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On the $L(2,1)$-labelling of block graphs

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The distance-two labelling problem of graphs was proposed by Griggs and Roberts in 1988, and it is a variation of the frequency assignment problem introduced by Hale in 1980. An $L(2,1)$-labelling of a graph G is an assignment of non-negative integers to the vertices of G such that vertices at distance two receive different numbers and adjacent vertices receive different and non-consecutive integers. The $L(2,1)$-labelling number of G, denoted by $\lambda(G)$, is the smallest integer k such that G has a $L(2,1)$-labelling in which no label is greater than k.

In this work, we study the $L(2,1)$-labelling problem on block graphs. We find upper bounds for $\lambda(G)$ in the general case and reduce those bounds for some particular cases of block graphs with maximum clique size equal to 3.

Keywords: block graphs; distance-two labelling problem; graph colouring

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1. Introduction

The distance-two labelling problem of graphs was proposed by Griggs and Roberts in 1988 (cf. [7]), and it is a variation of the frequency assignment problem introduced by Hale in 1980 [8]. Suppose we are given a number of transmitters or stations. The $L(2,1)$-labelling problem addresses the problem of assigning frequencies (non-negative integers) to the transmitters so that ‘close’ transmitters receive different frequencies and ‘very close’ transmitters receive frequencies that are at least two frequencies apart.

Let G be a simple, finite, undirected graph with a vertex set $V(G)$. Let $\Delta(G)$ denote the maximum degree of a vertex of G, $d_G(u,v)$ the distance in G between vertices u and v, and $\omega(G)$ the maximum size of a clique of G.

Let k be a non-negative integer. Denote by $[0,k]$ the set $\{x \in \mathbb{Z} : 0 \leq x \leq k\}$.

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An $L(2,1)$-*labelling* of a graph G is a function $f : V(G) \to \mathbb{Z}_{\geq 0}$ such that $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$ and $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$. For a non-negative integer k, a k-$L(2,1)$-*labelling* is an $L(2,1)$-labelling $f : V(G) \to [0,k]$. The $L(2,1)$-*labelling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labelling. It is not difficult to see that $\lambda(G) \geq \Delta(G) + 1$ and $\lambda(G) \geq 2\omega(G) - 2$. The $L(2,1)$-labelling problem has been studied widely. Griggs and Yeh [7] showed that the $L(2,1)$-labelling problem is NP-complete for general graphs. They proved that $\lambda(G) \leq \Delta^2(G) + 2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta^2(G)$ for general graphs different from K_2. Chang and Kuo [1] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G)$ and gave a linear-time algorithm for the $L(2,1)$-labelling problem on cographs. Král and Škrekovski [10] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 1$ for graphs different from K_2. More recently, Gonçalves [6] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$, giving the best-known upper bound for small values of $\Delta(G)$. In [9], the authors prove Griggs and Yeh's conjecture for graphs G with $\Delta(G)$ sufficiently large. For further studies on the $L(2,1)$-labelling and some generalizations, see [2–6,11,12].

A *block* of a graph is a maximal two-connected component. An *end-block* is a block containing exactly one cutpoint. A *block-cutpoint tree* of a graph G is a tree whose vertices are the cutpoints and the blocks of G, such that two vertices are adjacent if and only if they correspond to a block B of G and a cutpoint v of B. A graph is a *block graph* if it is connected and every block is a clique.

Block graphs with $\omega(G) = 2$ are trees. Griggs and Yeh [7] showed that $\Delta(G) + 1 \leq \lambda(G) \leq \Delta(G) + 2$ for trees, and Chang and Kuo [1] gave a polynomial-time algorithm for the $L(2,1)$-labelling problem on this class of graphs. However, there is no simple characterization distinguishing the cases $\lambda = \Delta + 1$ and $\lambda = \Delta + 2$. For the special case of paths, it is not difficult to see that $\lambda(P_1) = 0$, $\lambda(P_2) = 2$, $\lambda(P_3) = \lambda(P_4) = 3$, and $\lambda(P_n) = 4$ for $n \geq 5$.

The aim of this work is to study the $L(2,1)$-labelling problem on block graphs. We find upper bounds for $\lambda(G)$ in the general case and reduce those bounds for some particular cases with $\omega(G) = 3$.

2. Upper bounds

THEOREM 2.1 *Let G be a block graph with a maximum degree Δ and a maximum clique size ω. Then, $\lambda(G) \leq \max\{\Delta + 2, \min\{3\omega - 2, \Delta + \omega\}\}$.*

Proof Let G be a block graph with maximum degree Δ and maximum clique size ω, and let $k = \max\{\Delta + 2, \min\{3\omega - 2, \Delta + \omega\}\}$. We will prove that G has a k-$L(2,1)$-labelling by induction on the number of blocks. If G is a complete graph of n vertices, then $\omega = n$, $\Delta = n - 1$ and $k = \max\{n + 1, \min\{3n - 2, 2n - 1\}\} = \max\{n + 1, 2n - 1\} \geq 2n - 2 = \lambda(G)$. Suppose now that G is not a complete graph, and let v be a cutpoint of G such that all the blocks containing v but at most one are end-blocks. Denote by $B_1, B_2, \ldots, B_t$ the blocks containing v, where $B_2, \ldots, B_t$ are end-blocks with $|B_2| \geq \cdots \geq |B_t|$, let $B'_i = B_i - v$ for $i = 1, \ldots, t$ and let $B = \bigcup_{2 \leq i \leq t} B'_i$. Let $b_i = |B'_i|$ for $i = 1, \ldots, t$ and let $b = |B|$. By inductive hypothesis, there is a $L(2,1)$-labelling of $G \setminus B$ with labels in $[0,k]$. We will extend this labelling to B. From the set $[0,k]$, B can use neither the label used by v nor its previous or subsequent label, and it cannot use any of the labels used by the neighbours of v in $G \setminus B$, that is, vertices in B'_1. So, the available labels for B are at least $k + 1 - 3 - b_1$. One can observe that the labelling can be extended to B if there are at least $\max\{b, 2b_2 - 1\}$ available labels. In fact, let $c_1 < c_2 < \cdots < c_p$ be the available labels. The vertices in B_2, and subsequently in $B_3, \ldots, B_t$, can be labelled by using first labels with odd indices followed by the even ones, respecting the increasing ordering of labels. Since $p \geq 2b_2 - 1$, vertices in the same set B_i do not receive consecutive labels.

It holds that $k-2-b_1 \geq \max\{b, 2b_2-1\}$ if and only if $k \geq \max\{b+b_1+2, b_1+2b_2+1\}$. Note that $b_1+b \leq \Delta$, and for $i=1,\ldots,t$, $b_i \leq \omega-1$ since $B_i' \cup \{v\}$ is a clique. Thus, $b_1+b+2 \leq \Delta+2$, $b_1+2b_2+1 \leq 3(\omega-1)+1 = 3\omega-2$ and $b_1+2b_2+1 \leq (b_1+b_2)+b_2+1 \leq \Delta+b_2+1 \leq \Delta+\omega$. Therefore, $k = \max\{\Delta+2, \min\{3\omega-2, \Delta+\omega\}\} \geq \max\{b+b_1+2, b_1+2b_2+1\}$. This completes the proof. ■

Corollary 2.2 *Let G be a block graph different from K_2. Then, $\lambda(G) \leq 2\Delta(G)+1$ and $\lambda(G) \leq \Delta^2(G)$.*

Proof If G is a block graph different from K_2, then $\Delta(G) \geq 2$. So, $\Delta(G)+2 \leq \Delta^2(G)$. If $\Delta(G)=2$, then G is either a path or a triangle, and in both cases it is known that $\lambda(G) \leq 4$. If $\Delta(G) \geq 3$, then $\Delta(G)+\omega(G) \leq 2\Delta(G)+1 \leq \Delta^2(G)$. Thus, $\lambda(G) \leq \max\{\Delta(G)+2, \min\{3\omega(G)-2, \Delta(G)+\omega(G)\}\} \leq \min\{2\Delta(G)+1, \Delta^2(G)\}$. ■

Corollary 2.3 *Let G be a block graph with maximum degree Δ and maximum clique size at most 3. If $\Delta \geq 5$, then $\lambda(G) \leq \Delta+2$; if $\Delta = 4$, then $\lambda(G) \leq 7$; and if $\Delta \leq 3$, then $\lambda(G) \leq 6$.*

Proposition 2.4 *The bounds of Corollary 2.3 are tight for $\Delta = 3$, $\Delta = 4$ and $\Delta \geq 5$, and they are attained by graphs G_1, G_2 and $G_3(\Delta)$ of Figure 1, respectively.*

Proof Let us consider 5-$L(2,1)$-labellings and show that G_1 does not admit one. In a 5-$L(2,1)$-labelling of a graph, the set of possible triplets for a triangle is $\{0,2,4;\ 0,2,5;\ 0,3,5;\ 1,3,5\}$. Let $A_1 = v_1v_2v_3$ and $A_2 = v_4v_5v_6$ be two disjoint triangles in a graph, joined by the edge v_1v_4. If A_1 is labelled 0, 2, 5, then v_1 cannot receive number 2, otherwise v_4 must receive number 4 but A_2 cannot be labelled 0, 2, 4, and v_1 cannot receive number 5, otherwise v_4 must receive number 3 or 1, but A_2 can neither be labelled 0, 3, 5 nor 1, 3, 5. Analogously, if A_1 is labelled 0, 3, 5, then v_1 cannot receive number 0 or 3. Therefore, in a 5-$L(2,1)$-labelling of G_1, the triangles A, B and C should be labelled 0, 2, 4 or 1, 3, 5. If two of A, B, C use different labels, then there is no colour left to v. If all of them use the same labelling, the three neighbours of v must use different colours. Since none of 1, 3, 5 is at distance two of 0, 2 and 4, and conversely, there is no suitable colour for v. Consider the graph G_2 in Figure 1, and suppose there is an $L(2,1)$-labelling of it with labels in $[0,6]$. The set of possible triplets for labelling a triangle is $\{0,2,4;\ 0,2,5;\ 0,2,6;\ 0,3,5;\ 0,3,6;\ 0,4,6;\ 1,3,5;\ 1,3,6;\ 1,4,6;\ 2,4,6\}$. We say that two triplets are *compatible* if they share exactly one number, and we say that a triplet is *good* if it has three compatible triplets, each one sharing a different number with it. As it can be seen in Figure 2, the set of good triplets is $\{0,2,5;\ 0,3,6;\ 1,3,5;\ 1,4,6\}$. It is clear that the triangles A, B, C, D, A_1, A_2, B_1, B_2, C_1 and C_2 of G_2 should be labelled with good triplets. So, we call *very good* triplets those triplets having

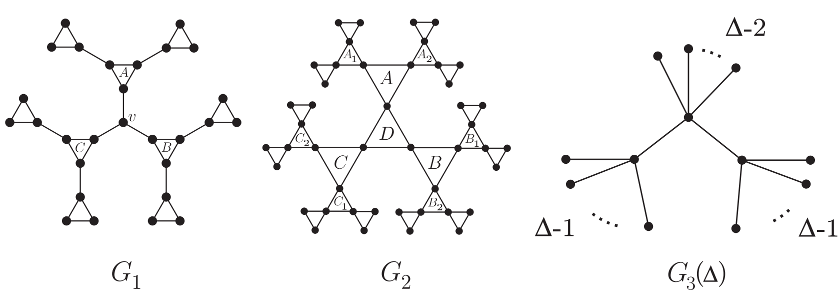

Figure 1. Examples showing tightness of the bounds of Corollary 2.3.

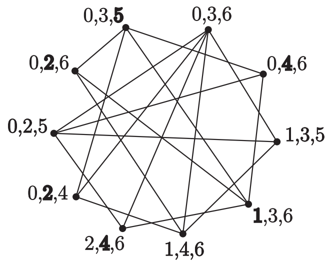

Figure 2. Compatibility between triplets on [0, 6].

three compatible good triplets, each one sharing a different number. As it can be seen in Figure 2, the only very good triplets are 0, 3, 6 and 1, 3, 5. The triangles A, B, C and D of G_2 should be labelled with very good triplets. Since the vertices of D are labelled by very good triplets and $\{0, 3, 6\} \cap \{1, 3, 5\} = \{3\}$, at least one of A, B or C is labelled by the same very good triplet as for D, a contradiction. Hence, the $L(2, 1)$-labelling number of G_2 is greater than 6. The family of graphs $G_3(\Delta)$ is a known example of trees with maximum degree Δ and $L(2, 1)$-labelling number $\Delta + 2$: in every $L(2, 1)$-labelling of a graph with labels in $[0, \Delta + 1]$, all the vertices of degree Δ must receive label 0 or $\Delta + 1$, but the three vertices of degree Δ in $G_3(\Delta)$ must receive pairwise distinct labels. ■

Let T_3 be the leftmost graph in Figure 3, that is, a graph with seven vertices $v_1, v_2, v_3, v_4, w_1, w_2, w_3$, where v_1, v_2, v_3, v_4 induce a path of length four, and w_i is adjacent to v_i and v_{i+1} for $i = 1, 2, 3$.

THEOREM 2.5 *Let G be a block graph with maximum degree* 4 *and maximum clique size* 3. *If G does not contain T_3, then $\lambda(G) \leq 6$.*

Proof We will construct a 6-$L(2, 1)$-labelling of G by labelling the vertices of G ordered by their distance to some vertex v_0 with degree 4. For $d \geq 0$, denote by V_d the set of vertices of G at distance d from v_0. Note that since G is a block graph with $\omega(G) = 3$, each vertex in V_d with $d \geq 1$ has exactly one neighbour in V_{d-1}, and at most one neighbour in V_d and, in that case, they share the neighbour in V_{d-1}. We call *type 1* the vertices belonging to two triangles. Note that a type 1 vertex belongs to a triangle formed by it and its neighbours in V_d and V_{d-1}, and to another triangle formed by it and its neighbours in V_{d+1}. We call *type 2* those vertices that are not of type 1. As in the proof of Proposition 2.4, we will consider the set of possible triplets for labelling a triangle. Recall that two triplets are *compatible* if they share exactly one number, and we say that a number a is *bad* for a triplet t if t has no compatible triplet sharing the number a. As it can be seen in Figure 2, every triplet has at most one bad number for it, which is boldfaced in each triplet.

First, we give to v_0 the label 0. Now, we will continue the labelling process in such a way that no type 1 vertex in V_d, with $d \geq 1$, is labelled with a bad colour for the triplet given to the triangle formed by it and its neighbours in V_d and V_{d-1}.

At most one of the four vertices in V_1 is of type 1, otherwise G would contain T_3 as an induced subgraph, so we can label vertices in V_1 following that rule. Let $d > 1$ and suppose that every

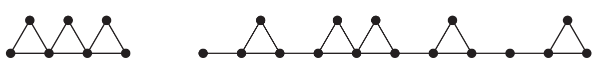

Figure 3. Paths of triangles.

vertex at distance at most $d-1$ from v_0 is labelled. Let v be a no-labelled vertex in V_d, and w its neighbour in V_{d-1}. Let W be the set of neighbours of w in V_d. We will label the vertices in W all at once. Since w has a neighbour in V_{d-2} and G has maximum degree 4, $|W| \leq 3$. If w has no neighbour in V_{d-1}, then the number of available colours for W is at least 3. If the vertices in W are pairwise non-adjacent, we can clearly label them. If two of them are adjacent, at most one is type 1, otherwise G would contain T_3 as an induced subgraph. So we can label those two vertices with two non-consecutive numbers within the available ones in such a way that the (possible) type 1 vertex does not receive a bad colour for the triplet formed by these two labels and the label of w. Finally, there is at least one remaining label for the (possible) third vertex. If w has a neighbour in V_{d-1}, then $|W| \leq 2$ and the number of available colours for W is at least 2. If the vertices in W are non-adjacent, then we can clearly label them. If they are adjacent, then w is type 1 and none of the vertices in W is, otherwise G would contain T_3 as an induced subgraph. Since w is not labelled with a bad colour for the triplet given to the triangle formed by it and its neighbours in V_{d-1} and V_{d-2}, there is a triplet compatible to that one in order to label W. ■

By now, the computational complexity of computing $\lambda(G)$ of a block graph G is open, even when $\omega(G) = 3$. Nevertheless, the proofs of the previous theorems are constructive and lead to algorithms to produce an $L(2,1)$-labelling of the graph with the showed upper bound.

3. The $L(2,1)$-labelling number for paths of triangles

We call *paths of triangles* the block graphs G with $\omega(G) = 3$ and such that the block-cutpoint tree of G is a path. Examples of paths of triangles can be seen in Figure 3. Note that since $\omega(G) = 3$, then $\lambda(G) \geq 4$.

For these kind of graphs we prove that $\lambda(G) \leq \Delta(G) + 2$ and give a complete characterization for each possible value of λ.

THEOREM 3.1 *Let G be a path of triangles. Then, $\lambda(G) \leq 6$. Moreover, $\lambda(G) = 6$ if and only if G contains T_3, and $\lambda(G) = 4$ if and only if G does not contain any of the graphs in Figure* 4.

Proof The scheme in Figure 5 shows that for every path of triangles G, $\lambda(G) \leq 6$.

It is easy to see also that T_3 does not admit a 5-$L(2,1)$-labelling because the only two pairs of compatible triplets are 0, 2, 5 and 1, 3, 5 sharing 5, and 0, 2, 4 and 0, 3, 5 sharing 0.

We will show that every path of triangles with no induced T_3 can be 5-$L(2,1)$-labelled. In order to do that, we will consider a path of triangles without T_3 as a sequence of pieces consisting of

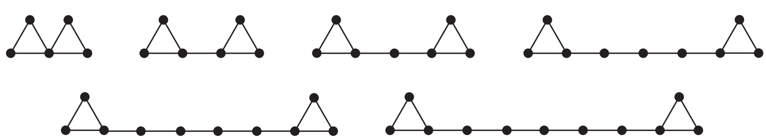

Figure 4. Paths of triangles with $\lambda = 5$.

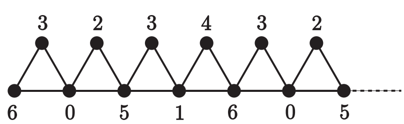

Figure 5. Scheme for the 6-$L(2,1)$-labelling on paths of triangles.

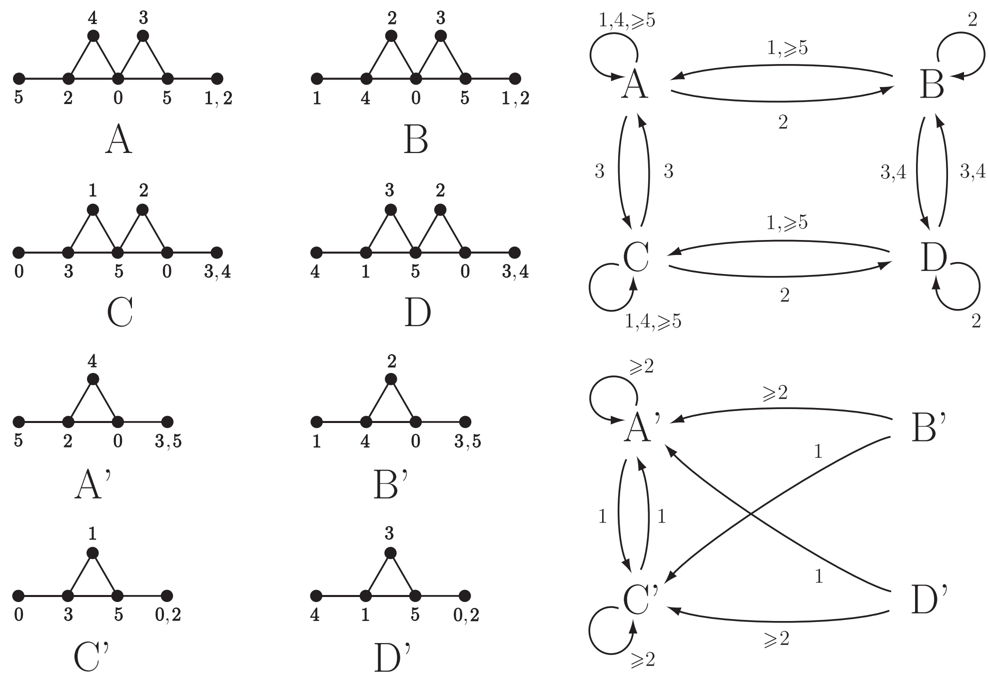

Figure 6. Scheme for the 5-$L(2, 1)$-labelling of paths of triangles without T_3.

one or two consecutive triangles joined by simple paths. We will consider four different ways of labelling a piece consisting on two consecutive triangles, namely A, B, C and D, and four different ways of labelling a piece consisting on a single triangle, namely A', B', C' and D'. Figure 6 shows A, B, C, D, A', B', C' and D', together with the possible labels for the last vertex in a path preceding the piece and the first vertex in a path succeeding the piece. In the scheme on the right of Figure 6, we have a node for every way of labelling a piece, and we join a node X and a node Y with a directed arc labelled t to mean that a piece labelled as X can be succeeded by a piece labelled as Y with a join path of length t. We omit the arcs between A, B, C, D and A', B', C', D', but note that the first two vertices of X and X' are identically labelled. Thus, if we have an arc (X, Y) labelled by t then we could add an arc (X, Y') labelled by t, and if we have an arc (X', Y') labelled by t then we could add an arc (X', Y) labelled by t. Since for every length t and for every node X there exists a directed arc labelled t joining X with one of A, B, C, D (and, consequently, with one of A', B', C', D'), every path of triangles without T_3 can be 5-$L(2, 1)$-labelled, subject to the correctness of the scheme. The arcs labelled up to 5 are easy to check by hand. It remains to prove that B and A (respectively D and C) can be joined to A (respectively C) by a path of arbitrary length at least 5, and that B' and A' (respectively D' and C') can be joined to A' (respectively C') by a path of arbitrary length at least 2.

To simplify the notation, we will enclose with brackets a subsequence of a sequence to mean that it can be either omitted or repeated as many times as necessary. For example, the sequence 1, 2, [3, 4], 5 will stand for any of the sequences 1, 2, 5; 1, 2, 3, 4, 5; 1, 2, 3, 4, 3, 4, 5; etc.

Case 1 We have to join 5, 1 or 5, 2 with 5, 2 by paths of length at least 5. We will use subsequences of the 5-periodic sequence [2, 4, 1, 3, 0] in the following way: if the length of the path is $5k$, $5k + 1$ or $5k + 2$, we will join 5, 2 with 5, 2 by using the subsequence beginning at 4 (e.g., 5, 2, 4, 1, 5, 2; 5, 2, 4, 1, 3, 5, 2; 5, 2, 4, 1, 3, 0, 5, 2; 5, 2, 4, 1, 3, 0, 2, 4, 1, 5, 2; etc.); if the length of the path is $5k + 3$ or $5k + 4$ we will join 5, 1 with 5, 2 by using the sequence beginning at 3 (e.g., 5, 1, 3, 0, 2, 4, 1, 5, 2; 5, 1, 3, 0, 2, 4, 1, 3, 5, 2; 5, 1, 3, 0, 2, 4, 1, 3, 0, 2, 4, 1, 5, 2; etc.).

Case 2 We have to join 0, 4 or 0, 3 with 0, 3 by paths of length at least 5. This case is symmetric of Case 1, considering the isometric bijection between labels $t \mapsto 5 - t$.

Case 3 We have to join 0, 3 or 0, 5 with 5, 2 by paths of length at least 2. The first cases are 0, 5, 2; 0, 3, 5, 2; 0, 3, 1, 5, 2. For paths of length greater than 5, we will join 0, 5 with 5, 2 by using subsequences of the 5-periodic sequence [2, 4, 1, 3, 0] in the following way: if the length of the path is $5k$ or $5k + 4$, we will use the sequence beginning at 1 (e.g., 0, 5, 1, 3, 5, 2; 0, 5, 1, 3, 0, 2, 4, 1, 5, 2; 0, 5, 1, 3, 0, 2, 4, 1, 3, 5, 2; etc.) and if the length of the path is $5k + 1$, $5k + 2$ or $5k + 3$, we will use the sequence beginning at 2 (e.g., 0, 5, 2, 4, 1, 5, 2; 0, 5, 2, 4, 1, 3, 5, 2; 0, 5, 2, 4, 1, 3, 0, 5, 2; etc.).

Case 4 We have to join 5, 2 or 5, 0 with 0, 3 by paths of length at least 2. This case is symmetric of Case 3, considering the isometric bijection between labels $t \mapsto 5 - t$.

Finally, we will characterize the paths of triangles G with $\lambda(G) = 4$. Since $\lambda(G) \geq \Delta(G) + 1$, then $\Delta(G) \leq 3$. In particular, G cannot contain the first graph in Figure 4. Moreover, if $\Delta(G) = 3$, then every vertex of degree 3 should be labelled with 0 or 4. The only possible triplet in this case is 0, 2, 4, and from the observation above, in every triangle the label 2 should be assigned to a vertex of degree 2. Considering these facts, it is not difficult to check that none of the graphs in Figure 4 admits a 4-$L(2, 1)$-labelling.

Since 4-$L(2, 1)$-labellings are preserved under the map $t \mapsto 4 - t$, we can assume that the cutpoint in the first triangle linked to the path is labelled 0. Two triangles joined by a path of length 3 can be labelled giving to the vertices of degree 3 numbers 0 and 4, and to their neighbours in the path numbers 3 and 1, respectively. If we have two triangles joined by a path of length at least 7, we have to join 0, 3 with 3, 0 or 1, 4 by a path. If the length of the path is $3k + 1$, we will join 0, 3 with 1, 4 by using the sequence 0, 3, 1, 4, 0, [2, 4, 0], 3, 1, 4. If the length of the path is $3k + 2$, we will join 0, 3 with 1, 4 by using the sequence 0, 3, 1, 4, 2, 0, [4, 2, 0], 3, 1, 4. Finally, if the length of the path is $3k$, we will join 0, 3 with 3, 0 by using the sequence 0, 3, 1, 4, 2, 0, [4, 2, 0], 4, 1, 3, 0. This completes the proof of the theorem. ∎

This characterization leads to an efficient algorithm to compute $\lambda(G)$ and an optimum $L(2, 1)$-labelling on paths of triangles.

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