

## TWO CLASSES OF ALGEBRAS WITH INFINITE HOCHSCHILD HOMOLOGY

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ABSTRACT. We prove without any assumption on the ground field that higher Hochschild homology groups do not vanish for two large classes of algebras whose global dimension is not finite.

### 1. INTRODUCTION

Let  $k$  be a fixed field. All the algebras we consider are associative unital  $k$ -algebras. We will denote  $\otimes = \otimes_k$ .

It is well known that the homological properties of an algebra are related to the properties of its Hochschild (co)homology groups. For example, if a finite dimensional algebra over an algebraically closed field has finite global dimension, then all its higher Hochschild cohomology groups vanish. In [12], D. Happel said that he did not know whether the converse was true or not. It has been shown in [5] that it does not hold for algebras of type  $A_q = k\langle x, y \rangle / (x^2, y^2, xy - qyx)$ , where  $q \in k$ .

In [11], Han proved that the total Hochschild homology of the algebras  $A_q$  is infinite dimensional. This fact led him to suggest the following conjecture:

**Conjecture (Han).** *Let  $A$  be a finite dimensional  $k$ -algebra. If the total Hochschild homology of  $A$  is finite dimensional, then  $A$  has finite global dimension.*

In the same paper, Han provided a proof of this statement for monomial finite dimensional algebras.

Avramov and Vigué's computations in [1] show that Han's conjecture holds in the commutative case not only for finite dimensional algebras but for essentially finitely generated ones; see also [18].

In [4], Han's conjecture is proved for graded local algebras, Koszul algebras and graded cellular algebras, provided the characteristic of the ground field is zero. The proof relies on the properties of the graded Cartan matrix and the logarithm and strongly uses the hypothesis on the characteristic of the field.

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In [3], the authors compute the Hochschild homology groups of quantum complete intersections; that is, algebras of type  $A = k\langle x, y \rangle / (x^a, y^b, xy - qyx)$ , where  $q \in k^*$  is not a root of unity and  $a, b \geq 2$  are fixed integers. In particular they prove Han's conjecture for this class of finite dimensional algebras.

The main purpose of this paper is to prove that higher Hochschild homology groups do not vanish for two large classes of algebras whose global dimension is not finite, without any assumption on the ground field.

In Theorem I, the algebras we consider are generalizations of quantum complete intersections, and they are not assumed to be finite dimensional.

On the other hand, the algebras satisfying the hypotheses of Theorem II are, in some sense, opposite of quantum complete intersections, since we assume that they have two generators  $x$  and  $y$  such that  $xy = yx = 0$ .

Now we state both main theorems.

**Theorem I.** *Let  $A = k\langle x_1, \dots, x_n \rangle / (f_1, \dots, f_p)$  be a finitely generated  $k$ -algebra, such that  $f_1$  belongs to  $k[x_1]$  and, for  $i \geq 2$ ,  $f_i$  belongs to the two-sided ideal  $(x_2, \dots, x_n)$ . If  $B = k[x_1] / (f_1)$  is not smooth, then the Hochschild homology groups  $HH_n(A)$  are not zero for all  $n \in \mathbb{N}$ .*

For example Theorem I is valid if  $f_1 = x_1^2 g_1$ , with  $g_1 \in k[x_1]$  and  $f_2, \dots, f_p$  satisfying the hypothesis of the theorem.

**Theorem II.** *Let  $A = \bigoplus_{n \geq 0} A^n$  be a finite dimensional graded  $k$ -algebra with  $A^0 = k$  and such that  $\bar{A} = \bigoplus_{n \geq 1} A^n$  is not zero. Assume that there exist two generators  $x$  and  $y$  of the algebra  $A$  verifying  $xy = yx = 0$ . Then the total Hochschild homology of  $A$  is not finite dimensional.*

*Remark 1.1.* This theorem is valid for very large classes of graded local algebras since relations between the other generators play no role.

The proof of Theorem I follows without any computation from the well known result for commutative algebras.

The methods used in the proof of Theorem II rely on differential homological algebra. In fact, we will work with the cobar construction on the graded coalgebra  $\bigoplus_{n \geq 0} \text{Hom}_k(A^n, k)$ . We denote it  $(\Omega^* A, d)$ . The Hochschild homology groups of the differential graded algebra  $(\Omega^* A, d)$  are dual, as vector spaces, to the Hochschild homology groups of the graded  $k$ -algebra  $A$ . Since  $(\Omega^* A, d)$  is a tensor algebra, a short complex is available to compute its Hochschild homology.

The paper is organized as follows:

- (1) Introduction
- (2) Proof of Theorem I
- (3) Interpretation in terms of quivers
- (4) Hochschild homology in the differential graded case
- (5) Proof of Theorem II

## 2. PROOF OF THEOREM I

Let  $A$  be an associative unital  $k$ -algebra. The definition of the Hochschild homology groups,  $HH_n(A)$ ,  $n \geq 0$ , is well known (see for example [13]). We have

$$HH_n(A) := \text{Tor}_n^{A^e}(A, A) = H_n(C_*(A), b),$$

where  $(C_*(A), b)$  is the Hochschild complex of  $A$ . Clearly,  $HH_n(A)$  is a  $k$ -vector space for all  $n \geq 0$ .

In this section we assume that  $A = k\langle x_1, \dots, x_n \rangle / (f_1, \dots, f_p)$ , where  $n, p \geq 1$ ; that  $f_1$ , which we may suppose is monic, belongs to  $k[x_1]$ ; and that, for  $i \geq 2$ ,  $f_i$  belongs to the two-sided ideal  $(x_2, \dots, x_n)$ . Let us consider the  $k$ -algebra  $B = k[x_1] / (f_1)$  and the maps

$$\begin{aligned} \iota : B &\rightarrow A \text{ with } \iota(x_1) = x_1, \\ \pi : A &\rightarrow B \text{ with } \pi(x_1) = x_1, \pi(x_i) = 0, \text{ for } i \geq 2. \end{aligned}$$

The following lemma is easy to prove.

**Lemma 2.1.** *The maps  $\iota$  and  $\pi$  are morphisms of  $k$ -algebras and satisfy  $\pi \circ \iota = id_B$ .*

Now, Theorem I is an immediate consequence of the following facts:

- the morphisms  $\iota$  and  $\pi$  induce by functoriality  $k$ -linear maps
 
$$HH_*(\iota) : HH_*(B) \rightarrow HH_*(A) \text{ and } HH_*(\pi) : HH_*(A) \rightarrow HH_*(B)$$
 satisfying  $HH_*(\pi) \circ HH_*(\iota) = id_{HH_*(B)}$ ,
- using a result of [1],  $HH_n(B)$  is nonzero for an infinite sequence of integers  $n$ .

Another proof can be given using the computations for  $HH_n(B)$  in [6]: if  $f_1$  and  $f'_1$  are not coprime, then  $HH_n(B) \neq 0$  for all  $n \in \mathbb{N}$ .

**Example 2.2.** If  $f_1 = x_1^a$ , with  $a \geq 2$ , and  $f_i \in (x_2, \dots, x_n)$ , then Theorem I holds. This covers the case of quantum complete intersections.

An interesting question is to ask if the algebras  $A$  considered in Theorem I have infinite global dimension. In the commutative case, it is well known that this is true. Also, if  $A = k\langle x_1, \dots, x_n \rangle / (f_1, \dots, f_p)$  is a finite dimensional  $k$ -vector space, Happel's result [12] implies that  $\text{gldim}(A) = \infty$ , where  $\text{gldim}$  denotes the global dimension of the algebra.

It follows from Serre's theorem on page 37 of [15] that if  $B$  is not smooth, then its global dimension is not finite. In the general case, we cannot ensure that if we have  $k$ -algebras  $A$  and  $B$  as above with  $\text{gldim}(B) = \infty$ , then  $\text{gldim}(A) = \infty$ .

However, we can use the algebra map  $\iota : B \rightarrow A$  to obtain that the global dimension of  $A$  is not finite in some cases: Suppose that  $\iota$  endows  $A$  with a structure of a flat  $B$ -module. In this situation, Corollary 4.4 of [2] says that  $\text{gldim}(A) = \infty$ . This is the case, for example, of quantum complete intersections.

### 3. INTERPRETATION IN TERMS OF QUIVERS

Let  $A$  be a  $k$ -algebra which is isomorphic to  $kQ^A / I^A$  for a given finite quiver  $Q^A$  and an admissible ideal  $I^A$ . In case  $k$  is an algebraically closed field and  $A$  is finite dimensional and basic, there always exist a quiver and an admissible ideal such that the above isomorphism holds.

Let us denote by  $Q_0^A = \{e_1, \dots, e_r\}$  the set of vertices of  $Q^A$  and by  $Q_1^A$  its set of arrows. Then  $kQ_0^A$  is an algebra,  $kQ_1^A$  is a  $kQ_0^A$  two-sided ideal and  $A = T_{kQ_0^A} kQ_1^A / I^A$ , where  $I^A \subseteq (kQ_1^A)^2$ .

Suppose that there exist  $e_i \in kQ_0^A$  and  $x \in e_i(kQ_1^A)e_i$ . In fact, since  $A$  is finite dimensional and  $I^A$  is admissible, if such a loop  $x$  exists, then  $x^n = 0$  for some integer  $n \geq 2$ .

Let  $B$  be the  $k$ -algebra  $k[x]/\langle x^n \rangle$ . Then  $B = T_{kQ_0^B} kQ_1^B / I^B$ , where  $Q_0^B = \{e_i\}$ ,  $Q_1^B = \{x\}$  and  $I^B = \langle x^n \rangle$ .

We may consider the morphisms of algebras of the previous section. In this case the map  $\iota$  is completely determined by its values on  $e_i$  and  $x$ . It sends  $e_i$  to  $e_1 + \cdots + e_r$  and  $x$  to  $x$ . Clearly, it is well defined.

On the other hand, the morphism  $\pi : A \rightarrow B$  is given as follows:  $\pi(e_j) = \delta_{ij}e_i$ , for  $1 \leq j \leq r$ , and the restriction of  $\pi$  to the arrows of  $A$  is given by  $\pi(y) = \delta_{yx}x$ , where  $\delta$  is the Kronecker delta. If we assume that  $I^A = \langle x^n, f_2, \dots, f_s \rangle$  is admissible and that  $f_i$  belongs to the two-sided ideal generated by  $Q_1^A - \{x\}$ , then it is straightforward to check that  $\pi$  is also well defined and  $\pi \circ \iota = id_B$ .

As a consequence of the results of Section 2, we see that the Hochschild homology dimension, denoted  $\text{hhdim}(B)$ , is infinite, and so the same holds for  $A$ . Being both  $k$ -finite dimensional, their global dimensions cannot be finite.

It is interesting to note that whenever  $A$  and  $B$  are finite dimensional  $k$ -algebras provided by morphisms  $\pi$  and  $\iota$  satisfying  $\pi \circ \iota = id$  as in the previous section and  $B$  has infinite Hochschild dimension,  $\text{hhdim}(A)$  will also be infinite. For example, let  $\text{char}(k) = 0$  and  $B$  be a finite dimensional monomial  $k$ -algebra with generators  $x_1, \dots, x_r$  such that  $\text{hhdim}(B) \neq 0$ , and let  $A$  be a finite dimensional  $k$ -algebra generated by  $x_1, \dots, x_r, \dots, x_n$  such that  $B \cong A/\langle f_1, \dots, f_s \rangle$  and  $f_i$  belongs to the two-sided ideal  $\langle x_{r+1}, \dots, x_n \rangle$ ,  $1 \leq i \leq s$ . Then  $\text{hhdim}(B) = \infty$  [11], and the same is true for  $A$ .

#### 4. HOCHSCHILD HOMOLOGY IN THE DIFFERENTIAL GRADED CASE

In this section we deal with finite dimensional algebras.

**4.1. Notation.** We use the methods of differential graded algebra of [7]. In particular an element of lower degree  $i \in \mathbb{Z}$  is, by the classical convention, of upper degree  $-i$ . All the algebras considered from now on are unital, associative, with a differential of degree  $-1$ . We recall that if  $V = \bigoplus_{i \in \mathbb{Z}} V_i$  is a graded  $k$ -vector space, then the suspended graded  $k$ -vector space  $sV$  has homogeneous components  $(sV)_i = V_{i-1}$ , for  $i \in \mathbb{Z}$ . The  $k$ -algebra  $TV$  will denote the tensor algebra on  $V$ . The degree of an element  $v \in V$  is denoted by  $|v|$ .

For any differential graded algebra  $A$ , let  $A^{op}$  be the opposite graded algebra, and  $A^e = A \otimes A^{op}$  be the enveloping algebra. The categories of graded  $A$ -bimodules and of left (or right) differential graded  $A^e$ -modules are equivalent.

**4.2. Bar resolution and Hochschild homology.** Let  $(A, d)$  be an augmented algebra and let  $\bar{A} = \text{Ker}(\epsilon : A \rightarrow k)$ . The normalized bar resolution of  $A$ , denoted by  $B(A, A, A)$ , is the differential graded  $A^e$ -module  $(A \otimes T(s\bar{A}) \otimes A, D_0 + D_1)$ , where  $D_0$  is the differential induced by  $d$  on the tensor product of complexes and  $D_1$  is defined as follows (see for example [9], 2.2):

$$\begin{aligned} D_1(a \otimes sa_1 \otimes \cdots \otimes sa_n \otimes b) &= (-1)^{|a|} aa_1 \otimes sa_2 \otimes \cdots \otimes sa_n \otimes b \\ &\pm \sum_{i=1}^{n-1} a \otimes sa_1 \otimes \cdots \otimes s(a_i a_{i+1}) \otimes \cdots \otimes sa_n \otimes b \\ &\pm a \otimes sa_1 \otimes \cdots \otimes sa_{n-1} \otimes a_n b. \end{aligned}$$

The Hochschild homology of the differential graded algebra  $(A, d)$  is, by definition, the graded vector space  $\mathcal{HH}_*(A) = \text{Tor}_*^{A^e}(A, A)$  in the differential sense of [14].

**Lemma 4.1** ([7]). *The canonical map  $m : B(A, A, A) \rightarrow A$  defined by 0 on  $A \otimes T^{\geq 1}(s\bar{A}) \otimes A$  and by multiplication on  $A \otimes A$  is a semifree resolution of  $A$  as an  $A^e$ -module.*

Consequently we have,

$$\mathcal{H}\mathcal{H}_*(A, d) = H_*(\mathcal{C}_*(A), \delta)$$

with

$$\mathcal{C}_*(A) = A \otimes_{A^e} B(A, A, A) = A \otimes T(s\bar{A}),$$

and  $\delta = \delta_0 + \delta_1$ , where  $\delta_0$  and  $\delta_1$  are obtained by tensorization. Explicitly,

$$\begin{aligned} \delta_1(a \otimes sa_1 \otimes \cdots \otimes sa_n) &= (-1)^{|a|} aa_1 \otimes sa_2 \otimes \cdots \otimes sa_n \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\epsilon_i} a \otimes sa_1 \otimes \cdots \otimes s(a_i a_{i+1}) \otimes \cdots \otimes sa_n \\ &\quad + (-1)^{\epsilon_n} a_n a \otimes sa_1 \otimes \cdots \otimes sa_{n-1}, \end{aligned}$$

where the  $\epsilon_i$ 's are integers depending on the degrees of the elements  $a_i$ . If all these degrees are even, then  $\epsilon_i = i$ .

In the rest of this paper we consider only differential graded algebras  $(A, d)$  satisfying either condition (a) or condition (b) below:

- (a)  $A_n = 0$  for  $n < 0$  and  $A_0 = k$ , so that  $\mathcal{C}_n(A) = 0$  for  $n < 0$ ;
- (b)  $A_n = 0$  for  $n > 0$ ,  $A_0 = k$ , and  $A_{-1} = 0$ , so that  $\mathcal{C}_n(A) = 0$  for  $n > 0$ .

In both cases, we have  $\mathcal{C}_0(A) = k$ .

**4.3. Cobar construction and duality construction in Hochschild homology.** We next recall the definition of the cobar construction described in Section 19 of [8]. Let  $(C, d_C)$  be a coaugmented differential graded coalgebra with comultiplication  $\Delta$ , and let  $\bar{C} = \text{Ker}(\epsilon : C \rightarrow k)$ . We denote  $(\Omega C, d)$  as the augmented differential graded algebra defined as follows:

- $\Omega C = T(s^{-1}\bar{C})$ , as an augmented graded algebra;
- $d = d_0 + d_1$ , with  $d_0(s^{-1}c) = -s^{-1}(d_C(c))$ , if  $c \in \bar{C}$ , and  $d_1$  is defined from  $\Delta$ .

Suppose now that  $(A, d_A)$  is a finite dimensional differential graded algebra. Then the graded dual  $A^\vee = \text{Hom}_k(A, k)$  is a differential graded coalgebra with differential  $d_A^\vee$ , the transpose of  $d_A$ .

**Definition 4.2.**  $(\Omega^* A, d) := (\Omega(A^\vee), d)$ , where  $d$  is defined from  $d_A^\vee$  and the comultiplication of  $A^\vee$  as above.

We have  $\Omega^* A = T(V)$  with  $V = \text{Hom}_k(s\bar{A}, k)$ . If  $(A, d_A)$  satisfies condition (b) above, then

$$V = \bigoplus_{n \geq 1} V_n, \text{ with } V_n = \text{Hom}_k(A_{-n-1}, k)$$

and then  $(\Omega^* A, d)$  satisfies condition (a). Similarly, if  $(A, d_A)$  satisfies condition (a), then  $(\Omega^* A, d)$  satisfies condition (b).

The first ingredient used to prove Theorem II is the following duality property.

**Theorem 4.3** ([10], [16]). *Let  $(A, d_A)$  be a finite dimensional algebra satisfying either condition (a) or (b) above. Then for all  $n \in \mathbb{Z}$  we have*

$$\text{Hom}_k(\mathcal{H}\mathcal{H}_{-n}(A), k) = \mathcal{H}\mathcal{H}_n(\Omega^* A).$$

Consequently, the computation of the graded vector space  $\mathcal{HH}_n(A)$  can be replaced by the computation of the Hochschild homology of a quasifree differential graded algebra  $(T(V), d)$ .

**4.4. A short complex for the computation of the Hochschild homology.** Now, we want to compute the Hochschild homology of  $(T(V), d)$ , with  $V = \bigoplus_{n \geq 1} V_n$ .

We recall here the main results of [17]. Put  $(T(V), d) = (B, d)$  and let  $P = (B \otimes B) \oplus (B \otimes (sV) \otimes B)$ . We define a differential  $D$  on  $P$ , which is the tensor product of the differentials on  $B \otimes B$ , and

$$D(a \otimes sv \otimes b) = da \otimes sv \otimes b \pm (av \otimes b - a \otimes vb) + S(a \otimes sv \otimes b),$$

where  $S(a \otimes sv \otimes b) \in B \otimes sV \otimes B$ , for  $a \in B, b \in B$  and  $v \in V$ .

**Proposition 4.4** (Thm. 1.4 in [17]). *The canonical map  $m : (P, D) \rightarrow B$  defined as 0 on  $B \otimes sV \otimes B$  and as multiplication on  $B \otimes B$  is a semifree resolution of  $B$  as a  $B^e$ -module.*

As a consequence,

$$\mathcal{HH}_*(T(V), d) = H_*(B \otimes_{B^e} P, \delta),$$

with differential  $\delta = d \otimes_{B^e} D$  that will be made precise in the next section. We have:

- $\delta|_{T(V)} = d$ ;
- $\delta(a \otimes sv) = da \otimes sv + (-1)^{|a|}(av - (-1)^{|v| \times |a|}va) - \sigma(a \otimes dv)$ , where  $\sigma(a \otimes dv)$  belongs to  $T(V) \otimes sV$ , for  $a \in T(V), v \in V$ .

Put  $Q_* := B \otimes_{B^e} P = T(V) \oplus (T(V) \otimes sV)$ .

**Theorem 4.5** (Thm. 1.5 of [17]). *With the above notation,*

$$\mathcal{HH}_*(T(V), d) = H_*(Q_*, \delta).$$

In the following section we will use the complex  $(Q_*, \delta)$  to compute the Hochschild homology of a finite dimensional graded algebra  $A = \bigoplus_{n \geq 0} A^n$ , with  $A^0 = k$ . In this case, the graded vector space  $V$  is also finite dimensional, and the differential  $\delta$  has good properties.

## 5. PROOF OF THEOREM II

We work with a finite dimensional graded algebra with  $A^0 = k$ . We may assume without loss of generality that  $A$  is graded in even degrees,  $A = k \oplus \left( \bigoplus_{n \geq 2} A^n \right)$ , and  $\bar{A} = \bigoplus_{n \geq 2} A^n$  is nonzero.

**5.1. Relations between  $HH_*(A)$  and  $\mathcal{HH}_*(A, 0)$ .** Using the conventions recalled at the beginning of the previous section, we consider  $A$  as a differential graded algebra with differential 0 and  $A_{-n} = A^n$ .

Since  $A$  is graded, the ordinary Hochschild homology  $HH_*(A)$  defined in Section 2 is graded, and there is a decomposition

$$HH_*(A) = \bigoplus_{p, q \geq 0} HH_p(A)^q.$$

Since  $A$  is finite dimensional,  $HH_p(A)$  is finite dimensional for all  $p$ .

**Lemma 5.1.** *Let  $A$  be an algebra as above. Then,*

- (1)  $\mathcal{H}\mathcal{H}_*(A, 0) = \bigoplus_{n \geq 0} \mathcal{H}\mathcal{H}_{-n}(A)$  and  $\mathcal{H}\mathcal{H}_{-n}(A) = \bigoplus_p HH_p(A)^{p+n}$ .
- (2)  $HH_p(A)^{p+n} = 0$  if  $p > n$  or  $p < \frac{n-N}{N-1}$ , where  $N = \sup\{n | A^n \neq 0\}$ .

**Corollary 5.2.** *If there exists an increasing sequence of integers  $n_i$  such that  $\mathcal{H}\mathcal{H}_{-n_i}(A) \neq 0$ , then  $HH_*(A)$  is not finite dimensional.*

The strategy now is to focus our attention on  $\mathcal{H}\mathcal{H}_*(\Omega^*A)$  using Theorem 4.3. But Theorem 4.5 allows us to use the short complex  $(Q_*, \delta)$  to compute  $\mathcal{H}\mathcal{H}_*(\Omega^*A)$ , so we will work with this last one.

**5.2. Description of  $(Q_*, \delta)$ .** Let  $A = k \oplus \left(\bigoplus_{n \geq 2} A^n\right)$  be a finite dimensional graded algebra. We fix a homogeneous linear basis  $(a_i)_{i \in I}$  for  $\bar{A} = \bigoplus_{n \geq 2} A^n$ . This choice determines the structure constants  $\alpha_{jk}^i$  by the equalities  $a_j a_k = \sum \alpha_{jk}^i a_i$ .

In this situation,  $(\bar{A})^\vee = \text{Hom}_k(\bar{A}, k)$  is endowed with the dual basis  $(b_i)_{i \in I}$  satisfying  $\langle b_i, a_j \rangle = \delta_{ij}$ . Notice that  $A^\vee$  is a graded coalgebra with comultiplication  $\Delta$ , and  $\Delta b_i = \sum_{j,k} \beta_i^{jk} b_j \otimes b_k$ , where  $\alpha_{jk}^i = (-1)^{|a_j| \times |a_k|} \beta_i^{jk}$ .

We have already defined  $(\Omega^*A, d) = (\Omega(A^\vee), d) = (T(V), d)$ . Now, put  $v_i = s^{-1}b_i$ ; then  $|v_i| = n - 1$  if  $a_i \in A^n$ . We check that

$$dv_i = \sum_{j,k} (-1)^{|a_j| + |a_j| \times |a_k|} \alpha_{jk}^i v_j \otimes v_k.$$

So  $(\Omega^*A, d) = (T(V), d)$  is a tensor algebra with a quadratic differential.

Furthermore, we have assumed without loss of generality that  $A$  is graded in even degrees, so that  $V$  is graded only in odd degrees. In this case, we give an explicit formula for the differential  $\delta$  on  $Q_*$  (cf. Subsection 4.4).

Put  $\bar{V} = sV$ ; then  $Q_* = T(V) \oplus T(V) \otimes \bar{V}$ . Let  $v$  be an element in  $V$ , and let  $dv = \sum_{j,k} \lambda_{jk} v_j \otimes v_k$  with  $\lambda_{jk} \in k$ . Let  $a$  be an element in  $T(V)$ .

We have

$$\delta(a \otimes \bar{v}) = da \otimes \bar{v} + (-1)^{|a|} (av - (-1)^{|a|} va) - \sigma(a \otimes dv),$$

where

$$\sigma(a \otimes dv) = -(-1)^{|a|} \sum_{j,k} \lambda_{jk} a v_j \otimes \bar{v}_k + \sum_{j,k} \lambda_{jk} v_k a \otimes \bar{v}_j.$$

**5.3. A nice homogeneous basis  $(a_i)$  for  $\bar{A}$ .** Since  $A = k \oplus \bar{A}$ , the projection  $\bar{A} \rightarrow \bar{A}/\bar{A}^2 = U$  has a section  $\rho$  that extends to a morphism of algebras  $T(U) \rightarrow A$  whose kernel is contained in  $T^{\geq 2}(U)$ . This implies that  $(x_i)_{1 \leq i \leq p}$  are generators of the algebra  $A$  if and only if their images in  $\bar{A}/\bar{A}^2$  form a basis of this vector space.

As vector spaces,  $\bar{A} = \bar{A}/\bar{A}^2 \oplus \bar{A}^2$ , and we will consider a homogeneous basis of  $\bar{A}/\bar{A}^2$  and a basis of  $\bar{A}^2$ . If  $a_i \in \bar{A}/\bar{A}^2$ , then the corresponding  $v_i$  in  $(\Omega^*A, d)$  satisfies  $dv_i = 0$ .

We will now prove the following result.

**Theorem 5.3.** *Let  $A = \bigoplus_{n \geq 0} A^n$  be a finite dimensional graded  $k$ -algebra with  $A^0 = k$ , such that  $\bar{A} = \bigoplus_{n \geq 1} A^n$  is not zero. Assume that there exist two generators  $x$  and  $y$  of the algebra  $A$  satisfying  $xy = yx = 0$ . Then  $H_{n_i}(Q_*, \delta) \neq 0$  for a strictly increasing sequence of integers  $(n_i)$ .*

*Proof.* We can associate to  $x$  and  $y$  two elements  $a_1$  and  $a_2$ , linearly independent in  $\bar{A}$ . We denote by  $v_1$  and  $v_2$  the corresponding elements in a dual basis of  $V$ . If  $(a_1, \dots, a_n)$  is a linear basis of  $\bar{A}$  and  $(v_1, \dots, v_n)$  is the corresponding basis of  $V$ , then we have  $dv_1 = 0$ ,  $dv_2 = 0$  and, for  $i \geq 3$ ,

$$dv_i = \sum_{j,k} \alpha_{jk}^i v_j \otimes v_k.$$

The fact that  $xy = yx = 0$  implies that, for  $i \geq 3$ ,  $\alpha_{12}^i = \alpha_{21}^i = 0$ .

For  $n \geq 1$ , consider

$$X_n = v_1 \otimes v_2 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_1 \otimes \bar{v}_2 - v_2 \otimes v_1 \otimes v_2 \otimes v_1 \otimes \cdots \otimes v_2 \otimes \bar{v}_1 \in V^{\otimes(2n-1)} \otimes \bar{V}.$$

It is easy to see that  $|X_n| = n(|v_1| + |v_2|) + 1$  and that  $\delta X_n = 0$ .

If  $X_n$  was a boundary, there should exist  $Y, b_i \in T(V)$  such that

$$X_n = \delta(Y + \sum_i b_i \otimes \bar{v}_i)$$

and

$$X_n = dY + \sum_i db_i \otimes \bar{v}_i + \sum_i (b_i v_i - v_i b_i) + \sum_i \alpha_{jk}^i b_i v_j \otimes \bar{v}_k - \sum_i \alpha_{jk}^i v_k b_i \otimes \bar{v}_j.$$

Such elements cannot exist since, for all  $i$ ,

$$dv_i = \sum_{j,k} \alpha_{jk}^i v_j \otimes v_k \text{ with } \alpha_{12}^i = \alpha_{21}^i = 0. \quad \square$$

**Example 5.4.** Let  $A = k\langle x, y, z \rangle / (xy, yx, x^2 - y^2, x^2 - z^2, xz - qzx, yz - qzy)$ , where  $q \in k$ ,  $q^2 \neq 1$  and  $-1$  is not a square in  $k$ . This example is not covered by Theorem I.

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