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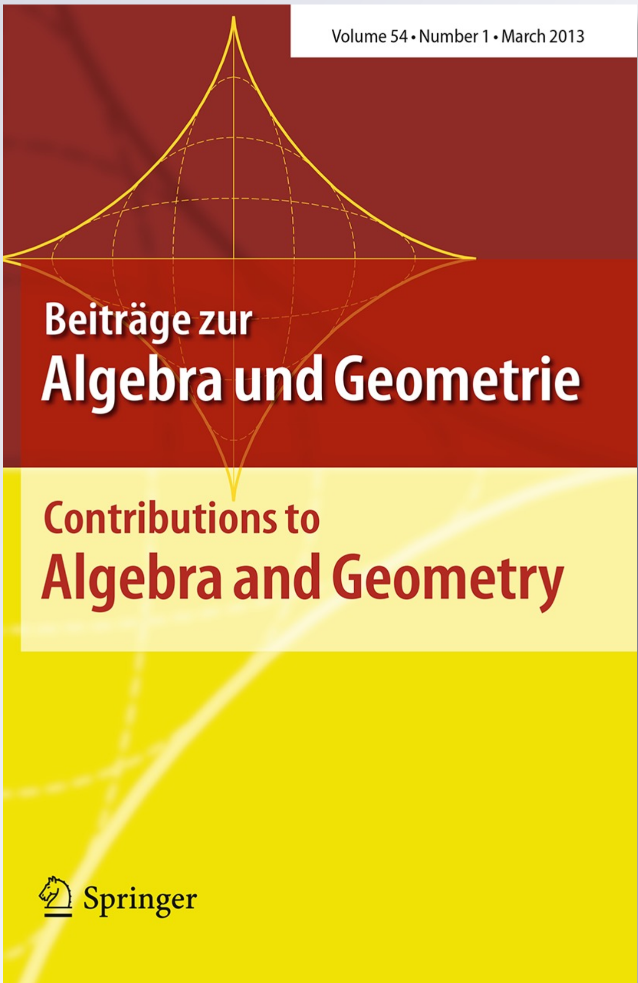

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ORIGINAL PAPER

Non-homogeneous combinatorial manifolds

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Abstract In this paper we extend the classical theory of combinatorial manifolds to the non-homogeneous setting. NH-manifolds are polyhedra which are locally like Euclidean spaces of varying dimensions. We show that many of the properties of classical manifolds remain valid in this wider context. NH-manifolds appear naturally when studying Pachner moves on (classical) manifolds. We introduce the notion of NH-factorization and prove that PL-homeomorphic manifolds are related by a finite sequence of NH-factorizations involving NH-manifolds.

Keywords Simplicial complexes · Combinatorial manifolds · Collapses · Shellability · Pachner moves

Mathematics Subject Classification (2000) 52B70 · 52B22 · 57Q10 · 57Q15

1 Introduction

The notion of manifold (piecewise linear, topological, differentiable) is central in mathematics. An n-manifold is an object which is locally like the Euclidean space $\mathbb{R}^n$. Concretely, in the piecewise linear setting a PL-manifold of dimension n is a polyhedron in which every point has a (closed) neighborhood which is a PL-ball of dimension n.

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The theory of combinatorial manifolds (which are the triangulations of PL-manifolds) has been widely developed during the last 90 years. J.W. Alexander's Theorem on regular expansions, Newman's result on the complement of an n-ball in an n-sphere, Whitehead's Regular Neighborhood theory and the s-cobordism theorem are some of its most important advances (see Alexander (1930); Glaser (1970); Hudson (1969); Lickorish (1999); Rourke and Sanderson (1972)). More recently Pachner (1991) studied a set of elementary combinatorial operations or *moves*, and showed that any combinatorial manifold can be transformed into any other PL-homeomorphic one by using these moves (see also Lickorish 1999).

It is well known that any combinatorial n-manifold is a homogeneous (or pure) simplicial complex, which means that all the maximal simplices have the same dimension. It is natural to ask whether it is possible to extend the theory of combinatorial manifolds to the non-homogeneous context. More concretely, the main goal of this article is to investigate the properties of those polyhedra which are locally like Euclidean spaces of varying dimensions (see Figs. 1, 2 below). In this paper we introduce the theory of *non-homogeneous manifolds* or NH-manifolds, for short. We will show that many of the basic properties of (classical) manifolds are also satisfied in this much wider setting.

We investigate shellability in the non-homogeneous context. It is well-known that any shellable complex is homotopy equivalent to a wedge of spheres and that the only shellable manifolds are balls and spheres (see Björner et al. 1999; Kozlov 2008). We prove that every shellable NH-manifold is in particular an NH-bouquet, which extends the classical result for manifolds. We also study the notion of regular expansion for NH-manifolds and prove a generalization of Alexander's Theorem.

Non-homogeneous manifolds appear naturally when studying Pachner moves between manifolds. We introduce the notion of NH-factorization and prove that any two PL-homeomorphic manifolds (with or without boundary) are related by a finite sequence of factorizations involving NH-manifolds. When the manifolds are closed, the converse also holds.

2 Preliminaries

We start by fixing some notation and terminology. In this paper, all the simplicial complexes that we deal with are assumed to be finite. If a simplex σ is a face of a simplex τ, we will write $\sigma < \tau$ and when σ is an immediate face we write $\sigma \prec \tau$. A *principal* or *maximal* simplex in K is a simplex which is not a proper face of any other simplex of K and a *ridge* in K is an immediate face of a maximal simplex. A complex is said to be *homogeneous* of dimension n if all of its principal simplices have dimension n. The boundary ∂K of an n-homogeneous complex K is the subcomplex generated by the mod 2 union of the $(n-1)$-simplices. The set of vertices of a complex K will be denoted by V_K.

The join of two simplices σ, τ with $\sigma \cap \tau = \emptyset$ will be denoted by $\sigma * \tau$. Also $K * L$ will denote the join of the complexes K and L. Given a simplex $\sigma \in K$, $lk(\sigma, K)$ will denote its *link*, which is the subcomplex $lk(\sigma, K) = \{\tau \in K : \tau \cap \sigma = \emptyset,\ \tau * \sigma \in K\}$,

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and $st(\sigma, K) = \sigma * lk(\sigma, K)$ will denote the (closed) *star* of σ in K. The union of two complexes K, L will be denoted by $K + L$.

Following Glaser (1970), arbitrary subdivisions of K will be denoted by αK, $\beta K, \ldots$ Derived subdivisions will be denoted by δK and the barycentric subdivision by K', as usual. If $\sigma \in K$ and $a \in \overset{\circ}{\sigma}$, the interior of σ, then $(\sigma, a)K$ will denote the elementary subdivision of K by starring σ in a; i.e. the replacing of $st(\sigma, K)$ by $a * \partial\sigma * lk(\sigma, K)$. A *stellar subdivision* sK of K is a finite sequence of elementary starrings. The operation inverse to an elementary starring is called an *elementary weld* and denoted by $(\sigma, a)^{-1}K$. Two complexes K and L are *stellar equivalent* if they are related by a sequence of starrings, welds and (simplicial) isomorphisms. In this case we write $K \sim L$. It is well known that the combinatorial and the stellar theories are equivalent (see for example Glaser 1970; Lickorish 1999), and therefore $K \sim L$ if and only if they are PL-homeomorphic. A class of complexes will be called *PL-closed* if it is closed under PL-homeomorphisms.

We recall now the basic definitions and properties of combinatorial manifolds. For a comprehensive exposition of the theory of combinatorial manifolds we refer the reader to Glaser (1970); Lickorish (1999) and Rourke and Sanderson (1972).

Δ^n will denote the n-simplex. A combinatorial n-ball is a complex which is PL-homeomorphic to Δ^n. A combinatorial n-sphere is a complex PL-homeomorphic to $\partial\Delta^{n+1}$. By convention, $\emptyset = \partial\Delta^0$ is considered a sphere of dimension -1. A combinatorial n-manifold is a complex M such that for every $v \in V_M$, $lk(v, M)$ is a combinatorial $(n-1)$-ball or $(n-1)$-sphere. It is easy to verify that n-manifolds are homogeneous complexes of dimension n. It is well known that the link of any simplex in a manifold is also a ball or a sphere and that the class of n-manifolds is PL-closed. It follows that combinatorial balls and spheres are combinatorial manifolds.

The boundary ∂M can be regarded as the set of simplices whose links are combinatorial balls. By a classical result of Newman (1926) (see also Glaser 1970; Hudson 1969; Lickorish 1999), if S is a combinatorial n-sphere containing a combinatorial n-ball B, then the closure $\overline{S - B}$ is a combinatorial n-ball.

Some global properties of combinatorial manifolds can be stated in terms of *pseudo manifolds*. An n-pseudo manifold is an n-homogeneous complex K satisfying the following two properties: (a) for every $(n-1)$-simplex σ, $lk(\sigma, K)$ is a combinatorial 0-ball or 0-sphere [or equivalently, every $(n-1)$-simplex is a face of at most two n-simplices], and (b) K is strongly connected, i.e. given two n-simplices σ, σ', there exists a sequence of n-simplices $\sigma = \sigma_0, \ldots, \sigma_k = \sigma'$ such that $\sigma_i \cap \sigma_{i+1}$ is $(n-1)$-dimensional for all $i = 0, \ldots, k-1$. It is well known that any connected combinatorial n-manifold, or more generally, any triangulated homological manifold, is an n-pseudo manifold.

A simplex τ of a complex K is said to be *collapsible* in K if it has a free face σ, i.e. a proper face which is not a face of any other simplex of K. Note that, in particular, τ is a maximal simplex and σ is a ridge. In this situation, the operation which transforms K into $K - \{\tau, \sigma\}$ is called an *elementary* (*simplicial*) *collapse*, and it is usually denoted by $K \searrow^e K - \{\tau, \sigma\}$. The inverse operation is called an *elementary* (*simplicial*) *expansion*. If there is a sequence $K \searrow^e K_1 \searrow^e \cdots \searrow^e L$ we say that K

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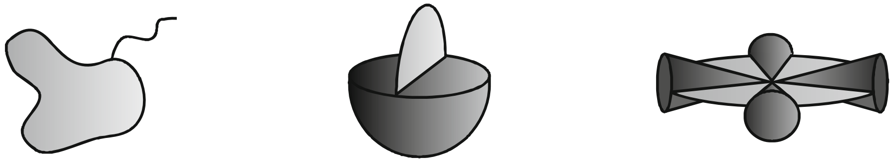

Fig. 1 The neighbourhoods of the points in the underlying topological space determined by an NH-manifold

collapses to L (or equivalently, L *expands* to K) and write $K \searrow L$ or $L \nearrow K$ respectively. A complex K is said to be *collapsible* if it has a subdivision which collapses to a single vertex. A celebrated theorem of J.H.C. Whitehead states that collapsible combinatorial n-manifolds are combinatorial n-balls (Glaser 1970, Corollary III.17).

A more general type of collapse is the *geometrical collapse*. If $K = K_0 + B^n$, where B^n is a combinatorial n-ball and $B^n \cap K_0 = B^{n-1}$ is a combinatorial $(n-1)$-ball contained in the boundary of B^n, then the move $K \to K_0$ it called an *elementary geometrical collapse*. A finite sequence of elementary geometrical collapses (resp. expansions) is a geometrical collapse (resp. expansion).

If M is an n-manifold, an elementary geometrical expansion $M \to N = M + B^n$ such that $M \cap B^n \subset \partial M$ is called an *elementary regular expansion*. By a Theorem of Alexander, an elementary regular expansion is a PL-equivalence (see Glaser 1970; Lickorish 1999). A sequence of elementary regular expansions (resp. collapses) is a regular expansion (resp. collapse). Note that the dimension of all the balls being expanded in such a sequence must be n.

If M is a combinatorial n-manifold with boundary and there is an n-simplex $\eta = \sigma * \tau \in M$ with $\dim \sigma, \dim \tau \geq 0$ such that $\sigma \in \overset{\circ}{M}$, the interior of M, and $\partial\sigma * \tau \subset \partial M$, then the move $M \overset{sh}{\longrightarrow} M_1 = \overline{M - \sigma * \tau}$ is called an *elementary shelling*. This operation produces again a combinatorial n-manifold. The inverse operation is called an *inverse shelling*. Pachner (1991) showed that two combinatorial n-manifolds with non-empty boundary are PL-homeomorphic if and only if one can obtain one from the other by a sequence of elementary shellings, inverse shellings and isomorphisms.

3 NH-manifolds

A *non-homogeneous manifold*, or NH-*manifold* for short, is a simplicial complex whose underlying topological space locally looks as in Fig. 1. We will define such complexes by induction on the dimension. We need first a definition.

Definition 3.1 Let K be a complex. A subcomplex $L \subseteq K$ is said to be *top generated* in K if it is generated by principal simplices of K, i.e. every maximal simplex of L is also maximal in K.

Definition 3.2 An NH-*manifold* (resp. NH-*ball*, NH-*sphere*) of dimension 0 is a manifold (resp. ball, sphere) of dimension 0. An NH-sphere of dimension -1 is, by convention, the empty set. For $n \geq 1$, we define by induction

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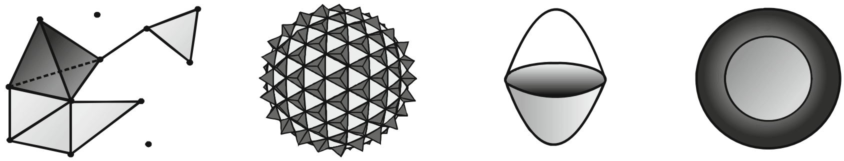

Fig. 2 From *left* to *right*: a non-connected NH-manifold of dimension 3; a homotopy 2-sphere triangulated by triangles and tetrahedra; the underlying space of an NH-sphere of dimension 2 and homotopy dimension 1; the underlying space of an NH-ball of dimension 3 constructed by gluing a solid torus with a 2-disk

- An NH-manifold of dimension n is a complex M of dimension n such that $lk(v, M)$ is an NH-ball of dimension $0 \leq k \leq n-1$ or an NH-sphere of dimension $-1 \leq k \leq n-1$ for all $v \in V_M$.
- An NH-ball of dimension n is a collapsible NH-manifold of dimension n.
- An NH-sphere of dimension n and homotopy dimension k is an NH-manifold S of dimension n such that there exist a top generated NH-ball B of dimension n and a top generated combinatorial k-ball L such that $B + L = S$ and $B \cap L = \partial L$. We say that $S = B + L$ is a *decomposition* of S.

Note that the definition of NH-ball is motivated by Whitehead's theorem on regular neighborhoods and the definition of NH-sphere by that of Newman's (see Glaser 1970; Lickorish 1999).

Remark 3.3 An NH-ball of dimension 1 is the same as a 1-ball. An NH-sphere of dimension 1 is either a 1-sphere (if the homotopy dimension is 1) or the disjoint union of a point and a combinatorial 1-ball (if the homotopy dimension is 0). In general, an NH-sphere of homotopy dimension 0 consists of a disjoint union of a point and an NH-ball. These are the only NH-spheres which are not connected.

Example 3.4 Figure 2 shows some examples of NH-manifolds and the underlying spaces of NH-manifolds.

Remark 3.5 Note that the decomposition of an NH-sphere need not be unique. However the homotopy dimension of the NH-sphere is well defined since the geometric realization of an NH-sphere of homotopy dimension k is a homotopy k-sphere.

We show now that the notion of NH-manifold is in fact an extension of the concept of combinatorial manifold to the non-homogeneous context.

Theorem 3.6 *A complex K is a homogeneous NH-manifold (resp. NH-ball, NH-sphere) of dimension n if and only if it is a combinatorial n-manifold (resp. n-ball, n-sphere).*

Proof Let $n \geq 1$. It is easy to see that the result holds for NH-manifolds of dimension n provided that it holds for NH-balls and NH-spheres of dimension less than n. Then it suffices to prove that the result holds for NH-balls and NH-spheres of dimension n if it holds for NH-manifolds of dimension n.

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For NH-balls the result is clear by the theorem of Whitehead (Glaser 1970, Corollary III.17). Suppose now that $S = B + L$ is a homogeneous NH-sphere of dimension n. It follows that B and L are combinatorial n-balls. Take $\sigma \in \partial L$ a maximal simplex. Since $lk(\sigma, S) = \{v\} + lk(\sigma, B)$ for some vertex $v \in L$ and S is an n-pseudo manifold, then $lk(\sigma, B)$ is also a single vertex. It follows that $\sigma \in \partial B$. Since both ∂L and ∂B are combinatorial $(n-1)$-spheres, this implies that $\partial L = \partial B$. This proves that S is a combinatorial n-sphere. Conversely, any n-simplex of a combinatorial n-sphere can play the role of L in its decomposition as an NH-sphere. The result then follows from Newman's Theorem. □

Following the same reasoning of (Glaser 1970, Theorem II.2) for combinatorial manifolds, one can show that the links of all simplices in an NH-manifold behave nicely. Concretely:

Proposition 3.7 *Let M be an NH-manifold of dimension n and let $\sigma \in M$ be a k-simplex. Then $lk(\sigma, M)$ is an NH-ball or an NH-sphere of dimension less than $n - k$.*

The property stated in the preceding proposition is often called *regularity*.

In order to show that the class of NH-manifolds is PL-closed, we will need the following lemma, which is somehow an analogue of (Glaser 1970, Proposition II.1). This result will be generalized in Corollary 3.10 and in Theorem 3.13.

Lemma 3.8 *Let K be an NH-ball or an NH-sphere and let σ be a simplex disjoint from K. Then,*

(1) *$\sigma * K$ is an NH-ball.*
(2) *$\partial\sigma * K$ is an NH-ball (if K is an NH-ball) or an NH-sphere (if K is an NH-sphere).*

Proof For the first part of the lemma, we proceed by double induction. Suppose first that $\dim \sigma = 0$, i.e. σ is a vertex v, and that the result holds for NH-balls and NH-spheres K of dimension less than n. Note that $v * K \searrow 0$, so we only have to verify that $v * K$ is an NH-manifold. Take $w \in V_K$. Since $lk(w, v * K) = v * lk(w, K)$, by induction applied to $lk(w, K)$, it follows that $lk(w, v * K)$ is an NH-ball. On the other hand, $lk(v, v * K) = K$, which is an NH-ball or an NH-sphere by hypothesis. This shows that $v * K$ is an NH-manifold and proves the case $\dim \sigma = 0$. Suppose now that $\dim \sigma = k \geq 1$. Write $\sigma = \tau * v$ for some $v \in \sigma$. Since $\sigma * K = \tau * (v * K)$, the results follows by induction applied to v and τ.

For the second part of the lemma, suppose that $\dim \sigma = k \geq 1$ and let K be an NH-ball or an NH-sphere of dimension n. It is easy to see that the result is valid if $n = 0$. Suppose now that $n \geq 1$ and that the result holds for $t < n$. For any vertex $v \in \partial\sigma * K$, we have

$$lk(v, \partial\sigma * K) = \begin{cases} \partial\sigma * lk(v, K) & v \notin \partial\sigma \\ lk(v, \partial\sigma) * K & v \in \partial\sigma \end{cases}$$

In the first case, by induction on n, it follows that $lk(v, \partial\sigma * K)$ is an NH-ball or sphere. In the second case, we use induction on k [note that $lk(v, \partial\sigma) = \partial lk(v, \sigma)$].

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This proves that $\partial\sigma * K$ is an NH-manifold. Now, if K is an NH-ball then $\partial\sigma * K \searrow 0$ and $\partial\sigma * K$ is again an NH-ball. If K is an NH-sphere write $K = B + L$ with B an NH-ball, L a combinatorial ball and $B \cap L = \partial L$. Since $\partial(\partial\sigma * L) = \partial\sigma * \partial L = \partial\sigma * B \cap \partial\sigma * L$, then $\partial\sigma * K = \partial\sigma * B + \partial\sigma * L$ is an NH-sphere by the previous case. This concludes the proof. □

In particular, from Lemma 3.8 we deduce that M is an NH-manifold if and only if $st(v, M)$ is an NH-ball for all $v \in V_M$.

Theorem 3.9 *The classes of NH-manifolds, NH-balls and NH-spheres are PL-closed.*

Proof It suffices to prove that K is an NH-manifold (resp. NH-ball, NH-sphere) if and only if any starring $(\tau, a)K$ is an NH-manifold (resp. NH-ball, NH-sphere). We suppose first that the result is valid for NH-manifolds of dimension n and prove that it is valid for NH-balls and NH-spheres of the same dimension. If $(\tau, a)K$ is an NH-ball of dimension n then K is also an NH-ball since it is an NH-manifold with $\alpha((\tau, a)K) \searrow 0$ for some subdivision α. On the other hand, if K is an NH-manifold of dimension n with $\alpha K \searrow 0$, by (Glaser 1970, Theorem I.2) we can find a stellar subdivision δ and an arbitrary subdivision β such that $\beta((\tau, a)K) = \delta(\alpha K)$. Since stellar subdivisions preserve collapses, $(\tau, a)K$ is collapsible and hence an NH-ball. Now, if K is an NH-sphere of dimension n with decomposition $B + L$ then the result holds by the previous case and the following identities.

$$(\tau, a)K = \begin{cases} (\tau, a)B + L, \text{ with } (\tau, a)B \cap L = \partial L & a \in B - L \\ B + (\tau, a)L, \text{ with } B \cap (\tau, a)L = \partial L & a \in L - B \\ (\tau, a)B + (\tau, a)L, \text{ with } (\tau, a)B \cap (\tau, a)L = (\tau, a)\partial L & \\ & a \in B \cap L = \partial L \end{cases}$$

Note that $(\tau, a)\partial L = \partial(\tau, a)L$. The converse follows by replacing (τ, a) with $(\tau, a)^{-1}$.

We assume now that the result is valid for NH-balls and NH-spheres of dimension n and prove that it is valid for NH-manifolds of dimension $n+1$. Suppose K is an NH-manifold of dimension $n+1$ and take $v \in (\tau, a)K$. If $v \neq a$ then $lk(v, (\tau, a)K)$ is PL-homeomorphic to an elementary starring of $lk(v, K)$. The inductive hypothesis on $lk(v, K)$ shows that $lk(v, (\tau, a)K)$ is also an NH-ball or NH-sphere. On the other hand, $lk(a, (\tau, a)K) = \partial\tau * lk(\tau, K)$, which is an NH-ball or an NH-sphere by Lemma 3.8. Once again, the converse follows by replacing (τ, a) with $(\tau, a)^{-1}$. □

Corollary 3.10 *Let B be a combinatorial n-ball, S a combinatorial n-sphere and K an NH-ball or NH-sphere. Then,*

(1) *$B * K$ is an NH-ball.*
(2) *$S * K$ is an NH-ball (if K is an NH-ball) or an NH-sphere (if K is an NH-sphere).*

Proof Follows from Lemma 3.8 and Theorem 3.9. □

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Proposition 3.11 *Let K be an n-dimensional complex and let B be a combinatorial r-ball. Suppose $K + B$ is an NH-manifold such that*

(1) $K \cap B \subset \partial B$ *is homogeneous of dimension* $r - 1$ *and*
(2) $lk(\sigma, K)$ *is collapsible for all* $\sigma \in K \cap B$

Then, K is an NH-manifold.

Proof We show first that $K, B \subset K + B$ are top generated. Clearly, B is top generated since it intersects K in dimension $r - 1$. On the other hand, a principal simplex in K which is not principal in $K + B$ must lie in $K \cap B$. Then, by hypothesis, it has a collapsible link in K. But this contradicts the fact that it is principal in K. Therefore $K, B \subset K + B$ are top generated and, in particular, $r \leq n$.

We prove the result by induction on r. For $r = 0$ the result is trivial. Let $r \geq 1$ and $v \in K$. If $v \notin B$ then $lk(v, K) = lk(v, K+B)$, which is an NH-ball or NH-sphere by hypothesis. Suppose now that $v \in K \cap B$. If $r = 1$, then $lk(v, K+B) = lk(v, K)+*$. It follows that $lk(v, K)$ is an NH-ball. Suppose $r \geq 2$ (and hence $n \geq 2$). We will see that the pair $lk(v, K), lk(v, B)$ also satisfies the conditions of the theorem. Note that $lk(v, K) + lk(v, B) = lk(v, K + B)$ is an NH-manifold by hypothesis and $lk(v, K) \cap lk(v, B) = lk(v, K \cap B)$ is homogeneous of dimension $r - 2$. On the other hand, if $\eta \in lk(v, K) \cap lk(v, B)$ then, $v * \eta \in K \cap B$, so $lk(\eta, lk(v, K)) = lk(v*\eta, K)$ is collapsible. By induction, it follows that $lk(v, K)$ is an NH-manifold, and, since it is also collapsible, it is an NH-ball. This shows that K is an NH-manifold. □

Lemma 3.12 *Suppose $S_1 = G_1 + L_1$ and $S_2 = G_2 + L_2$ are two disjoint NH-spheres. Then, $G_1 * S_2 + L_1 * G_2$ is collapsible.*

Proof Since G_1 and G_2 are collapsible, there exist subdivisions ϵ_1, ϵ_2 such that $\epsilon_1 G_1 \searrow 0$ and $\epsilon_2 G_2 \searrow 0$. We can extend these subdivisions to S_1 and S_2 and then suppose without loss of generality that $G_1 \searrow 0$ and $G_2 \searrow 0$. Note that

$$G_1 * S_2 \cap L_1 * G_2 = \partial L_1 * G_2.$$

We will show that some subdivision of $L_1 * G_2$ collapses to (the induced subdivision of) $\partial L_1 * G_2$. Let α be an arbitrary subdivision of L_1 and δ a derived subdivision of Δ^r such that $\alpha L_1 = \delta \Delta^r$. Then, $\alpha(L_1 * G_2) = \delta(\Delta^r * G_2)$. Since $G_2 \searrow 0$, then $\Delta^r * G_2 \searrow \partial \Delta^r * G_2$ (Glaser 1970, Corollary III.4). Therefore

$$\alpha(L_1 * G_2) = \delta(\Delta^r * G_2) \searrow \delta(\partial \Delta^r * G_2) = \alpha(\partial L_1 * G_2).$$

We extend α to $(G_1 * S_2 + L_1 * G_2)$ and then

$$\alpha(G_1 * S_2 + L_1 * G_2) = \alpha(G_1 * S_2) + \alpha(L_1 * G_2) \searrow \alpha(G_1 * S_2) + \alpha(\partial L_1 * G_2) = \alpha(G_1 * S_2).$$

By (Glaser 1970, Theorem III.6) there is a stellar subdivision s such that $s\alpha G_1 \searrow 0$ and therefore

$$s\alpha(G_1 * S_2 + L_1 * G_2) \searrow s\alpha(G_1 * S_2) = s\alpha G_1 * s\alpha S_2 \searrow 0.$$

□

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Theorem 3.13 *Let B_1, B_2 be NH-balls and S_1, S_2 be NH-spheres. Then,*

(1) *$B_1 * B_2$ and $B_1 * S_2$ are NH-balls.*
(2) *$S_1 * S_2$ is an NH-sphere.*

Proof Let K_1 represent B_1 or S_1 and let K_2 represent B_2 or S_2. We must show that $K_1 * K_2$ is an NH-ball or an NH-sphere. We proceed by induction on $s = \dim K_1 + \dim K_2$. If $s = 0, 1$ the result follows from Lemma 3.8. Let $s \geq 2$. We show first that $K_1 * K_2$ is an NH-manifold. Let $v \in K_1 * K_2$ be a vertex. Then,

$$lk(v, K_1 * K_2) = \begin{cases} lk(v, K_1) * K_2 & v \in K_1 \\ K_1 * lk(v, K_2) & v \in K_2 \end{cases}$$

Since $\dim lk(v, K_1) + \dim K_2 = \dim K_1 + \dim lk(v, K_2) = s - 1$, then by induction, $lk(v, K_1 * K_2)$ is an NH-ball or an NH-sphere. It follows that $K_1 * K_2$ is an NH-manifold. Now, if $K_1 = B_1$ or $K_2 = B_2$, then $K_1 * K_2 \searrow 0$ and $K_1 * K_2$ is an NH-ball.

We prove now that $S_1 * S_2$ is an NH-sphere. Decompose $S_1 = G_1 + L_1$ and $S_2 = G_2 + L_2$. Note that $S_1 * S_2 = (G_1 * S_2 + L_1 * G_2) + L_1 * L_2$ and that $(G_1 * S_2 + L_1 * G_2) \cap (L_1 * L_2) = \partial(L_1 * L_2)$, then it suffices to show that $(G_1 * S_2 + L_1 * G_2)$ is an NH-ball. By Lemma 3.12 it is collapsible, so we only need to check that $(G_1 * S_2 + L_1 * G_2)$ is an NH-manifold. In order to prove this, we apply Proposition 3.11 to the complex $G_1 * S_2 + L_1 * G_2$ and the combinatorial ball $L_1 * L_2$. The only non-trivial fact is that $lk(\sigma, G_1 * S_2 + L_1 * G_2)$ is collapsible for $\sigma \in \partial(L_1 * L_2)$. To see this, take $\eta \in \partial(L_1 * L_2) = \partial L_1 * L_2 + L_1 * \partial L_2$ and write $\eta = l_1 * l_2$ with $l_1 \in L_1, l_2 \in L_2$. Then,

$$lk(\eta, G_1 * S_2 + L_1 * G_2) = lk(l_1, G_1) * lk(l_2, S_2) + lk(l_1, L_1) * lk(l_2, G_2).$$

Now, if $l_1 \in L_1 - \partial L_1$ then $lk(l_1 * l_2, G_1 * S_2) = \emptyset$ and $lk(\eta, G_1 * S_2 + L_1 * G_2) = lk(l_1, L_1) * lk(l_2, G_2) \searrow 0$. By a similar argument, the same holds if $l_2 \in L_2 - \partial L_2$. If $l_1 \in \partial L_1$ and $l_2 \in \partial L_2$ then $lk(l_1, S_1) = lk(l_1, G_1) + lk(l_1, L_1)$ and $lk(l_2, S_2) = lk(l_2, G_2) + lk(l_2, L_2)$ are NH-spheres (by Lemma 4.8). By Lemma 3.12, it follows that $lk(\eta, G_1 * S_2 + L_1 * G_2)$ is also collapsible. By Proposition 3.11, we conclude that $G_1 * S_2 + L_1 * G_2$ is an NH-manifold. □

The following result will be used in the next section. First we need a definition.

Definition 3.14 Two principal simplices $\sigma, \tau \in M$ are said to be *adjacent* if the intersection $\tau \cap \sigma$ is an immediate face of σ or τ.

Lemma 3.15 *Let M be a connected NH-manifold. Then*

(1) *For each ridge $\sigma \in M$, $lk(\sigma, M)$ is either a point or an NH-sphere of homotopy dimension* 0.
(2) *Given any two principal simplices $\sigma, \tau \in M$, there exists a sequence $\sigma = E_1, \ldots, E_s = \tau$ of principal simplices of M such that E_i is adjacent to E_{i+1} for every $1 \leq i \leq s - 1$.*

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By analogy with the homogeneous case, a complex K satisfying properties (1) and (2) of this lemma will be called an *NH-pseudo manifold*. For more details on (homogeneous) pseudo manifolds we refer the reader to Munkres (1984) (see also Spanier 1966). The proof of Lemma 3.15 will follow from the next result.

Lemma 3.16 *If K is a connected complex such that $st(v, K)$ is an NH-pseudo manifold for all $v \in V_K$ then K is an NH-pseudo manifold.*

Proof We will show that K satisfies properties (1) and (2) of Lemma 3.15. Let $\sigma \in K$ be a ridge and let $v \in \sigma$ be any vertex. Then σ is also a ridge in $st(v, K)$ and $lk(\sigma, K) = lk(\sigma, st(v, K))$. Therefore K satisfies property (1).

Let $\nu, \tau \in K$ be maximal simplices and let $v \in \nu,\ w \in \tau$. Take an edge path from v to w. We will prove that K satisfies property (2) by induction on the length r of the edge path. If $r = 0$, then $v = w$. In this case, $\nu, \tau \in st(v, K)$ and the results follows by hypothesis. Suppose now that $\psi_1, \dots, \psi_r$ is an edge path from v to w of length $r \geq 1$. Take maximal simplices E_i such that $\psi_i \leq E_i$. Note that $E_1 \cap E_2$ contains the vertex $\psi_1 \cap \psi_2$. By hypothesis, $st(\psi_1 \cap \psi_2, K)$ satisfies property (2) and therefore we can join E_1 with E_2 by a sequence of adjacent maximal simplices. Now the result follows by induction. □

Proof of Lemma 3.15 We proceed by induction on the dimension n of M. By Lemma 3.16, it suffices to prove that $st(v, M)$ is an NH-pseudo manifold for every vertex v. The case $n = 0$ is trivial. Suppose that $n \geq 1$ and that the result is valid for $k \leq n - 1$. Now, if $lk(v, M)$ is an NH-ball or a connected NH-sphere then, by induction, it is an NH-pseudo manifold. It follows that $st(v, M)$ is also an NH-pseudo manifold since it is a cone of an NH-pseudo manifold. In the other case, $lk(v, M)$ is an NH-sphere of homotopy dimension 0 of the form $B + *$, for some NH-ball B. Since vB is an NH-pseudo manifold, it follows that $st(v, M)$ is also an NH-pseudo manifold. □

4 Boundary, pseudo boundary and the anomaly complex

The concept of boundary is not defined in the non-homogeneous setting and, in fact, it is not clear what a boundary of a general complex could be. However, the characterization of the boundary of combinatorial manifolds allows us to extend this notion to the class of NH-manifolds.

Definition 4.1 Let M be an NH-manifold. The *pseudo boundary* of M is the set of simplices $\tilde{\partial} M$ whose links are NH-balls. The *boundary* of M is the subcomplex ∂M spanned by $\tilde{\partial} M$. In other words, ∂M is the closure $\overline{\tilde{\partial} M}$.

It is clear that $\tilde{\partial} M = \partial M$ for any combinatorial manifold M. We will see that, in fact, this is the only case where this happens. The result will follow from the next lemma.

Lemma 4.2 *Let M be an NH-manifold and let $\sigma \in M$. If σ is a face of two principal simplices of different dimensions then $\sigma \in \partial M$.*

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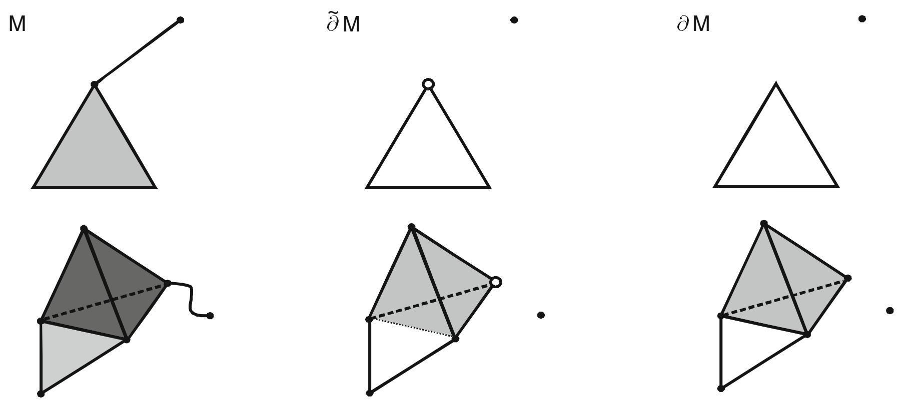

Fig. 3 Boundary and pseudo boundary

Proof Let $\tau_1 = \sigma * \eta_1$ and $\tau_2 = \sigma * \eta_2$ be principal simplices such that $\dim \tau_1 \neq \dim \tau_2$. By Lemma 3.15 we may assume that τ_1 and τ_2 are adjacent. Let $\rho = \tau_1 \cap \tau_2$ and suppose $\rho \prec \tau_1$. Then, $lk(\rho, M)$ is an NH-sphere of homotopy dimension 0 with decomposition $lk(\rho, M - \tau_1) + *$. Since $\dim lk(\rho, M - \tau_1) \geq 1$ then $\tilde{\partial} lk(\rho, M - \tau_1) = \tilde{\partial} lk(\rho, M)$ is non-empty. For any simplex ν in $\tilde{\partial} lk(\rho, M)$, $\nu * \rho \in \tilde{\partial} M$. Thus $\sigma \in \partial M$. □

Proposition 4.3 *If M is a connected NH-manifold such that $\tilde{\partial} M = \partial M$ then M is a combinatorial manifold. In particular, NH-manifolds without boundary (or pseudo boundary) are combinatorial manifolds.*

Proof If M is non-homogeneous, by Lemma 3.15 there exist two adjacent principal simplices τ_1, τ_2 of different dimensions. By Lemma 4.2, $\rho = \tau_1 \cap \tau_2 \in \partial M - \tilde{\partial} M$. □

The following result will be used in the next sections. It is the non-homogeneous version of the well-known fact that any n-homogeneous subcomplex of an n-combinatorial manifold with non-empty boundary has also a non-empty boundary (see Glaser 1970).

Lemma 4.4 *Let M be a connected NH-manifold with non-empty boundary and let $L \subseteq M$ be a top generated NH-submanifold. Then, $\partial L \neq \emptyset$.*

Proof We may assume $L \neq M$. We proceed by induction on $n = \dim M$. The 1-dimensional case is clear. Let $n \geq 2$. Take adjacent principal simplices $\sigma \in L$ and $\tau \in M - L$ and let $\rho = \sigma \cap \tau$. If $\dim \sigma = \dim \tau$ then $lk(\rho, M) = S^0$ and therefore, $\rho \in \partial L$. If $\dim \sigma \neq \dim \tau$ then $lk(\rho, M) = B + *$ is a non-homogeneous NH-sphere of homotopy dimension 0. We analyze both cases: $\rho \prec \sigma$ and $\rho \prec \tau$. If $\rho \prec \sigma$ then $lk(\rho, L)$ is either a 0-ball, which implies $\rho \in \tilde{\partial} L$, or a non-homogeneous NH-sphere of homotopy dimension 0. In this case, by Proposition 4.3 $\tilde{\partial} lk(\rho, L) \neq \emptyset$. If $\rho \prec \tau$ then $\tilde{\partial} lk(\rho, L) \neq \emptyset$ by induction applied to $lk(\rho, L) \subset B$. In any case, if $\eta \in \tilde{\partial} lk(\rho, L)$ then $\eta * \rho \in \tilde{\partial} L$. □

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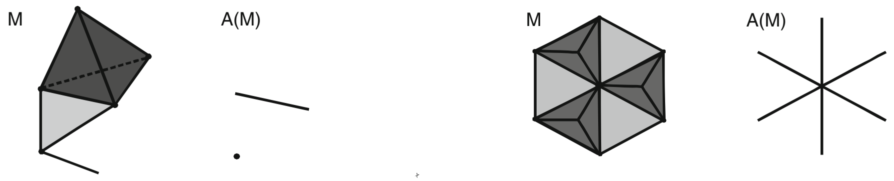

Fig. 4 Anomaly complex

Corollary 4.5 *If M is a connected NH-manifold of dimension $n \geq 1$ containing a top generated combinatorial manifold L without boundary then $M = L$.*

Note that if $S = B + *$ is a non-homogeneous NH-sphere of homotopy dimension 0 and M is a non-trivial top generated combinatorial n-manifold contained in S, then $M \subseteq B$. This implies that $\partial M \neq \emptyset$ by Corollary 4.5. We state this fact in the following

Corollary 4.6 *A non-homogeneous NH-sphere $S = B + *$ of homotopy dimension 0 cannot contain a non-trivial top generated combinatorial manifold without boundary.*

In contrast to the classical situation, the boundary of an NH-manifold is not in general an NH-manifold (see Fig. 3). However, similarly as in the homogeneous setting, if M is an NH-manifold and $\eta \in M$ is any simplex, then $lk(\eta, \partial M) = \partial lk(\eta, M)$. Moreover, it is well-known that the boundary of a combinatorial manifold has no boundary. The following result generalizes this fact to the non-homogeneous setting.

Proposition 4.7 *The boundary of an NH-manifold M has no collapsible simplices.*

Proof Let σ be a ridge in ∂M. Since $lk(\sigma, \partial M) = \partial lk(\sigma, M)$, it suffices to show that the boundary of any NH-ball or NH-sphere cannot be a singleton. This is clear for classical balls and spheres and, by Proposition 4.3, the same is true for NH-balls and NH-spheres. □

A simplex $\sigma \in M$ will be called *internal* if $lk(\sigma, M)$ is an NH-sphere, i.e. if $\sigma \notin \tilde{\partial} M$. We denote by $\overset{\circ}{M}$ the relative interior of M, which is the set of its internal simplices.

Lemma 4.8 *Let S be an NH-sphere with decomposition $B + L$. Then, every $\sigma \in L$ is internal in S. In particular, $\tilde{\partial} S = \tilde{\partial} B - L$.*

Proof This is a particular case of Lemma 5.7. □

Definition 4.9 Let M be an NH-manifold. The *anomaly complex* of M is the subcomplex

$$A(M) = \{\sigma \in M : lk(\sigma, M) \text{ is not homogeneous}\}.$$

The fact that $A(M)$ is a simplicial complex follows from the equation $lk(\sigma * \eta, M) = lk(\sigma, lk(\eta, M))$. Figure 4 shows examples of anomaly complexes.

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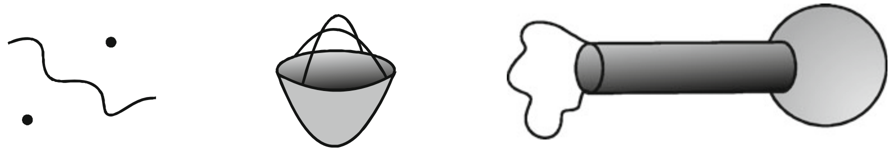

Fig. 5 The underlying spaces of NH-bouquets of index 2. Note that the rightmost example is obtained from a solid cylinder by attaching a 1-disk to one side and a 2-disk to the other

Proposition 4.10 *For any NH-manifold M, $\partial M = \tilde{\partial} M + A(M)$.*

Proof If $\sigma \in A(M)$ then σ is face of two principal simplices of M of different dimensions. Therefore $\sigma \in \partial M$ by Lemma 4.2. For the other inclusion, let $\sigma \in \partial M - \tilde{\partial} M$. Then $lk(\sigma, M)$ is an NH-sphere and $\sigma < \tau$ with $\tau \in \tilde{\partial} M$. Write $\tau = \sigma * \eta$, thus $lk(\tau, M) = lk(\eta, lk(\sigma, M))$. If $\sigma \notin A(M)$ then $lk(\sigma, M)$ is a combinatorial sphere and so is $lk(\tau, M)$, contradicting the fact that $\tau \in \tilde{\partial} M$. □

5 NH-bouquets and shellability

Recall that, similarly as in the homogeneous setting, an NH-sphere is obtained by "gluing" a combinatorial ball to an NH-ball along its entire boundary. In the homogeneous case one can no longer glue another ball to a sphere for it would produce a complex which is not a manifold (not even a pseudo manifold). The existence of boundary in non-homogeneous NH-spheres allows us to glue balls and obtain again an NH-manifold. This is the idea behind the notion of NH-bouquet. This concept arises naturally when studying shellability of non-homogeneous manifolds.

Definition 5.1 We define an *NH-bouquet* G of dimension n and *index* k by induction on k.

- If $k = 0$ then G is an NH-ball of dimension n.
- If $k \geq 1$ then G is an NH-manifold of dimension n such that there exist a top generated NH-bouquet S of dimension n and index $k - 1$ and a top generated combinatorial ball L, such that $G = S + L$ and $S \cap L = \partial L$.

We will show below that the index k is well defined since an NH-bouquet of index k is homotopy equivalent to a bouquet of k spheres (of different dimensions). In fact, the index is the number of balls that are glued to an NH-ball. A *decomposition* $G = B + L_1 + \cdots + L_k$ of an NH-bouquet G consists of top generated subcomplexes of G such that B is an NH-ball, L_i is a combinatorial ball for each $i = 1, \ldots, k$ and $(B + \cdots + L_i) \cap L_{i+1} = \partial L_{i+1}$. Of course, a decomposition is not unique.

Example 5.2 Figure 5 shows some examples of NH-bouquets of low dimensions.

Remark 5.3 Clearly an NH-bouquet of index 1 is an NH-sphere. Note also that for every $n \geq 0$ and every $k \geq 0$ there exists an NH-bouquet G of dimension n and index k.

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Similarly as in Theorem 3.9, it can be proved that the class of NH-bouquets is PL-closed.

Lemma 5.4 *If $G = B + L_1 + \cdots + L_k$ is a decomposition of an NH-bouquet of index $k \geq 2$, then $L_i \cap L_j = \partial L_i \cap \partial L_j$ for all $1 \leq j < i \leq k$.*

Proof $L_i \cap L_j \subseteq \partial L_i$ by definition. Suppose that $L_i \cap L_j \nsubseteq \partial L_j$. Then there exists a simplex $\sigma \in L_i \cap L_j$ such that $lk(\sigma, L_j)$ is a sphere. By Corollaries 4.5 and 4.6, $lk(\sigma, L_j) = lk(\sigma, G)$. In particular $lk(\sigma, L_i) \subseteq lk(\sigma, L_j)$, but if $\nu \in lk(\sigma, L_i)$ is maximal, then $\sigma * \nu$ is a maximal simplex in G and it is contained in $L_i \cap L_j \subseteq \partial L_i$ which is a contradiction. □

Proposition 5.5 *If $G = B + L_1 + \cdots + L_k$ is a decomposition of an NH-bouquet, then $\partial L_i \subseteq B$ for every $i = 1, \ldots, k$. In particular, an NH-bouquet of index k is homotopy equivalent to a bouquet of spheres of dimensions* $\dim L_i$, *for $1 \leq i \leq k$.*

Proof $\partial L_1 \subseteq B$ by definition. For $i \geq 2$ the result follows immediately by induction and Lemma 5.4.

For the second statement, note that, since $\partial L_i \subseteq B$ for every i, G is homotopy equivalent to a CW-complex obtained by attaching cells of dimensions $\dim L_i$ to a point. □

Remark 5.6 It is not hard to see that a homogeneous NH-bouquet of dimension $n \geq 1$ is a combinatorial n-ball or n-sphere. This follows from Theorem 3.6 and Corollary 4.5.

The following result extends Lemma 4.8 and will be used below.

Lemma 5.7 *Let $G = B + L_1 + \cdots + L_k$ be a decomposition of an NH-bouquet. Then every simplex in each L_i is internal in G. Furthermore, if $\sigma \in \partial L_i$ then $lk(\sigma, G)$ is an NH-sphere with decomposition $lk(\sigma, B) + lk(\sigma, L_i)$. In particular, $\tilde{\partial} G = \tilde{\partial} B - \cup_i L_i$.*

Proof It is clear that every simplex internal in L_i is internal in G. Given $\sigma \in \partial L_i$, by Proposition 5.5 $lk(\sigma, G) = lk(\sigma, B) + lk(\sigma, L_i)$. Also $lk(\sigma, L_i) \cap lk(\sigma, B) = \partial lk(\sigma, L_i)$. □

Shellings are structure-preserving moves that transform a combinatorial manifold into another one. They were first studied by Newman 1926 (see also Lickorish 1999; Rourke and Sanderson 1972; Whitehead 1939) and they turned out to be central in the theory. At the beginning of the 1990s Pachner (1991) showed that two (connected) combinatorial manifolds with boundary are PL homeomorphic if and only if one can obtain one from the other by a sequence of elementary shellings, inverse shellings and simplicial isomorphisms (see also Lickorish 1999).

An *elementary shelling* on a combinatorial n-manifold M is the move $M \overset{sh}{\rightarrow} M' = \overline{M - \tau}$, where $\tau = \sigma * \eta$ is an n-simplex of M with $\sigma \in \overset{\circ}{M}$ and $\partial\sigma * \eta \subset \partial M$. The opposite move is called an *inverse shelling*. It is not hard to see that these moves are special cases of regular collapses and expansions and therefore, they preserve the structure of the manifold.

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A combinatorial n-manifold which can be transformed into a single n-simplex by a sequence of elementary shellings is said to be *shellable*. Shellable combinatorial n-manifolds are collapsible and, hence, combinatorial n-balls. The definition of shellability can also be extended to combinatorial n-spheres by declaring S to be shellable if for some n-simplex σ, $\overline{S-\sigma}$ is a shellable n-ball.

The alternative, and more constructive, definition of shellability by means of inverse shellings requires the existence of a linear order $F_1, \ldots, F_t$ of all the n-simplices such that $F_k \cap (F_1 + \cdots + F_{k-1})$ is $(n-1)$-homogeneous for all $2 \leq k \leq t$. This formulation can be used to define the concept of shellability in arbitrary n-homogeneous complexes. It is not difficult to see that shellable pseudo manifolds are necessarily combinatorial balls (Björner et al. 1999, Proposition 4.7.22). It is also known that every ball of dimension less than or equal to 2 is shellable. Examples of non-shellable 3-balls abound in the bibliography, the first one was discovered by Furch in 1924 (see Ziegler 1998 for a survey of non-shellable 3-balls). A way for constructing non-shellable balls for every $n \geq 3$ was presented by Lickorish in (1991).

Shellability in the non-homogeneous context was first considered by Björner and Wachs (1996) in the 1990s. A finite (non-necessarily homogeneous) simplicial complex is shellable if there is a linear order $F_1, \ldots, F_t$ of its maximal simplices such that $F_k \cap (F_1 + \cdots + F_{k-1})$ is $(\dim F_k - 1)$-homogeneous for all $2 \leq k \leq t$. A simplex F_k is said to be a *spanning simplex* if $F_k \cap (F_1 + \cdots + F_{k-1}) = \partial F_k$. It is not hard to see that the spanning simplices may be moved to any later position in the shelling order (see Kozlov 2008). It is known that a shellable complex is homotopy equivalent to a wedge of spheres, which are indexed by the spanning simplices (see Kozlov 2008, Theorem 12.3). In particular, shellable NH-balls cannot have spanning simplices and shellable NH-spheres have exactly one spanning simplex. In general, a shellable NH-bouquet of index k must have exactly k spanning simplices.

Theorem 5.8 *Let M be a shellable NH-manifold. Then, for every shelling order $F_1, \ldots, F_t$ of M and every $0 \leq l \leq t$, $\mathcal{F}_l(M) = F_1 + \cdots + F_l$ is an NH-manifold. Moreover, $\mathcal{F}_l(M)$ is an NH-bouquet of index $\sharp\{F_j \in \mathcal{T} \mid j \leq l\}$, where $\mathcal{T}$ is the set of spanning simplices. In particular, M is an NH-bouquet of index $\sharp\mathcal{T}$.*

Proof We proceed by induction on $n = \dim M$. Suppose $n \geq 1$ and fix a shelling order $F_1, \ldots, F_t$. Let $1 \leq l \leq t$ and let $v \in M$ be a vertex. Since $lk(v, M)$ is a shellable NH-ball or NH-sphere with shelling order $lk(v, F_1), \ldots, lk(v, F_t)$ (some of them possibly empty), then by induction $\mathcal{F}_j(lk(v, M))$ is an NH-bouquet of index at most 1 for all $1 \leq j \leq l$. Since $lk(v, \mathcal{F}_l(M)) = \mathcal{F}_l(lk(v, M))$ then $\mathcal{F}_l(M)$ is an NH-manifold. To see that $\mathcal{F}_l(M)$ is actually an NH-bouquet, reorder $F_1, \ldots, F_l$ so that the spanning simplices are placed at the end of the order. If F_{p+1} is the first spanning simplex in the order, then $\mathcal{F}_p(M)$ is a collapsible NH-manifold (see Kozlov 2008, Theorem 12.3) and hence an NH-ball. Then, $\mathcal{F}_l(M) = \mathcal{F}_p(M) + F_{p+1} + \cdots + F_l$ is an NH-bouquet of index $\sharp\{F_j \in \mathcal{T} | j \leq l\}$ by definition. □

6 Regular collapses, elementary shellings and Pachner moves

Recall that a regular expansion in an n-combinatorial manifold M is a geometrical expansion $M \to N = M + B^n$ such that $M \cap B^n \subset \partial M$. As we mentioned before,

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this move produces a new combinatorial n-manifold. In this section we prove a general version of this result for NH-manifolds. We start with some preliminary results.

Lemma 6.1 *Let B be a combinatorial n-ball and let $L \subset \partial B$ be a combinatorial $(n-1)$-ball. Then, there exists a stellar subdivision s such that $sB \searrow sL$.*

Proof By (Glaser 1970, Lemma III.8) there exists a derived subdivision δ and a subdivision α such that $\delta B = \alpha\Delta^n$ and $\delta L = \alpha\Delta^{n-1}$, where Δ^{n-1} is an $(n-1)$-face of Δ^n. Now, by (Glaser 1970, Lemma III.7) there exists a stellar subdivision $\tilde{s}$ such that $\tilde{s}\alpha\Delta^n \searrow \tilde{s}\alpha\Delta^{n-1}$ and therefore $\tilde{s}\delta B \searrow \tilde{s}\delta L$.

Corollary 6.2 *Let B be a combinatorial n-ball and let $K \subset \partial B$ be a collapsible complex. Then, there exists a stellar subdivision s such that $sB \searrow sK$.*

Proof Subdivide B baricentrically twice and consider a regular neighborhood N of K'' in $\partial B''$ (see Glaser 1970, Corollary III.17). Since K'' is collapsible, then N is an $(n-1)$-ball. Since $N \subset \partial B''$, by the previous lemma, there is a stellar subdivision $\tilde{s}$ such that $\tilde{s}B'' \searrow \tilde{s}N$. We conclude that $\tilde{s}B'' \searrow \tilde{s}N \searrow \tilde{s}K''$. □

Theorem 6.3 *Let M be an NH-manifold and B^r a combinatorial r-ball. Suppose $M \cap B^r \subseteq \partial B^r$ is an NH-ball or an NH-sphere generated by ridges of M or B^r and that $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$. Then $M + B^r$ is an NH-manifold. Moreover, if M is an NH-bouquet of index k and $M \cap B^r \neq \emptyset$ for $r \neq 0$, then $M + B^r$ is an NH-bouquet of index k (if $M \cap B^r$ is an NH-ball) or $k+1$ (if $M \cap B^r$ is an NH-sphere).*

Proof We note first that $M, B^r \subset M + B^r$ are top generated. Since $M \cap B^r \subseteq \partial B^r$ then B^r is top generated. On the other hand, if σ is a principal simplex in M which is not principal in $M + B^r$ then σ must be in $M \cap B^r$. Since $\sigma \notin \tilde{\partial} M$ then $\sigma \notin (M \cap B^r)^\circ$. Hence, σ is not principal in $M \cap B^r$, which contradicts the maximality of σ in M.

We shall prove the result by induction on r. The case $M \cap B^r = \emptyset$ is clear, so let $r \geq 1$ and assume $M \cap B^r \neq \emptyset$. We need to prove that every vertex in $M + B^r$ is regular. It is clear that the vertices in $(M - B^r) + (B^r - M)$ are regular since B^r and M are NH-manifolds. Consider then a vertex $v \in M \cap B^r$. We claim that the pair $lk(v, M), lk(v, B^r)$ fulfills the hypotheses of the theorem. Note that $lk(v, M)$ is an NH-ball or NH-sphere, $lk(v, B^r)$ is a combinatorial ball, since $v \in M \cap B^r \subseteq \partial B^r$, and $lk(v, M \cap B^r)$ is an NH-ball or NH-sphere contained in $\partial lk(v, B^r)$. Note also that the inclusion $(M \cap B^r)^\circ \subseteq \tilde{\partial} M$ implies that $lk(v, M \cap B^r)^\circ \subseteq \tilde{\partial} lk(v, M)$. We now check that $lk(v, M \cap B^r)$ is generated by ridges of $lk(v, M)$ or $lk(v, B^r)$. This is easily seen if $lk(v, M \cap B^r) \neq \emptyset$. For the case $lk(v, M \cap B^r) = \emptyset$ we need to show that there is a principal 0-simplex in $lk(v, M)$ or $lk(v, B^r)$. Now, $lk(v, M \cap B^r) = \emptyset$ implies that v is principal in $M \cap B^r$, so $v \in (M \cap B^r)^\circ \subseteq \tilde{\partial} M$ and $lk(v, M)$ is an NH-ball (and hence, collapsible). And since $v \in M \cap B^r \subseteq \partial B^r$ then $lk(v, B^r)$ is a ball. Now, if v is a ridge in B^r then $r = 1$ and, hence, $lk(v, B^1) = *$. If, on the other hand, v is a ridge of M then there exists a principal 1-simplex σ with $v \prec \sigma$. Since σ is principal in M, $* = lk(v, \sigma)$ is principal in $lk(v, M)$. Since $lk(v, M)$ is collapsible, then $lk(v, M) = *$.

Therefore, by induction, $lk(v, M + B^r)$ is an NH-manifold. Now, if $lk(v, M \cap B^r) \neq \emptyset$, then $lk(v, M + B^r)$ is an NH-ball or an NH-sphere if $lk(v, M)$ is an

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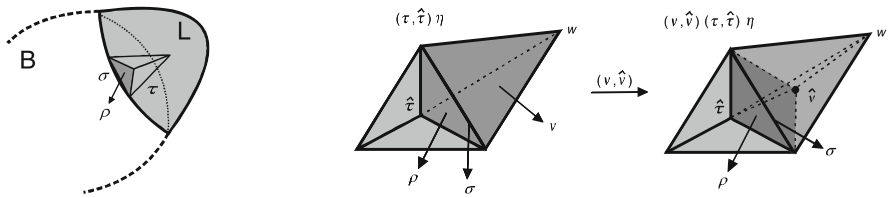

Fig. 6 The starrings of Theorem 6.3

NH-ball and it is an NH-sphere if $lk(v, M)$ is an NH-sphere. If $lk(v, M \cap B^r) = \emptyset$, we showed above that $lk(v, M) = *$ and $lk(v, B^r)$ is a ball or $lk(v, B^r) = *$ and $lk(v, M)$ is an NH-ball. In either case, $lk(v, M + B^r)$ is an NH-sphere of homotopy dimension 0. This proves that $M + B^r$ is an NH-manifold.

We prove now the second part of the statement. We proceed by induction on the index k. Suppose first that $k = 0$, i.e. M is an NH-ball. Let α be a subdivision such that $\alpha M \searrow 0$, and extend α to all $M + B^r$. If $M \cap B^r$ is an NH-ball we can apply Corollary 6.2 to $\alpha(M \cap B^r) \subset \alpha\partial B^r$ and find a stellar subdivision s such that $s\alpha B^r \searrow s\alpha(M \cap B^r)$. This implies that $s\alpha(M + B^r) \searrow s\alpha M \searrow 0$ and therefore $M + B^r$ is an NH-ball. If $M \cap B^r$ is an NH-sphere S with decomposition $S = G + L$, take any maximal simplex $\tau \in L$ with an immediate face σ in ∂L and consider the starring $(\tau, \hat{\tau})S$ of S (see Fig. 6). Let $\rho = \hat{\tau} * \sigma \in (\tau, \hat{\tau})S$. We claim that $(\tau, \hat{\tau})S - \{\rho\}$ is an NH-ball. On one hand, it is clear that $((\tau, \hat{\tau})S - \{\rho\}) \cap \rho = \partial\rho$. On the other hand, $(\tau, \hat{\tau})L - \{\rho, \sigma\}$ is a combinatorial ball because it is PL-homeomorphic to L. Since G is an NH-ball, $(\tau, \hat{\tau})L - \{\rho, \sigma\}$ is a combinatorial ball and $G \cap ((\tau, \hat{\tau})L - \{\rho, \sigma\}) = \partial L - \{\sigma\}$, which is a combinatorial ball by Newman's Theorem, it follows that $(\tau, \hat{\tau})S - \{\rho\}$ is an NH-ball, as claimed. Now, since $\tau \in L \subset M \cap B^r$ is principal then it must be a ridge of M or of B^r. We analyze both cases. Suppose τ is a ridge of B^r and let $\tau \prec \eta \in B^r$. Write $\eta = w * \tau$ (see Fig. 6). Note that the starring $(\tau, \hat{\tau})S$ performed earlier also subdivides η and the simplex ρ lies in the boundary of $(\tau, \hat{\tau})\eta$. Consider the simplex $\nu = w * \rho$, which is one of the principal simplices in which η has been subdivided. Now make the starring $(\nu, \hat{\nu})$ in $(\tau, \hat{\tau})\eta$ (see Fig. 6). By removing the simplex $\hat{\nu} * \rho$ from $(\nu, \hat{\nu})(\tau, \hat{\tau})B^r$, we obtain a complex which is PL-homeomorphic to B^r. Then

$$(\nu, \hat{\nu})(\tau, \hat{\tau})B^r - \{\hat{\nu} * \rho\}$$

is a combinatorial ball and it intersects M in $(\tau, \hat{\tau})S - \{\rho\}$, which is an NH-ball. It follows that

$$(\nu, \hat{\nu})(\tau, \hat{\tau})(M + B^r) - \{\hat{\nu} * \rho\} = (\tau, \hat{\tau})M + (\nu, \hat{\nu})(\tau, \hat{\tau})B^r - \{\hat{\nu} * \rho\}$$

is again an NH-ball. If we now plug the simplex $\hat{\nu} * \rho$, $(\nu, \hat{\nu})(\tau, \hat{\tau})(M + B^r)$ is an NH-sphere by definition. This completes the case where τ is a ridge of B^r. The case that τ is a ridge of M is analogous.

Suppose now that M is an NH-bouquet of index $k \geq 1$. Write $M = G + L$ with G an NH-bouquet of index $k - 1$ and L a combinatorial ball glued to G along its entire boundary. If $r = 0$ we obtain an NH-bouquet. Suppose then that $M \cap B^r \neq \emptyset$. We claim that $B^r \cap L \subseteq \partial L$. Suppose $(L - \partial L) \cap B^r \neq \emptyset$ and let $\eta \in (L - \partial L) \cap B^r$.

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Now, $lk(\eta, M) = lk(\eta, L)$ is a combinatorial sphere and Corollaries 4.5 and 4.6 imply that $lk(\eta, B^r) \subset lk(\eta, M)$. But if $\tau \in B^r$ is a principal simplex containing η then $lk(\eta, \tau) \in lk(\eta, M)$ and $\tau \in M \cap B^r \subseteq \partial B^r$, contradicting the maximality of τ in B^r. This proves that $B^r \cap L \subseteq \partial L$ and, therefore $M \cap B^r = G \cap B^r$. Also, $(G \cap B^r)^\circ \subseteq \tilde{\partial} M = \tilde{\partial} G - L \subset \tilde{\partial} G$. By induction, $G + B^r$ is an NH-bouquet of index $k - 1$ (if $G \cap B^r = M \cap B^r$ is an NH-ball) or k (if $G \cap B^r = M \cap B^r$ is an NH-sphere). In either case, $M + B^r = G + L + B^r = (G + B^r) + L$ with $(G + B^r) \cap L = G \cap L + B^r \cap L = \partial L$. Thus, $M + B^r$ is an NH-bouquet of index k or $k + 1$. This completes the proof. □

Note that the previous theorem generalizes Alexander's Theorem on regular expansions (Lickorish 1999, Theorem 3.9) to the non-homogeneous setting. The condition $(M \cap B)^\circ \subset \tilde{\partial} M$ corresponds to $M \cap B \subset \partial M$ in the homogeneous case. We next extend the notion of regular expansion to the non-homogeneous context. This will be used to characterize the notion of shelling on NH-manifolds similarly as in the case of manifolds.

Definition 6.4 A *regular expansion* on an NH-manifold M is a geometrical expansion $M \to M + B$ (i.e. B is a ball and $M \cap B \subset \partial B$ is a ball of dimension $\dim B - 1$) such that $(M \cap B)^\circ \subset \tilde{\partial} M$.

Recall that an inverse shelling in a combinatorial n-manifold M corresponds to a (classical) regular expansion $M \to M + \sigma$ involving a single n-simplex σ. An elementary shelling is the inverse move (Whitehead 1939). We investigate now shellable NH-balls. First we need the following result.

Proposition 6.5 *Let $M \to M + B$ be a geometrical expansion in an NH-manifold M. If $M + B$ is an NH-manifold and $M, B \subset M + B$ are top generated then $(M \cap B)^\circ \subset \tilde{\partial} M$ (i.e. $M \to M + B$ is a regular expansion).*

Proof Take $\rho \in (M \cap B)^\circ$. Since $lk(\rho, M \cap B)$ is a sphere contained in the sphere $\partial lk(\rho, B)$, then $lk(\rho, M \cap B) = \partial lk(\rho, B)$. Suppose $\rho \notin \tilde{\partial} M$. Then $lk(\rho, M + B) = lk(\rho, M) + lk(\rho, B)$ is an NH-bouquet of index 2 since $lk(\rho, M), lk(\rho, B) \subset lk(\rho, M + B)$ are top generated by hypothesis. This contradicts the fact that $M + B$ is an NH-manifold. □

Definition 6.6 Let M be an NH-manifold. An *inverse shelling* is a regular expansion $M \to M + \sigma$ where σ is a single simplex. An *elementary shelling* is the inverse move.

By Proposition 6.5 and Theorem 5.8, we obtain the following characterization of shellable NH-balls in terms of elementary shellings.

Corollary 6.7 *An NH-ball B is shellable if and only if B can be transformed into a single maximal simplex by a sequence of elementary shellings.*

A *stellar exchange* $\kappa(\sigma, \tau)$ is the move that transforms a complex M into a new complex $\kappa(\sigma, \tau)M$ by replacing $st(\sigma, M) = \sigma * \partial\tau * L$ with $\partial\sigma * \tau * L$, for $\sigma \in M$ and $\tau \notin M$ (see Lickorish 1999; Pachner 1991). Note that elementary starrings and

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welds are particular cases of stellar exchanges (when τ or σ is a vertex). When $L = \emptyset$ the stellar exchange is called a *bistellar move*. Also, since $\kappa(\sigma, \tau) = (\tau, b)^{-1}(\sigma, a)$, two simplicial complexes are PL-homeomorphic if and only if they are related by a sequence of stellar exchanges. In the case of PL-homeomorphic combinatorial manifolds without boundary, all the moves in this sequence can be taken to be bistellar moves (see Lickorish 1999; Pachner 1991 for more details). This discussion motivates the following definition.

Definition 6.8 Let M be a combinatorial n-manifold and let $\sigma \in M$ be a simplex such that $lk(\sigma, M) = \partial\tau * L$ with $\tau \notin M$. An *NH-factorization* is the move $M \to M + \sigma * \tau * L$. We write $F(\sigma, \tau)M = M + \sigma * \tau * L$. When $L = \emptyset$, we call it a *bistellar factorization*.

Note that, in fact, NH-factorizations can be defined for arbitrary complexes. When τ is a single vertex $b \notin M$, we will denote $M^+_\sigma = F(\sigma, b)M$. Note that M^+_σ is the simplicial cone of the inclusion $st(\sigma, M) \subseteq M$. Note also that, since $st(\sigma, M)$ is collapsible, $M^+_\sigma \searrow M$.

By definition, the following diagram commutes (this justifies the term “factorization”).

$$\begin{tikzcd} M \arrow[rr, "\kappa(\sigma,\tau)"] \arrow[dr, "F(\sigma,\tau)"'] & & \kappa(\sigma, \tau)M \arrow[dl, "F(\tau,\sigma)"] \\ & M + \sigma * \tau * L & \end{tikzcd}$$

Proposition 6.9 *Let M be a combinatorial n-manifold and let $M \longrightarrow N = F(\sigma, \tau)M$ be an NH-factorization. Then N is an NH-manifold.*

Proof Let $N = M + \sigma * \tau * L$ with $\tau \notin M$. Since $(\tau, b)N = M + b * \partial\tau * \sigma * L = M^+_\sigma$, by Theorem 3.9 it suffices to prove that M^+_σ is an NH-manifold. We prove by induction on n that the simplicial cone M^+_B of the inclusion of any combinatorial ball $B \subseteq M$ is an NH-manifold.

Denote $M^+_B = M + b * B$ and let v be a vertex of M^+_B. If $v \notin B$, then $lk(v, M^+_B) = lk(v, M)$. If $v \in \mathring{B}$ then $lk(v, M^+_B) = b * lk(v, M)$, which is a combinatorial n-ball. If $v \in \partial B$ then $lk(v, M^+_B) = lk(v, M) + b * lk(v, B)$ is an NH-manifold by induction. Since $lk(v, B)$ is collapsible then $lk(v, M^+_B) \searrow lk(v, M)$, so $lk(v, M^+_B)$ is an NH-ball if $v \in \partial M$. If $v \notin \partial M$ then $lk(v, B)$ is strictly contained in $lk(v, M)$. It follows that there is an n-simplex $\eta \in M - B$ containing v. By Newman's Theorem, $lk(v, M) - lk(v, \eta)$ is an $(n-1)$-ball. It follows that $lk(v, M^+_B)$ is an NH-sphere with decomposition

$$(lk(v, M - \eta) + b * lk(v, B)) + lk(v, \eta)$$

since $lk(v, M - \eta) + b * lk(v, B)$ is an NH-ball by the previous case and

$$\begin{aligned}(lk(v, M - \eta) + b * lk(v, B)) \cap lk(v, \eta) &= (lk(v, M) - lk(v, \eta)) \cap lk(v, \eta) \\ &= \partial lk(v, \eta).\end{aligned}$$

$\square$

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Lemma 6.10 *Let M_1, M_2 be combinatorial n-manifolds without boundary and let $B_i \subset M_i$ be combinatorial n-balls. Suppose $\overline{M_1 - B_1} = \overline{M_2 - B_2}$. Then, $M_1 \simeq_{PL} M_2$.*

Proof Note that $\overline{M_i - B_i}$ is a combinatorial n-manifold and that $\partial B_i = \overline{M_i - B_i} \cap B_i$. Since $\partial B_2 = \overline{M_2 - B_2} \cap B_2 = \overline{M_1 - B_1} \cap B_2$ and $M_2 = \overline{M_2 - B_2} + B_2 = \overline{M_1 - B_1} + B_2$, then $B_2 \cap \overline{M_1 - B_1} \subseteq \partial(\overline{M_1 - B_1}) = \partial B_1$. Hence, $\partial B_2 \subseteq \partial B_1$. Analogously, $\partial B_1 \subseteq \partial B_2$. The result now follows from the fact that every ball may be starred (see Glaser 1970, Theorem II.11). □

Theorem 6.11 *Let $M, \tilde{M}$ be combinatorial n-manifolds (with or without boundary). If M and $\tilde{M}$ are PL-homeomorphic then there exists a sequence*

$$M = M_1 \to N_1 \leftarrow M_2 \to N_2 \leftarrow M_3 \to \cdots \leftarrow M_{r-1} \to N_{r-1} \leftarrow M_r = \tilde{M}$$

where the N_i's are NH-manifolds, the M_i's are n-manifolds, and $M_i, M_{i+1} \to N_i$ are NH-factorizations. Moreover, if M and $\tilde{M}$ are closed then the converse holds. Also, in this case the NH-factorizations may be taken to be bistellar factorizations.

Proof Let $\kappa(\sigma_1, \tau_1), \ldots, \kappa(\sigma_r, \tau_r)$ be a sequence of stellar exchanges taking M to $\tilde{M}$. Then for each i, the sequence

$$M_i \overset{F(\sigma_i,\tau_i)}{\longrightarrow} N_i \overset{F(\tau_i,\sigma_i)}{\longleftarrow} M_{i+1} = \kappa(\sigma_i, \tau_i) M_i$$

is a factorization and N_i is an NH-manifold by Lemma 6.9.

For the second part of the proof, assume that $M \overset{F(\sigma,\tau)}{\longrightarrow} N \overset{F(\rho,\eta)}{\longleftarrow} \tilde{M}$ are NH-factorizations, with M and $\tilde{M}$ n-manifolds and M closed. Since $M + \sigma * \tau * L = \tilde{M} + \rho * \eta * T$, by a dimension argument and the homogeneity of M and $\tilde{M}$, it follows that $\sigma * \tau * L = \rho * \eta * T$. Hence, $\overline{M - \sigma * \partial\tau * L} = \overline{N - \sigma * \tau * L} = \overline{N - \rho * \eta * T} = \overline{\tilde{M} - \rho * \partial\eta * T}$. The result now follows from Lemma 6.10. □

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