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Combinatorics of binomial primary decomposition

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Abstract An explicit lattice point realization is provided for the primary components of an arbitrary binomial ideal in characteristic zero. This decomposition is derived from a characteristic-free combinatorial description of certain primary components of binomial ideals in affine semigroup rings, namely those that are associated to faces of the semigroup. These results are intimately connected to hypergeometric differential equations in several variables.

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1 Introduction

A binomial is a polynomial with at most two terms; a binomial ideal is an ideal generated by binomials. Binomial ideals abound as the defining ideals of classical algebraic varieties,

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particularly because equivariantly embedded affine or projective toric varieties correspond to prime binomial ideals.

In fact, the zero set of any binomial ideal is a union of (translated) toric varieties. Thus, binomial ideals are "easy" in geometric terms, and one may hope that their algebra is simple as well. This is indeed the case: the associated primes of a binomial ideal are essentially toric ideals, and their corresponding primary components can be chosen binomial as well. These results, due to Eisenbud and Sturmfels [5], are concrete when it comes to specifying associated primes, but less so when it comes to primary components themselves, in part because of difficulty in identifying the monomials therein.

The main goal of this article is to provide explicit lattice-point combinatorial realizations of the primary components of an arbitrary binomial ideal in a polynomial ring over an algebraically closed field of characteristic zero; this is achieved in Theorems 3.2 and 4.13. These are proved by way of our other core result, Theorem 2.15, which combinatorially characterizes, in the setting of an affine semigroup ring over an arbitrary field (not required to be algebraically closed or of characteristic zero), primary binomial ideals whose associated prime comes from a face of the semigroup.

The hypotheses on the field $\Bbbk$ are forced upon us, when they occur. Consider the univariate case: in the polynomial ring $\Bbbk[x]$, primary decomposition is equivalent to factorization of polynomials. Factorization into binomials in this setting is the fundamental theorem of algebra, which requires $\Bbbk$ to be algebraically closed. On the other hand, $\Bbbk$ must have characteristic zero because of the slightly different behavior of binomial primary decomposition in positive characteristic [5]: in characteristic zero, every primary binomial ideal contains all of the non-monomial (i.e., two-term binomial) generators of its associated prime, but this is false in positive characteristic.

The motivation and inspiration for this work came from the theory of hypergeometric differential equations, and the results here are used heavily in the companion article [3] (see Sect. 5 for an overview of these applications). In fact, these two projects began with a conjectural expression for the non fully supported solutions of a Horn hypergeometric system; its proof reduced quickly to the statement of Corollary 4.14, which directed all of the developments here. Our consequent use of M-subgraphs (Definition 2.6), and more generally the application of commutative monoid congruences toward the primary decomposition of binomial ideals, serves as an advertisement for hypergeometric intuition as inspiration for developments of independent interest in combinatorics and commutative algebra.

The explicit lattice-point binomial primary decompositions in Sects. 3 and 4 have potential applications beyond hypergeometric systems. Consider the special case of monomial ideals: certain constructions at the interface between commutative algebra and algebraic geometry, such as integral closure and multiplier ideals, admit concrete convex polyhedral descriptions. The path is now open to attempt analogous constructions for binomial ideals.

1.1 Overview of results

The central combinatorial idea behind binomial primary decomposition is elementary and concrete, so we describe it geometrically here. The notation below is meant to be intuitive, but in any case it coincides with the notation formally defined later.

Fix an arbitrary binomial ideal I in a polynomial ring, or more generally in any affine semigroup ring $\Bbbk[Q]$. Then I determines an equivalence relation (*congruence*) on $\Bbbk[Q]$ in which two elements $u, v \in Q$ are congruent, written $u \sim v$, if $\mathbf{t}^u - \lambda \mathbf{t}^v \in I$ for some $\lambda \neq 0$. If one makes a graph with vertex set Q by drawing an edge from u to v when $u \sim v$, then the congruence classes of $\sim$ are the connected components of this graph. For example, for an integer matrix A with n columns, the *toric ideal*

$$I_A = \langle \mathbf{t}^u - \mathbf{t}^v \mid u, v \in \mathbb{N}^n \text{ and } Au = Av \rangle \subseteq \mathbb{C}[t_1, \ldots, t_n] = \mathbb{C}[\mathbf{t}] = \mathbb{C}[\mathbb{N}^n]$$

determines the congruence in which the class of $u \in \mathbb{N}^n$ consists of the set $(u + \ker(A)) \cap \mathbb{N}^n$ of lattice points in the polyhedron $\{\alpha \in \mathbb{R}^n \mid A\alpha = Au \text{ and } \alpha \geq 0\}$.

The set of congruence classes for I_A can be thought of as a periodically spaced archipelago whose islands have roughly similar shape (but steadily grow in size as the class moves interior to $\mathbb{N}^n$). With this picture in mind, the congruence classes for any binomial ideal I come in a finite family of such archipelagos, but instead of each island coming with a shape predictable from its archipelago, its boundary becomes fragmented into a number of surrounding skerries. The extent to which a binomial ideal deviates from primality is measured by which bridges must be built—and in which directions—to join the islands to their skerries.

When $Q = \mathbb{N}^n$ is an orthant as in the previous example, each prime binomial ideal in $\mathbb{C}[Q] = \mathbb{C}[\mathbb{N}^n]$ equals the sum $\mathfrak{p} + \mathfrak{m}_J$ of a prime binomial ideal $\mathfrak{p}$ containing no monomials and a prime monomial ideal $\mathfrak{m}_J$ generated by the variables whose indices lie outside of a subset $J \subseteq \{1, \ldots, n\}$. The ideal $\mathfrak{p}$ is a toric ideal after rescaling the variables, so its congruence classes are parallel to a sublattice $L = L_{\mathfrak{p}} \subseteq \mathbb{Z}^J$; in the notation above, $L = \ker(A)$. Now suppose that $\mathfrak{p} + \mathfrak{m}_J$ is associated to our binomial ideal I. Joining the aforementioned skerries to their islands is accomplished by considering congruences defined by $I + \mathfrak{p}$: a bridge is built from u to v whenever $u - v \in L$.

To be more accurate, just as $I + \mathfrak{p}$ determines a congruence on $\mathbb{N}^n$, it determines one—after inverting the variables outside of $\mathfrak{m}_J$—on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$, where $\mathbb{Z}^J = \operatorname{span}_{\mathbb{Z}}\{e_j \mid j \in J\}$, and $\mathbb{N}^{\bar{J}}$ is defined analogously for the index subset $\bar{J}$ complementary to J. Each resulting class in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ is acted upon by L and hence is a union of cosets of L. The key observation is that, when $\mathfrak{p} + \mathfrak{m}_J$ is an associated prime of I, some of these classes consist of finitely many cosets of L; let us call these *L-bounded* classes. The presence of L-bounded classes signals that L is "sufficiently parallel" to the congruence determined by I, and this is how we visualize the manner in which $\mathfrak{p} + \mathfrak{m}_J$ is associated to I.

Intersecting the L-bounded $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ congruence classes with $\mathbb{N}^n$ yields *L-bounded* classes in $\mathbb{N}^n$; again, these are constructed more or less by building bridges in directions from L to join the classes defined by I. When the prime $\mathfrak{p} + \mathfrak{m}_J$ is minimal over I, there are only finitely many L-bounded classes in $\mathbb{N}^n$, up to translation by $\mathbb{N}^J$. In this case, the primary component $C_{\mathfrak{p}+\mathfrak{m}_J}$ of I is well-defined, as reflected in its combinatorics: the congruence defined on $\mathbb{N}^n$ by $C_{\mathfrak{p}+\mathfrak{m}_J}$ has one huge class consisting of the lattice points in $\mathbb{N}^n$ lying in no L-bounded class, and each of its remaining classes is L-bounded in $\mathbb{N}^n$; this is the content of Theorem 3.2.1. The only difference for a nonminimal associated prime of I is that the huge class is inflated by swallowing all but a sufficiently large finite number of the orbits under translation by $\mathbb{N}^J$ of L-bounded classes; this is the content of the remaining parts of Theorem 3.2. Here, "sufficiently large" means that every swallowed L-bounded class contains a lattice point u with $\mathbf{t}^u$ lying in a fixed high power of $\mathfrak{m}_J$.

In applications, binomial ideals often arise in the presence of multigradings. (One reason for this is that binomial structures are closely related to algebraic tori, whose actions on varieties induce multigradings on coordinate rings.) In this context, the matrix A above induces the grading in which two monomials $\mathbf{t}^u$ and $\mathbf{t}^v$ have equal degree if and only if $Au = Av$. Theorem 4.13 expounds on the observation that if $L = \ker(A) \cap \mathbb{Z}^J$, then a congruence class for $I + \mathfrak{p}$ in $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ is L-bounded if and only if its image in $\mathbb{N}^{\bar{J}}$ is finite. This simplifies the description of the primary components because, to describe the set of monomials in a primary component, it suffices to refer to lattice point geometry in $\mathbb{N}^{\bar{J}}$, without mentioning $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$.

When it comes to proofs, the crucial insight is that the geometry of L-bounded classes for the congruence determined by $I + \mathfrak{p}$ gives rise to simpler algebra when $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ is reduced modulo the action of L. Equivalently, instead of considering the associated prime $\mathfrak{p} + \mathfrak{m}_J$ of an arbitrary binomial ideal I in $\mathbb{C}[t_1, \dots, t_n]$, consider the prime image of the monomial ideal $\mathfrak{m}_J$ associated to the image of I in $\mathbb{C}[Q] = \mathbb{C}[\mathbf{t}]/\mathfrak{p}$, where $Q = \mathbb{N}^n/L$. Since monomial primes in an affine semigroup ring $\mathbb{C}[Q]$ correspond to faces of Q, the lattice point geometry is clearer in this setting, and the algebra is sufficiently uncomplicated that it works over an arbitrary field in place of $\mathbb{C}$. If Φ is a face of an arbitrary affine semigroup Q whose corresponding prime $\mathfrak{p}_\Phi$ is minimal over a binomial ideal I in $\Bbbk[Q]$, then I determines a congruence on the semigroup $Q + \mathbb{Z}\Phi$ obtained from Q by allowing negatives for Φ. The main result in this context, Theorem 2.15, says that the monomials $\mathbf{t}^u$ in the $\mathfrak{p}_\Phi$-primary component of I are precisely those corresponding to lattice points $u \in Q$ not lying in any finite congruence class of $Q + \mathbb{Z}\Phi$. This, in turn, is proved by translating lattice point geometry and combinatorics into semigroup-graded commutative algebra in Proposition 2.13.

2 Binomial ideals in affine semigroup rings

Our eventual goal is to analyze the primary components of binomial ideals in polynomial rings over the complex numbers $\mathbb{C}$ or any algebraically closed field of characteristic zero. Our principal result along these lines (Theorem 3.2) is little more than a rephrasing of a statement (Theorem 2.15) about binomial ideals in arbitrary affine semigroup rings in which the associated prime comes from a face, combined with results of Eisenbud and Sturmfels [5]. The developments here stem from the observation that quotients by binomial ideals are naturally graded by noetherian commutative monoids. Our source for such monoids is Gilmer's excellent book [9]. For the special case of affine semigroups, by which we mean finitely generated submonoids of free abelian groups, see [14, Chapter 7]. We work in this section over an arbitrary field $\Bbbk$, so it might be neither algebraically closed nor of characteristic zero.

Definition 2.1 A *congruence* on a commutative monoid Q is an equivalence relation $\sim$ with

$$u \sim v \quad \Longrightarrow \quad u+w \sim v+w \quad \text{for all } w \in Q.$$

The *quotient monoid* $Q/{\sim}$ is the set of equivalence classes under addition.

Definition 2.2 The *semigroup algebra* $\Bbbk[Q]$ is the direct sum $\bigoplus_{u\in Q} \Bbbk \cdot \mathbf{t}^u$, with multiplication $\mathbf{t}^u \mathbf{t}^v = \mathbf{t}^{u+v}$. Any congruence $\sim$ on Q induces a $(Q/{\sim})$-grading on $\Bbbk[Q]$ in which the *monomial* $\mathbf{t}^u$ has degree $\Gamma \in Q/{\sim}$ whenever $u \in \Gamma$. A *binomial ideal* $I \subseteq \Bbbk[Q]$ is an ideal generated by *binomials* $\mathbf{t}^u - \lambda \mathbf{t}^v$, where $\lambda \in \Bbbk$ is a scalar, possibly equal to zero.

Example 2.3 A *pure difference* binomial ideal is generated by differences of monic monomials. Given an integer matrix M with q rows, we call $I(M) \subseteq \Bbbk[t_1, \dots, t_q] = \Bbbk[\mathbb{N}^q]$ the pure difference binomial ideal

$$\begin{aligned} I(M) &= \langle \mathbf{t}^u - \mathbf{t}^v \mid u - v \text{ is a column of } M,\ u, v \in \mathbb{N}^q \rangle \\ &= \langle \mathbf{t}^{w_+} - \mathbf{t}^{w_-} \mid w = w_+ - w_- \text{ is a column of } M \rangle. \end{aligned} \tag{2.1}$$

Here and in the remainder of this article we adopt the convention that, for an integer vector $w \in \mathbb{Z}^q$, the vector w_+ has ith coordinate w_i if $w_i \geq 0$ and 0 otherwise. The vector $w_- \in \mathbb{N}^q$ is defined by $w_+ - w_- = w$, or equivalently, $w_- = (-w)_+$. If the columns of M are linearly independent, the ideal $I(M)$ is called a *lattice basis ideal* (cf. Example 4.10). An ideal of $\Bbbk[t_1, \ldots, t_q]$ has the form described in (2.1) if and only if it is generated by differences of monomials with disjoint support.

The equality of the two definitions in (2.1) is easy to see: the ideal in the first line of the display contains the ideal in the second line by definition; and the disjointness of the supports of w_+ and w_- implies that whenever $u - v = w$ is a column of M, and denoting by $\alpha := u - w_+ = u - w_-$, we have that the corresponding generator of the first ideal $\mathbf{t}^u - \mathbf{t}^v = \mathbf{t}^\alpha(\mathbf{t}^{w_+} - \mathbf{t}^{w_-})$, lies in the second ideal.

Proposition 2.4 *A binomial ideal $I \subseteq \Bbbk[Q]$ determines a congruence $\sim_I$ under which*

$$u \sim_I v \text{ if } \mathbf{t}^u - \lambda \mathbf{t}^v \in I \quad \text{for some scalar } \lambda \neq 0.$$

The ideal I is graded by the quotient monoid $Q_I = Q/\sim_I$, and $\Bbbk[Q]/I$ has Q_I-graded Hilbert function 1 on every congruence class except the class $\{u \in Q \mid \mathbf{t}^u \in I\}$ of monomials.

Proof That $\sim_I$ is an equivalence relation is because $\mathbf{t}^u - \lambda\mathbf{t}^v \in I$ and $\mathbf{t}^v - \lambda'\mathbf{t}^w \in I$ implies $\mathbf{t}^u - \lambda\lambda'\mathbf{t}^w \in I$. It is a congruence because $\mathbf{t}^u - \lambda\mathbf{t}^v \in I$ implies that $\mathbf{t}^{u+w} - \lambda\mathbf{t}^{v+w} \in I$. The rest is similarly straightforward. □

Example 2.5 In the case of a pure difference binomial ideal $I(M)$ as in Example 2.3, the congruence classes under $\sim_{I(M)}$ from Proposition 2.4 are the M-subgraphs in the following definition, which—aside from being a good way to visualize congruence classes—will be useful later on (see Example 2.12 and Corollary 4.14, as well as Sect. 5).

Definition 2.6 Any integer matrix M with q rows defines an undirected graph $\Gamma(M)$ having vertex set $\mathbb{N}^q$ and an edge from u to v if $u - v$ or $v - u$ is a column of M. An *M-path* from u to v is a path in $\Gamma(M)$ from u to v. A subset of $\mathbb{N}^q$ is *M-connected* if every pair of vertices therein is joined by an M-path passing only through vertices in the subset. An *M-subgraph* of $\mathbb{N}^q$ is a maximal M-connected subset of $\mathbb{N}^q$ (a connected component of $\Gamma(M)$). An M-subgraph is *bounded* if it has finitely many vertices, and *unbounded* otherwise. (See Example 5.4 for a concrete computation and an illustrative figure).

These M-subgraphs bear a marked resemblance to the concept of *fiber* in [18, Chapter 4]. The interested reader will note, however, that even if these two notions have the same flavor, their definitions have mutually exclusive assumptions, since for a square matrix M, the corresponding matrix A in [18] is empty.

Definition 2.7 Let Q be an affine semigroup in $\mathbb{Z}^n$ and let $H \subset \mathbb{R}^n$ be a hyperplane. If Q is contained in one of the two (closed) half-spaces determined by H, then $\Phi = Q \cap H$ is called a *face* of Q. A subset $S \subseteq Q$ is an *ideal* if $Q + S \subseteq S$. In that case, we have that

$$\operatorname{span}_{\Bbbk} S = \langle \mathbf{t}^u \mid u \in S \rangle$$

is the monomial ideal in $\Bbbk[Q]$ having S as its $\Bbbk$-basis.

Given a face Φ of an affine semigroup $Q \subseteq \mathbb{Z}^\ell$, the *localization* of Q along Φ is the affine semigroup $Q + \mathbb{Z}\Phi$ obtained from Q by adjoining negatives of the elements in Φ. The algebraic version of this notion is a common tool for affine semigroup rings [14, Chap. 7]:

for each $\Bbbk[Q]$-module V, let $V[\mathbb{Z}\Phi]$ denote its *homogeneous localization* along Φ, obtained by inverting $\mathbf{t}^\phi$ for all $\phi \in \Phi$. For example, $\Bbbk[Q][\mathbb{Z}\Phi] \cong \Bbbk[Q + \mathbb{Z}\Phi]$. Writing

$$\mathfrak{p}_\Phi = \operatorname{span}_\Bbbk\{\mathbf{t}^u \mid u \in Q \smallsetminus \Phi\} \subseteq \Bbbk[Q]$$

for the prime ideal of the face Φ, so that $\Bbbk[Q]/\mathfrak{p}_\Phi = \Bbbk[\Phi]$ is the affine semigroup ring for Φ, we find, as a consequence, that $\mathfrak{p}_\Phi[\mathbb{Z}\Phi] = \mathfrak{p}_{\mathbb{Z}\Phi} \subseteq \Bbbk[Q + \mathbb{Z}\Phi]$, because

$$\Bbbk[Q + \mathbb{Z}\Phi]/\mathfrak{p}_\Phi[\mathbb{Z}\Phi] = (\Bbbk[Q]/\mathfrak{p}_\Phi)[\mathbb{Z}\Phi] = \Bbbk[\Phi][\mathbb{Z}\Phi] = \Bbbk[\mathbb{Z}\Phi].$$

(We write equality signs to denote canonical isomorphisms.) For any ideal $I \subseteq \Bbbk[Q]$, the localization $I[\mathbb{Z}\Phi]$ equals the extension $I\,\Bbbk[Q + \mathbb{Z}\Phi]$ of I to $\Bbbk[Q + \mathbb{Z}\Phi]$, and we write

$$(I : \mathbf{t}^\Phi) = I[\mathbb{Z}\Phi] \cap \Bbbk[Q], \tag{2.2}$$

the intersection taking place in $\Bbbk[Q + \mathbb{Z}\Phi]$. Equivalently, $(I : \mathbf{t}^\Phi)$ is the usual colon ideal $(I : \mathbf{t}^\phi)$ for any element ϕ sufficiently interior to Φ (for example, take ϕ to be a high multiple of the sum of the generators of Φ); in particular, $(I : \mathbf{t}^\Phi)$ is a binomial ideal when I is.

For the purpose of investigating $\mathfrak{p}_\Phi$-primary components, the ideal $(I : \mathbf{t}^\Phi)$ is as good as I itself, since this colon operation does not affect such components, or better, since the natural map from $\Bbbk[Q]/(I : \mathbf{t}^\Phi)$ to its homogeneous localization along Φ is injective. Combinatorially, what this means is the following.

Lemma 2.8 *A subset $\Gamma' \subseteq Q$ is a congruence class in $Q_{(I:\mathbf{t}^\Phi)}$ determined by $(I : \mathbf{t}^\Phi)$ if and only if $\Gamma' = \Gamma \cap Q$ for some class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under the congruence $\sim_{I[\mathbb{Z}\Phi]}$.*

Lemma 2.9 *If a congruence class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under $\sim_{I[\mathbb{Z}\Phi]}$ has two distinct elements whose difference lies in $Q + \mathbb{Z}\Phi$, then for all $u \in \Gamma$ the monomial $\mathbf{t}^u$ maps to 0 in the (usual inhomogeneous) localization $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$ inverting all elements not in $\mathfrak{p}_\Phi$.*

Proof Suppose $v \neq w \in \Gamma$ with $w - v \in Q + \mathbb{Z}\Phi$. The images in $\Bbbk[Q]/I$ of the monomials $\mathbf{t}^u$ for $u \in \Gamma$ are nonzero scalar multiples of each other, so it is enough to show that $\mathbf{t}^v$ maps to zero in $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. Since $w - v \in Q + \mathbb{Z}\Phi$, we have $\mathbf{t}^{w-v} \in \Bbbk[Q + \mathbb{Z}\Phi]$. Therefore $1 - \lambda\mathbf{t}^{w-v}$ lies outside of $\mathfrak{p}_{\mathbb{Z}\Phi}$ for all $\lambda \in \Bbbk$, because its image in $\Bbbk[\mathbb{Z}\Phi] = \Bbbk[Q + \mathbb{Z}\Phi]/\mathfrak{p}_{\mathbb{Z}\Phi}$ is either $1 - \lambda\mathbf{t}^{w-v}$ or 1, according to whether or not $w - v \in \mathbb{Z}\Phi$. (The assumption $v \neq w$ was used here: if $v = w$, then for $\lambda = 1$, we have $1 - \lambda\mathbf{t}^{w-v} = 0$.) Hence $1 - \lambda\mathbf{t}^{w-v}$ maps to a unit in $(\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. It follows that $\mathbf{t}^v$ maps to 0, since $(1 - \lambda_{vw}\mathbf{t}^{w-v})\mathbf{t}^v = \mathbf{t}^v - \lambda_{vw}\mathbf{t}^w$ maps to 0 in $\Bbbk[Q]/I$ whenever $\mathbf{t}^v - \lambda_{vw}\mathbf{t}^w \in I$. □

Lemma 2.10 *A congruence class $\Gamma \subseteq Q + \mathbb{Z}\Phi$ under $\sim_{I[\mathbb{Z}\Phi]}$ is infinite if and only if it contains two distinct elements whose difference lies in $Q + \mathbb{Z}\Phi$.*

Proof Let $\Gamma \subseteq Q + \mathbb{Z}\Phi$ be a congruence class. If $v, w \in \Gamma$ and $v - w \in Q + \mathbb{Z}\Phi$, then $w + \epsilon(v - w) \in \Gamma$ for all positive $\epsilon \in \mathbb{Z}$. On the other hand, assume Γ is infinite. There are two possibilities: either there are $v, w \in \Gamma$ with $v - w \in \mathbb{Z}\Phi$, or not. If so, then we are done, so assume not. Let $\mathbb{Z}^q$ be the quotient of $\mathbb{Z}^\ell/\mathbb{Z}\Phi$ modulo its torsion subgroup. (Here $\mathbb{Z}^\ell$ is the ambient lattice of Q.) The projection $\mathbb{Z}^\ell \to \mathbb{Z}^q$ induces a map from Γ to its image $\overline{\Gamma}$ that is finite-to-one. More precisely, if Γ' is the intersection of Γ with a coset of $\mathbb{Z}\Phi$ in $\ker(\mathbb{Z}^\ell \to \mathbb{Z}^q)$, then Γ' maps bijectively to its image $\overline{\Gamma}'$. There are only finitely many cosets, so some Γ' must be infinite, along with $\overline{\Gamma}'$. But $\overline{\Gamma}'$ is a subset of the affine semigroup $\overline{Q/\Phi}$, defined as the image of $Q + \mathbb{Z}\Phi$ in $\mathbb{Z}^q$. As $\overline{Q/\Phi}$ has unit group zero, every infinite subset contains two points whose difference lies in $\overline{Q/\Phi}$, and the corresponding lifts of these to Γ' have their difference in $Q + \mathbb{Z}\Phi$. □

Definition 2.11 Fix a face Φ of an affine semigroup Q. An ideal S is $\mathbb{Z}\Phi$-*closed* if $S = Q \cap (S + \mathbb{Z}\Phi)$. If $\sim$ is a congruence on $Q + \mathbb{Z}\Phi$, then the *unbounded ideal* $U \subseteq Q$ is the ($\mathbb{Z}\Phi$-closed) ideal of elements $u \in Q$ with infinite congruence class under $\sim$ in $Q + \mathbb{Z}\Phi$. We write $\mathcal{B}(Q + \mathbb{Z}\Phi)$ for the set of bounded (i.e. finite) congruence classes of $Q + \mathbb{Z}\Phi$ under $\sim$.

Example 2.12 Let M be as in Definition 2.6 and consider the congruence $\sim_{I(M)}$ on $Q = \mathbb{N}^q$. If $\Phi = \{0\}$, then the unbounded ideal $U \subseteq \mathbb{N}^q$ is the union of the unbounded M-subgraphs of $\mathbb{N}^q$, while $\mathcal{B}(\mathbb{N}^q)$ is the union of the bounded M-subgraphs.

Proposition 2.13 *Fix a face Φ of an affine semigroup Q, a binomial ideal $I \subseteq \Bbbk[Q]$, and a $\mathbb{Z}\Phi$-closed ideal $S \subseteq Q$ containing U under the congruence $\sim_{I[\mathbb{Z}\Phi]}$. Write $\mathcal{B} = \mathcal{B}(Q + \mathbb{Z}\Phi)$ for the bounded classes, J for the binomial ideal $(I : \mathbf{t}^{\Phi}) + \operatorname{span}_{\Bbbk} S$, and $\overline{Q} = (Q + \mathbb{Z}\Phi)_{I[\mathbb{Z}\Phi]}$.*

1. *$\Bbbk[Q]/J$ is graded by $\overline{Q}$, and its set of nonzero degrees is contained in $\mathcal{B}$.*
2. *The group $\mathbb{Z}\Phi \subseteq \overline{Q}$ acts freely on $\mathcal{B}$, and the $\Bbbk[\Phi]$-submodule $(\Bbbk[Q]/J)_T \subseteq \Bbbk[Q]/J$ in degrees from any orbit $T \subseteq \mathcal{B}$ is 0 or finitely generated and torsion-free of rank 1.*
3. *The quotient Q_J/Φ of the monoid $(Q + \mathbb{Z}\Phi)_{J[\mathbb{Z}\Phi]}$ by its subgroup $\mathbb{Z}\Phi$ is a partially ordered set if we define $\zeta \preceq \eta$ whenever $\zeta + \xi = \eta$ for some $\xi \in Q_J/\Phi$.*
4. *$\Bbbk[Q]/J$ is filtered by $\overline{Q}$-graded $\Bbbk[Q]$-submodules with associated graded module*

$$\operatorname{gr}(\Bbbk[Q]/J) = \bigoplus_{T \in \mathcal{B}/\Phi} (\Bbbk[Q]/J)_T, \quad \textit{where } \mathcal{B}/\Phi = \{\mathbb{Z}\Phi\text{-orbits } T \subseteq \mathcal{B}\},$$

   the canonical isomorphism being as $\mathcal{B}$-graded $\Bbbk[\Phi]$-modules, although the left-hand side is naturally a $\Bbbk[Q]$-module annihilated by $\mathfrak{p}_\Phi$.
5. *If $(\Bbbk[Q]/J)_T \neq 0$ for only finitely many orbits $T \in \mathcal{B}/\Phi$, then J is a $\mathfrak{p}_\Phi$-primary ideal.*

Proof The quotient $\Bbbk[Q]/(I : \mathbf{t}^{\Phi})$ is automatically $\overline{Q}$-graded by Proposition 2.4 applied to $Q + \mathbb{Z}\Phi$ and $I[\mathbb{Z}\Phi]$, given (2.2). The further quotient by $\operatorname{span}_{\Bbbk} S$ is graded by $\mathcal{B}$ because $S \supseteq U$.

$\mathbb{Z}\Phi$ acts freely on $\mathcal{B}$ by Lemmas 2.9 and 2.10: if $\phi \in \mathbb{Z}\Phi$ and Γ is a bounded congruence class, then the translate $\phi + \Gamma$ is, as well; and if $\phi \neq 0$ then $\phi + \Gamma \neq \Gamma$, because each coset of $\mathbb{Z}\Phi$ intersects Γ at most once. Combined with the $\mathbb{Z}\Phi$-closedness of S, this shows that $\Bbbk[Q]/J$ is a $\Bbbk[\Phi]$-submodule of the free $\Bbbk[\mathbb{Z}\Phi]$-module whose basis consists of the $\mathbb{Z}\Phi$-orbits $T \subseteq \mathcal{B}$. Hence $(\Bbbk[Q]/J)_T$ is torsion-free (it might be zero, of course, if S happens to contain all of the monomials corresponding to congruence classes of Q arising from $\sim_{I[\mathbb{Z}\Phi]}$ classes in T). For item 2, it remains to show that $(\Bbbk[Q]/J)_T$ is finitely generated. Let $\mathcal{T} = \bigcup_{\Gamma \in T} \Gamma \cap Q$. By construction, $\mathcal{T}$ is the (finite) union of the intersections $Q \cap (\gamma + \mathbb{Z}\Phi)$ of Q with cosets of $\mathbb{Z}\Phi$ in $\mathbb{Z}^\ell$ for γ in any fixed $\Gamma \in T$. Such an intersection is a finitely generated Φ-set (a set closed under addition by Φ) by [12, Eq. (1) and Lemma 2.2] or [14, Theorem 11.13], where the $\Bbbk$-vector space it spans is identified as the set of monomials annihilated by $\Bbbk[\Phi]$ modulo an irreducible monomial ideal of $\Bbbk[Q]$. The images in $\Bbbk[Q]/J$ of the monomials corresponding to any generators for these Φ-sets generate $(\Bbbk[Q]/J)_T$.

The point of item 3 is that the monoid Q_J/Φ acts sufficiently like an affine semigroup whose only unit is the trivial one. To prove it, observe that Q_J/Φ consists, by item 1, of the (possibly empty set of) orbits $T \in \mathcal{B}$ such that $(\Bbbk[Q]/J)_T \neq 0$ plus one congruence class $\overline{S}$ for the monomials in J (if there are any). The proposed partial order has $T \prec \overline{S}$ for all orbits $T \in Q_J/\Phi$, and also $T \prec T + v$ if and only if $v \in (Q + \mathbb{Z}\Phi) \smallsetminus \mathbb{Z}\Phi$. This relation $\prec$ a priori defines a directed graph with vertex set Q_J/Φ, and we need it to have no directed cycles. The terminal nature of $\overline{S}$ implies that no cycle can contain $\overline{S}$, so suppose that $T = T + v$. For some $\phi \in \mathbb{Z}\Phi$ and $u \in T$, the translate $u + \phi$ lies in the same congruence class under $\sim_{I[\mathbb{Z}\Phi]}$ as $u + v$. Lemma 2.10 implies that $v - \phi$, and hence v itself, does not lie in $Q + \mathbb{Z}\Phi$.

For item 4, it suffices to find a total order $T_0, T_1, T_2, \dots$ on $\mathcal{B}/\Phi$ such that $\oplus_{j \geq k}(\Bbbk[Q]/J)_{T_j}$ is a $\Bbbk[Q]$-submodule for all $k \in \mathbb{N}$. Use the partial order of $\mathcal{B}/\Phi$ via its inclusion in the monoid Q_J/Φ in item 3 for $S = U$. Any well-order refining this partial order will do.

Item 5 follows from items 2 and 4 because the associated primes of $\mathrm{gr}(\Bbbk[Q]/J)$ contain every associated prime of J for any finite filtration of $\Bbbk[Q]/J$ by $\Bbbk[Q]$-submodules. □

For connections with toral modules (Definition 4.3), we record the following.

Corollary 2.14 *Fix notation as in Proposition 2.13. If I is homogeneous for a grading of $\Bbbk[Q]$ by a group $\mathcal{A}$ via a monoid morphism $Q \to \mathcal{A}$, then $\Bbbk[Q]/J$ and $\mathrm{gr}(\Bbbk[Q]/J)$ are $\mathcal{A}$-graded via a natural coarsening $\mathcal{B} \to \mathcal{A}$ that restricts to a group homomorphism $\mathbb{Z}\Phi \to \mathcal{A}$.*

Proof The morphism $Q \to \mathcal{A}$ induces a morphism $\pi_{\mathcal{A}} : Q + \mathbb{Z}\Phi \to \mathcal{A}$ by the universal property of monoid localization. The morphism $\pi_{\mathcal{A}}$ is constant on the non-monomial congruence classes in Q_I precisely because I is $\mathcal{A}$-graded. It follows that $\pi_{\mathcal{A}}$ is constant on the non-monomial congruence classes in $(Q + \mathbb{Z}\Phi)_{I[\mathbb{Z}\Phi]}$. In particular, $\pi_{\mathcal{A}}$ is constant on the bounded classes $\mathcal{B}(Q + \mathbb{Z}\Phi)$, which therefore map to $\mathcal{A}$ to yield the natural coarsening. The group homomorphism $\mathbb{Z}\Phi \to \mathcal{A}$ is induced by the composite morphism $\mathbb{Z}\Phi \to (Q + \mathbb{Z}\Phi) \to \mathcal{A}$, which identifies the group $\mathbb{Z}\Phi$ with the $\mathbb{Z}\Phi$-orbit in $\mathcal{B}$ containing (the class of) 0. □

Theorem 2.15 *Fix a face Φ of an affine semigroup Q and a binomial ideal $I \subseteq \Bbbk[Q]$. If $\mathfrak{p}_\Phi$ is minimal over I, then the $\mathfrak{p}_\Phi$-primary component of I is $(I : \mathbf{t}^\Phi) + \mathrm{span}_\Bbbk U$, where $(I : \mathbf{t}^\Phi)$ is the binomial ideal (2.2) and $U \subseteq Q$ is the unbounded ideal (Definition 2.11) for $\sim_{I[\mathbb{Z}\Phi]}$. Furthermore, the only monomials in $(I : \mathbf{t}^\Phi) + \mathrm{span}_\Bbbk U$ are those of the form $\mathbf{t}^u$ for $u \in U$.*

Proof The $\mathfrak{p}_\Phi$-primary component of I is the kernel of the localization homomorphism $\Bbbk[Q] \to (\Bbbk[Q]/I)_{\mathfrak{p}_\Phi}$. As this factors through the homogeneous localization $\Bbbk[Q + \mathbb{Z}\Phi]/I[\mathbb{Z}\Phi]$, we find that the kernel contains $(I : \mathbf{t}^\Phi)$. Lemmas 2.9 and 2.10 imply that the kernel contains $\mathrm{span}_\Bbbk U$. But already $(I : \mathbf{t}^\Phi) + \mathrm{span}_\Bbbk U$ is $\mathfrak{p}_\Phi$-primary by Proposition 2.13.5; the finiteness condition there is satisfied by minimality of $\mathfrak{p}_\Phi$ applied to the filtration in Proposition 2.13.4. Thus the quotient of $\Bbbk[Q]$ by $(I : \mathbf{t}^\Phi) + \mathrm{span}_\Bbbk U$ maps injectively to its localization at $\mathfrak{p}_\Phi$. To prove the last sentence of the theorem, observe that under the $\overline{Q}$-grading from Proposition 2.13.1, every monomial $\mathbf{t}^u$ outside of $\mathrm{span}_\Bbbk U$ maps to a $\Bbbk$-vector space basis for the (1-dimensional) graded piece corresponding to the bounded congruence class containing u. □

Example 2.16 One might hope that when $\mathfrak{p}_\Phi$ is an embedded prime of a binomial ideal I, the $\mathfrak{p}_\Phi$-primary components, or even perhaps the irreducible components, would be unique, if we require that they be finely graded (Hilbert function 0 or 1) as in Proposition 2.13. However, this fails even in simple examples, such as $\Bbbk[x, y]/\langle x^2 - xy, xy - y^2\rangle$. In this case, $I = \langle x^2 - xy, xy - y^2\rangle = \langle x^2, y\rangle \cap \langle x - y\rangle = \langle x, y^2\rangle \cap \langle x - y\rangle$ and Φ is the face $\{0\}$ of $Q = \mathbb{N}^2$, so that $I = (I : \mathbf{t}^\Phi)$ by definition. The monoid Q_I, written multiplicatively, consists of $1, x, y$, and a single element of degree i for each $i \geq 2$ representing the congruence class of the monomials of total degree i. Our two choices $\langle x^2, y\rangle$ and $\langle x, y^2\rangle$ for the irreducible component with associated prime $\langle x, y\rangle$ yield quotients of $\Bbbk[x, y]$ with different Q_I-graded Hilbert functions, the first nonzero in degree x and the second nonzero in degree y.

3 Primary components of binomial ideals

In this section, we express the primary components of binomial ideals in polynomial rings over the complex numbers as explicit sums of binomial and monomial ideals. We formulate our main result, Theorem 3.2, after recalling some essential results from [5]. Henceforth, we work with the complex polynomial ring $\mathbb{C}[\mathbf{t}]$ in variables $\mathbf{t} = t_1, \ldots, t_n$.

If $L \subseteq \mathbb{Z}^n$ is a sublattice, then with notation as in Example 2.3, the *lattice ideal* of L is

$$I_L = \langle \mathbf{t}^{u_+} - \mathbf{t}^{u_-} \mid u = u_+ - u_- \in L \rangle,$$

More generally, any *partial character* $\rho : L \to \mathbb{C}^*$ of $\mathbb{Z}^n$, which includes the data of both its domain lattice $L \subseteq \mathbb{Z}^n$ and the map to $\mathbb{C}^*$, determines a binomial ideal

$$I_\rho = \langle \mathbf{t}^{u_+} - \rho(u)\mathbf{t}^{u_-} \mid u = u_+ - u_- \in L \rangle.$$

(The ideal I_ρ is called $I_+(\rho)$ in [5].) The ideal I_ρ is prime if and only if L is a *saturated* sublattice of $\mathbb{Z}^n$, meaning that L equals its *saturation*, in general defined as

$$\mathrm{sat}(L) = (\mathbb{Q}L) \cap \mathbb{Z}^n,$$

where $\mathbb{Q}L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ is the rational vector space spanned by L in $\mathbb{Q}^n$. In fact, writing $\mathfrak{m}_J = \langle t_j \mid j \notin J \rangle$ for any $J \subseteq \{1, \ldots, n\}$, every binomial prime ideal in $\mathbb{C}[\mathbf{t}]$ has the form

$$I_{\rho,J} = I_\rho + \mathfrak{m}_J \tag{3.1}$$

for some *saturated* partial character ρ (i.e., whose domain is a saturated sublattice) and subset J such that the binomial generators of I_ρ only involve variables t_j for $j \in J$ (some of which might actually be absent from the generators of I_ρ) [5, Corollary 2.6].

Remark 3.1 A rank m lattice $L \subseteq \mathbb{Z}^n$ is saturated if and only if there exists an $(n - m) \times n$ integer matrix A of full rank such that $L = \ker_{\mathbb{Z}}(A)$. In this case, if ρ is the trivial character, the ideal I_ρ is denoted by I_A and called a *toric ideal*. Note that

$$I_A = \langle \mathbf{t}^u - \mathbf{t}^v \mid Au = Av \rangle. \tag{3.2}$$

If ρ is not the trivial character, then I_ρ becomes isomorphic to I_A when the variables are rescaled via $t_i \mapsto \rho(e_i)t_i$, which induces the rescaling $\mathbf{t}^u \mapsto \rho(u)\mathbf{t}^u$ on general monomials.

The characteristic zero part of the main result in [5, Theorem 7.1′], says that an irredundant primary decomposition of an arbitrary binomial ideal $I \subseteq \mathbb{C}[\mathbf{t}]$ is given by

$$I = \bigcap_{I_{\rho,J} \in \mathrm{Ass}(I)} \mathrm{Hull}(I + I_\rho + \mathfrak{m}_J^e) \tag{3.3}$$

for any large integer e, where Hull means to discard the primary components for embedded (i.e. nonminimal associated) primes, and $\mathfrak{m}_J^e = \langle t_j \mid j \notin J \rangle^e$. In other words, $\mathrm{Hull}(I + I_\rho + \mathfrak{m}_J^e)$ is the localization of $I + I_\rho + \mathfrak{m}_J^e$ at $I_{\rho,J}$. Our goal in this section is to be explicit about the Hull operation. The salient feature of (3.3) is that $I + I_\rho + \mathfrak{m}_J^e$ contains I_ρ. In contrast, (3.3) is false in positive characteristic, where $I_\rho + \mathfrak{m}_J^e$ should be replaced by a Frobenius power of $I_{\rho,J}$ [5, Theorem 7.1′].

Our notation in the next theorem is as follows. Given a subset $J \subseteq \{1, \ldots, n\}$, let $\bar{J} = \{1, \ldots, n\} \smallsetminus J$ be its complement, and use these sets to index coordinate subspaces of $\mathbb{N}^n$ and $\mathbb{Z}^n$; in particular, $\mathbb{N}^n = \mathbb{N}^J \times \mathbb{N}^{\bar{J}}$. Adjoining additive inverses for the elements in $\mathbb{N}^J$ yields $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$, whose semigroup ring we denote by $\mathbb{C}[\mathbf{t}][\mathbf{t}_J^{-1}]$, with $\mathbf{t}_J = \prod_{j \in J} t_j$. As in

Definition 2.7, $\mathrm{span}_{\mathbb{C}} S$ is the monomial ideal in $\mathbb{C}[\mathbf{t}]$ having $\mathbb{C}$-basis S. Finally, for a saturated sublattice $L \subseteq \mathbb{Z}^J$, we write $\mathbb{N}^J/L$ for the image of $\mathbb{N}^J$ in the torsion-free group $\mathbb{Z}^J/L$.

Theorem 3.2 *Fix a binomial ideal $I \subseteq \mathbb{C}[\mathbf{t}]$ and an associated prime $I_{\rho,J}$ of I, where $\rho : L \to \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. Set $\Phi = \mathbb{N}^J/L$, and write $\sim$ for the congruence on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by the ideal $(I + I_\rho)[\mathbb{Z}^J] = (I + I_\rho)\mathbb{C}[\mathbf{t}][\mathbf{t}_J^{-1}]$.*

1. *If $I_{\rho,J}$ is a minimal prime of I and $\widetilde{U}$ is the set of $u \in \mathbb{N}^n$ whose congruence classes in $(\mathbb{Z}^J \times \mathbb{N}^{\bar{J}})/\sim$ have infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$, then the $I_{\rho,J}$-primary component of I is*

$$\mathcal{C}_{\rho,J} = \big((I + I_\rho) : \mathbf{t}_J^\infty\big) + \mathrm{span}_{\mathbb{C}}\widetilde{U}.$$

Fix a monomial ideal $K \subseteq \mathbb{C}[t_j \mid j \in \bar{J}]$ containing a power of each available variable, and let $\approx$ be the congruence on $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}}$ determined by $(I + I_\rho + K)\mathbb{C}[\mathbf{t}][\mathbf{t}_J^{-1}]$. Write $\widetilde{U}_K$ for the set of $u \in \mathbb{N}^n$ whose congruence classes in $(\mathbb{Z}^J \times \mathbb{N}^{\bar{J}})/\approx$ have infinite image in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$.

2. *The $I_{\rho,J}$-primary component of $\langle I + I_\rho + K\rangle \subseteq \mathbb{C}[\mathbf{t}]$ is $\big((I + I_\rho + K) : \mathbf{t}_J^\infty\big) + \mathrm{span}_{\mathbb{C}}\widetilde{U}_K$.*
3. *If K is contained in a sufficiently high power of $\mathfrak{m}_J$, then*

$$\mathcal{C}_{\rho,J} = ((I + I_\rho + K) : \mathbf{t}_J^\infty) + \mathrm{span}_{\mathbb{C}}\widetilde{U}_K$$

is a valid choice of $I_{\rho,J}$-primary component for I.

The only monomials in the above primary components are those in $\mathrm{span}_{\mathbb{C}}\widetilde{U}$ or $\mathrm{span}_{\mathbb{C}}\widetilde{U}_K$.

Proof First suppose $I_{\rho,J}$ is a minimal prime of I. We may, by rescaling the variables t_j for $j \in J$, harmlessly assume that ρ is the trivial character on its lattice L, so that $I_\rho = I_L$ is the lattice ideal for L. The quotient $\mathbb{C}[\mathbf{t}]/I_L$ is the affine semigroup ring $\mathbb{C}[Q]$ for $Q = (\mathbb{N}^J/L) \times \mathbb{N}^{\bar{J}} = \Phi \times \mathbb{N}^{\bar{J}}$. Now let us take the whole situation modulo I_L. The image of $I_{\rho,J} = I_L + \mathfrak{m}_J$ is the prime ideal $\mathfrak{p}_\Phi \subseteq \Bbbk[Q]$ for the face Φ. The image in $\mathbb{C}[Q]$ of the binomial ideal I is a binomial ideal I', and $((I + I_L) : \mathbf{t}_J^\infty)$ has image $(I' : \mathbf{t}^\Phi)$, as defined in (2.2). Finally, the image of $\widetilde{U}$ in Q is the unbounded ideal $U \subseteq Q$ (Definition 2.11) by construction. Now we can apply Theorem 2.15 to I' and obtain a combinatorial description of the component associated to $I_{\rho,J}$.

The second and third items follow from the first by replacing I with $I + K$, given the primary decomposition in (3.3). □

Remark 3.3 One of the mysteries in [5] is why the primary components $\mathcal{C}$ of binomial ideals turn out to be generated by monomials and binomials. From the perspective of Theorem 3.2 and Proposition 2.13 together, this is because the primary components are *finely graded*: under some grading by a free abelian group, namely $\mathbb{Z}\Phi$, the vector space dimensions of the graded pieces of the quotient modulo the ideal $\mathcal{C}$ are all 0 or 1 [5, Proposition 1.11]. In fact, via Lemma 2.9, fine gradation is the root cause of primaryness.

Remark 3.4 Theorem 3.2 easily generalizes to arbitrary binomial ideals in arbitrary commutative noetherian semigroup rings over $\mathbb{C}$: simply choose a presentation as a quotient of a polynomial ring modulo a pure difference binomial ideal [9, Theorem 7.11].

Remark 3.5 The methods of Sect. 2 work in arbitrary characteristic—and indeed, over a field $\Bbbk$ that can fail to be algebraically closed, and can even be finite—because we assumed that a prime ideal $\mathfrak{p}_\Phi$ for a face Φ is associated to our binomial ideal. In contrast, this section and the next work only over an algebraically closed field of characteristic zero. However, it might be possible to produce similarly explicit binomial primary decompositions in positive characteristic by reducing to the situation in Sect. 2; this remains an open problem.

4 Associated components and multigradings

In this section, we turn our attention to interactions of primary components with various gradings on $\mathbb{C}[\mathbf{t}]$. These played crucial roles already in the proof of Theorem 3.2: taking the quotient of its statement by the toric ideal I_ρ put us in the situation of Proposition 2.13 and Theorem 2.15, which provide excellent control over gradings. The methods here can be viewed as aids for clarification in examples, as we shall see in the case of lattice basis ideals (Example 4.10). However, this theory was developed with applications in mind [3]; see Sect. 5.

Generally speaking, given a grading of $\mathbb{C}[\mathbf{t}]$, there are two kinds of graded modules: those with bounded Hilbert function (the *toral* case below) and those without. The main point is Theorem 4.13: if $\mathbb{C}[\mathbf{t}]/\mathfrak{p}$ has bounded Hilbert function for some graded prime $\mathfrak{p}$, then the $\mathfrak{p}$-primary component of any graded binomial ideal is easier to describe than usual.

To be consistent with notation, we adopt the following conventions for this section.

Convention 4.1 $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ denotes an integer $d \times n$ matrix of rank d whose columns $a_1, \ldots, a_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of A is pointed (contains no lines), and all of the columns a_i are nonzero. We also assume that $\mathbb{Z}A = \mathbb{Z}^d$; that is, the columns of A span $\mathbb{Z}^d$ as a lattice.

Convention 4.2 Let $B = (b_{jk}) \in \mathbb{Z}^{n\times m}$ be an integer matrix of full rank $m \le n$. Assume that every nonzero element of the column span $\mathbb{Z}B$ of B over the integers $\mathbb{Z}$ is *mixed*, meaning that it has at least one positive and one negative entry; in particular, the columns of B are mixed. We write $b_1, \ldots, b_n$ for the rows of B. Having chosen B, we set $d = n - m$ and pick a matrix $A \in \mathbb{Z}^{d\times n}$ such that $AB = 0$ and $\mathbb{Z}A = \mathbb{Z}^d$.

If $d \neq 0$, the mixedness hypothesis on B is equivalent to the pointedness assumption for A in Convention 4.1. We do allow $d = 0$, in which case A is the empty matrix.

The $d \times n$ integer matrix A in Convention 4.1 determines a $\mathbb{Z}^d$-grading on $\mathbb{C}[\mathbf{t}]$ in which the degree $\deg(t_j) = a_j$ is defined[1] to be the jth column of A. Our conventions imply that $\mathbb{C}[\mathbf{t}]$ has finite-dimensional graded pieces, like any finitely generated module [14, Chap. 8].

Definition 4.3 Let $V = \bigoplus_{\alpha\in\mathbb{Z}^d} V_\alpha$ be an A-graded module over the polynomial ring $\mathbb{C}[\mathbf{t}]$. The *Hilbert function* $H_V : \mathbb{Z}^d \to \mathbb{N}$ takes the values $H_V(\alpha) = \dim_{\mathbb{C}} V_\alpha$. If V is finitely generated, we say that the module V is *toral* if the Hilbert function H_V is bounded above. A graded prime $\mathfrak{p}$ is a *toral prime* if $\mathbb{C}[\mathbf{t}]/\mathfrak{p}$ is a toral module. Similarly, a graded primary component $\mathcal{C}$ of an ideal I is a *toral component* of I if $\mathbb{C}[\mathbf{t}]/\mathcal{C}$ is a toral module.

Example 4.4 The toric ideal I_A for the grading matrix A is always an A-graded toral prime, since the quotient $\mathbb{C}[\mathbf{t}]/I_A$ is always toral: its Hilbert function takes only the values 0 or 1. In contrast, $\mathbb{C}[\mathbf{t}]$ itself is not a toral module unless $d = n$ (which forces A to be invertible over $\mathbb{Z}$, by Convention 4.1).

We will be most interested in the quotients of $\mathbb{C}[\mathbf{t}]$ by prime and primary binomial ideals. To begin, here is a connection between the natural gradings from Sect. 2 and the A-grading.

Lemma 4.5 *Let $I \subseteq \mathbb{C}[\mathbf{t}]$ be an A-graded binomial ideal and $\mathcal{C}_{\rho,J}$ a primary component, with $\rho : L \to \mathbb{C}^*$ for $L \subseteq \mathbb{Z}^J$. The image $\mathbb{Z}A_J$ of the homomorphism $\mathbb{Z}^J/L = \mathbb{Z}\Phi \to \mathbb{Z}^d$ induced by Corollary 2.14 (with $\mathcal{A} = \mathbb{Z}A = \mathbb{Z}^d$) is generated by the columns a_j of A indexed by $j \in J$, as is the monoid image of $\Phi = \mathbb{N}^J/L$, which we denote by $\mathbb{N}A_J$.*

[1] In noncommutative settings, such as [3,13], the variables are written $\partial_1, \ldots, \partial_n$, and the degree of ∂_j is usually defined to be $-a_j$ instead of a_j.

To make things a little more concrete, let us give one more perspective on the homomorphism $\mathbb{Z}\Phi \to \mathbb{Z}^d$. Simply put, the ideal $I_{\rho,J}$ is naturally graded by $\mathbb{Z}^J/L = \mathbb{Z}\Phi$, and the fact that it is also A-graded means that $L \subseteq \ker(\mathbb{Z}^n \to \mathbb{Z}^d)$, the map to $\mathbb{Z}^d$ being given by A. (The real content of Corollary 2.14 lies with the action on the rest of $\mathcal{B}$).

Example 4.6 Let $\rho : L \to \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. If $\mathcal{C}_{\rho,J}$ is an $I_{\rho,J}$-primary binomial ideal, then $\mathbb{C}[\mathbf{t}]/\mathcal{C}_{\rho,J}$ has a finite filtration whose successive quotients are torsion-free modules of rank 1 over the affine semigroup ring $R = \mathbb{C}[\mathbf{t}]/I_{\rho,J}$. This follows by applying Proposition 2.13 to Theorem 3.2.1 and its proof. If, in addition, $I_{\rho,J}$ is A-graded, then some A-graded translate of each successive quotient admits a $\mathbb{Z}^J/L$-grading refining the A-grading via $\mathbb{Z}^J/L \to \mathbb{Z}^d = \mathbb{Z}A$; this follows by conjointly applying Corollary 2.14.

The next three results provide alternate characterizations of toral primary binomial ideals. In what follows, A_J is the submatrix of A on the columns indexed by J.

Proposition 4.7 *Every A-graded toral prime is binomial. In the situation of Lemma 4.5, $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ and $\mathbb{C}[\mathbf{t}]/\mathcal{C}_{\rho,J}$ are toral if and only if the homomorphism $\mathbb{Z}\Phi \to \mathbb{Z}^d$ is injective.*

Proof To prove the first part of the statement, fix a toral prime $\mathfrak{p}$, and let $h \in \mathbb{N}$ be the maximum of the Hilbert function of $\mathbb{C}[\mathbf{t}]/\mathfrak{p}$. It is enough, by [5, Proposition 1.11], to show that $h = 1$. Let R be the localization of $\mathbb{C}[\mathbf{t}]/\mathfrak{p}$ by inverting all nonzero homogeneous elements. Because of the homogeneous units in R, all of its graded pieces have the same dimension over $\mathbb{C}$; and since R is a domain, this dimension is at least h. Thus we need only show that $R_0 = \mathbb{C}$. For any given finite-dimensional subspace of R_0, multiplication by a common denominator maps it injectively to some graded piece of $\mathbb{C}[\mathbf{t}]/\mathfrak{p}$. Therefore every finite-dimensional subspace of R_0 has dimension at most h. It follows that $H_R(0) \le h$, so R_0 is artinian. But R_0 is a domain because $R_0 \subseteq R$, so $R_0 = \mathbb{C}$.

For the second part, $\mathbb{C}[\mathbf{t}]/\mathcal{C}_{\rho,J}$ has a finite filtration whose associated graded pieces are A-graded translates of quotients of $\mathbb{C}[\mathbf{t}]$ by A-graded primes, at least one of which is $I_{\rho,J}$ and all of which contain it. By additivity of Hilbert functions, $\mathbb{C}[\mathbf{t}]/\mathcal{C}_{\rho,J}$ is toral precisely when all of these are toral primes. However, if a graded prime $\mathfrak{p}$ contains a toral prime, then $\mathfrak{p}$ is itself a toral prime. Therefore, we need only treat the case of $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$. But $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ is naturally graded by $\mathbb{Z}\Phi$, with Hilbert function 0 or 1, so injectivity immediately implies that $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ is toral. On the other hand, if $\mathbb{Z}\Phi \to \mathbb{Z}^d$ is not injective, then $\mathbb{N}A_J$ is a proper quotient of the affine semigroup Φ, and such a proper quotient has fibers of arbitrary cardinality. □

Corollary 4.8 *Let $\rho : L \to \mathbb{C}^*$ for a saturated lattice $L \subseteq \mathbb{Z}^J \cap \ker_{\mathbb{Z}}(A) = \ker_{\mathbb{Z}}(A_J)$. The quotient $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ by an A-graded prime $I_{\rho,J}$ is toral if and only if $L = \ker_{\mathbb{Z}}(A_J)$.*

Lemma 4.9 *Every A-graded binomial prime ideal $I_{\rho,J}$ satisfies*

$$\dim(I_{\rho,J}) \ge \operatorname{rank}(A_J),$$

with equality if and only if $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ is toral.

Proof Rescale the variables and assume that $I_{\rho,J} = I_L$, the lattice ideal for a saturated lattice $L \subseteq \ker_{\mathbb{Z}}(A_J)$. The rank of L is at most $\#J - \operatorname{rank}(A_J)$; thus $\dim(I_L) = \#J - \operatorname{rank}(L) \ge \operatorname{rank}(A_J)$. Equality holds exactly when $L = \ker_{\mathbb{Z}}(A)$, i.e. when $\mathbb{C}[\mathbf{t}]/I_{\rho,J}$ is toral. □

Example 4.10 Fix matrices A and B as in Convention 4.2. This identifies $\mathbb{Z}^d$ with the quotient of $\mathbb{Z}^n/\mathbb{Z}B$ modulo its torsion subgroup. Consider the *lattice basis ideal*

$$I(B) = \langle \mathbf{t}^{u_+} - \mathbf{t}^{u_-} \mid u = u_+ - u_- \text{ is a column of } B\rangle \subseteq \mathbb{C}[t_1, \dots, t_n]. \qquad (4.1)$$

The toric ideal I_A from (3.2) is an associated prime of $I(B)$, the primary component being I_A itself. More generally, all of the minimal primes of the lattice ideal $I_{\mathbb{Z}B}$, one of which is I_A, are minimal over $I(B)$ with multiplicity 1; this follows from [5, Theorem 2.1] by inverting the variables. That result also implies that the minimal primes of $I_{\mathbb{Z}B}$ are precisely the ideals I_ρ for partial characters $\rho : \mathrm{sat}(\mathbb{Z}B) \to \mathbb{C}^*$ of $\mathbb{Z}^n$ extending the trivial partial character on $\mathbb{Z}B$, so the lattice ideal $I_{\mathbb{Z}B}$ is the intersection of these prime ideals. Hence $I_{\mathbb{Z}B}$ is a radical ideal, and every irreducible component of its zero set is isomorphic, as a subvariety of $\mathbb{C}^n$, to the variety of I_A.

In complete generality, each of the minimal primes of $I(B)$ arises, after row and column permutations, from a block decomposition of B of the form

$$\left[\begin{array}{c|c} N & B_J \\ \hline M & 0 \end{array}\right], \tag{4.2}$$

where M is a mixed submatrix of B of size $q \times p$ for some $0 \le q \le p \le m$ [11]. (Matrices with $q = 0$ rows are automatically mixed; matrices with $q = 1$ row are never mixed.) We note that not all such decompositions correspond to minimal primes: the matrix M has to satisfy another condition which Hoşten and Shapiro call irreducibility [11, Definition 2.2 and Theorem 2.5]. If $I(B)$ is a complete intersection, then only square matrices M will appear in the block decompositions (4.2), by a result of Fischer and Shapiro [6].

For each partial character $\rho : \mathrm{sat}(\mathbb{Z}B_J) \to \mathbb{C}^*$ extending the trivial character on $\mathbb{Z}B_J$, the prime $I_{\rho,J}$ is associated to $I(B)$, where $J = J(M) = \{1, \dots, n\} \smallsetminus \mathrm{rows}(M)$ indexes the $n-q$ rows not in M. We reiterate that the symbol ρ here includes the specification of the sublattice $\mathrm{sat}(\mathbb{Z}B_J) \subseteq \mathbb{Z}^n$. The corresponding primary component $\mathcal{C}_{\rho,J} = \mathrm{Hull}\big(I(B) + I_\rho + \mathfrak{m}_J^e\big)$ of the lattice basis ideal $I(B)$ is simply I_ρ if $q = 0$, but will in general be non-radical when $q \ge 2$ (recall that $q = 1$ is impossible).

The quotient $\mathbb{C}[\mathbf{t}]/\mathcal{C}_{\rho,J}$ is toral if and only if M is square and satisfies either $\det(M) \neq 0$ or $q = 0$. To check this statement, observe that $I(B)$ has $m = n - d$ generators, so the dimension of any of its associated primes is at least d. But since A_J has rank at most d, Lemma 4.9 implies that toral primes of $I(B)$ have dimension exactly d (and are therefore minimal). If $I_{\rho,J}$ is a toral associated prime of $I(B)$ arising from a decomposition of the form (4.2), where M is $q \times p$, then the dimension of $I_{\rho,J}$ is $n - p - (m - q) = d + q - p$, and from this we conclude that M is square. That M is invertible follows from the fact that $\mathrm{rank}(\ker_{\mathbb{Z}}(A_J)) = d$. The same arguments show that if M is not square invertible, then $I_{\rho,J}$ is not toral.

Example 4.11 A binomial ideal $I \subseteq \mathbb{C}[\mathbf{t}]$ may be A-graded for different matrices A; in this case, which of the components of I are toral will change if we alter the grading. For instance, the prime ideal $I = \langle t_1t_4 - t_2t_3 \rangle \subseteq \mathbb{C}[t_1, \dots, t_4]$ is homogeneous for both the matrix $[1\,1\,1\,1]$ and the matrix $\left[\begin{smallmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\end{smallmatrix}\right]$. But $\mathbb{C}[t_1, \dots, t_4]/I$ is toral in the $\left[\begin{smallmatrix}1&1&1&1\\0&1&0&1\\0&0&1&1\end{smallmatrix}\right]$-grading, while it is not toral in the $[1\,1\,1\,1]$-grading.

Example 4.12 Let $I = \langle bd - de, bc - ce, ab - ae, c^3 - ad^2, a^2d^2 - de^3, a^2cd - ce^3, a^3d - ae^3 \rangle$ be a binomial ideal in $\mathbb{C}[\mathbf{t}]$, where we write $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5) = (a, b, c, d, e)$, and let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & -1 & 0 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

One easily verifies that the binomial ideal I is graded by $\mathbb{Z}A = \mathbb{Z}^2$. If ω is a primitive cube root of unity ($\omega^3 = 1$), then I, which is a radical ideal, has the prime decomposition

$$\begin{aligned} I = \langle a, c, d\rangle &\cap \langle bc - ad, b^2 - ac, c^2 - bd, b - e\rangle \\ &\cap \langle \omega bc - ad, b^2 - \omega ac, \omega^2 c^2 - bd, b - e\rangle \\ &\cap \langle \omega^2 bc - ad, b^2 - \omega^2 ac, \omega c^2 - bd, b - e\rangle. \end{aligned}$$

The intersectand $\langle a, c, d\rangle$ equals the prime ideal $I_{\rho,J}$ for $J = \{2, 5\}$ and $L = \{0\} \subseteq \mathbb{Z}^J$. The homomorphism $\mathbb{Z}^J \to \mathbb{Z}^2$ is not injective since it maps both basis vectors to $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$; therefore the prime ideal $\langle a, c, d\rangle$ is not a toral component of I. In contrast, the remaining three intersectands are the prime ideals $I_{\rho,J}$ for the three characters ρ that are defined on $\ker(A)$ but trivial on its index 3 sublattice $\mathbb{Z}B$ spanned by the columns of B, where $J = \{1, 2, 3, 4, 5\}$. These prime ideals are all toral by Corollary 4.8, with $\mathbb{Z}A_J = \mathbb{Z}A$.

Toral components can be described more simply than in Theorem 3.2, as we do not need to pass to $\widetilde{U}$ and can work with $\overline{U}$ (and fewer variables) instead.

Theorem 4.13 *Fix an A-graded binomial ideal $I \subseteq \mathbb{C}[\mathbf{t}]$ and a toral associated prime $I_{\rho,J}$ of I. Define the binomial ideal $\bar{I} = I \cdot \mathbb{C}[\mathbf{t}]/\langle t_j - 1 \mid j \in J\rangle$ by setting $t_j = 1$ for $j \in J$.*

1. *Fix a minimal prime $I_{\rho,J}$ of I. If $\overline{U} \subseteq \mathbb{N}^{\bar{J}}$ is the set of elements with infinite congruence class in $\mathbb{N}^{\bar{J}}_{\bar{I}}$ (Proposition 2.4), and $\mathbf{t}_J = \prod_{j\in J} t_j$, then I has $I_{\rho,J}$-primary component*

$$\mathcal{C}_{\rho,J} = ((I + I_\rho) : \mathbf{t}_J^\infty) + \langle \mathbf{t}^u \mid u \in \overline{U}\rangle.$$

Let $K \subseteq \mathbb{C}[t_j \mid j \in \bar{J}]$ be a monomial ideal containing a power of each available variable, and let $\overline{U}_K \subseteq \mathbb{N}^{\bar{J}}$ be the set of elements with infinite congruence class in $\mathbb{N}^{\bar{J}}_{\bar{I}+K}$.

2. *The $I_{\rho,J}$-primary component of $\langle I + I_\rho + K\rangle \subseteq \mathbb{C}[\mathbf{t}]$ is equal to*

$$\big((I + I_\rho + K) : \mathbf{t}_J^\infty\big) + \langle \mathbf{t}^u \mid u \in \overline{U}_K\rangle.$$

3. *If K is contained in a sufficiently high power of $\mathfrak{m}_J$, then*

$$\mathcal{C}_{\rho,J} = ((I + I_\rho) : \mathbf{t}_J^\infty) + \langle \mathbf{t}^u \mid u \in \overline{U}_K\rangle$$

   is a valid choice of $I_{\rho,J}$-primary component for I.

The only monomials in the above primary components are in $\langle \mathbf{t}^u \mid u \in \overline{U}\rangle$ or $\langle \mathbf{t}^u \mid u \in \overline{U}_K\rangle$.

Proof Resume the notation from the statement and proof of Theorem 3.2. As in that proof, it suffices here to deal with the first item. In fact, the only thing to show is that $\widetilde{U}$ in Theorem 3.2 is the same as $\mathbb{N}^J \times \overline{U}$ here.

Recall that $I' \subseteq \mathbb{C}[Q]$ is the image of I modulo I_ρ. The congruence classes of $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ determined by $I'[\mathbb{Z}\Phi]$ are the projections under $\mathbb{Z}^J \times \mathbb{N}^{\bar{J}} \to \mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ of the $\sim$ congruence classes. Further projection of these classes to $\mathbb{N}^{\bar{J}}$ yields the congruence classes determined by the ideal $I'' \subseteq \mathbb{C}[\mathbb{N}^{\bar{J}}]$, where I'' is obtained from $I'[\mathbb{Z}\Phi]$ by setting $\mathbf{t}^\phi = 1$ for all $\phi \in \mathbb{Z}\Phi$. This ideal I'' is just $\bar{I}$. Hence we are reduced to showing that a congruence class in $\Phi \times \mathbb{N}^{\bar{J}}$ determined by $I'[\mathbb{Z}\Phi]$ is infinite if and only if its projection to $\mathbb{N}^{\bar{J}}$ is infinite. This is clearly true for the monomial congruence class in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$. For any other congruence class $\Gamma \subseteq \mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$, the homogeneity of I (and hence that of I') under the A-grading implies that Γ is contained within a coset of $\mathcal{K} = \ker(\mathbb{Z}\Phi \times \mathbb{Z}^{\bar{J}} \to \mathbb{Z}^d = \mathbb{Z}A)$. This kernel $\mathcal{K}$ intersects

$\mathbb{Z}\Phi$ only at 0 because $I_{\rho,J}$ is toral. Therefore the projection of any coset of $\mathcal{K}$ to $\mathbb{Z}^{\bar{J}}$ is bijective onto its image. In particular, Γ is infinite if and only if its bijective image in $\mathbb{N}^{\bar{J}}$ is infinite. □

Corollary 4.14 *Resume the notation of Example 4.10. If $I_{\rho,J}$ is a toral minimal prime of the lattice basis ideal $I(B)$ given by a decomposition as in (4.2), so $J = J(M)$, then*

$$\mathcal{C}_{\rho,J} = I(B) + I_{\rho,J} + U_M,$$

where $U_M \subseteq \mathbb{C}[t_j \mid j \in \bar{J}]$ is the ideal $\mathbb{C}$-linearly spanned by all monomials whose exponent vectors lie in the union of the unbounded M-subgraphs of $\mathbb{N}^{\bar{J}}$, as in Definition 2.6. The only monomials in $\mathcal{C}_{\rho,J}$ belong to U_M.

Remark 4.15 Theorem 4.13 need not always be false for a component that is not toral, but it can certainly fail: there can be congruence classes in $\mathbb{Z}\Phi \times \mathbb{N}^{\bar{J}}$ that are infinite only in the $\mathbb{Z}\Phi$ direction, so that their projections to $\mathbb{N}^{\bar{J}}$ are finite.

5 Applications, examples, and further directions

In this section, we give a brief overview of the connection between binomial primary decomposition and hypergeometric differential equations, study some examples, and discuss computational issues.

From the point of view of complexity, primary decomposition is hard: even in the case of zero dimensional binomial complete intersections, counting the number of associated primes (with or without multiplicity) is a $\#P$-complete problem [2]. However, the primary decomposition algorithms implemented in Singular [15] or Macaulay2 [10] work very well in reasonably sized examples, and in fact, they provide the only implemented method for computing bounded congruence classes or M-subgraphs as in Sect. 3. We remind the reader that [5, Section 8] contains specialized algorithms for binomial primary decomposition, whose main feature is that they preserve binomiality at each step. These algorithms were improved and made more explicit by Ojeda and Sánchez [16]. We also mention that the results in the present article require more development before a combinatorial method for primary decomposition can be obtained. Even if we bypass the difficulty of computing congruence classes in monoids, it is an open problem to combinatorially determine the associated primes of a binomial ideal; see [1] for a study of this question in the special case of circuit ideals.

In the case that $q = 2$, we can study M-subgraphs directly by combinatorial means [4, Sect. 6]. The relevant result is the following.

Proposition 5.1 *Let M be a mixed invertible 2×2 integer matrix. Without loss of generality, write $M = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix}$, where a, b, c, d are positive integers. Then the number of bounded M-subgraphs is $\min(ad, bc)$. Moreover, if*

$$R = \begin{cases} \{(s,t) \in \mathbb{N}^2 \mid s < b \text{ and } t < c\} & \text{if } ad > bc, \\ \{(s,t) \in \mathbb{N}^2 \mid s < a \text{ and } t < d\} & \text{if } ad < bc, \end{cases}$$

then every bounded M-subgraph passes through exactly one of the points in R.

If $q > 2$, a method for computing M-subgraphs may be obtained through a link to differential equations. To make this evident, we change the notation for the ambient ring.

Notation 5.2 All binomial ideals in the remainder of this article are ideals in the polynomial ring $\mathbb{C}[\partial] = \mathbb{C}[\partial_1, \ldots, \partial_n]$.

The following result for M-subgraphs can be adapted to fit the more general context of congruences.

Proposition 5.3 *Let M be a $q \times q$ mixed invertible integer matrix, and assume that $q > 0$. Given $\gamma \in \mathbb{N}^q$, denote by Γ the M-subgraph containing γ. Think of the ideal $I(M) \subseteq \mathbb{C}[\partial]$ as a system of linear partial differential equations with constant coefficients.*

1. *The system of differential equations $I(M)$ has a unique formal power series solution of the form $G_\gamma = \sum_{u \in \Gamma} \lambda_u x^u$ in which $\lambda_\gamma = 1$.*
2. *The other coefficients λ_u of G_γ for $u \in \Gamma$ are all nonzero.*
3. *The set $\{G_\gamma \mid \gamma$ runs over a set of representatives for the M-subgraphs of $\mathbb{N}^q\}$ is a basis for the space of all formal power series solutions of $I(M)$.*
4. *The set $\{G_\gamma \mid \gamma$ runs over a set of representatives for the* bounded *M-subgraphs of $\mathbb{N}^q\}$ is a basis for the space of polynomial solutions of $I(M)$.*

The straightforward proof of this proposition can be found in [3, Sect. 7].

The following example illustrates the correspondence between M-subgraphs and solutions of $I(M)$.

Example 5.4 Consider the 3×3 matrix

$$M = \begin{bmatrix} 1 & -5 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}.$$

A basis of solutions (with minimal support under inclusion) of $I(M)$ is easily computed:

$$\left\{ 1, \quad x+y+z, \quad (x+y+z)^2, \quad (x+y+z)^3, \quad \sum_{n \geq 4} \frac{(x+y+z)^n}{n!} \right\}.$$

The M-subgraphs of $\mathbb{N}^3$ are the four slices $\{(a, b, c) \in \mathbb{N}^3 \mid a + b + c = n\}$ for $n \leq 3$; for $n \geq 4$, two consecutive slices are M-connected by $(-5, 1, 3)$, yielding one unbounded M-subgraph (see Fig. 1).

A direct combinatorial algorithm for producing the bounded M-subgraphs for $q > 2$, or even for finding their number would be interesting and useful, as the number of bounded M-subgraphs gives the dimension of the polynomial solution space of a hypergeometric system, and also the multiplicity of an associated prime of a lattice basis ideal. In the case where $I(M)$ is a zero-dimensional complete intersection, such an algorithm can be produced from the results in [2]. The combinatorial computation of the number of bounded congruence classes determined by a binomial ideal in a semigroup ring is open.

The system of differential equations $I(M) \subseteq \mathbb{C}[\partial]$ is a special case in the class of *Horn hypergeometric systems*. That class of systems takes center stage in our companion article [3], in the more general setting of *binomial D-modules* that are introduced there. The input data for these consist of a binomial ideal I and a vector β of complex parameters. The special case where I is prime corresponds to the *A-hypergeometric* or *GKZ hypergeometric systems*, after Gelfand et al. [7,8]; see also [17]. Binomial primary decomposition is crucial for the study of Horn systems, and their more general binomial relatives, because the numerics, algebra,

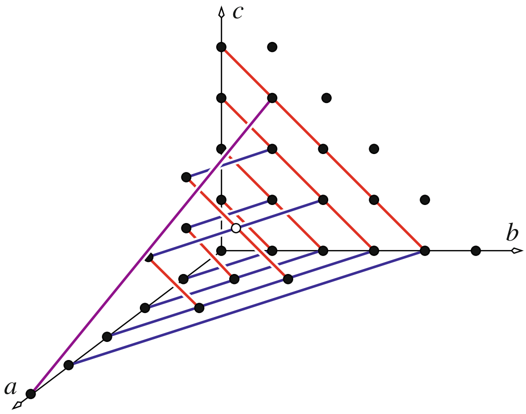

Fig. 1 The M-subgraphs of $\mathbb{N}^3$ for Example 5.4

and combinatorics of their solutions are directly governed by the corresponding features of the input binomial ideal. The dichotomy between components that are toral or not, for example, distinguishes between the choices of parameters yielding finite- or infinite-dimensional solution spaces; and in the finite case, the multiplicities of the toral components enter into simple formulas for the dimension. The use of binomial primary decomposition to extract invariants and reduce to the A-hypergeometric case underlies the entirety of [3].

We will state two of the main results in that article to illustrate this point, but first we need a definition.

Definition 5.5 Let A be as in Convention 4.1, and let $I \subseteq \mathbb{C}[\partial]$ be an A-graded binomial ideal. We think of $\mathbb{C}[\partial]$ as a (commutative) subring of the Weyl algebra D_n of linear partial differential operators (with polynomial coefficients) in $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. For $\beta \in \mathbb{C}^d$, consider the left D_n-ideal

$$H_A(I;\beta) = I + \left\langle \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i : i = 1, \dots, d \right\rangle \subseteq D_n.$$

The quotient $D_n/H_A(I;\beta)$ is called a *binomial D-module*.

The importance of binomial primary decomposition in the study of hypergeometric systems and binomial D-modules is evident in the following result.

Theorem 5.6 [3, Theorem 6.8] *Let I be an A-graded binomial ideal, and suppose that $I = \cap C_{\rho,J}$ is a (binomial) primary decomposition. If β is generic, then*

$$\frac{D_n}{H_A(I;\beta)} = \bigoplus \frac{D_n}{H_A(C_{\rho,J};\beta)}.$$

The following theorem showcases the difference between toral and Andean behavior in the differential setting.

Theorem 5.7 [3, Theorems 4.9 and 6.10, Corollary 5.7] *Let C be a primary A-graded binomial ideal, whose radical is $I_{\rho,J}$.*

- *If C is toral, then $H_A(C;\beta)$ has a finite dimensional solution space for all $\beta \in \mathbb{C}^d$ of dimension at least $\mu_{\rho,J}\mathrm{vol}(A_J)$, where $\mu_{\rho,J}$ is the multiplicity of $I_{\rho,J}$ as an associated* *prime of C, and* $\mathrm{vol}(A_J)$ *is the volume of the convex hull of the origin in* $\mathbb{C}^d$ *and the columns of* A_J*, normalized so that the unit simplex has volume* 1*. For generic* β*, the dimension of the solution space is exactly* $\mu_{\rho,J}\mathrm{vol}(A_J)$.

- *If C is Andean, then for generic* β*,* $H_A(C;\beta) = D_n$*, and therefore* $H_A(C;\beta)$ *has no nonzero solutions. If* $H_A(C;\beta) \neq D_n$*, then the solutions of* $H_A(C;\beta)$ *have uncountable dimension as a vector space over* $\mathbb{C}$.

Remark 5.8 The results in [3] referenced in Theorems 5.6 and 5.7 contain very specific descriptions of the genericity requirements for the parameters β.

Theorems 5.6 and 5.7 imply that, for generic parameters β, the solution space of a system $H_A(I;\beta)$ splits as the direct sum of the solutions of the systems $H_A(C_{\rho,J};\beta)$ where $C_{\rho,J}$ runs over only the dimension d toral components of I, thereby providing a formula for the generic dimension of this system. Our detailed information regarding such components can be utilized to write a basis for the solution space of $H_A(C_{\rho,J};\beta)$ in terms of the solutions of the underlying GKZ-type system $H_{A_J}(I_\rho;\beta)$. Since the solutions of GKZ systems for generic parameters can be explicitly written down, the basis mentioned above is also very explicit.

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