Sigma$.

Let $(t, x) \in (0, T) \times \Omega$ and $\Phi \in C^{1,2}((0, T) \times \Omega))$ such that $\underline{w} - \Phi$ have a global strict minimum in (t, x).. Following a similar analysis to that one used for $\overline{w}$, we obtain that $\exists$ a sequence ε_ν such that $\varepsilon_\nu \to 0$ and there exist $h_\nu \leq \varepsilon_\nu$, $k_\nu \leq \varepsilon_\nu$, $(t_\nu, x_\nu) \in \Sigma_{h_\nu k_\nu}$, $|t - t_\nu| \leq \varepsilon_\nu$, $||x - x_\nu|| \leq \varepsilon_\nu$ such that

$$\lim_{\nu \to \infty} w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) = \underline{w}(t, x). \tag{27}$$

Moreover there exists $(\hat{t}_\nu, \widehat{x}_\nu) \in \Sigma_{h_\nu k_\nu}$ which realizes a global minimum of $w_{k_\nu}^{h_\nu} - \Phi$ in $\Sigma_{h_\nu k_\nu}$ and they verify

$$\left| \begin{array}{l} (\hat{t}_\nu, \widehat{x}_\nu) \to (t, x) \\[2ex] \lim\limits_{\nu \to \infty} w_{k_\nu}^{h_\nu}(\widehat{t}_\nu, \widehat{x}_\nu) = \underline{w}(t, x)\,. \end{array} \right.$$

Then, $\forall(t_\nu, x_\nu) \in \Sigma_{h_\nu k_\nu}$

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) - \Phi(\hat{t}_\nu, \widehat{x}_\nu) \leq w_{k_\nu}^{h_\nu}(t_\nu, x_\nu) - \Phi(t_\nu, x_\nu)\,. \tag{28}$$

As function $w_{k_l}^{h_l}$ verifies

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) = \min(\widehat{w}_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu), \widehat{\widehat{w}}\,_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu))\,, \tag{29}$$

we have two possibilities:

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) = \widehat{w}_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) \tag{30}$$

or

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) = \widehat{\widehat{w}}\,_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu)\,. \tag{31}$$

Then, there is a subsequence of ν such that one of the two conditions is true. Let us suppose that for such a subsequence (30) is satisfied. As U is a finite set, there are a control $\bar{u}$ and a subsequence (that, w.l.g., we call ν) such that it verifies

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) =$$

$$\sum_{0 \neq i = -d}^{d} \frac{1}{2d} w_{k_\nu}^{h_\nu}\left((n_\nu + 1)h_\nu, \widehat{x}_\nu + h_\nu b(\hat{t}_\nu, \widehat{x}_\nu, \bar{u}) + e_i\sqrt{dh_\nu}\right) + h_\nu f(\hat{t}_\nu, \widehat{x}_\nu, \bar{u})\ .$$

Let η_j^i be the barycentric components of $(\widehat{x}_\nu^+)^i$ such that

$$(\widehat{x}_\nu^+)^i := \widehat{x}_\nu + h_\nu b(\hat{t}_\nu, \widehat{x}_\nu, \bar{u}) + e_i\sqrt{dh_\nu} = \sum_j \eta_j^i (\hat{x}_\nu^+)_j^i\,.$$

By using the definition of $w_{k_\nu}^{h_\nu}$, we have

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) = h_\nu f(\hat{t}_\nu, \widehat{x}_\nu, \bar{u}) + \sum_{0 \neq i = -d}^{d} \sum_j \eta_j^i \frac{1}{2d} w_{k_\nu}^{h_\nu}((n_\nu + 1)h_\nu, (\hat{x}_\nu)_j^i\,)\ . \tag{32}$$

By (28) and (32), using the fact that $\Phi \in C^{1,2}(\Sigma)$ and taking limit when ν goes to ∞, we get

$$(\widetilde{\mathcal{L}}\Phi + f)(t, x, \bar{u}) \leq 0\,,$$

then the minimum over U verifies

$$\min_U((\widetilde{\mathcal{L}}\Phi + f)(t, x, u)) \leq 0\,.$$

Now, let us suppose that (31) is true, i.e. there is a subsequence v_ν and a control $\bar{v}_{\bar{\imath}}$ such that $\forall\nu$

$$w_{k_\nu}^{h_\nu}(\hat{t}_\nu, \widehat{x}_\nu) = w_{k_\nu}^{h_\nu}((t_\nu + 1)h_\nu, \widehat{x}_\nu + g\bar{v}_{\bar{\imath}}\xi(h_\nu)) + c\bar{v}_{\bar{\imath}}\xi(h_\nu)\,.$$

By using the same argument, we arrive to the inequality

$$(g^*\nabla\Phi(t,x)+c)^{\bar{\imath}} \le 0$$

and then the minimum over the set $\{i = 1, ..., n\}$ verifies

$$\min_{i=1,\dots,n} (g^*\nabla\Phi(t,x)+c)^{i} \le 0\,.$$

In consequence, we conclude that $\underline{w}$ is a supersolution.

□

5 Example

We have solve numerically an example given by the following data:

- $\Sigma = [0, 0.3] \times [-20, 20]$, $d = 1$, $g = 1$, $b = 0$, $c = 0.1042$, $\xi(h) = 0.2$,
- $f(t,x,u) = \max(0,\, \cos(\min(\max(0.25x, -\pi/2), \pi/2)))^{0.01}$,
- step of time discretization $h = 0.0065$
- step of space discretization $k = 0.2$,
- $U = \{-1, 1\}$

In Figure 1 we show the function $w_k^h(0,\cdot)$ corresponding to the given data, the gradient function and, as an inverted characteristic function, the region where the impulsive control is applied.

We can observe that while the impulsive control v is applied, the value of the gradient function approaches the constant function c

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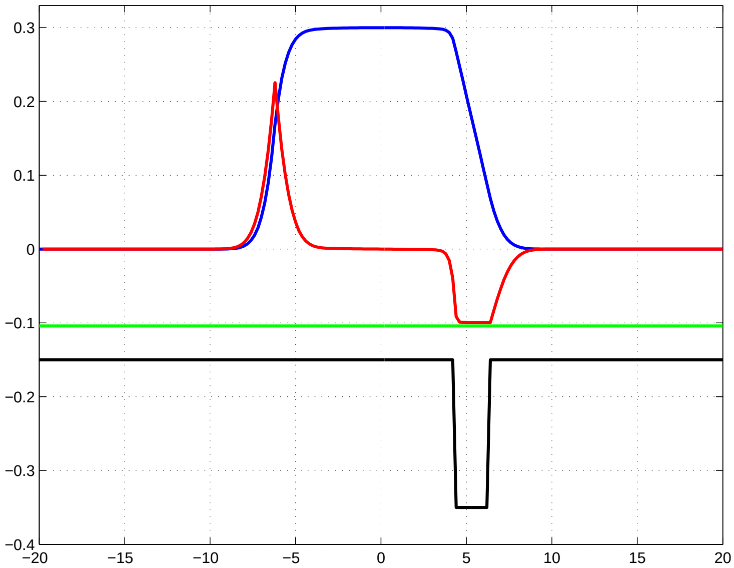

Figure 1: Function w and its gradient

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