> "Common sense would, I think, lead us to believe that, though the velocity of advance might be temporarily enhanced by this method, yet ultimately, the velocity of advance would adjust itself so as to be the same irrespective of the initial conditions. If this is so, this equation must omit some essential element of the problem, and it is indeed clear that while a coefficient of diffusion may represent the biological conditions adequately in places where large numbers of individuals of both types are available, it cannot do so at the extreme front and back of the advancing wave, where the numbers of the mutant and the parent gene respectively are small, and where their distribution must be largely sporadic".

Fisher proposed a way to overcome this difficulty, related to the probabilistic representation given later on by McKean [35], weaving links between solutions to (1) and Branching Brownian Motion. The general principle behind is that microscopic effects should be taken into account to properly describe the physical phenomena. With a similar point of view in mind, Brunet, Derrida and coauthors [13, 14, 11, 12] started in the nineties a study of the effect of microscopic noise in front propagation for equation (1) and related models, which resulted in a huge number of works that study the change in the behavior of the front when microscopic effects are taken into account. These works include both numerical and heuristic arguments [13, 14, 11, 12, 26] as well as rigorous proofs [4, 5, 16, 31, 32]. Before that, Bramson et.al [10] gave the first rigorous proof of a microscopic model for (1) that has a unique velocity for every initial condition. They also prove that these velocities converge in the macroscopic scale to the minimum velocity of (1), and call this fact a *microscopic selection principle*, as opposed to the macroscopic selection principle stated above, that holds for solutions of the hydrodynamic equation.

The theory of quasi-stationary distributions (QSD) has their own counterpart. It is a typical situation that there is an infinite number of quasi-stationary distributions, but the *Yaglom limit* (the limit of the conditioned evolution of the process started from a deterministic initial condition) selects the minimal one, i.e. the one with minimal expected time of absorption.

Up to our knowledge, despite of a shared feeling that similar principles do occur in the context of QSD and of traveling waves, this relation has never been stated precisely. The purpose of this note is to show that they are two faces of the same coin. We first explain this link through the example of Brownian motion. Then we show how to extend these results to more general Lévy processes.

The paper is organized as follows. In Section 2, we introduce traveling waves and QSDs as macroscopic models. We focus in particular on the KPP equation and the links between its traveling waves and the QSDs of a drifting Brownian motion. In Section 3, we introduce particle systems enlightening the selection principles observed for the macroscopic models. Finally in Section 4 we export these observations to more general models. We consider general Lévy processes under suitable assumptions and analyze selection principles in this context.

2   Macroscopic models

We elaborate on the two macroscopic models we study: front propagation and QSD.

2.1 Front propagation in the KPP

Since the seminal papers [20, 28], equation (1) has received a huge amount of attention for several reasons. Among them, it is one of the simplest models explaining several phenomena that are expected to be universal. For instance, it admits a continuum of traveling wave solutions that can be parametrized by their velocity c. More precisely, for each $c \in [\sqrt{2r}, +\infty)$ there exists a function $w_c \colon \mathbb{R} \to [0, 1]$ such that

$$v(t, x) = w_c(x - ct)$$

is a solution to (1). For $c < \sqrt{2r}$, there is no traveling wave solution, [1, 28]. Hence $c^* = \sqrt{2r}$ represents the minimal velocity and w_{c^*} the minimal traveling wave. Moreover, if v_0 verifies for some $0 < b < \sqrt{2r}$

$$\lim_{x \to \infty} e^{bx}(1 - v_0(x)) = a > 0,$$

then

$$\lim_{t \to \infty} v(t, x + ct) = w_c(x), \quad \text{for } c = r/b + \frac{1}{2}b, \tag{2}$$

see [35, 36]. If the initial measure has compact support (or fast enough decay at infinity), the solution converges to the minimal traveling wave and the domain of attraction and velocity of each traveling wave is determined by the tail of the initial distribution [1, 35, 39]. A smooth traveling wave solution of (1) that travels at velocity c is a solution to

$$\frac{1}{2}w'' + cw + r(w^2 - w) = 0. \tag{3}$$

The behavior at infinity of these traveling waves is given by

$$1 - w_c(x) \sim \begin{cases} c_1 e^{-bx} & c > \sqrt{2r} \\ c_2 x e^{-x\sqrt{2r}} & c = \sqrt{2r}. \end{cases}$$

This behavior is determined by the linearization of (3) at $w = 1$, i.e. the solution of

$$\frac{1}{2}w'' + cw' + rw = 0. \tag{4}$$

See [25, 39]. We come back to this equation when dealing with quasi-stationary distributions.

2.2 Quasi-stationary distributions

Quasi-stationary distributions have been extensively studied since the pioneering work of Kolmogorov (1938), Yaglom (1947) and Sevastyanov (1951) on the behavior of Galton-Watson processes.

The beginning of this theory and an important part of the research in the area has been motivated by models on genetics and population biology, where the notion of quasi-stationarity is completely natural to describe the behavior of populations that are expected to get extinct, conditioned on the event that extinction has not yet occurred, on large time scales.

Being more precise, consider a Markov process $Z = (Z_t,\ t \geq 0)$, killed at some state or region that we call 0. The absorption time is defined by $\tau = \inf\{t > 0 \colon Z_t \in 0\}$. The conditioned evolution at time t is defined by

$$\mu_t^\gamma(\cdot) := \mathbb{P}_\gamma(Z_t \in \cdot | \tau > t).$$

Here γ denotes the initial distribution of the process. A probability measure ν is said to be a quasi-stationary distribution (QSD) if $\mu_t^\nu = \nu$ for all $t \geq 0$.

For Markov chains in finite state spaces, the existence and uniqueness of QSDs as well as the convergence of the conditioned evolution to this unique QSD for every initial measure follows from Perron-Frobenius theory. The situation is more delicate for unbounded spaces as there can be 0, 1 or an infinite number of QSD. Among those distributions, the *minimal* QSD is the one that minimizes $\mathbb{E}_\nu(\tau)$.

The *Yaglom limit* is a probability measure ν defined by

$$\nu := \lim_{t \to \infty} \mu_t^{\delta_x},$$

if it exists and does not depend on x. It is known that if the Yaglom limit exists, then it is a QSD. A general principle is that the Yaglom limit *selects* the minimal QSD, i.e. the Yaglom limit is the QSD with minimal mean absorption time. This fact has been proved for a wide class of processes that include birth and death process, Galton-Watson processes, random walks, Brownian motion, more general Lévy processes, etc.

It can also be proved for R-positive processes by means of the theory of R-positive matrices [38]. To give a flavor of the results that hold in this situation, consider a discrete time Markov chain in $\mathbb{N}$ that it is absorbed at 0. Denote $p = (p(i,j),\ i, j \in \mathbb{N})$ its transition matrix so that p is sub-stochastic. We use $p^{(n)}$ for the n-th power of p. We say that p is R-positive if one (and hence both) of the following equivalent statements hold

1. For some i and j, the sequence $R^n p^{(n)}(i,j)$ tends to a finite non-zero limit as $n \to \infty$.

2. There exist non-negative, non-zero eigenvectors $\nu = (\nu(k))_{k \in \mathbb{N}}$, $\beta = (\beta(k))_{k \in \mathbb{N}}$ associated to the eigenvalue $1/R$ such that $\sum_{k=1}^\infty \nu(k)\beta(k) < \infty$.

In 1966, Seneta and Vere-Jones proved the following theorem

Theorem 2.1 (Seneta and Vere-Jones, [38]). *Assume that the matrix p is R-positive, then the conditioned evolution converges to ν as $n \to \infty$ if one of the following conditions hold*

1. *The left eigenvector ν satisfies $\sum \nu(i) < \infty$ and the initial distribution μ is dominated (pointwise) by a multiple of ν.*

2. *The right eigenvector β is bounded away from zero and $\sum \mu(j)\beta(j) < \infty$*

Observe that in both situations we have that ν is the minimal QSD and the Yaglom limit. Also every initial distribution with tail light enough is in the domain of attraction of ν. So, in the R-positive case, the situation is pretty clear. These results can be applied for instance to the Galton-Watson process. In that case, a detailed study of the domain of attraction of the other QSDs (which are parametrized by an interval) is given in [37] where it can be seen that the limiting conditional distribution is given by the tail of the initial distribution.

Unfortunately, on the one hand R-positivity is a property difficult to check and on the other hand, there is a lot of interesting processes that are not R-positive. For example, a birth and death process with constant drift towards the origin, has a continuum of QSDs and initial distributions with light tails are attracted by the minimal QSD, which can be computed explicitly [19], but this process is not R-positive and hence Theorem 2.1 does not apply.

The presence of an infinite number of quasi-stationary distributions is something anomalous from the modeling point of view, in the sense that no physical nor biological meaning has been attributed to them. The reason for their presence here and in the front propagation context is similar: when studying for instance population or genes dynamics through the conditioned evolution of a Markov process, we are implicitly considering an infinite population and microscopic effects are lost.

So, as Fisher suggests, in order to avoid the undesirable infinite number of QSD, we should take into account microscopic effects. A natural way to do this is by means of interacting particle systems. We discuss this in Section 3.

Brownian Motion with drift. Quasi-stationary ditributions for Brownian Motion with constant drift towards the origin are studied in [33, 34]. We briefly review here some of the results of these papers and refer to them for the details.

For $c > 0$ we consider a one-dimensional Brownian Motion $X = (X_t)_{t \geq 0}$ with drift $-c$ defined by $X_t = B_t - ct$. Here B_t is a one dimensional Wiener process defined in the standard Wiener space. We use $\mathbb{P}_x$ for the probability defined in this space such that B_t is Brownian Motion started at x and $\mathbb{E}_x$ for expectation respect to $\mathbb{P}_x$. Define the hitting time of zero, when the process is started at $x > 0$ by $\tau_x(c) = \inf\{t > 0 \colon X_t = 0\}$ and denote with P_t^c the submarkovian semigroup defined by

$$P_t^c f(x) = \mathbb{E}_x(f(X_t)\mathbf{1}_{\{\tau_x(c) > t\}}). \tag{5}$$

In this case, differentiating (5) and after some manipulation it can be seen that the conditioned evolution μP_t has a density $u(t, \cdot)$ for every $t > 0$ and verifies

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial^2 x}(t,x) + c\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\frac{\partial u}{\partial x}(t,0)u(t,x), \quad t > 0, x > 0, \\
u(t,0) &= u(t,+\infty) = 0, \quad t > 0,
\end{aligned}
\tag{6}
$$

Recall now that a probability measure ν in $\mathbb{R}_+$ is a QSD if

$$\mathbb{P}_\nu(X_t \in \cdot | X_t > 0) = \nu(\cdot).$$

It is easy to check that if ν is a QSD, the hitting time of zero, started with ν is an exponential variable of parameter r and hence ν is a QSD if and only if there exists $r > 0$ such that

$$\nu P_t^c = e^{-rt}\nu, \quad \text{for any} \quad t > 0.$$

Differentiating (5) and using the semigroup property we get that ν is a QSD if and only if

$$\int (\frac{1}{2}f'' - cf')\, d\nu = -r \int f\, d\nu, \quad \text{for all } f \in C_0^\infty(\mathbb{R}_+). \tag{7}$$

Integrating by parts we get that the density w of ν must verify

$$\frac{1}{2}w'' + cw' + rw = 0. \tag{8}$$

Solutions to this equation with initial condition $w(0) = 0$ are given by

$$w(x) = \begin{cases} me^{-cx}\sin(\sqrt{c^2 - 2r}\,x) & r > \frac{c^2}{2}, \\ mxe^{-cx} & r = \frac{c^2}{2}, \\ me^{-cx}\sinh(\sqrt{c^2 - 2r}\,x) & r < \frac{c^2}{2}. \end{cases}$$

Observe that w defines an integrable density function if and only if $0 < r \le c^2/2$ (or equivalently, $c \ge \sqrt{2r}$). One can thus parametrize the set of QSDs by their eigenvalues r, $\{\nu_r : 0 < r \le c^2/2\}$. For each r, the distribution function of ν_r, $v(x) = \int_0^x w(y)\,dy$ is a monotone solution of (8) with boundary conditions

$$v(0) = 0, \qquad v(+\infty) = 1, \tag{9}$$

which is the same equation (4) but in a different domain. The following theorem characterizes the domain of attraction of each QSD.

Theorem 2.2 (Martínez, Picco, San Martín, [33])**.** *Let γ be a probability measure on $(0, +\infty)$ with density ρ. If*

$$\lim_{x \nearrow \infty} -\frac{1}{x}\log\rho(x) = b < c, \tag{10}$$

then $\lim_t \mu_t^\gamma = \nu_{r(b)}$, where $r(b) = cb - b^2/2$.

Observe that this last equation is equivalent to $c = r/b + \frac{1}{2}b$, which should be compared with (2). Also remark that if (10) holds then

$$\lim_{x \nearrow \infty} -\frac{1}{x}\log\mu([x, +\infty)) = b.$$

We come back to equation (8) later, shedding light on the links between QSD and traveling waves.

Finally let us mention that a similar result holds for the discrete-space analog, i.e. birth and death processes with constant drift towards the origin [17, 19]

3 Particle systems

In this section we introduce two particle systems. The first one is known as Branching Brownian Motion (BBM) with selection of the N right-most particles ($N-$BBM). As a consequence of the link between BBM and F-KPP that we describe below, this process can be thought of as a microscopic version of F-KPP. The second one is called Fleming-Viot and was introduced by Burdzy, Ingemar, Holyst and March [15], in the context of Brownian Motion in a d-dimensional bounded domain. It is a slight variation of the original one introduced by Fleming and Viot [21]. The first interpretation of this process as a microscopic version of a conditioned evolution is due to Ferrari and Maric [18].

3.1 BBM and F-KPP equation

One-dimensional supercritical Branching Brownian Motion is a well-understood object. Particles diffuse following standard Brownian Motion started at the origin and branch at rate 1 according to an offspring distribution that we assume for simplicity to be δ_2. When a particle branches, it has two children and then dies. As already underlined, its connection with the F-KPP equation and traveling waves was pointed out by McKean in the seminal paper [35]. Denote with N_t the number of particles alive at time $t \geq 0$ and $\xi_t(1) \leq \cdots \leq \xi_t(N_t)$ the position of the particles enumerated from left to right. McKean's representation formula states that if $0 \leq v_0(x) \leq 1$ and we start the process with one particle at 0 (i.e. $N(0) = 1$, $\xi_0(1) = 0$), then

$$v(t, x) := \mathbb{E}\left(\prod_{i=1}^{N_t} v_0(\xi_t(i) + x)\right)$$

is the solution of (1). Of special interest is the case where the initial condition is the Heaviside function $v_0 = \mathbf{1}\{[0, +\infty)\}$ since in this case

$$v(t, x) = \mathbb{P}(\xi_t(1) + x > 0) = \mathbb{P}(\xi_t(N_t) < x).$$

This identity as well as various martingales obtained as functionals of this process have been widely exploited to obtain the precise behavior of solutions of (1), using analytic as well as probabilistic tools [8, 9, 24, 25, 35, 39].

3.2 $N-$BBM and Durrett-Remenik equation

Consider now a variant of BBM where the N right-most particles are selected. In other words, each time a particle branches, the left-most one is killed, keeping the total number of particles constant.

This process was introduced by Brunet and Derrida [11, 12] as part of a family of models of branching-selection particle systems to study the effect of microscopic noise in front propagation. By means of numerical simulations and heuristic arguments, they conjectured that the linear speed of $N-$BBM differs from the speed of standard BBM by $(\log N)^{-2}$ and in a series of papers with coauthors they study various statistics of the process [13, 14, 11, 12]. Bérard and Gouéré proved this shift in the velocity for a similar process. We refer to the work of Maillard [31] for the most detailed study of this process.

Durret and Remenik [16] considered a slightly different process in the class of Brunet and Derrida: $N-$BRW. The system starts with N particles. Each particle gives rise to a child at rate one. The position of the child of a particle at $x \in \mathbb{R}$ is $x + y$, where y is chosen according to a probability distribution with density ρ, which is assumed symmetric and with finite expectation. After each birth, the $N+1$ particles are sorted and the left-most one is deleted, in order to keep always N particles. They prove that the empirical measure of this system converges to a deterministic probability measure ν_t for every t, which is absolutely continuous with density $u(t, \cdot)$, a solution of the

following free-boundary problem

Find (γ, u) such that

$$\frac{\partial u}{\partial t}(t, x) = \int_{-\infty}^{\infty} u(t, y) \rho(x - y) \, dy \quad \forall x > \gamma(t), \tag{11}$$

$$\int_{\gamma(t)}^{\infty} u(t, y) \, dy = 1, \quad u(t, x) = 0, \quad \forall x \le \gamma(t),$$

$$u(0, x) = u_0(x).$$

They also find all the traveling wave solutions for this equation. Just as for the BBM, there exists a minimal velocity $c^* \in \mathbb{R}$ such that for $c \ge c^*$ there is a unique traveling wave solution with speed c and no traveling wave solution with speed c for $c < c^*$. The value c^* and the behavior at infinity of the traveling waves can be computed explicitly in terms of the Laplace transform of the random walk. In Section 4 we show that these traveling waves correspond to QSDs of drifted random walks.

It follows from renewal arguments that for each N, the process seen from the left-most particle is ergodic, which in turn implies the existence of a velocity v_N at which the empirical measure travels for each N. Durrett and Remenik prove that these velocities are increasing and converge to c^* as N goes to infinity.

We can interpret this fact as a *weak selection principle*: the microscopic system has a unique velocity for each N (as opposed to the limiting equation) and the velocities converge to the minimal velocity of the macroscopic equation. The word "weak" here refers to the fact that only convergence of the velocities is proved, but not convergence of the empirical measures in equilibrium.

In view of these results, the same theorem is expected to hold for a $N-$BBM that branches at rate r. In this case the limiting equation is conjectured to be given by

Find (γ, u) such that

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 u}{\partial^2 x}(t, x) + ru(t, x) \quad \forall x > \gamma(t), \tag{12}$$

$$\int_{\gamma(t)}^{\infty} u(t, y) \, dy = 1, \quad u(t, x) = 0, \quad \forall x \le \gamma(t),$$

$$u(0, x) = u_0(x).$$

The empirical measures in equilibrium are also expected to converge to the minimal traveling wave. More precisely,

Conjecture 3.1. *Both N-BBM and N-BRW are ergodic, with (unique) invariant measure λ^N and the empirical measure distributed according to λ^N converges to the delta measure supported on the minimal quasi-stationary distribution.*

Traveling waves. Let us look at the traveling wave solutions $u(t, x) = w(x - ct)$ of (12). Plugging-in in (12) we see that they must verify

$$\frac{1}{2} w'' + cw' + rw = 0, \quad w(0) = 0, \quad \int_0^{\infty} w(y) \, dy = 1. \tag{13}$$

Which is exactly (8). Note nevertheless that in (13) the parameter r is part of the data of the problem (the branching rate) and c is part of the unknown (the velocity), while

in (8) the situation is reversed: c is data (the drift) and r unknown (the absorption rate under the QSD). However, we have the following relation

$$c \text{ is a minimal velocity for } r \quad\Longleftrightarrow\quad r \text{ is a maximal absorption rate for } c$$
$$\text{in (13)} \qquad\qquad\qquad\qquad \text{in (8)}$$

Observe also that $1/r$ is the mean absorption time for the QSD associated to r and hence, if r is maximal, the associated QSD is minimal. So the minimal QSD for Brownian Motion in $\mathbb{R}_+$ and the minimal velocity traveling wave of (12) are one and the same. They are given by

$$u_{c^*(r)}(x) = u_{r^*(c)} = 2r^* x e^{-\sqrt{2r^*}} = (c^*)^2 x e^{-c^* x},$$

which is the one with fastest decay at infinity.

Again, the distribution function v of u is a monotone solution to the same problem but with boundary conditions given by $v(0) = 0$, $v(+\infty) = 1$.

It is worth noting that although the solutions to (8) and (4) are not the same since they are defined in different domains, there is a natural way to identify them (and also with solutions of (3)). Given positive constants c and r, there is a solution w of (3) if and only if there is solution $\tilde{w}$ of (8). Moreover, we have

$$\lim_{x \to \infty} \frac{w(x)}{\tilde{w}(x)} = 1.$$

In this case, the proof of this statement is immediate since solutions of (8) and (4) are explicit and the relation among solutions of (4) and (3) is very well understood [25]. We conjecture that the same situation holds in much more generality.

3.3 Fleming-Viot and QSD

The Fleming-Viot process can be thought of as a microscopic version of conditioned evolutions. Its dynamics are built with a continuous time Markov process $Z = (Z_t, t \geq 0)$ taking values in the metric space $\Lambda \cup \{0\}$, that we call the *driving process*. We assume that 0 is absorbing in the sense that

$$\mathbb{P}(Z_t = 0 | Z_0 = 0) = 1, \qquad \forall t \geq 0.$$

We use τ for the absorption time

$$\tau = \inf\{t > 0 \colon Z_t \notin \Lambda\}.$$

As, before, we use P_t for the submarkovian semigroup defined by

$$P_t f(x) = \mathbb{E}_x(f(Z_t)\mathbf{1}\{\tau > t\}).$$

For a given $N \geq 2$, the Fleming-Viot process is an interacting particle system with N particles. We use $\xi_t = (\xi_t(1), \dots, \xi_t(N)) \in \Lambda^N$ to denote the state of the process, $\xi_t(i)$ denotes the position of particle i at time t. Each particle evolves according to Z and independently of the others unless it hits 0, at which time, it chooses one of the $N - 1$ particles in Λ uniformly and takes its position. The guenine definition of this process is

not obvious and in fact is not true in general. It can be easily constructed for processes with bounded jumps to 0, but is much more delicate for diffusions in bounded domains [6, 22] and it does not hold for diffusions with a strong drift close to the boundary of Λ, [7].

Here we are also interested in the empirical measure of the process

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi_t(i)}. \tag{14}$$

Its evolution is mimicking the conditioned evolution: the mass lost from Λ, is redistributed in Λ proportionally to the mass at each state. Hence, as N goes to infinity, we expect to have a deterministic limit given by the conditioned evolution of the driving process Z, i.e.

$$\mu_t^N(A) \to \mathbb{P}(Z_t \in A | \tau > t) \qquad (N \to \infty).$$

This is proved in [40] by the Martingale method in great generality. See also [23] for a proof based on sub and super-solutions and correlations inequalities. A much more subtle question is the ergodicity of the process for fixed N and the behavior of these invariant measures as $N \to \infty$. As a general principle it is expected that

Conjecture 3.2. *If the driving process Z has a Yaglom limit ν, then the Fleming-Viot process driven by Z is ergodic, with (unique) invariant measure λ^N and the empirical measures (14) distributed according to λ^N converge to ν.*

We refer to [23] for an extended discussion on this issue. This conjecture has been proved for subcritical Galton-Watson processes, where a continuum of QSDs arises [2].

We have again here a *microscopic selection principle*: whereas there exists an infinite number of QSDs, when microscopic effects are taken into account (through the dynamics of the Fleming-Viot process), there is a unique stationary distribution for the empirical measure, which selects asymptotically the minimal QSD of the macroscopic model.

When the driving process is a one dimensional Brownian motion with drift $-c$ towards the origin as in Section 2.2, the proof of the whole picture remains open, but the ergodicity of FV for fixed N has been recently proved [3, 27].

So, from [40, Theorem 2.1] we have that for every $t > 0$, μ_t^N converges as $N \to \infty$ to a measure μ_t with density $u(t, \cdot)$ satisfying (6). The open problem is to prove a similar statement in equilibrium. Observe that u is a stationary solution of (6) if and only if it solves (8) for some $r > 0$. Hence, although equations (12) and (6) are pretty different, stationary solutions to (6) coincide with traveling waves of (12).

3.4 Summing up

1. The link between $N-$BBM and Fleming Viot, in the Brownian Motion case is clear. Both processes evolve according to N independent Brownian Motions and branch into two particles. At branching times, the left-most particle is eliminated (selection) to keep the population size constant. The difference is that while $N-$BBM branches at a constant rate Nr, Fleming-Viot branches each time a particle hits 0. This explains why in the limiting equation for $N-$BBM the branching rate is data and the velocity is determined by the system while in the

hydrodynamic equation for Fleming-Viot the velocity is data and the branching rate is determined by the system.

2. The empirical measure of $N-$BBM is expected to converge in finite time intervals to the solution of (12). This is supported by the results of [16] where the same result is proved for random walks.

3. The empirical measure of Fleming-Viot driven by Brownian Motion converges in finite time intervals to the solution of (6).

4. Both N-BBM seen from the left-most particle and FV are ergodic and their empirical measure in equilibrium is expected to converge to the deterministic measure given by the minimal solution of (13).

   Note though that while for $N-$BBM r is data and minimality refers to c, for Fleming-Viot c is data and minimality refers to $1/r$ (microscopic selection principle).

5. $u(t, x) = w(x - ct)$ is a traveling wave solution of (12) if and only if w is the density of a QSD for Brownian Motion with drift $-c$ and eigenvalue $-r$.

6. c is minimal for r (in (13)) if and only if $1/r$ is minimal for c. So, we can talk of a "minimal solution of (13)", which is both a minimal QSD and a minimal velocity traveling wave.

7. The microscopic selection principle is conjectured to hold in both cases, with the same limit, but a proof is still unavailable.

4   Traveling waves and QSD for Lévy processes

Let $Z = (Z_t, t \geq 0)$ be a Lévy process with values in $\mathbb{R}$, defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and Laplace exponent $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbb{E}(e^{\theta Z_t}) = e^{\psi(\theta)t}$$

such that

$$\psi(\theta) = b\theta + \sigma^2 \frac{\theta^2}{2} + g(\theta),$$

where $b \in \mathbb{R}$, $\sigma > 0$ (which ensures that Z is non-lattice) and g is defined in terms of the jump measure Π supported in $\mathbb{R} \setminus \{0\}$ by

$$g(\theta) = \int_x (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx), \qquad \int_{\mathbb{R}} (1 \wedge x^2)\Pi(dx) < \infty.$$

Let $\theta^\star = \sup\{\theta \colon |\psi(\theta)| < \infty\}$ and recall that ψ is strictly convex on $(0, \theta^\star)$ and by monotonicity $\psi(\theta^\star) = \psi(\theta^\star-)$ and $\psi'(\theta^\star) = \psi'(\theta^\star-)$ are well defined as well as the right derivative at zero $\psi'(0) = \psi'(0+) = \mathbb{E}(Z_1)$, that we assume to be zero. We also assume that $\theta^\star > 0$. In this case we can relate ψ to the characteristic function $\Psi(\lambda) = -\log \mathbb{E}(e^{i\lambda Z_1})$ by $\psi(\theta) = -\Psi(-i\theta)$ for $0 \leq \theta < \theta^\star$.

This centered Lévy process plays the role of Brownian Motion in the previous sections.

The generator of Z applied to a function $f \in C_0^2$, the class of compactly supported functions with continuous second derivatives, gives

$$\mathcal{L}f(x) = \frac{1}{2}\sigma^2 f''(x) + bf'(x) + \int_{\mathbb{R}} (f(x + y) - f(x) - yf'(x)\mathbf{1}\{|y| \leq 1\})\Pi(dy).$$

The adjoint of $\mathcal{L}$ is also well defined in C_0^2 and has the form

$$\mathcal{L}^* f(x) = \frac{1}{2}\sigma^2 f''(x) - bf'(x) + \int_{\mathbb{R}} (f(x-y) - f(x) + yf'(x)\mathbf{1}\{|y| \le 1\})\Pi(dy).$$

Now, for $c > 0$ we consider the *drifted process* Z^c given by

$$Z_t^c = Z_t - ct$$

It is immediate to see that the Laplace exponent of Z^c is given by $\psi_c(\theta) = \psi(\theta) - c\theta$ for $\theta \in [0, \theta^\star]$, that C_0^2 is contained in the domain of the generator $\mathcal{L}_c$ of Z^c, and that $\mathcal{L}_c f = \mathcal{L}f - cf'$. Recall that the forward Kolmogorov equation for Z is given by

$$\frac{d}{dt} E^x(f(Z_t)) = \mathcal{L}f(x),$$

while the forward Kolmogorov (or Fokker-Plank) equation for the density u (which exists since $\sigma > 0$) is given by

$$\frac{d}{dt} u(t, x) = \mathcal{L}^* u(t, \cdot)(x).$$

As in the Brownian case, we consider

- A branching Lévy process (BLP) $(N_t, (\xi_t(1), \ldots, \xi_t(N_t)))$ driven by $\mathcal{L}$.
- A branching Lévy process with selection of the N rightmost particles ($N-$BLP), also driven by $\mathcal{L}$.
- A Fleming-Viot process driven by $\mathcal{L}_c$ (FV).

We focus on the last two processes. For a detailed account on BLP, we refer to [29]. Let us just mention that the KPP equation can be generalized in this context to:

$$\frac{\partial v}{\partial t} = \mathcal{L}v + rf(v), \quad x \in \mathbb{R}, \, t > 0,$$

$$v(0, x) = v_0(x), \quad x \in \mathbb{R}. \tag{15}$$

A characterization of the traveling waves as well as sufficient conditions of existence are then provided in [29].

For $N-$BLP we expect (but a proof is lacking) that the empirical measure converges to a deterministic measure whose density is the solution of the generalized Durrett-Remenik equation

Find (γ, u) such that

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t, x) &= \mathcal{L}^* u(t, x) + ru(t, x), \quad x > \gamma(t), \\
\int_{\gamma(t)}^{\infty} u(t, y)\, dy &= 1, \quad u(t, x) = 0, \quad x \le \gamma(t), \\
u(0, x) &= u_0(x), \quad x \ge 0.
\end{aligned}
\tag{16}
$$

Existence and uniqueness of solutions to this problem have to be examined, but we believe that the proof presented in [16] can be extended to this context.

We show below the existence of traveling wave solutions for this equation under mild conditions on $\mathcal{L}$ based on the existence of QSDs.

Concerning FV, it is known [40] that the empirical measure converges to the deterministic process given by the conditioned evolution of the process, which has a density for all times and verifies

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \mathcal{L}^* u(t,x) + c\frac{\partial u}{\partial x}(t,x) - u(t,x)\int_{\mathbb{R}} \mathcal{L}^* u(t,y,)dy \quad t>0, x>0, \\
u(t,0) &= u(t,+\infty) = 0, \quad t>0,
\end{aligned}
\tag{17}
$$

4.1 Traveling waves vs. QSDs

We now establish the link between traveling waves and QSD.

Proposition 4.1. *The following statements are equivalent:*

- *The probability measure ν with density w is a QSD for Z^c with eigenvalue $-r$,*

- *$u(t,x) = w(x-ct)$ is a traveling wave solution with speed c for the free-boundary problem (16), with parameter r.*

Proof. Denote $\langle f, g\rangle = \int f(x)g(x)dx$. A QSD ν for Z^c with eigenvalue $-r$ is a solution of the equation

$$
\langle \nu\mathcal{L}_c + r\nu, f\rangle = 0, \forall f \in C_0^2.
$$

Using that $\mathcal{L}_c^* f = \mathcal{L}^* f + cf'$ and writing that ν has density w, we obtain that

$$
\mathcal{L}^* w + cw' + rw = 0,
$$

which in turn is clearly equivalent to w being a traveling wave solution with speed c for (16). ∎

4.2 Minimal QSD and minimal velocities

We now study the relation between minimal traveling waves and minimal QSD. Using known results on QSDs for Lévy processes, we can describe the traveling waves for the corresponding set of equations. Since we rely on results of Kyprianou and Palmowski [30], we assume that Z is non-lattice and consider the following two classes.

Definition 4.1. *The Lévy process Z belongs to class* C1 *if there exists $0 < \theta_0 < \theta^\star$ such that $\psi'(\theta_0) > 0$ and the process $(Z, \mathbb{P}^{\theta_0})$ is in the domain of attraction of a stable law with index $1 < \alpha \le 2$, where the probability $\mathbb{P}^\theta$ is given by*

$$
\left.\frac{d\mathbb{P}_z^\theta}{d\mathbb{P}_z}\right|_{\mathcal{F}_t} = e^{\theta(Z_t - z) - \psi(\theta)t}.
$$

Definition 4.2. *The process Z is in class* C2 *if $-\infty < \psi'(\theta^\star) < 0$, and the function $x \to \Pi_{\theta^\star}\big([x,\infty)\big)$ is regularly varying at infinity with index $-\beta < -2$, where $\Pi_\theta(dx) = e^{\theta x}\Pi(dx)$.*

Recall that we are also assuming $\psi'(0) = 0$ and hence there exists a critical c^*, possibly infinity, such that for $c \leq c^*$, ψ_c attains its negative infimum at a point $\theta_c \leq \theta^\star$ with $\psi'_c(\theta_c) = 0$ and for $c > c^*$, the negative infimum is attained at $\theta^\star$. In this case we write $\theta_c = \theta^\star$. Since $\psi_c(\theta) = \psi(\theta) - c\theta$, observe that $e^{\theta(Z_t^c - z) - \psi_c(\theta)t} = e^{\theta(Z_t - z) - \psi(\theta)t}$. Hence, if Z is in class C1, then for $c \leq c^*$, Z^c is in class A in the sense of Bertoin and Doney [12, 30]. Similarly if Z is in class C2 and $c > c^*$, then Z^c is in class B in the sense of Bertoin and Doney.

The union of class A and B represents a very large family of Lévy processes including for instance Brownian motion and spectrally negative (positive) processes, as well as many others. The following theorem is proved in [30].

Theorem 4.1 (Kyprianou and Palmowski, [30]). *Assume $c \leq c^*$ and Z is in class* C1 *or $c > c^*$ and Z is in class* C2. *Then the Yaglom limit of Z^c exists and is given by*

$$\nu(dx) = \theta_0 \kappa_{\theta_c}(0, \theta_c) e^{-\theta_c x} V_{\theta_c}(x) \, dx.$$

Here κ_θ and V_θ are respectively the Laplace exponent of the ascending ladder process and the renewal function of the ladder heights process corresponding to $(Z^c, \mathbb{P}^\theta)$.

The proof is based on a careful control of the asymptotics of the process as $t \to \infty$ that in particular yields

$$\mathbb{P}_x(\tau > t) \sim H(x, \theta_c)\ell(t)e^{\psi_c(\theta_c)t}. \tag{18}$$

Here H is a function that depends on the characteristic exponent of the process and ℓ is a function regularly varying at infinity. This allows us to prove the following

Proposition 4.2. *The probability measure ν defined in Theorem 4.1 is the minimal QSD.*

Proof. Recall that under a QSD, the hitting time of zero is exponentially distributed. Since ν is the Yaglom limit, for any bounded function f

$$\int f d\nu = \lim_{t \to \infty} \frac{\mathbb{E}_x(f(X_t), \tau > t)}{\mathbb{P}_x(\tau > t)}.$$

Applying this to $f(y) = \mathbb{P}_y(\tau > s)$ and using Corollary 4 in [30], one obtains that

$$\mathbb{P}_\nu(\tau > s) = \lim_{t \to \infty} \frac{\mathbb{P}_x(\tau > t + s)}{\mathbb{P}_x(\tau > t)} = \exp(\psi_c(\theta_c)s).$$

Hence the parameter of ν equals $-\psi_c(\theta_c)$. If ν is not minimal, there exists another QSD $\tilde{\nu}$, with parameter $\tilde{r} > -\psi_c(\theta_c)$. Then $\mathbb{E}_{\tilde{\nu}}(e^{-\psi_c(\theta_c)\tau}) < \infty$ and as a consequence there exists $x > 0$ such that $\mathbb{E}_x(e^{-\psi_c(\theta_c)\tau}) < \infty$, but this contradicts (18). ∎

Let $\mathcal{C}$ be the subset of $\mathbb{R}^2$ such that $(c, r) \in \mathcal{C}$ if and only if there exists a QSD ν for Z^c with eigenvalue $-r$. Proposition 4.1 states that this set coincides with the set of pairs (c, r) such that there exists a traveling wave for (16) with velocity c.

Definition 4.3. *We say that r is maximal for c if $r = \max\{r' \colon (c, r') \in \mathcal{C}\}$. In the same way, c is minimal for r if $c = \min\{c' \colon (c', r) \in \mathcal{C}\}$.*

Proposition 4.3. *Under the same hypotheses of Theorem 4.1 we have that c is minimal for r if and only if r is maximal for c.*

Proof. For each $c > 0$, the maximal absorption rate is given by $r(c) = -\psi_c(\theta_c)$. The proof follows by observing that the function $c \mapsto \psi_c(\theta_c)$ is strictly decreasing and continuous. ∎

5 Conclusions and general conjectures

We emphasized the direct links between quasi stationary distributions QSDs for space invariant Markov processes in $\mathbb{R}$ and traveling waves of Durrett-Remenik equation, that are closely related to the generalized F-KPP equation. We proved also that minimal QSDs correspond to minimal velocity traveling waves. As a general fact both macroscopic and microscopic selection principles are expected to hold for QSDs and for traveling waves. They have been proved for a series of models. This suggests that the selection principles in front propagation and in QSDs are one and the same. For random walks in $\mathbb{R}$, the microscopic selection principle is an open problem in both cases (traveling waves and QSD).

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