

# LARGE TIME BEHAVIOR FOR A NONLOCAL DIFFUSION EQUATION WITH ABSORPTION AND BOUNDED INITIAL DATA

JOANA TERRA AND NOEMI WOLANSKI

ABSTRACT. We study the large time behavior of nonnegative solutions of the Cauchy problem  $u_t = \int J(x-y)(u(y,t) - u(x,t)) dy - u^p$ ,  $u(x,0) = u_0(x) \in L^\infty$ , where  $|x|^\alpha u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . One of our main goals is the study of the critical case  $p = 1 + 2/\alpha$  for  $0 < \alpha < N$ , left open in previous articles, for which we prove that  $t^{\alpha/2}|u(x,t) - U(x,t)| \rightarrow 0$  where  $U$  is the solution of the heat equation with absorption with initial datum  $U(x,0) = C_{A,N}|x|^{-\alpha}$ . Our proof, involving sequences of rescalings of the solution, allows us to establish also the large time behavior of solutions having more general nonintegrable initial data  $u_0$  in the supercritical case and also in the critical case ( $p = 1 + 2/N$ ) for bounded and integrable  $u_0$ .

## 1. INTRODUCTION

Consider the following nonlocal evolution problem with absorption

$$(1.1) \quad \begin{cases} u_t = \int J(x-y)(u(y,t) - u(x,t)) dy - u^p(x,t) & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where  $J \in C_0^\infty(\mathbb{R}^N)$  is radially symmetric,  $J \geq 0$  with  $\int J = 1$  and  $u_0 \geq 0$  and bounded.

Equation (1.1) can be seen as a model for the density of a population at a certain point and given time. In fact, let  $u$  represent such density and the kernel  $J(x-y)$  represent the probability distribution density of jumping from a point  $x$  to a point  $y$ . Then, using the symmetry of the kernel  $J$ , the diffusion term  $\int J(x-y)(u(y,t) - u(x,t)) dy$  represents the difference between the rate at which the population is arriving at the point  $x$  and the rate at which it is leaving  $x$ . The absorption term  $-u^p$  represents a rate of consumption due to an internal reaction.

This diffusion operator has been used to model several nonlocal diffusion processes in the last few years. See for instance [1, 2, 3, 4, 10, 20]. In particular, nonlocal diffusions are of interest in biological and biomedical problems. Recently, these kind of nonlocal operators have also been used for image enhancement [11].

We are interested in the large time behavior of the solutions to (1.1) and how the space dimension  $N$ , the absorption exponent  $p$  and the assumptions on the initial data  $u_0$  influence the result.

These kind of problems have been widely studied for the heat equation with absorption or, more generally, the porous medium equation or other diffusion equations. (See, for instance, [6, 12, 14, 15, 16, 21]).

---

*Key words and phrases.* Nonlocal diffusion, Large time behavior.  
2000 *Mathematics Subject Classification* 35K57, 35B40.

In the case of semilinear problems, one possible approach is the direct use of the variations of constants formula associated to the knowledge of a fundamental solution of the purely diffusive linear part. This approach has been considered for problem (1.1) in [18, 19] and it allowed to study the supercritical cases, in which the presence of the absorption term does not influence the asymptotic behavior.

Another possible approach is to consider rescalings that leave the purely diffusive equation unchanged. This is done, for instance, for the porous medium or the evolutionary  $p$ -laplace equations.

At first sight, this last approach is not possible for (1.1) since the nonlocal diffusion equation is not invariant under any rescaling. Anyhow, it is easy to see that if  $u$  is a solution to

$$(1.2) \quad u_t = \int J(x-y)(u(y,t) - u(x,t)) dy = Lu,$$

and, for  $k > 0$  and  $f(k)$  any function of the parameter  $k$ ,

$$u^k(x,t) = f(k)u(kx, k^2t),$$

then  $u^k$  is a solution to the following equation

$$(1.3) \quad v_t = k^2 \int J_k(x-y)(v(y,t) - v(x,t)) dy,$$

where  $J_k(x) = k^N J(kx)$ . It is not difficult to prove that for a fixed smooth function  $v$ , the right hand side in (1.3) converges, as  $k$  goes to infinity, to  $\mathfrak{a} \Delta v$  where  $\mathfrak{a}$  is a constant that depends only on the kernel  $J$  of the nonlocal operator.

This fact has already been used –with  $\varepsilon = k^{-1} \rightarrow 0$ – in order to prove that the solutions of the rescaled problems set in a fixed bounded domain converge to the solution of the heat equation with diffusivity  $\mathfrak{a}$ . (See, for instance, [8, 9]).

Therefore, since  $k$  going to infinity for  $u^k(x,1)$  amounts to  $t$  going to infinity for  $u(x,t)$ , it is not at all striking that the asymptotic behavior as  $t$  goes to infinity of the solution of (1.1) is the same as that of the solution of the equation obtained by replacing the nonlocal operator by  $\mathfrak{a} \Delta$ , as was proved in [18] when  $u_0 \in L^\infty \cap L^1$  and  $p > 1 + 2/N$  and in [19] when  $u_0$  is bounded and  $|x|^\alpha u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha \leq N$  and  $p > 1 + 2/\alpha$ .

Both in [18] and [19] no rescaling was proposed and instead, the variations of constants formula was used. Nevertheless, such method only led to the study of the supercritical cases.

As stated above, the idea of the rescaling method is that the behavior of  $u(x,t)$  as  $t \rightarrow \infty$  is that of  $u^k(x,1)$  as  $k \rightarrow \infty$  (for a suitable choice of  $f(k)$ ).

In order to prove the convergence of the functions  $u^k(x,1)$  on compact sets of  $\mathbb{R}^N$ , it is necessary to establish compactness of a family of uniformly bounded solutions to the equation satisfied by  $u^k$ . In the case of the heat or the porous medium equations, this compactness follows from their regularizing effect.

The purpose of this paper is twofold. On one hand, to establish the large time behavior in the critical case  $p = 1 + 2/\alpha$ ,  $0 < \alpha < N$  that was left open in [19]. On the other hand, to show how to use the rescaling method in the present situation in which each  $u^k$  is a solution of a different equation. And moreover, how to obtain, in the present situation, the compactness of the family  $\{u^k\}$  from uniform  $L^\infty$  bounds.

One very important issue that we had to overcome is the lack of a regularizing effect of equation (1.2) and its rescalings. In fact,  $u(x, t)$  is exactly as smooth as  $u_0(x)$ . This problem is overcome by the observation that the fundamental solution of equation (1.2) can be decomposed as  $e^{-t\delta} + W(x, t)$  with  $W(x, t)$  smooth, as proved in [5]. Then, the variations of constants formula for the rescaled equations allowed us to decompose  $u^k(x, t) = v^k(x, t) + h^k(x, t)$  with  $v^k \rightarrow 0$  and  $h^k$  smooth. So, the limit as  $k \rightarrow \infty$  of  $u^k(x, t)$  for  $t > 0$  is that of the smooth functions  $h^k$ , for which we can prove compactness by establishing uniform Holder estimates.

One of the main contributions of the present paper is a very sharp estimate on the space-time behavior of  $W(x, t)$  (as well as its space and time derivatives). This estimate is invariant under the rescaling  $W_k(x, t) = k^N W(kx, k^2t)$ , thus leading to estimates for the rescalings  $u^k$  of the solution  $u$ .

This study of the good part of the fundamental solution and the idea of how to use the rescaling method in order to study asymptotics related to the nonlocal diffusion equation (1.2) give insight into how to attack other semilinear problems related to this equation like blow up or quenching problems (determine blow up and quenching profiles, for instance) and thus they are of an independent interest.

In the present paper we apply this method in order to study the asymptotic behavior as time goes to infinity of the solution to (1.1) for general bounded initial data  $u_0$ . We find that this behavior is determined by the way  $u_0$  decays at infinity, retrieving for (1.1) results that were known for the heat equation (see [12, 14, 16]). In particular, this method allows to treat the critical case  $p = 1 + 2/\alpha$ ,  $0 < \alpha < N$  left open in our previous paper [19]. Moreover, we also complete the results of [18] for integrable initial data by proving that –in the critical case  $p = 1 + 2/N$ – there holds that  $t^{N/2}u(x, t) \rightarrow 0$  as  $t \rightarrow \infty$  (as compared to the supercritical case  $p > 1 + 2/N$  where a nontrivial limit is achieved).

Moreover, since our approach allows for very general initial data, we find results for both integrable and nonintegrable initial data that do not behave as a negative power at infinity, and we give some examples of application of our results to such cases at the end of this article.

Before presenting some examples of initial data to which our results apply let us introduce some notation.

**Notation.** We consider rescalings that depend on the behavior of  $u_0$  at infinity. For  $k > 0$  we denote by  $v_k(x, t) = k^N v(kx, k^2t)$  and, following the notation in [16], we denote by  $v^k(x, t) = f(k)v(kx, k^2t)$  where

$$f(k) := \frac{k^N}{\int_{B_k} u_0}.$$

**Remark 1.1.** *Since  $u_0 \geq 0$ ,  $u_0 \neq 0$ , there exist  $\kappa > 0$  and  $x_0 \in \mathbb{R}^N$  such that  $\int_{B_k(x_0)} u_0(x) dx \geq \kappa k^N$  for  $k > 0$  small. Without loss of generality we will assume that  $x_0 = 0$ .*

We assume further.

### Conditions on $f$

**(F1)**  $u_0 \in L^\infty(\mathbb{R}^N)$  and there exists  $B > 0$  such that  $f(|x|)u_0(x) \leq B$ .

**(F2)** For every  $\delta > 0$ , there exists  $C_\delta > 0$  such that  $f(k) \leq C_\delta f(l)$  if  $k_0 \leq k \leq l\delta^{-1}$ .

**(F3)** There exists  $c_0 \geq 0$  such that  $F(k) := f(k)^{1-p}k^2 \rightarrow c_0$  as  $k \rightarrow \infty$ .

**Remark 1.2.** Assumption **(F3)** implies that  $f(k) \geq c_1 k^{2/(p-1)}$  if  $k \geq k_0$  and  $k_0$  is large. Therefore,  $f(k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

**Examples** By including the function  $f$  in our rescaled sequence we are able to deal with rather general initial conditions. It is also interesting to note that, for  $u_0 \in L^\infty$  satisfying

$$(1.4) \quad |x|^\alpha u_0(x) \rightarrow A > 0 \quad \text{as } |x| \rightarrow \infty \quad \text{with } 0 < \alpha < N,$$

the function  $f$  behaves like  $k^\alpha$  as  $k$  tends to infinity. This is then, a rather usual rescaling in this case.

The following is a list of examples of initial data satisfying our assumptions.

**Ex. 1** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $|x|^\alpha u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Then,

$$f(k) \sim k^\alpha \quad \text{so we take it to be equal}$$

$$F(k) = k^{-\alpha(p-1)+2} \rightarrow \begin{cases} 0 & \text{if } p > 1 + \frac{2}{\alpha} \\ 1 & \text{if } p = 1 + \frac{2}{\alpha} \end{cases}$$

**Ex. 2** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $|x|^N u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Then,

$$f(k) \sim \frac{k^N}{\log k} \quad \text{so we take it to be equal}$$

$$F(k) = k^{-N(p-1)+2} (\log k)^{p-1} \rightarrow 0 \in \mathbb{R} \quad \text{if } p > 1 + \frac{2}{N}.$$

**Ex. 3** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $\frac{|x|^\alpha}{\log |x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Then,

$$f(k) \sim \frac{k^\alpha}{\log k} \quad \text{so we take it to be equal}$$

$$F(k) = k^{-\alpha(p-1)+2} (\log k)^{p-1} \rightarrow 0 \in \mathbb{R} \quad \text{if } p > 1 + \frac{2}{\alpha}.$$

**Ex. 4** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $|x|^\alpha (\log |x|) u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Then,

$$f(k) \sim k^\alpha \log k \quad \text{so we take it to be equal}$$

$$F(k) = \frac{k^{-\alpha(p-1)+2}}{(\log k)^{p-1}} \rightarrow 0 \quad \text{if } p \geq 1 + \frac{2}{\alpha}.$$

**Ex. 5** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $\frac{|x|^N}{\log |x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Then,

$$f(k) \sim \frac{k^N}{\log^2 k} \quad \text{so we take it to be equal}$$

$$F(k) = k^{-N(p-1)+2} (\log k)^{2(p-1)} \rightarrow 0 \quad \text{if } p > 1 + \frac{2}{N}.$$

**Ex. 6** Assume  $u_0 \in L^\infty(\mathbb{R}^N)$  and  $|x|^N(\log|x|)u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Then,

$$f(k) \sim \frac{k^N}{\log \log k} \quad \text{so we take it to be equal}$$

$$F(k) = k^{-N(p-1)+2}(\log \log k)^{p-1} \rightarrow 0 \quad \text{if } p > 1 + \frac{2}{N}.$$

**Ex. 7** Assume  $u_0 \in L^1(\mathbb{R}^N)$ . Then,

$$f(k) \sim k^N \quad \text{so we take it to be equal}$$

$$F(k) = k^{-N(p-1)+2} \rightarrow \begin{cases} 0 & \text{if } p > 1 + \frac{2}{N} \\ 1 & \text{if } p = 1 + \frac{2}{N} \end{cases}$$

Let us just mention that the function  $f(k)$  is related to the rate of decay in time of the solution whereas  $c_0 = \lim_{k \rightarrow \infty} F(k)$  turns out to be the coefficient in front of the absorption term in the equation satisfied by the limiting profile.

For our main results we refer to Section 4.

The paper is organized as follows. In Section 2 we construct barriers for the good part  $W$  of the fundamental solution of (1.2), as well as for its space and time derivatives. By using these barriers we obtain an upper bound for the solution  $u_L$  to (1.2) for all times. This result improves the one we had established for finite time intervals in [19] in case  $u_0$  satisfies (1.4). Also, this bound on  $u_L$  implies that the rescaled functions  $u^k$  are uniformly bounded in  $\mathbb{R}^N \times [\tau, \infty)$  for every  $\tau > 0$ .

In Section 3 we analyze the rescaled problems satisfied by the  $u^k$ 's and prove that, under sequences,  $\{u^k\}$  converges uniformly on compact sets to a solution  $U$  of the equation  $U_t - \mathbf{a}\Delta U = -c_0 U^p$ , where  $c_0 = \lim_{k \rightarrow \infty} F(k)$ .

Moreover, we obtain a general result stating that we can determine the limit function  $U$ . In fact, we prove that if  $u_0^k(x) \rightarrow \phi(x)$  as  $k$  tends to infinity in the sense of distributions and, for every  $R > 0$  there holds that

$$\|u^k\|_{L^1(B_R \times (0, \tau))} \leq C(R)\tau$$

and either there exists  $\gamma > 0$  such that

$$\|u^k\|_{L^p(B_R \times (0, \tau))} \leq C(R)\tau^\gamma$$

or, for every  $\tau \leq 1$ ,  $R > 0$ ,

$$\lim_{k \rightarrow \infty} \|u^k\|_{L^p(B_R \times (0, \tau))} = 0,$$

then  $U$  satisfies moreover,

$$U(x, 0) = \phi(x).$$

Then we establish, for the different cases of nonintegrable initial data  $u_0$  considered in the examples, the desired  $L^1$  and  $L^p$  bounds.

At the end of the section we analyze the case where  $u_0$  is integrable (in which case it does not satisfy an  $L^p$  estimate as above) and determine  $U$ , both for  $p$  supercritical and for  $p$  critical.

Finally, in Section 4 we prove our main results on the asymptotic behavior of the solution for initial data comprised in the examples above. In particular, we obtain the behavior in the

critical cases  $p = 1 + 2/\alpha$  when  $u_0$  satisfies (1.4), and  $p = 1 + 2/N$  when  $u_0 \in L^1 \cap L^\infty$ , that were left open in [19] and [18] respectively.

## 2. BARRIERS AND FIRST ASYMPTOTIC ESTIMATES

In this section we analyze in detail the smooth part  $W$  of the fundamental solution to the linear equation

$$(2.5) \quad u_t = \int J(x-y)(u(y,t) - u(x,t)) dy = Lu.$$

We establish upper bounds for  $W$  and also for its space and time derivatives. These will be essential in Section 3, in order to prove the convergence of the rescaled sequence.

In [5] the authors observed that the fundamental solution of (2.5) can be written as

$$U(x,t) = e^{-t}\delta + W(x,t)$$

where  $\delta$  is the Dirac measure and  $W(x,t)$  is a smooth function. Then, in [13] the authors showed that

$$|W(x,t)| \leq C_0 t^{-N/2}.$$

In [19] we established that  $W$  is a solution to the problem

$$(2.6) \quad \begin{cases} W_t(x,t) = \int J(x-y)(W(y,t) - W(x,t)) dy + e^{-t}J(x) \\ W(x,0) = 0, \end{cases}$$

and used this fact to prove that  $W \geq 0$  and to obtain estimates in  $L^q(\mathbb{R}^N)$ , which in turn allowed us to establish the asymptotic behavior of solutions to the nonlocal problem with absorption in the supercritical case.

In order to deal with the critical case, we will use the method of rescaled sequences for which we need the knowledge of the behavior of  $W(x,t)$  as  $|x| \rightarrow \infty$ . We obtain this behavior from sharp barriers. We have,

**Theorem 2.1.** *Let  $W$  as above. There exists a constant  $C > 0$  depending only on  $J$  and  $N$  such that*

$$(2.7) \quad W(x,t) \leq C \frac{t}{|x|^{N+2}}.$$

*Proof.* First observe that  $W(x,t) \leq v_1(x,t) := \|J\|_\infty t$ . In fact, in any finite time interval, the function  $v_1$  is a bounded supersolution of the problem (2.6) satisfied by  $W$ . Thus, the inequality follows from the comparison principle.

Now, let  $v_2(x,t) = C \frac{t}{|x|^{N+2}}$ . We will show that there is a constant  $C$  depending only on  $J$  and  $N$  such that  $v_2$  is a supersolution of the Dirichlet problem

$$\begin{cases} v_t - Lv = e^{-t}J(x) & \text{in } \mathcal{A} := \{|x| \geq K\sqrt{t}\} \cap \{|x| \geq 2R\} \\ v = W & \text{in the complement of } \mathcal{A} \\ v(x,0) = 0 \end{cases}$$

satisfied by  $W$ . Here  $R$  is large enough and such that  $B_R$  contains the support of  $J$ , and  $K$  is a large enough constant to be determined.

In fact,  $v_2 \geq W$  in  $\mathcal{A}^c$  if  $C$  is large since  $W \leq \|J\|_\infty t$  and  $W \leq C_0 t^{-N/2}$ .

On the other hand,  $v_{2t} = \frac{C}{|x|^{N+2}}$ . In order to estimate  $Lv_2$  we use Taylor's expansion to get,

$$\begin{aligned} \frac{1}{|y|^{N+2}} - \frac{1}{|x|^{N+2}} &= -\frac{N+2}{|x|^{N+4}} x \cdot (y-x) - \frac{1}{2} \frac{N+2}{|x|^{N+4}} |y-x|^2 \\ &+ \frac{1}{2} \frac{(N+2)(N+4)}{|x|^{N+6}} |x \cdot (y-x)|^2 + \int_0^1 O\left(\frac{|y-x|^3}{|x+s(y-x)|^{N+5}}\right) ds. \end{aligned}$$

Now, since  $J$  is radially symmetric,

$$\begin{aligned} -\frac{N+2}{|x|^{N+4}} \sum_{i=1}^N x_i \int J(x-y)(y_i-x_i) dy &= 0, \\ \frac{1}{2} \frac{N+2}{|x|^{N+4}} \int J(x-y)|y-x|^2 dy &= \frac{(N+2)N}{|x|^{N+4}} \mathbf{a}, \end{aligned}$$

and

$$\frac{1}{2} \sum_{i,j=1}^N \frac{x_i x_j}{|x|^{N+6}} (-(N+2)(N+4)) \int J(x-y)(y_i-x_i)(y_j-x_j) dy = -\frac{(N+2)(N+4)}{|x|^{N+4}} \mathbf{a},$$

where  $\mathbf{a} = \frac{1}{2N} \int J(x)|x|^2 dx$ .

On the other hand,  $x + s(y-x) \geq |x| - |y-x| \geq \frac{1}{2}|x|$  if  $|x| \geq 2R$  and  $|y-x| \leq R$ . Thus, if  $|x| \geq 2R$ ,

$$\int \int_0^1 J(x-y) O\left(\frac{|y-x|^3}{|x+s(y-x)|^{N+5}}\right) ds dy \leq \frac{C_1}{|x|^{N+5}}.$$

Putting everything together we get, if  $|x| \geq 2R$  and  $R$  is large enough,

$$v_{2t} - Lv_2 \geq \frac{C}{|x|^{N+2}} \left(1 - C_2 \frac{t}{|x|^2}\right).$$

So that, if  $|x| \geq K\sqrt{t}$  with  $K$  large enough,

$$v_{2y} - Lv_2 \geq \frac{C/2}{|x|^{N+2}} \geq e^{-t} J(x)$$

if  $C$  is large enough since  $J$  is bounded and has compact support.  $\square$

Now we find a barrier for the space derivatives of  $W$ .

**Theorem 2.2.** *Let  $W$  be as above. There exists a constant  $C > 0$  depending only on  $J$  and  $N$  such that*

$$(2.8) \quad |\nabla W(x, t)| \leq C \frac{t}{|x|^{N+3}}.$$

*Proof.* We proceed as above. First, by differentiating the equation satisfied by  $W$  we find that  $V_i = W_{x_i}$  is the solution to

$$(2.9) \quad \begin{cases} V_{it} - LV_i = e^{-t} J_{x_i} \\ V_i(x, 0) = 0 \end{cases}$$

As with  $W$  we find immediately that  $|V_i| \leq \|\nabla J\|_\infty t$ . On the other hand, from the Fourier characterization of  $W$  it can be seen that  $\|\nabla W(\cdot, t)\|_\infty \leq C_0 t^{-\frac{N+1}{2}}$ . (See, for instance, [13]). Therefore, for every  $K > 0$  there exists a constant  $C$  such that

$$\bar{v}(x, t) := C \frac{t}{|x|^{N+3}} \geq V_i \quad \text{in } \mathcal{A}^c.$$

On the other hand, the same type of computation as the one in Theorem 2.1 yields that, if  $K$  is large enough, there exists  $C$  such that  $\bar{v}$  is a supersolution to the Dirichlet problem

$$\begin{cases} v_t - Lv = e^{-t} J_{x_i}(x) & \text{in } \mathcal{A} := \{|x| \geq K\sqrt{t}\} \cap \{|x| \geq 2R\} \\ v = V_i & \text{in the complement of } \mathcal{A} \\ v(x, 0) = 0 \end{cases}$$

satisfied by  $V_i$ .

We conclude that  $V_i \leq \bar{v}$ .

Analogously,  $-\bar{v}$  is a subsolution to this problem. Thus,  $|V_i| \leq \bar{v}$  and the theorem is proved.  $\square$

The estimate in (2.8) allows to derive estimates of the  $L^1$  norm of  $\nabla W$ . In fact,

$$\begin{aligned} \int |\nabla W(x, t)| dx &= \int_{|x| \leq \sqrt{t}} |\nabla W(x, t)| dx + \int_{|x| \geq \sqrt{t}} |\nabla W(x, t)| dx \\ &\leq C_0 \int_{|x| \leq \sqrt{t}} t^{-\frac{N+1}{2}} dx + C \int_{|x| \geq \sqrt{t}} \frac{t}{|x|^{N+3}} dx \\ &= C_{N,J} \left[ t^{-\frac{N+1}{2}} t^{\frac{N}{2}} + t t^{-\frac{3}{2}} \right] \\ &= C_{N,J} t^{-\frac{1}{2}}. \end{aligned}$$

So, we obtain a first estimate that is suitable for large times

$$(2.10) \quad \int |\nabla W(x, t)| dx \leq C_1 t^{-\frac{1}{2}}$$

with  $C_1$  depending only on  $J$  and  $N$ .

On the other hand, since  $|\nabla W| \leq Ct$ , there holds that

$$(2.11) \quad |\nabla W(x, t)| \leq C \frac{t}{(1 + |x|)^{N+3}}.$$

Thus, we have the following estimate, which is better than the previous one for small  $t$ .

$$(2.12) \quad \int |\nabla W(x, t)| dx \leq C_1 t.$$

We obtain similar results for  $W_t$ . We have,

**Theorem 2.3.** *Let  $W$  be as above. There exists a constant  $C > 0$  depending only on  $J$  and  $N$  such that*

$$(2.13) \quad |W_t(x, t)| \leq e^{-t} J(x) + C \frac{t}{(1 + |x|)^{N+4}}.$$



*Proof.* By differentiating the equation satisfied by  $W$  we obtain

$$(W_t)_t(x, t) = LW_t(x, t) - e^{-t}J(x).$$

On the other hand, from the equation for  $W$  we get

$$W_t(x, 0) = J(x).$$

Let  $V(x, t) = W_t(x, t) - e^{-t}J(x)$ . Then,

$$(2.14) \quad \begin{cases} V_t - LV = e^{-t}(J * J - J) \\ V(x, 0) = 0 \end{cases}$$

Since  $|J * J - J| \leq 2\|J\|_\infty$  we get a first estimate for  $V$ :  $|V(x, t)| \leq 2\|J\|_\infty t$ .

We can derive the estimate  $|V(x, t)| \leq Ct^{-\frac{N+2}{2}}$  by differentiating  $W$  with respect to time in its Fourier representation.

Now, proceeding as above we see that there exist  $C$  and  $K$  large so that the function  $C\frac{t}{|x|^{N+4}}$  is a supersolution of the following problem satisfied by  $V$ .

$$\begin{cases} v_t - Lv = e^{-t}(J * J - J) & \text{in } \mathcal{A} := \{|x| \geq K\sqrt{t}\} \cap \{|x| \geq 2R\} \\ v = V & \text{in the complement of } \mathcal{A} \\ v(x, 0) = 0 \end{cases}$$

Analogously,  $-C\frac{t}{|x|^{N+4}}$  is a subsolution to this problem. Therefore,

$$|V(x, t)| \leq C\frac{t}{|x|^{N+4}}.$$

So that,

$$|W_t(x, t)| \leq e^{-t}J(x) + |V(x, t)| \leq e^{-t}J(x) + C\frac{t}{|x|^{N+4}}.$$

Since,  $|V(x, t)| \leq Ct$  we prove (2.13). □

From estimate (2.13) we obtain the following estimates

$$(2.15) \quad \|W_t(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq Ct^{-1}$$

$$(2.16) \quad \|W_t(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq e^{-t} + Ct.$$

Finally, we construct a barrier for the solution  $u_L$ . We have,

**Proposition 2.1.** *Let  $u_L$  be the solution to (2.5) with initial datum  $u_0$  satisfying our assumptions. There exists a constant  $C$  depending only on  $\|u_0\|_\infty$ ,  $B$ ,  $N$ ,  $J$  such that*

$$(2.17) \quad \begin{aligned} f(t^{1/2})u_L(x, t) &\leq C, \\ f(|x|)u_L(x, t) &\leq C. \end{aligned}$$

*In particular, if  $\mu > 0$  is such that  $k^\mu \leq Cf(k)$  if  $k \geq 1$  there holds that*

$$(2.18) \quad u_L(x, t) \leq \frac{C}{(1 + t^{1/2} + |x|)^\mu}$$

Let us recall that this is always the case when  $\mu = \frac{2}{p-1}$ .

**Remark 2.1.** (2.18) improves the estimate found in [19], Proposition 2.1 where the estimate was proved in finite time intervals when  $u_0$  satisfies (1.4) with  $\alpha > 0$ .

*Proof.* We already know that  $u_L(x, t) \leq \|u_0\|_\infty$ . Let us begin with the time estimate. We have –by the estimates on  $W$  and our assumptions on  $u_0$ – that for  $t$  large,

$$\begin{aligned} f(t^{1/2})u_L(x, t) &= f(t^{1/2})e^{-t}u_0(x) + f(t^{1/2}) \int W(x-y, t)u_0(y) dy \\ &\leq C + \frac{1}{\int_{B_{t^{1/2}}} u_0} \int_{|y| < t^{1/2}} t^{N/2} W(x-y, t)u_0(y) dy + \int_{|y| > t^{1/2}} W(x-y, t)f(t^{1/2})u_0(y) dy \\ &\leq C \left(1 + \int W(x-y, t) dy\right) \leq C. \end{aligned}$$

On the other hand, since  $u_L$  is bounded and  $f$  is locally bounded,

$$f(t^{1/2})u_L(x, t) \leq C$$

if  $t$  is bounded.

Now, we estimate,

$$\begin{aligned} f(|x|)u_L(x, t) &= e^{-t}f(|x|)u_0(x) + f(|x|) \int W(x-y, t)u_0(y) dy \\ &\leq B + f(|x|) \int_{|y| < \frac{1}{2}|x|} W(x-y, t)u_0(y) dy + f(|x|) \int_{|y| \geq \frac{1}{2}|x|} W(x-y, t)u_0(y) dy \\ &= B + I + II. \end{aligned}$$

Observe that  $f(2k) = \frac{2^N k^N}{\int_{B_{2k}} u_0} \leq 2^N f(k)$ . Thus,

$$II \leq C \int W(x-y, t)f(2|y|)u_0(y) dy \leq C2^N B.$$

In order to estimate  $I$  we use the barrier of  $W$ . We have, since  $|y| < \frac{1}{2}|x|$  implies that  $|x-y| > \frac{1}{2}|x|$ ,

$$\begin{aligned} I &= f(|x|) \int_{|y| < \frac{1}{2}|x|} W(x-y, t)u_0(y) dy \leq C|x|^{-2t} \frac{1}{\int_{B_{|x|}} u_0} \int_{|y| < \frac{1}{2}|x|} u_0(y) dy \\ &\leq C \frac{t}{|x|^2} \leq C \quad \text{if } |x|^2 > t. \end{aligned}$$

On the other hand, in the region  $k_0^2 \leq |x|^2 \leq t$  there holds that

$$f(|x|)u_L(x, t) \leq C_1 f(t^{1/2})u_L(x, t) \leq C.$$

Finally, if  $|x| \leq k_0$ , there holds that  $f(|x|)u_L(x, t) \leq C$ . So, the proposition is proved.  $\square$

**Remark 2.2.** When  $u_0 \geq 0$  the solution  $u_L$  of the homogeneous equation (2.5) with initial data  $u_0$  is non-negative. Thus,  $u_L$  is a supersolution to (1.2) and 0 is a subsolution to (1.2). By the comparison principle we deduce that

$$0 \leq u(x, t) \leq u_L(x, t),$$

for every solution  $u$  of (1.2). Hence, the estimates of the previous proposition hold with  $u_L$  replaced by  $u$ , that is,

$$f(|x|)u(x, t) \leq C \quad \text{and} \quad f(t^{1/2})u(x, t) \leq C.$$

### 3. THE RESCALED PROBLEM

In this section we analyze the rescaled problem. This is, the one satisfied by the rescaled functions  $u^k$ . Using the bounds obtained in the previous section we are able to prove that the rescaled sequence  $\{u^k\}$  has a convergent subsequence to a function  $U$ . We establish the equation satisfied by the limit function  $U$ , as well as the initial datum  $U(x, 0)$ , depending on the conditions assumed on  $u_0$ . In the case of nonintegrable initial data  $u_0$ , in order to completely determine  $U$ , it is necessary to establish certain bounds on the  $L^1$  and  $L^p$  norms of  $u^k$ . On the other hand, if  $u_0$  is integrable, we proceed in a different way, as can be seen at the end of this section.

Let  $u$  be a solution of

$$(3.1) \quad \begin{cases} u_t = Lu - u^p & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

As defined in the introduction we denote by

$$u^k(x, t) = f(k)u(kx, k^2t), \quad \text{where } f(k) = \frac{k^N}{\int_{B_R} u_0}.$$

By Remark 2.2 we have that  $f(t^{1/2})u(x, t) \leq C$ . So that, by our assumption **(F2)** on  $f$ , if  $t \geq t_0$  there holds that

$$u^k(x, t) \leq C_{t_0} f(k\sqrt{t})u(kx, k^2t) \leq C_{t_0}.$$

The function  $u^k$  satisfies the following equation,

$$(3.2) \quad u_t^k = k^2 L_k u^k - F(k)(u^k)^p$$

where  $F(k) = f(k)^{1-p}k^2 \rightarrow c_0 \geq 0$  as  $k \rightarrow \infty$  by our assumptions, and the operator  $L_k$  is defined by

$$(3.3) \quad L_k v(x) = (J_k * v)(x) - v(x) = k^N \int J(k(x-y))(v(y) - v(x)) dy.$$

Our goal is to study the behavior of the sequence  $\{u^k\}$ . To do so, we will decompose  $u^k$  into an exponentially small part and another one,  $h^k$ , depending on  $W$ , the smooth part of the fundamental solution of the homogeneous linear problem.

Take  $t_0 > 0$  and write  $u(x, t) = e^{-(t-k^2t_0)}u(x, k^2t_0) + z(x, t)$ . Then,

$$z_t - Lz = e^{-(t-k^2t_0)}(J * u(\cdot, k^2t_0)) - u^p$$

and  $z(x, k^2t_0) = 0$ . After rescaling we have that  $u^k(x, t) = e^{-k^2(t-t_0)}u^k(x, t_0) + z^k(x, t)$ . Observe that

$$e^{-k^2(t-t_0)}u^k(x, t_0) \leq C_{t_0}e^{-k^2(t-t_0)} \rightarrow 0,$$

as  $k \rightarrow \infty$  uniformly in  $t - t_0 \geq c > 0$ , so that the asymptotic behavior of  $u^k$  for  $k \rightarrow \infty$  is that of  $z^k$ .

By the variations of constants formula we get

$$z(x, t) = \int_{k^2 t_0}^t S(t-s) \left[ e^{-(s-k^2 t_0)} (J * u(\cdot, k^2 t_0))(x) - u^p(x, s) \right] ds$$

where  $S(t)$  is the semigroup associated to the homogeneous equation  $u_t - Lu = 0$ .

Thus,

$$\begin{aligned} z(x, t) &= \int_{k^2 t_0}^t e^{-(t-s)} \left[ e^{-(s-k^2 t_0)} (J * u(\cdot, k^2 t_0))(x) - u^p(x, s) \right] ds \\ &+ \int_{k^2 t_0}^t \int W(x-y, t-s) \left[ e^{-(s-k^2 t_0)} (J * u(\cdot, k^2 t_0))(y) - u^p(y, s) \right] dy ds \\ &= \int_{k^2 t_0}^t e^{-(t-k^2 t_0)} (J * u(\cdot, k^2 t_0))(x) ds - \int_{k^2 t_0}^t e^{-(t-s)} u^p(x, s) ds + h(x, t) \\ &= (t - k^2 t_0) e^{-(t-k^2 t_0)} (J * u(\cdot, k^2 t_0))(x) - \int_{k^2 t_0}^t e^{-(t-s)} u^p(x, s) ds + h(x, t) \end{aligned}$$

with

$$h(x, t) = \int_{k^2 t_0}^t \int W(x-y, t-s) \left[ e^{-(s-k^2 t_0)} (J * u(\cdot, k^2 t_0))(y) - u^p(y, s) \right] dy ds.$$

Therefore,

$$\begin{aligned} z^k(x, t) &= k^2(t-t_0) e^{-k^2(t-t_0)} f(k) (J * u(\cdot, k^2 t_0))(kx) - \int_{k^2 t_0}^{k^2 t} e^{-(k^2 t-s)} f(k) u^p(kx, s) ds \\ &+ h^k(x, t) \\ &= k^2(t-t_0) e^{-k^2(t-t_0)} f(k) (J * u(\cdot, k^2 t_0))(kx) - F(k) \int_{t_0}^t e^{-k^2(t-s)} (u^k)^p(x, s) ds \\ &+ h^k(x, t). \end{aligned}$$

There holds that,

$$\begin{aligned} f(k) (J * u(\cdot, k^2 t_0))(kx) &= \int J(kx-y) f(k) u(y, k^2 t_0) dy \\ &= (J_k * u^k(\cdot, t_0))(x). \end{aligned}$$

So that,

$$\begin{aligned} z^k(x, t) &= k^2(t-t_0) e^{-k^2(t-t_0)} (J_k * u^k(\cdot, t_0))(x) - F(k) \int_{t_0}^t e^{-k^2(t-s)} (u^k)^p(x, s) ds \\ &+ h^k(x, t). \end{aligned}$$

With similar computations we find that

$$\begin{aligned} h^k(x, t) &= k^2 \int_{t_0}^t \int W_k(x-y, t-s) e^{-k^2(s-t_0)} (J_k * u^k(\cdot, t_0))(y) dy ds \\ &- F(k) \int_{t_0}^t \int W_k(x-y, t-s) (u^k)^p(y, s) dy ds, \end{aligned}$$

where  $W_k(x, t) = k^N W(kx, k^2t)$ .

Let us now estimate the first term in the expansion of  $z^k$ . We have

$$k^2(t-t_0)e^{-k^2(t-t_0)}(J_k * u^k(\cdot, t_0))(x) \leq Ck^2(t-t_0)e^{-k^2(t-t_0)}C_{t_0} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

uniformly in  $t-t_0 \geq c > 0$ .

For the second term, since  $F(k)$  is bounded, we have the estimate

$$\begin{aligned} F(k) \int_{t_0}^t e^{-k^2(t-s)}(u^k)^p(x, s) ds &\leq C_{t_0} \int_{t_0}^t e^{-k^2(t-s)} ds \\ &\leq C_{t_0} k^{-2} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Therefore, the asymptotic behavior as  $k \rightarrow \infty$  of  $u^k$  is that of  $h^k$ .

Since the functions  $h^k$  are smooth, we can show that the family  $\{h^k\}$  is precompact in  $C(\mathcal{K})$  with  $\mathcal{K} \subset \subset \mathbb{R}^N \times (t_0, \infty)$  by finding uniform Holder estimates.

We begin with estimates in space.

**Proposition 3.1.** *Let  $t_0 > 0$  and  $T > 2t_0$ . There exists a constant  $L > 0$  such  $|\nabla h^k(x, t)| \leq L$  for  $x \in \mathbb{R}^N$  if  $t \in [2t_0, T]$ .*

*Proof.* Recall that we have obtained estimates for the  $L^1$  norm of  $\nabla W(\cdot, t)$  (cf. (2.10) and (2.12)). Now we derive estimates for the  $L^1$  norm of  $\nabla W_k(x, t) = k^N \nabla W(kx, k^2t)$ . We have,

$$\|\nabla W_k(\cdot, t)\|_{L^1(\mathbb{R}^N)} = k \|\nabla W(\cdot, k^2t)\|_{L^1(\mathbb{R}^N)}.$$

Therefore,

$$\|\nabla W_k(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq Ck(k^2t) = Ck^3t.$$

On the other hand,

$$\|\nabla W_k(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq Ck(k^2t)^{-1/2} = Ct^{-1/2}.$$

Differentiating the function  $h^k$  we obtain,

$$\begin{aligned} |\nabla h^k(x, t)| &\leq k^2 \int_{t_0}^t \int |\nabla W_k(x-y, t-s)| e^{-k^2(s-t_0)}(J_k * u^k(\cdot, t_0))(y) dy ds \\ &\quad + F(k) \int_{t_0}^t \int |\nabla W_k(x-y, t-s)|(u^k)^p(y, s) dy ds = I + II. \end{aligned}$$

There holds,

$$\begin{aligned} I &\leq k^2 C_{t_0} \int_{t_0}^{3t/4} e^{-k^2(s-t_0)}(t-s)^{-1/2} ds + k^2 C_{t_0} \int_{3t/4}^t e^{-k^2(s-t_0)} k^3(t-s) ds \\ &\leq C_{t_0}(t/4)^{-1/2} + C_{t_0}(t/4)k^3 e^{-k^2(\frac{3t}{4}-t_0)} \leq C_{t_0, T} \end{aligned}$$

On the other hand,

$$II \leq C_{t_0} \int_{t_0}^t (t-s)^{-1/2} ds \leq C_{t_0, T}.$$

□

Finally we prove one of our main results, namely, that the sequence  $\{h^k\}$  and therefore, the sequence  $\{u^k\}$  is uniformly convergent on compact sets.

**Theorem 3.1.** *There exists a subsequence that we still call  $h^k$  which is uniformly convergent on every compact subset of  $\mathbb{R}^N \times [2t_0, \infty)$ .*

*Proof.* In order to prove the result, let us split  $h^k$  into two terms.

$$\begin{aligned} h^k(x, t) &= k^2 \int_{t_0}^t \int W_k(x - y, t - s) e^{-k^2(s-t_0)} (J_k * u^k(\cdot, t_0))(y) dy ds \\ &\quad - F(k) \int_{t_0}^t \int W_k(x - y, t - s) (u^k)^p(y, s) dy ds \\ &= H_0^k(x, t) + H^k(x, t). \end{aligned}$$

with

$$H_0^k(x, t) = k^2 \int_{t_0}^t \int W_k(x - y, t - s) e^{-k^2(s-t_0)} (J_k * u^k(\cdot, t_0))(y) dy ds$$

and

$$H^k(x, t) = F(k) \int_{t_0}^t \int W_k(x - y, t - s) (u^k)^p(y, s) dy ds.$$

By the estimates of  $\|W_t(\cdot, t)\|_{L^1(\mathbb{R}^N)}$  (cf. (2.15)) we get

$$\|W_{kt}(\cdot, t)\|_{L^1} \leq Ct^{-1}$$

and

$$\|W_{kt}(\cdot, t)\|_{L^1} \leq k^2 e^{-k^2 t} + Ck^4 t.$$

Therefore,

$$\begin{aligned} |H_0^k(x, t)| &\leq k^2 \int_{t_0}^t \int |W_{kt}(x - y, t - s)| e^{-k^2(s-t_0)} (J_k * u^k(\cdot, t_0))(y) dy ds \\ &\leq Ck^2 \int_{t_0}^{3t/4} (t - s)^{-1} C_{t_0} e^{-k^2(s-t_0)} ds + Ck^2 \int_{3t/4}^t k^2 e^{-k^2(t-s)} C_{t_0} e^{-k^2(s-t_0)} ds \\ &\quad + Ck^2 \int_{3t/4}^t k^4 (t - s) C_{t_0} e^{-k^2(s-t_0)} ds \\ &\leq C_{t_0} (t/4)^{-1} + C_{t_0} (t/4) k^4 e^{-k^2(t-t_0)} + C_{t_0} (t/4) k^4 e^{-k^2(\frac{3t}{4}-t_0)} \\ &\leq C_{t_0, T}. \end{aligned}$$

Therefore, the sequence  $H_0^k$  has a subsequence that converges uniformly on compact subsets of  $\mathbb{R}^N \times [2t_0, \infty)$ .

In order to see that the same conclusion holds for the sequence  $H^k$ , let us define

$$R^k(x, t) := F(k) \int_{t_0}^t \int U_{\mathbf{a}}(x - y, t - s) (u^k)^p(y, s) dy ds,$$

with  $U_{\mathbf{a}}$  the fundamental solution of the heat equation with diffusivity  $\mathbf{a}$ . Then, for every  $T > 0$ ,

$$\|R^k\|_{L^\infty(\mathbb{R}^N \times [t_0, T])} \leq C_{t_0, T}$$

and

$$\|H^k - R^k\|_{L^\infty(\mathbb{R}^N \times [t_0, T])} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

In fact,

$$\|H^k(\cdot, t) - R^k(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq F(k) \int_{t_0}^t \|W_k(\cdot, t-s) - U_{\mathbf{a}}(\cdot, t-s)\|_{q'} \|(u^k)^p(\cdot, s)\|_q ds.$$

Recall that  $f(k) \geq ck^\beta$  with  $\beta = 2/(p-1)$ . Let us take  $q > N/\beta p$ . So that there holds,

$$\|(u^k)^p(\cdot, s)\|_q \leq \left( \int \frac{dx}{(s^{1/2} + |x|)^{\beta pq}} \right)^{1/q} \leq C_{t_0, q} \quad \text{for } s \geq t_0.$$

On the other hand, since  $U_{\mathbf{a}}(x, t) = U_{\mathbf{a}k}(x, t)$ ,

$$\begin{aligned} \|W_k(\cdot, t) - U_{\mathbf{a}}(\cdot, t)\|_{q'} &= \|W_k(\cdot, t) - U_{\mathbf{a}k}(\cdot, t)\|_{q'} \\ &= k^{N/q} \|W(\cdot, k^2 t) - U_{\mathbf{a}}(\cdot, k^2 t)\|_{q'} \\ &\leq k^{N/q} (k^2 t)^{-(N+1)/2q} = k^{-1/q} t^{-(N+1)/2q} \quad (\text{cf. [19]}). \end{aligned}$$

Let us choose  $q$  big enough so that we also have  $q > (N+1)/2$ . Then,

$$\begin{aligned} \|H^k(\cdot, t) - R^k(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} &\leq F(k) C_{q, t_0} k^{-1/q} \int_{t_0}^t (t-s)^{-(N+1)/2q} ds \\ &\leq C_{q, t_0, T} k^{-\frac{1}{q}} \quad \text{if } t_0 \leq t \leq T. \end{aligned}$$

Hence,

$$\|H^k - R^k\|_{L^\infty(\mathbb{R}^N \times [t_0, T])} \leq C_{q, t_0, T} k^{-\frac{1}{q}} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Finally, observe that  $R^k$  is a solution of the heat equation with diffusivity  $\mathbf{a}$  and uniformly bounded right hand side. In fact,

$$R_t^k(x, t) - \mathbf{a} \Delta R^k(x, t) = F(k) (u^k)^p(x, t)$$

and

$$0 \leq F(k) (u^k)^p(x, t) \leq C_{t_0} \quad \text{if } t \geq t_0.$$

Therefore, the family  $R^k$  is uniformly Holder continuous in  $\mathbb{R}^N \times [2t_0, T]$  for every  $T > 2t_0$ . We conclude that there exists a subsequence that is uniformly convergent on every compact subset of  $\mathbb{R}^N \times [2t_0, \infty)$ . And the same conclusion then holds for the family  $H^k$ .  $\square$

Now that we know that  $\{u^k\}$  has a convergent subsequence, we proceed in identifying the limit function  $U$ . As a first step, and using the assumptions on  $F(k)$  in the introduction we establish the equation satisfied by  $U$ .

**Proposition 3.2.** *Let  $k_n \rightarrow \infty$  be such that  $u^{k_n} \rightarrow U$  uniformly on compact sets of  $\mathbb{R}^N \times (0, \infty)$ . Then,  $U$  is a solution to*

$$(3.4) \quad U_t - \mathbf{a} \Delta U = -c_0 U^p$$

where  $c_0 = \lim_{k \rightarrow \infty} F(k)$ .

*Proof.* For simplicity we drop the subscript  $n$ . Let  $\varphi \in C_0^\infty(\mathbb{R}^N \times (0, \infty))$  and  $\mathcal{K}$  a compact set containing its support. Then,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^N} U(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt &= \int_0^\infty \int_{\mathbb{R}^N} u^k(x, t)(\varphi_t + k^2 L_k \varphi)(x, t) dx dt + \\
+ \int_0^\infty \int_{\mathbb{R}^N} (U - u^k)(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt &- \int_0^\infty \int_{\mathbb{R}^N} u^k(x, t)(k^2 L_k \varphi - \mathbf{a}\Delta\varphi)(x, t) dx dt \\
&= - \int_0^\infty \int_{\mathbb{R}^N} (u_t^k - k^2 L_k u^k)(x, t)\varphi(x, t) dx dt + \\
+ \int_0^\infty \int_{\mathbb{R}^N} (U - u^k)(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt &- \int_0^\infty \int_{\mathbb{R}^N} u^k(x, t) O(k^{-3})\chi_{\mathcal{K}} dx dt \\
&= F(k) \int_0^\infty \int_{\mathbb{R}^N} (u^k)^p(x, t)\varphi(x, t) dx dt \\
+ \int_0^\infty \int_{\mathbb{R}^N} (U - u^k)(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt &- \int_0^\infty \int_{\mathbb{R}^N} u^k(x, t) O(k^{-3})\chi_{\mathcal{K}} dx dt \\
&\rightarrow c_0 \int_0^\infty \int_{\mathbb{R}^N} U^p(x, t)\varphi(x, t) dx dt \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

So that,

$$\int_0^\infty \int_{\mathbb{R}^N} U(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt = c_0 \int_0^\infty \int_{\mathbb{R}^N} U^p(x, t)\varphi(x, t) dx dt$$

for every  $\varphi \in C_0^\infty(\mathbb{R}^N \times (0, \infty))$ . Thus,

$$U_t - \mathbf{a}\Delta U = -c_0 U^p.$$

□

Finally, in order to identify the initial datum  $U(x, 0)$ , we need to take into account the behavior of  $u_0$  at infinity. Moreover, it is necessary to have some control on the  $L^1$  and  $L^p$  norms of  $u^k$  on sets of the form  $B_R \times (0, \tau)$  for  $R, \tau > 0$ .

**Proposition 3.3.** *Let  $u$  be a solution to (3.1) and  $u^k$  its rescaling. Let  $k_n \rightarrow \infty$  and assume  $u^{k_n} \rightarrow U$  as  $n \rightarrow \infty$ . Assume that for every  $R > 0$  there exists  $C_R$  such that*

$$\int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_R \tau$$

and either there exists  $\gamma > 0$  such that

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C_R \tau^\gamma$$

or else, for every  $R, \tau > 0$ ,

$$\lim_{k \rightarrow \infty} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt = 0.$$

Assume further that

$$u_0^k(x) \rightarrow \phi(x) \quad (k \rightarrow \infty) \quad \text{in the sense of distributions.}$$



Then, there holds that  $U$  is the solution to

$$\begin{aligned} U_t - \mathbf{a}\Delta U &= -c_0 U^p \\ U(x, 0) &= \phi(x) \end{aligned}$$

*Proof.* We proceed as in the proof of Proposition 3.2. We drop the subscript  $n$ . Let  $\varphi \in C_0^\infty(\mathbb{R}^N \times [0, \infty))$ ,  $R > 0$  such that  $\varphi(x, t) = 0$  if  $|x| > R$  and let  $\varepsilon > 0$ . Let  $\tau > 0$ ,  $k_0 > 0$  be such that for  $k \geq k_0$ ,

$$\begin{aligned} \int_0^\tau \int_{B_R} u^k(x, t) dx dt + F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &< \varepsilon \\ \int_0^\tau \int_{B_R} U(x, t) dx dt + c_0 \int_0^\tau \int_{B_R} U^p(x, t) dx dt &< \varepsilon \end{aligned}$$

Then,

$$\begin{aligned} &\left| \int_0^\infty \int_{\mathbb{R}^N} U(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt - \int_0^\infty \int_{\mathbb{R}^N} c_0 U^p(x, t)\varphi(x, t) dx dt \right. \\ &+ \int_{\mathbb{R}^N} \phi(x)\varphi(x, 0) dx \left. \leq C \left| \int_0^\tau \int_{B_R} U(x, t) dx dt \right| + C \left| c_0 \int_0^\tau \int_{B_R} U^p(x, t) dx dt \right| \right. \\ &+ C \int_\tau^\infty \int_{B_R} |U - u^k|(x, t) dx dt + \int_\tau^\infty \int_{B_R} |c_0 U^p - F(k)(u^k)^p|(x, t) dx dt \\ &+ \int_\tau^\infty \int_{B_R} u^k(x, t)O(k^{-3}) dx dt + C \left| \int_0^\tau \int_{B_R} u^k(x, t) dx dt \right| + C \left| F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \right| \\ &+ \left| \int_{\mathbb{R}^N} u_0^k(x)\varphi(x, 0) dx - \int_{\mathbb{R}^N} \phi(x)\varphi(x, 0) dx \right| \leq 2C\varepsilon + C \int_\tau^\infty \int_{B_R} |U - u^k|(x, t) dx dt \\ &+ \int_\tau^\infty \int_{B_R} |c_0 U^p - F(k)(u^k)^p|(x, t) dx dt + \int_\tau^\infty \int_{B_R} u^k(x, t)O(k^{-3}) dx dt \\ &+ \left| \int_{\mathbb{R}^N} u_0^k(x)\varphi(x, 0) dx - \int_{\mathbb{R}^N} \phi(x)\varphi(x, 0) dx \right| \end{aligned}$$

Therefore, taking  $\limsup_{k \rightarrow \infty}$  we obtain,

$$\begin{aligned} &\left| \int_0^\infty \int_{\mathbb{R}^N} U(x, t)(\varphi_t + \mathbf{a}\Delta\varphi)(x, t) dx dt - \int_0^\infty \int_{\mathbb{R}^N} c_0 U^p(x, t)\varphi(x, t) dx dt + \right. \\ &\quad \left. + \int_{\mathbb{R}^N} \phi(x)\varphi(x, 0) dx \right| \leq 2C\varepsilon. \end{aligned}$$

As  $\varepsilon$  is arbitrary, the proposition is proved.  $\square$

In order to prove the bounds assumed in the previous proposition, and hence be able to completely determine  $U$ , we need to consider separate cases, according to the behavior of  $u_0$  at infinity. We begin with the case where  $u_0$  behaves as a power  $-\alpha > -N$ .

**Lemma 3.1.** *Let  $u$  be the solution to (3.1). Assume  $(1+|x|)^\alpha u(x, t) \leq B$  and  $(1+t)^{\alpha/2} u(x, t) \leq B$  with  $0 < \alpha < N$ . Then, for every  $R > 0$  there exists  $C_R$  such that, for  $k \geq \tau^{-1/2}$ ,*

$$\begin{aligned}
& \int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_R \tau \\
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt & \leq C_R k^{-\alpha(p-1)+2} \tau \leq C_R \tau && \text{if } N - \alpha p > 0 \\
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt & \leq C_R k^{-\alpha(p-1)+2} \tau |\log \tau| \leq C_R \tau |\log \tau| && \text{if } N - \alpha p = 0 \\
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt & \leq C k^{-\alpha(p-1)+2} \tau^{\frac{N-\alpha p+2}{2}} \leq C \tau^{\frac{N-\alpha p+2}{2}} && \text{if } 0 > N - \alpha p > -2 \\
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt & \leq C k^{\alpha-N} \leq C_R \tau^{\frac{N-\alpha}{2}} && \text{if } N - \alpha p < -2 \\
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt & \leq C k^{\alpha-N} \log(1 + k^2 \tau) \leq C \tau^{(N-\alpha)/2} && \text{if } N - \alpha p = -2.
\end{aligned}$$

*Proof.* We begin with the estimate of the integral of  $u^k$ .

$$\begin{aligned}
(3.5) \quad \int_0^\tau \int_{B_R} u^k(x, t) dx dt &= f(k) k^{-N-2} \int_0^{k^2 \tau} \int_{B_{Rk}} u(x, t) dx dt \\
&\leq C k^{\alpha-N-2} \int_0^{k^2 \tau} \int_{B_{Rk}} \frac{1}{(1+|x|)^\alpha} dx dt \\
&\leq C k^{\alpha-N-2} k^2 \tau (Rk)^{N-\alpha} \leq C_R \tau.
\end{aligned}$$

Let us now estimate the integral of  $(u^k)^p$ . There holds,

$$\begin{aligned}
(3.6) \quad F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &= F(k) f(k)^p k^{-N-2} \int_0^{k^2 \tau} \int_{B_{Rk}} u^p(x, t) dx dt \\
&= f(k) k^{-N} \int_0^{k^2 \tau} \int_{B_{Rk}} u^p(x, t) dx dt.
\end{aligned}$$

We consider several cases.

**Case 1:**  $N - \alpha p > 0$

We have,

$$\begin{aligned}
F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C F(k) k^{\alpha p - N - 2} \int_0^{k^2 \tau} \int_{B_{Rk}} \frac{1}{(1+|x|)^{\alpha p}} dx dt \\
&\leq C F(k) k^{\alpha p - N - 2} k^2 \tau (Rk)^{N - \alpha p} \\
&= C_R k^{-\alpha(p-1)+2} \tau \leq C_R \tau.
\end{aligned}$$

**Case 2:**  $N - \alpha p < 0$

First,

$$(3.7) \quad \int_{B_{Rk}} u^p(x, t) dx \leq C \int_{|x| \leq \sqrt{t}} (1+t)^{-\alpha p/2} dx + C \int_{\sqrt{t} < |x| < Rk} \frac{1}{(1+|x|)^{\alpha p}} dx \\ \leq C(1+t)^{\frac{N-\alpha p}{2}}.$$

Assume  $k \geq \tau^{-1/2}$ . We consider 3 subcases.

(i)  $-2 < N - \alpha p < 0$ .

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq Ck^{\alpha-N} \int_0^{k^2\tau} (1+t)^{\frac{N-\alpha p}{2}} dt \\ \leq Ck^{\alpha-N} (k^2\tau)^{\frac{N-\alpha p+2}{2}} \\ = Ck^{-\alpha(p-1)+2} \tau^{\frac{N-\alpha p+2}{2}} \\ \leq C\tau^{\frac{N-\alpha p+2}{2}}.$$

(ii)  $N - \alpha p = -2$ .

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq Ck^{\alpha-N} \int_0^{k^2\tau} (1+t)^{-1} dt \\ = Ck^{\alpha-N} \log(1+k^2\tau) \\ = C(k^2\tau)^{(\alpha-N)/2} \log(1+k^2\tau) \tau^{(N-\alpha)/2} \\ \leq C\tau^{(N-\alpha)/2}.$$

(iii)  $N - \alpha p < -2$ .

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq Ck^{\alpha-N} \int_0^{k^2\tau} (1+t)^{(N-\alpha p)/2} dt \\ \leq Ck^{\alpha-N} \leq C\tau^{(N-\alpha)/2}.$$

**Case 3:**  $N - \alpha p = 0$

Instead of (3.7) we have,

$$\int_{B_{Rk}} u^p(x, t) dx \leq C \left(1 + \log \frac{Rk}{\sqrt{t}}\right).$$

Thus,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq Ck^{\alpha-N} \int_0^{k^2\tau} \left(1 + \log \frac{Rk}{\sqrt{t}}\right) dt \\ = Ck^{\alpha-N+2} \int_0^\tau \left(1 + \log \frac{R}{\sqrt{t}}\right) dt \\ \leq C_R k^{-\alpha(p-1)+2} \tau |\log \tau| \leq C_R \tau |\log \tau|.$$

□

**Corollary 3.1.** *Assume  $|x|^\alpha u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Let  $u$  be the solution to (3.1). Then,  $u$  satisfies the conclusions of Lemma 3.1*

*Proof.* By Remark 2.2, the estimates of Proposition 2.1 hold for  $u$ . In this case, since  $f(k) \sim k^\alpha$  we take it to be equal and hence,

$$(1 + |x|)^\alpha u(x, t) \leq C \quad \text{and} \quad (1 + t)^{\alpha/2} u(x, t) \leq C.$$

Thus,  $u$  satisfies the assumptions of Lemma 3.1.  $\square$

Using Lemma 3.1, we are able to prove with almost no computations, the desired estimates in the other examples considered in the introduction. We use the ideas of Kamin-Ughi in [16].

Let us introduce some notation. For  $\mu > 0$ , let

$$(3.8) \quad u^{k,\mu}(x, t) = k^\mu u(kx, k^2 t).$$

Then,

$$(3.9) \quad F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt = f(k) k^{-\mu} k^{-\mu(p-1)+2} \int_0^\tau \int_{B_R} (u^{k,\mu})^p(x, t) dx dt.$$

We begin with the case where  $u_0$  behaves as  $|x|^{-N}$  at infinity.

**Lemma 3.2.** *Assume  $|x|^N u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Let  $p > 1 + 2/N$ . Then, for every  $R > 0$  there exists  $C_R$  such that,*

$$(3.10) \quad \int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_R \tau$$

$$(3.11) \quad F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq \frac{C}{\log k}.$$

*Proof.* Recall that in this case  $f(k) \sim \frac{k^N}{\log k}$  for  $k \geq 2$ . Let  $0 < \mu < N$  to be chosen later. Then, for  $|x| \geq 1, t \geq 1$ ,

$$|x|^\mu u(x, t) \leq C \frac{|x|^N}{\log(1 + |x|)} u(x, t) \leq B, \quad t^{\mu/2} u(x, t) \leq C \frac{t^{N/2}}{\log(1 + t)} u(x, t) \leq B.$$

So that, we can apply the results of Lemma 3.1 to  $u^{k,\mu}$ . Let us choose  $\mu$  so close to  $N$  so that  $\mu p > N + 2 (> \mu + 2)$ . By (3.9) and Lemma 3.1, Case 2, (iii), there holds,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C f(k) k^{-\mu} k^{\mu-N} = \frac{C}{\log k}.$$

On the other hand,

$$\int_0^\tau \int_{B_R} u^k(x, t) dx dt = \frac{k^{-2}}{\log k} \int_0^{k^2 \tau} \int_{B_{Rk}} u(x, t) dx dt.$$

Now,

$$\begin{aligned} \int_{B_{Rk}} u(x, t) dx &\leq e^{-t} \int_{B_{Rk}} u_0(x) dx + \int \int_{B_{Rk}} W(y, t) u_0(x - y) dx dy \\ &= e^{-t} \int_{B_{Rk}} u_0(x) dx + \int_{|y| < 2Rk} \int_{B_{Rk}} W(y, t) u_0(x - y) dx dy \\ &\quad + \int_{|y| > 2Rk} \int_{B_{Rk}} W(y, t) u_0(x - y) dx dy \leq C e^{-t} \log(Rk) + I + II. \end{aligned}$$

There holds,

$$I \leq \int_{|y| < 2Rk} W(y, t) \int_{|x| < 3Rk} u_0(x) dx dy \leq C \log(3Rk).$$

On the other hand,

$$\begin{aligned} II &\leq \int_{|y| > 2Rk} W(y, t) \int_{|x| < Rk} \frac{C}{|y|^N} dx dy \\ &\leq C(Rk)^N \int_{|y| > 2Rk} \frac{t}{|y|^{2N+2}} dy \leq C_K k^{-2} t. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^\tau \int_{B_R} u^k(x, t) dx dt &\leq C \frac{\log(Rk)}{\log k} k^{-2} \int_0^{k^2\tau} e^{-t} dt \\ &\quad + C \frac{\log(3Rk)}{\log k} k^{-2} k^2 \tau + C_K k^{-4} \int_0^{k^2\tau} t dt \\ &\leq C_{R\tau} \quad \text{if } \tau < 1. \end{aligned}$$

□

With similar computations we can prove

**Lemma 3.3.** *Assume  $|x|^N (\log |x|) u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Let  $p > 1 + \frac{2}{N}$ . Then, for every  $R > 0$  there exists  $C_R$  such that*

$$(3.12) \quad \int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_K \tau$$

$$(3.13) \quad F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq \frac{C}{\log \log k}.$$

Also,

**Lemma 3.4.** *Assume  $\frac{|x|^N}{\log |x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Let  $p > 1 + \frac{2}{N}$ . Then, for every  $R > 0$  there exists  $C_R$  such that*

$$(3.14) \quad \int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_K \tau$$

$$(3.15) \quad F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq \frac{C}{\log^2 k}.$$

In a similar way we obtain,

**Lemma 3.5.** *Assume  $\frac{|x|^\alpha}{\log|x|}u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Let  $p > 1 + \frac{2}{\alpha}$ . Then, for every  $R > 0$  there exists  $C_R$  such that, for  $k \geq \tau^{-1/2}$ ,*

$$\begin{aligned} \int_0^\tau \int_{B_R} u^k(x, t) dx dt &\leq C_R \tau \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C_R \tau \quad \text{if } N - \alpha p \geq 0 \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C \tau^{\frac{N - \mu p + 2}{2}} \quad \text{for a certain } \mu \in (0, \alpha) \text{ if } 0 > N - \alpha p \geq -2 \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C_R \tau^{\frac{N - \alpha}{2}} \quad \text{if } N - \alpha p < -2. \end{aligned}$$

*Proof.* In this case  $f(k) = \frac{k^\alpha}{\log(1+k)}$ . We will chose  $\mu \in (0, \alpha)$  according to the relative sizes of  $\alpha$  and  $p$ . For any such  $\mu$  there holds that  $|x|^\mu u(x, t) \leq C \frac{|x|^\alpha}{\log(1+|x|)} u(x, t) \leq B$  if  $|x| \geq 1$ ,  $t^{\mu/2} u(x, t) \leq C \frac{t^{\alpha/2}}{\log(1+t)} u(x, t)$  if  $t \geq 1$ . So that, we can apply the results of Lemma 3.1 to  $u^{k, \mu}$ .

Recall that  $\alpha p > \alpha + 2$ . We consider several cases.

**Case 1:**  $N - \alpha p \geq 0$

Chose  $\mu \in (0, \alpha)$  so that  $\alpha + 2 < \mu p < N$ . Then, by (3.9) and Lemma 3.1, Case (i),

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C_R f(k) k^{-\mu} k^{-\mu(p-1)+2} \tau = C_R \frac{k^{\alpha+2-\mu p}}{\log k} \tau \leq C_R \tau.$$

**Case 2:**  $N - \alpha p < 0$

(i)  $-2 \leq N - \alpha p < 0$ . Chose  $\mu \in (0, \alpha)$  so that  $-2 < N - \mu p < 0$  and  $\mu p > \alpha + 2$ . Then, by (3.9) and Lemma 3.1, Case 2, (ii),

$$\begin{aligned} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C_R f(k) k^{-\mu} k^{-\mu(p-1)+2} \tau^{\frac{N - \mu p + 2}{2}} \\ &= C_R \frac{k^{\alpha+2-\mu p}}{\log k} \tau^{\frac{N - \mu p + 2}{2}} \leq C_R \tau^{\frac{N - \mu p + 2}{2}}. \end{aligned}$$

(ii)  $N - \alpha p < -2$ . Choose  $\mu \in (0, \alpha)$  so that  $N - \mu p < -2$ . Then, by (3.9) and Lemma 3.1, Case 2, (iii),

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C f(k) k^{-\mu} k^{\mu-N} = C \frac{k^{\alpha-N}}{\log k} \leq C \tau^{\frac{N-\alpha}{2}}.$$

On the other hand, computations similar to those in (3.5) give

$$\int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_R \tau.$$

This time we use that

$$\int_{B_{Rk}} \frac{dx}{(2+|x|)^\alpha \log(2+|x|)} \sim (Rk)^N (f(Rk))^{-1} \sim (Rk)^{N-\alpha} \log(Rk) \quad \text{as } k \rightarrow \infty.$$

□

With similar computations we can prove,

**Lemma 3.6.** *Assume  $|x|^\alpha(\log|x|)u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Let  $p > 1 + \frac{2}{\alpha}$ . Then, for every  $R > 0$  there exists  $C_R$  such that, for  $k \geq \tau^{-1/2}$ ,*

$$\begin{aligned} \int_0^\tau \int_{B_R} u^k(x, t) dx dt &\leq C_R \tau \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C_R \tau \quad \text{if } N - \alpha p \geq 0 \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C \tau^{\frac{N - \mu p + 2}{2}} \quad \text{for a certain } \mu \in (0, \alpha) \text{ if } 0 > N - \alpha p \geq -2 \\ F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C_R k^{\alpha - N} \log k \leq C_R \tau^{\frac{N - \alpha}{4}} \quad \text{if } N - \alpha p < -2. \end{aligned}$$

In order to consider the critical case  $p = 1 + 2/\alpha$  we need to perform different computations. There holds,

**Lemma 3.7.** *Assume  $|x|^\alpha(\log|x|)u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Let  $p = 1 + \frac{2}{\alpha}$ . Then, for every  $\tau, R > 0$  there exists  $C_R$  such that,*

$$\int_0^\tau \int_{B_R} u^k(x, t) dx dt \leq C_R \tau$$

and

$$\lim_{k \rightarrow \infty} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt = 0.$$

*Proof.* The  $L^1$  estimate follows as in the previous lemmas. For the  $L^p$  estimate we consider several cases.

**Case 1:**  $N - \alpha p > 0$ . There holds,

$$\int_{B_{Rk}} u^p(x, t) dx \leq C \int_{B_{Rk}} \frac{dx}{(2 + |x|)^{\alpha p} \log^p(2 + |x|)} \leq C \int_0^{Rk} \frac{(2 + r)^{N - \alpha p - 1}}{\log^p(2 + r)} dr$$

where the last integral goes to infinity as  $k$  goes to infinity. Moreover,

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\int_0^{Rk} \frac{(2+r)^{N-\alpha p-1}}{\log^p(2+r)} dr}{\frac{k^{N-\alpha p}}{\log^p k}} &= \lim_{k \rightarrow \infty} \frac{\frac{R(2+Rk)^{N-\alpha p-1}}{\log^p(2+Rk)}}{\frac{(N-\alpha p)k^{N-\alpha p-1}}{\log^p k} - \frac{pk^{N-\alpha p-1}}{\log^{p+1} k}} \\ &= \lim_{k \rightarrow \infty} \frac{(2+Rk)^{N-\alpha p-1}}{k^{N-\alpha p-1}} \frac{\log^p k}{\log^p(2+Rk)} \frac{R}{N - \alpha p - \frac{p}{\log k}} \\ &= C_{N,R,\alpha,p} < \infty. \end{aligned}$$

Therefore, for  $k$  large,

$$\begin{aligned} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq C \tau k^{\alpha+2-N} \log k \int_0^{Rk} \frac{(2+r)^{N-\alpha p-1}}{\log^p(2+r)} dr \\ &\leq C \tau k^{\alpha+2-N} \log k \frac{k^{N-\alpha p}}{\log^p k} \\ &= C \tau \frac{k^{-\alpha(p-1)+2}}{\log^{p-1} k} = C \tau \frac{1}{\log^{p-1} k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

**Case 2:**  $N - \alpha p < 0$ . There holds,

$$\begin{aligned}
\int_{B_{Rk}} u^p(x, t) dx &\leq C \int_{|x| \leq \sqrt{t}} \frac{dx}{(2 + t^{1/2})^{\alpha p} \log^p(2 + t^{1/2})} + \\
&\quad + C \int_{\sqrt{t} \leq |x| \leq Rk} \frac{dx}{(2 + |x|)^{\alpha p} \log^p(2 + |x|)} \\
&\leq C \frac{(2 + t^{1/2})^{N - \alpha p}}{\log^p(2 + t^{1/2})} + C \int_{\sqrt{t}}^{Rk} \frac{(2 + r)^{N - \alpha p - 1}}{\log^p(2 + r)} dr \\
&\leq C \frac{(2 + t^{1/2})^{N - \alpha p}}{\log^p(2 + t^{1/2})} + C \frac{1}{\log^p(2 + t^{1/2})} \int_{\sqrt{t}}^{Rk} (2 + r)^{N - \alpha p - 1} dr \\
&\leq C \frac{(2 + t^{1/2})^{N - \alpha p}}{\log^p(2 + t^{1/2})}
\end{aligned}$$

Since  $\alpha p = \alpha + 2$ , there holds that  $N - \alpha p = N - \alpha - 2 > -2$ . We have,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C k^{\alpha - N} \log k \int_0^{k^2 \tau} \frac{(2 + t^{1/2})^{N - \alpha p}}{\log^p(2 + t^{1/2})} dt,$$

and this last integral goes to infinity as  $k$  goes to infinity. Thus,

$$\begin{aligned}
\lim_{k \rightarrow \infty} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq \lim_{k \rightarrow \infty} \frac{C \int_0^{k^2 \tau} \frac{(2 + t^{1/2})^{N - \alpha p}}{\log^p(2 + t^{1/2})} dt}{\frac{k^{N - \alpha}}{\log k}} \\
&= \lim_{k \rightarrow \infty} \frac{\frac{C 2k\tau (2 + k\tau^{1/2})^{N - \alpha p}}{\log^p(2 + k\tau^{1/2})}}{\frac{(N - \alpha)k^{N - \alpha - 1}}{\log k} - \frac{k^{N - \alpha - 1}}{\log^2 k}} \\
&= \lim_{k \rightarrow \infty} \frac{C 2k\tau (2 + k\tau^{1/2})^{N - \alpha - 2}}{k^{N - \alpha - 1}} \frac{\log k}{\log^p(2 + k\tau^{1/2})} \frac{1}{N - \alpha - \frac{1}{\log k}} = 0
\end{aligned}$$

**Case 3:**  $N - \alpha p = 0$ . We begin as in Case 2.

$$\begin{aligned}
\int_{B_{Rk}} u^p(x, t) dx &\leq C \int_{|x| \leq \sqrt{t}} \frac{dx}{(2 + t^{1/2})^{\alpha p} \log^p(2 + t^{1/2})} + \\
&\quad + C \int_{\sqrt{t} \leq |x| \leq Rk} \frac{dx}{(2 + |x|)^{\alpha p} \log^p(2 + |x|)} \\
&\leq C \frac{1}{\log^p(2 + t^{1/2})} + C \int_{\sqrt{t}}^{Rk} \frac{(2 + r)^{-1}}{\log^p(2 + r)} dr \\
&\leq C \frac{1}{\log^p(2 + t^{1/2})} \left( 1 + \log \frac{2 + Rk}{2 + t^{1/2}} \right)
\end{aligned}$$

Thus,

$$F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt \leq C k^{\alpha - N} \log k \int_0^{k^2 \tau} \frac{1}{\log^p(2 + t^{1/2})} \left( 1 + \log \frac{2 + Rk}{2 + t^{1/2}} \right) dt,$$



and this last integral goes to infinity as  $k$  goes to infinity. Therefore, since  $N - \alpha - 1 = 1$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} F(k) \int_0^\tau \int_{B_R} (u^k)^p(x, t) dx dt &\leq \lim_{k \rightarrow \infty} \frac{C \int_0^{k^2\tau} \frac{1}{\log^p(2+t^{1/2})} \left(1 + \log \frac{2+Rk}{2+t^{1/2}}\right) dt}{\frac{k^{N-\alpha}}{\log k}} \\ &= \lim_{k \rightarrow \infty} \frac{\frac{C2k\tau}{\log^p(2+k\tau^{1/2})} \left(1 + \log \frac{2+Rk}{2+k\tau^{1/2}}\right)}{\frac{(N-\alpha)k^{N-\alpha-1}}{\log k} - \frac{k^{N-\alpha-1}}{\log^2 k}} \\ &= \lim_{k \rightarrow \infty} 2C\tau \frac{\log k}{\log^p(2+k\tau^{1/2})} \left(1 + \log \frac{2+Rk}{2+k\tau^{1/2}}\right) \frac{1}{N-\alpha-\frac{1}{\log k}} = 0 \end{aligned}$$

□

If  $u_0 \in L^1(\mathbb{R}^N)$ , the limit  $U$  of  $u^k$  does not satisfy that  $U(x, 0) = \lim u_0^k = M_0\delta$  with  $M_0 = \int u_0$ . Therefore, estimates as the ones we have just stated do not hold. In order to establish our result we need the following lemma.

**Lemma 3.8.** *Let  $u_0 \in L^\infty(\mathbb{R}^N)$ . Assume further that  $|x|^{N+2}u_0(x) \leq B$ . Let  $u$  be the solution to (3.1) and  $u^k$  its rescaling. Assume  $u^{k_n} \rightarrow U$  with  $k_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Then, there exists a constant  $C > 0$  such that*

$$U(x, t) \leq C \frac{t}{|x|^{N+2}}.$$

In particular, for every  $\mu > 0$ ,

$$\lim_{t \rightarrow 0} U(x, t) = 0 \quad \text{uniformly in } |x| \geq \mu.$$

*Proof.* We proceed as in the proof of Theorem 2.1. There holds that  $u(x, t) \leq e^{-t}u_0(x) + z(x, t)$  with  $z$  the solution to

$$(3.16) \quad \begin{cases} z_t - Lz = e^{-t}(J * u_0)(x) \\ z(x, 0) = 0 \end{cases}$$

By the assumption on the growth of  $u_0$  at infinity there holds that  $v(x, t) = C \frac{t}{|x|^{N+2}}$  is a supersolution to (3.16) in  $\frac{t}{|x|^2} \leq K$ ,  $|x| \geq 2$ . On the other hand,  $z(x, t) \leq v(x, t)$  in the complement of this set since  $z(x, t) \leq Ct$  and  $t^{N/2}z(x, t) \leq C$  in this case ( $u_0 \in L^1(\mathbb{R}^N)$ ).

Therefore,  $u(x, t) \leq e^{-t}u_0(x) + C \frac{t}{|x|^{N+2}}$ . Rescaling, and recalling that in the present case  $f(k) = k^N$  we have that  $u^k(x, t) \leq e^{-k^2t}u_0^k(x) + C \frac{t}{|x|^{N+2}}$ . Passing to the limit as  $k \rightarrow \infty$  with  $x \neq 0$  we get  $U(x, t) \leq C \frac{t}{|x|^{N+2}}$ . And the result follows. □

**Proposition 3.4.** *Let  $u_0 \in L^1 \cap L^\infty$ . Let  $u$  be the solution to (3.1) and  $u^k$  its rescaling. Assume for some sequence  $k_n \rightarrow \infty$  there holds that  $u^{k_n} \rightarrow U$  uniformly on compact sets of  $\mathbb{R}^N \times (0, \infty)$ .*

*Assume  $p > 1 + 2/N$ . Then,  $U(x, t) = MU_{\mathbf{a}}(x, t)$  where  $U_{\mathbf{a}}$  is the fundamental solution of the heat equation with diffusivity  $\mathbf{a}$  and  $M = \int u_0(x) dx - \int_0^\infty \int u^p(x, t) dx dt$ .*

*When  $p = 1 + 2/N$  there holds that  $U \equiv 0$ .*

*Proof.* As before we drop the subscript  $n$ . We already know that  $U$  is a solution to

$$U_t - \mathbf{a}\Delta U = -c_0 U^p.$$

Assume for simplicity that  $u_0$  has compact support. Then, by Lemma 3.8 there exists  $C > 0$  such that  $U(x, t) \leq C \frac{t}{|x|^{N+2}}$ . Let  $M(t) = \int U(x, t) dx$ . We will see that  $M(t) \equiv M$  is constant, and  $U(x, t) \rightarrow M\delta$  as  $t \rightarrow 0$  in the sense of distributions where  $\delta$  is the Dirac delta. In fact, assume we already proved that  $M(t)$  is constant  $M$ . Let  $\varphi \in C_0^\infty(\mathbb{R}^N)$ . Then, given  $\varepsilon > 0$ , if  $\mu$  is small we get,

$$\begin{aligned} \left| \int U(x, t)\varphi(x) dx - M\varphi(0) \right| &= \left| \int U(x, t)[\varphi(x) - \varphi(0)] dx \right| \\ &\leq \int_{|x| < \mu} U(x, t)|\varphi(x) - \varphi(0)| dx + C \int_{|x| > \mu} \frac{t}{|x|^{N+2}} |\varphi(x) - \varphi(0)| dx \\ &\leq \varepsilon \int U(x, t) dx + 2C\|\varphi\|_\infty \int_{|x| > \mu} \frac{t}{|x|^{N+2}} dx \\ &\leq M\varepsilon + \bar{C}_\mu t < (M + 1)\varepsilon \quad \text{if } t \text{ is small.} \end{aligned}$$

Now, integrating the equation satisfied by  $u^k$  and using the symmetry of  $J$  we obtain

$$\begin{aligned} (3.17) \quad \int u^k(x, t) dx &= \int u_0^k(x) dx - F(k) \int_0^t \int (u^k)^p(x, s) dx ds \\ &= \int u_0(x) - \int_0^{k^2 t} \int u^p(x, s) dx ds. \end{aligned}$$

Moreover

$$(3.18) \quad \int_{B_K} u^k(x, t) dx \rightarrow \int_{B_K} U(x, t) dx \quad \text{for every } K > 0$$

and

$$(3.19) \quad \begin{aligned} \int_{|x| > K} u^k(x, t) dx &\leq e^{-k^2 t} \int_{|x| > Kk} u_0(x) dx + Ct \int_{|x| > K} \frac{dx}{|x|^{N+2}} < \varepsilon \\ \int_{|x| > K} U(x, t) dx &\leq Ct \int_{|x| > K} \frac{dx}{|x|^{N+2}} < \varepsilon \end{aligned}$$

if  $K$  is large.

Hence, we have that

$$M(t) = \int U(x, t) dx = \int u_0(x) dx - \int_0^\infty \int u^p(x, t) dx dt = M.$$

When  $p > 1 + 2/N$  there holds that  $c_0 = 0$ , so that  $U$  is  $M$  times the fundamental solution of the heat equation with diffusivity  $\mathbf{a}$ . On the other hand, if  $p = 1 + 2/N$  there holds that  $c_0 = 1$ , so that

$$\int U(x, t) dx = \int U(x, \tau) dx - \int_\tau^t \int U^p(x, s) dx ds.$$

Therefore,  $M(t)$  cannot be constant unless  $U \equiv 0$ . And the proposition is proved when  $u_0$  has compact support.

Let now  $u_0 \in L^\infty \cap L^1$  be arbitrary. We have to prove that  $u^k$  converges to  $U = MU_{\mathbf{a}}$  uniformly on compact subsets of  $\mathbb{R}^N \times (0, \infty)$  with  $M = \int u_0(y) dy - \int_0^\infty \int u^p(y, t) dy dt$ .

Let  $u_0^n = u_0 \chi_{|x| < n}$  and  $u_n$  the solution to (3.1) with initial datum  $u_0^n$ . Then,  $u_n \leq u$ . Let  $v_n = u - u_n$ . There holds that  $v_n$  is a nonnegative solution to

$$v_{nt} - Lv_n = -u^p + u_n^p \leq 0.$$

Therefore,

$$0 \leq v_n(x, t) \leq e^{-t} \{u_0(x) - u_0^n(x)\} + \int W(x - y, t) \{u_0(y) - u_0^n(y)\} dy.$$

Thus, if  $t \geq \tau > 0$ ,

$$\begin{aligned} 0 \leq v_n^k(x, t) &\leq k^N e^{-k^2 t} \{u_0(kx) - u_0^n(kx)\} + \int W_k(x - y, t) \{u_0^k(y) - (u_0^n)^k(y)\} dy \\ &\leq C k^N e^{-k^2 t} + C_\tau \int \{u_0(y) - u_0^n(y)\} dy \end{aligned}$$

Then, if  $t \geq \tau > 0$  we find,

$$\begin{aligned} U_n(x, t) &\leq \liminf_{k \rightarrow \infty} u^k(x, t) \leq \limsup_{k \rightarrow \infty} u^k(x, t) \\ &\leq U_n(x, t) + C_\tau \int \{u_0(y) - u_0^n(y)\} dy \\ &\leq U_n(x, t) + \varepsilon \quad \text{if } n \geq n_0(\tau, \varepsilon). \end{aligned}$$

Therefore, if there exists  $\lim_{n \rightarrow \infty} U_n$  there holds that there exists  $U = \lim_{k \rightarrow \infty} u^k = \lim_{n \rightarrow \infty} U_n$ . Moreover, it is easy to prove from the estimates above that this limit is uniform on compact sets of  $\mathbb{R}^N \times (0, \infty)$ .

On the other hand, by similar arguments we find that for  $m > n$ ,

$$\begin{aligned} 0 &\leq \int \{U_m(x, t) - U_n(x, t)\} dx \\ &\leq \limsup_{k \rightarrow \infty} e^{-k^2 t} \int \{u_0^m(x) - u_0^n(x)\} dx + \int \{u_0^m(y) - u_0^n(y)\} dy \rightarrow 0 \quad \text{as } m, n \rightarrow \infty. \end{aligned}$$

Therefore,  $M_n = \int U_n(x, t) dx \rightarrow M$  and, since  $U_n = M_n U_{\mathbf{a}}$ , we deduce that  $U = M U_{\mathbf{a}}$ .

So, it only remains to prove that  $M = \int u_0(y) dy - \int_0^\infty \int u^p(y, t) dy dt$ .

The estimates (3.17), (3.18) hold in the present situation and, instead of the estimate (3.19) on the mass of  $u^k$  and  $U$  outside a large ball, we get

$$\begin{aligned}
\int_{|x|>K} u^k(x, t) dx &\leq \int_{|x|>Kk} u_0(x) dx + \int_{|x|>K} \int_{|y|<K/2} W_k(x-y, t) u_0^k(y) dy dx \\
&\quad + \int_{|x|>K} \int_{|y|>K/2} W_k(x-y, t) u_0^k(y) dy dx \\
(3.20) \quad &\leq \int_{|x|>Kk} u_0(x) dx + \int_{|x|>K} \int_{|y|<K/2} C \frac{t}{|x-y|^{N+2}} u_0^k(y) dy dx \\
&\quad + \int_{|y|>K/2} u_0^k(y) \int_{|x|>K} W_k(x-y, t) dx dy \\
&\leq \int_{|x|>Kk} u_0(x) dx + Ct \int_{|x|>K} \frac{1}{|x|^{N+2}} \int_{|y|<Kk/2} u_0(y) dy dx \\
&\quad + \int_{|y|>Kk/2} u_0(y) dy < \varepsilon \quad \text{if } K \text{ is large independently of } k \geq 1
\end{aligned}$$

So, that we also have

$$\int_{|x|>K} U(x, t) dx \leq \liminf_{k \rightarrow \infty} \int_{|x|>K} u^k(x, t) dx \leq \varepsilon$$

if  $K$  is large. So, we get again that

$$M = \int U(x, t) dx = \int u_0(x) dx - \int_0^\infty \int u^p(x, t) dx dt$$

and the proposition is proved by using again that, in the critical case,  $\int U(x, t) dx$  cannot be constant unless  $U$  is identically zero.  $\square$

#### 4. MAIN RESULTS

In this section we prove our main results. Namely, we establish the asymptotic behavior of solutions to

$$(4.1) \quad \begin{cases} u_t(x, t) = \int J(x-y)(u(y, t) - u(x, t)) dy - u^p(x, t) & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

depending on the values of  $p$ ,  $N$  and the coefficient  $\alpha$  of non-integrability of the initial data  $u_0$ . At the end of this section we also address the case where  $u_0$  is integrable and bounded. As mentioned before, we have to distinguish between several cases.

We begin with the case  $0 < \alpha < N$ . We further divide the analysis between the supercritical case  $p > 1 + 2/\alpha$  and the critical case  $p = 1 + 2/\alpha$ . In the previous paper [19], we were able to establish the asymptotic behavior only for the supercritical case. Now we present a different proof that actually allows us to obtain the result both in the critical and supercritical cases. Moreover, in the supercritical case, we prove the same type of result for more general non-integrable initial data.

**Theorem 4.1.** *Let  $u_0 \in L^\infty$  be such that  $|x|^\alpha u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ . Let  $p \geq 1 + 2/\alpha$  and  $u$  the solution to (4.1). Then, for every  $R > 0$ ,*

$$t^{\alpha/2} |u(x, t) - U(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

where  $U$  is the solution to

$$(4.2) \quad \begin{cases} U_t - \mathbf{a}\Delta U = -c_0 U^p \\ U(x, 0) = \frac{C_{A,N}}{|x|^\alpha} \end{cases}$$

with  $c_0 = 0$  if  $p > 1 + 2/\alpha$  and  $c_0 = 1$  if  $p = 1 + 2/\alpha$ .

In a similar way we get the following results: When  $\frac{|x|^\alpha}{\log|x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$  and  $p > 1 + 2/\alpha$ , for every  $R > 0$  there holds that

$$t^{\alpha/2} \left| \frac{u(x, t)}{\log t^{1/2}} - U(x, t) \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

with  $U$  as above.

When  $|x|^\alpha (\log|x|) u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$  and  $p \geq 1 + 2/\alpha$ , for every  $R > 0$  there holds that

$$t^{\alpha/2} \left| u(x, t) \log t^{1/2} - U(x, t) \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

with  $U$  the solution to (4.2) with  $c_0 = 0$  even in the critical case  $p = 1 + 2/\alpha$ .

*Proof.* We know that the family  $u^k$  is precompact in  $C(\mathcal{K})$  for every compact set  $\mathcal{K} \subset \mathbb{R}^N \times (0, \infty)$ . Therefore, for every sequence  $k_n \rightarrow \infty$  there exists a subsequence that we still call  $k_n$  such that  $u^{k_n}$  converges uniformly on every compact subset of  $\mathbb{R}^N \times (0, \infty)$  to a function  $U$ .

On the other hand, it is easy to see that  $u_0^k \rightarrow \frac{C_{A,N}}{|x|^\alpha}$  in the sense of distributions. Moreover, we are in the situation of Proposition 3.3. So that,  $U$  is the solution to (4.2). Thus, the whole family  $u^k$  converges to  $U$  as  $k \rightarrow \infty$ . In particular,  $u^k(x, 1) \rightarrow U(x, 1)$  uniformly on compact sets of  $\mathbb{R}^N$ .

As the solution of (4.2) is invariant under the present rescaling. There holds that  $U(y, 1) = U^k(y, 1)$ . Thus, for every  $R > 0$ ,

$$k^\alpha |u(ky, k^2) - U(ky, k^2)| \rightarrow 0 \quad \text{uniformly for } |y| \leq R.$$

By calling  $x = y\sqrt{t}$ ,  $t = k^2$  we get the result.

When  $\frac{|x|^\alpha}{\log|x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ , it is easy to see that we still have that  $u_0^k \rightarrow \frac{C_{A,N}}{|x|^\alpha}$  in the sense of distributions. So the result follows also in this case.

Analogously, when  $|x|^\alpha (\log|x|) u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  with  $0 < \alpha < N$ , we also have that  $u_0^k \rightarrow \frac{C_{A,N}}{|x|^\alpha}$  in the sense of distributions, and the result follows.  $\square$

We now analyze the case  $\alpha = N$ . Once again, we prove the result for more general non-integrable initial data than the one considered in [19].

**Theorem 4.2.** *Let  $u_0 \in L^\infty$  be such that  $|x|^N u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$ . Let  $p > 1 + 2/N$ . Then, for every  $R > 0$ ,*

$$t^{N/2} \left| \frac{u(x,t)}{\log t^{1/2}} - U(x,t) \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

where  $u$  is the solution to (4.1) and  $U$  is the solution to

$$\begin{cases} U_t - \mathbf{a}\Delta U = 0 \\ U(x,0) = C_{A,N}\delta \end{cases}$$

with  $\delta$  Dirac's delta.

In a similar way we get the following result when  $\frac{|x|^N}{\log|x|} u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  and  $p > 1 + 2/N$ : for every  $R > 0$ ,

$$t^{N/2} \left| \frac{u(x,t)}{\log^2 t^{1/2}} - U(x,t) \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

with  $U$  as above.

When  $|x|^N (\log|x|) u_0(x) \rightarrow A > 0$  as  $|x| \rightarrow \infty$  and  $p > 1 + 2/N$  we get: for every  $R > 0$ ,

$$t^{N/2} \left| \frac{u(x,t)}{\log \log t^{1/2}} - U(x,t) \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

with  $U$  as above.

*Proof.* We proceed as in the previous theorem. This time,  $u_0^k \rightarrow C_{A,N}\delta$  in the sense of distributions. In fact,

$$\int u_0^k(x) \varphi(x) dx = \frac{1}{\log k} \int_{|x| < Kk} u_0(x) \varphi(x/k) dx$$

where  $K$  is such that  $\varphi(x) = 0$  if  $|x| > K$ . Since  $\varphi(x/k) \rightarrow \varphi(0)$  uniformly in  $\mathbb{R}^N$  the result easily follows from this formula and the fact that

$$\frac{1}{\log Kk} \int_{|x| < Kk} u_0(x) dx \rightarrow C_{A,N}, \quad \frac{\log Kk}{\log k} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

The other cases follow similarly. □

Finally we consider the case where  $u_0$  is integrable and bounded. Such case has been studied by Pazoto and Rossi in [18] for non-critical values of  $p$ . Here we present a new proof the includes the critical case and therefore settles the question as far as integrable data is concerned.

**Theorem 4.3.** *Let  $u_0 \in L^1 \cap L^\infty$  and  $p \geq 1 + 2/N$ . Let  $u$  be the solution to (4.1).*

*First, assume  $p > 1 + 2/N$ . Then, for every  $R > 0$ ,*

$$t^{N/2} |u(x,t) - U(x,t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

where  $U$  is the solution to

$$(4.3) \quad \begin{cases} U_t - \mathbf{a}\Delta U = 0 \\ U(x,0) = M\delta \end{cases}$$

with  $\delta$  Dirac's delta and  $M = \int u_0(x) dx - \int_0^\infty \int u^p(x,t) dx dt$ .

Now, let  $p = 1 + 2/N$ . For every  $R > 0$  there holds that

$$t^{N/2}u(x, t) \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \text{uniformly in } |x| \leq R\sqrt{t}$$

*Proof.* As in the previous theorems we know that  $u^k(y, 1) \rightarrow U(y, 1)$  uniformly on compact sets of  $\mathbb{R}^N \times (0, \infty)$  as  $k \rightarrow \infty$ . In the present situation, if  $p > 1 + 2/N$ , we know that  $U$  is the solution to (4.3). In particular,  $U$  is invariant under the present rescaling. When  $p = 1 + 2/N$  we know that  $U \equiv 0$ . So, we get the result as in the previous theorems.  $\square$

**Remark 4.1.** *The same method allows to study the asymptotic behavior of the solution  $u_L$  of the equation without absorption.*

#### REFERENCES

- [1] P. Bates, A. Chmaj, *An integrodifferential model for phase transitions: Stationary solutions in higher dimensions*, J. Statistical Phys. **95**, 1999, 1119–1139.
- [2] P. Bates, A. Chmaj, *A discrete convolution model for phase transitions*, Arch. Rat. Mech. Anal. **150**, 1999, 281–305.
- [3] P. Bates, P. Fife, X. Ren, X. Wang, *Travelling waves in a convolution model for phase transitions*, Arch. Rat. Mech. Anal. **138**, 1997, 105–136.
- [4] C. Carrillo, P. Fife, *Spatial effects in discrete generation population models*, J. Math. Biol. **50(2)**, 2005, 161–188.
- [5] M. Chaves, E. Chasseigne, J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, Adv. Differential Equations, **2**, 2006, 271–291.
- [6] X. Chen, Y. W. Qi, M. Wang, *Long time behavior of solutions to  $p$ -laplacian equation with absorption*, SIAM Jour. Math. Anal. **35(1)**, 2003, 123–134.
- [7] C. Cortazar, M. Elgueta, F. Quiros, N. Wolanski, *Large time behavior of the solution to the Dirichlet problem for a nonlocal diffusion equation in an exterior domain*, in preparation.
- [8] C. Cortazar, M. Elgueta, J. D. Rossi, *Nonlocal diffusion problems that approximate the heat equation with Dirichlet boundary conditions*, Israel Journal of Mathematics. **170(1)**, 2009, 53–60.
- [9] C. Cortazar, M. Elgueta, J. D. Rossi, N. Wolanski, *How to approximate the heat equation with Neumann boundary conditions by nonlocal diffusion problems*, Arch. Rat. Mech. Anal. **187(1)**, 2008, 137–156.
- [10] P. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions*, Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.
- [11] G. Gilboa, S. Osher, *Nonlocal operators with application to image processing*, Multiscale Model. Simul., **7(3)**, 2008, 1005, 1028.
- [12] L. Herraiz, *Asymptotic behavior of solutions of some semilinear parabolic problems*, Ann. Inst. Henri Poincaré, **16(1)**, 1999, 49–105.
- [13] Ignat, J. D. Rossi, *Refined asymptotic expansions for nonlocal diffusion equations*, J. Evolution Equations. **8**, 2008, 617–629.
- [14] S. Kamin, L. A. Peletier, *Large time behavior of solutions of the heat equation with absorption*, Anal. Scuola. Norm. Sup. Pisa Serie 4, **12**, 1985, 393–408.
- [15] S. Kamin, L. A. Peletier, *Large time behavior of solutions of the porous media equation with absorption*, Israel J. Math., **55**, 1986, 129–146.
- [16] S. Kamin, M. Ughi, *On the behavior as  $t \rightarrow \infty$  of the solutions of the Cauchy problem for certain nonlinear parabolic equations*, J. Math. Anal. Appl. **128**, 1997, 456–469.
- [17] C. Lederman, N. Wolanski, *Singular perturbation in a nonlocal diffusion model*, Communications in PDE **31(2)**, 2006, 195–241.
- [18] A. Pazoto, J. D. Rossi, *Asymptotic behavior for a semilinear nonlocal equation*. Asymptotic Analysis. **52(1-2)**, 2007, 143–155.
- [19] J. Terra, N. Wolanski, *Asymptotic behavior for a nonlocal diffusion equation with absorption and nonintegrable initial data. The supercritical case*, submitted.
- [20] L. Zhang, *Existence, uniqueness and exponential stability of traveling wave solutions of some integral differential equations arising from neuronal networks*, J. Differential Equations **197(1)**, 2004, 162–196.

- [21] J. Zhao, *The Asymptotic Behavior of solutions of a quasilinear degenerate parabolic equation*, J. Differential Equations, **102**, 1993, 33-52.

JOANA TERRA  
DEPARTAMENTO DE MATEMÁTICA, FCEYN  
UBA (1428) BUENOS AIRES, ARGENTINA.  
*E-mail address:* `jterra@dm.uba.ar`

NOEMI WOLANSKI  
DEPARTAMENTO DE MATEMÁTICA, FCEYN  
UBA (1428) BUENOS AIRES, ARGENTINA.  
*E-mail address:* `wolanski@dm.uba.ar`