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A SINGULAR PERTURBATION PROBLEM FOR THE $p(x)$-LAPLACIAN

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Abstract: We present results for the following singular perturbation problem:

$$\Delta_{p(x)}u^\varepsilon := \text{div}(|\nabla u^\varepsilon(x)|^{p(x)-2}\nabla u^\varepsilon) = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0 \qquad (P_\varepsilon(f^\varepsilon))$$

in $\Omega \subset \mathbb{R}^N$, where $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz function satisfying $\beta > 0$ in $(0,1)$, $\beta \equiv 0$ outside $(0,1)$ and $\int \beta(s)\,ds = M$. The functions u^ε and f^ε are uniformly bounded. We prove uniform Lipschitz regularity, we pass to the limit $(\varepsilon \to 0)$ and we show that limit functions are weak solutions to a free boundary problem.

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1 INTRODUCTION

The $p(x)$-Laplacian, defined as

$$\Delta_{p(x)}u := \text{div}(|\nabla u(x)|^{p(x)-2}\nabla u),$$

extends the Laplacian, where $p(x) \equiv 2$, and the p-Laplacian, where $p(x) \equiv p$ with $1 < p < \infty$. This operator has been used in the modelling of electrorheological fluids ([16]) and in image processing ([6], [1]).

We will present results for the following singular pertubation problem for the $p(x)$-Laplacian:

$$\Delta_{p(x)}u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0 \qquad (P_\varepsilon(f^\varepsilon))$$

in a domain $\Omega \subset \mathbb{R}^N$. Here $\varepsilon > 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, with β a Lipschitz function satisfying $\beta > 0$ in $(0,1)$, $\beta \equiv 0$ outside $(0,1)$ and $\int \beta(s)\,ds = M$.

By a solution to $P_\varepsilon(f^\varepsilon)$ we mean a nonnegative function $u^\varepsilon \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega |\nabla u^\varepsilon(x)|^{p(x)-2}\nabla u^\varepsilon \cdot \nabla\varphi\,dx = -\int_\Omega \varphi\,(\beta_\varepsilon(u^\varepsilon) + f^\varepsilon)\,dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

We assume that the functions u^ε and f^ε are uniformly bounded. We prove uniform Lipschitz regularity, we pass to the limit $(\varepsilon \to 0)$ and we show that limit functions are weak solutions to the following free boundary problem:

$$\begin{cases} \Delta_{p(x)}u = f & \text{in } \{u > 0\} \\ u = 0,\ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\}, \end{cases} \qquad (1)$$

where $u = \lim u^\varepsilon$, $f = \lim f^\varepsilon$, $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1}\,M\right)^{1/p(x)}$ and $M = \int \beta(s)\,ds$.

When $p(x) \equiv 2$ and $f^\varepsilon \equiv 0$, this problem arises in combustion theory to describe the propagation of curved premixed equi-diffusional deflagration flames. The study of the limit $(\varepsilon \to 0)$ was proposed in the 1930s and was first rigorously studied in the pioneering work [2]. Since then, much research has been done on this problem, see [5, 3, 4, 10, 7, 17, 15]. The inhomogeneous case, $f^\varepsilon \not\equiv 0$, allows the treatment of more general combustion models with nonlocal diffusion and/or transport (see [11], [12]).

For previous results in the literature on free boundary problems for the $p(x)$-Laplacian we refer to [9] and [8].

2 Basic Definitions and Assumptions

Assumptions on $p(x)$

We will assume that the function $p(x)$ verifies

$$1 < p_{\min} \leq p(x) \leq p_{\max} < \infty, \qquad x \in \Omega.$$

We also assume that $p(x)$ is continuous up to the boundary and that it has a modulus of continuity $\omega : \mathbb{R} \to \mathbb{R}$, i.e. $|p(x) - p(y)| \leq \omega(|x - y|)$ if $|x - y|$ is small. For some results we need to assume further that $p(x)$ is Lipschitz continuous in Ω.

Assumptions on β_ε

We will assume that the functions β_ε are defined by scaling of a single function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying:

i) β is a Lipschitz continuous function,

ii) $\beta > 0$ in $(0, 1)$ and $\beta \equiv 0$ otherwise,

iii) $\int_0^1 \beta(s)\, ds = M$.

And then $\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$.

Definition 1 *We call u a weak solution of* (1) *in Ω if*

1. *u is continuous and nonnegative in Ω and $\Delta_{p(x)} = f$ in $\Omega \cap \{u > 0\}$.*

2. *For $D \subset\subset \Omega$ there are constants $0 < c_{\min} \leq C_{\max}$ such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$*

$$c_{\min} \leq \frac{1}{r} \fint_{B_r(x)} u dx \ \leq C_{\max}.$$

3. *For $\mathcal{H}^{N-1}$ a.e $x_0 \in \partial\{u > 0\}$ where there is a unit interior normal $\nu(x_0)$ to $\partial\{u > 0\}$ in the measure theoretic sense, u has the asymptotic development*

$$u(x) = \lambda^*(x_0)\langle x - x_0, \nu(x_0)\rangle^- + o(|x - x_0|)$$

where $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1}\, M\right)^{1/p(x)}$ and $\int \beta(s)\, ds = M$.

4. *For every $x_0 \in \Omega \cap \partial\{u > 0\}$,*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| \leq \lambda^*(x_0).$$

If there is a ball $B \subset \{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at x_0 then,

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} \frac{u(x)}{dist(x, B)} \geq \lambda^*(x_0).$$

Here $\lambda^*(x)$ is as above.

3 Main Results

We will present the following results:

Theorem 1 *Let u^ε be a solution of*

$$\Delta_{p(x)} u^\varepsilon = \beta_\varepsilon(u^\varepsilon) + f^\varepsilon, \quad u^\varepsilon \geq 0 \quad \text{in } \Omega, \qquad (P_\varepsilon(f^\varepsilon))$$

with $\|u^\varepsilon\|_{L^\infty(\Omega)} \leq L_1$, $\|f^\varepsilon\|_{L^\infty(\Omega)} \leq L_2$. Then, for $\Omega' \subset\subset \Omega$, we have

$$|\nabla u^\varepsilon(x)| \leq C \quad \text{in } \Omega'$$

with $C = C(N, L_1, L_2, \|\beta\|_\infty, p, dist(\Omega', \partial\Omega))$, if $\varepsilon \leq \varepsilon_0(\Omega, \Omega')$.

Proof. An essential tool in the proof is a Harnack Inequality result for the inhomogenous $p(x)$-Laplacian equation. We refer to [13] for the detailed proof of this theorem. □

Theorem 2 *Let u^{ε_j} be a family of solutions to $P_{\varepsilon_j}(f^{\varepsilon_j})$ in a domain $\Omega \subset \mathbb{R}^N$ such that $u^{\varepsilon_j} \to u$ uniformly on compact subsets of Ω, $f^{\varepsilon_j} \to f$ $*$-weakly in $L^\infty(\Omega)$ and $\varepsilon_j \to 0$. Then, under suitable assumptions, u is a weak solution to the following free boundary problem:*

$$\begin{cases} \Delta_{p(x)} u = f & \text{in } \{u > 0\} \\ u = 0, \ |\nabla u| = \lambda^*(x) & \text{on } \partial\{u > 0\} \end{cases}$$

where $\lambda^*(x) = \left(\frac{p(x)}{p(x)-1} M\right)^{1/p(x)}$ and $M = \int \beta(s)\, ds$.

Proof. In the proof we make use of an important auxiliary result, which says that, when u^{ε_j} are solutions to $P_{\varepsilon_j}(f^{\varepsilon_j})$ with power $p_j(x)$, $u^{\varepsilon_j} \to \alpha x_1^+$ with $\alpha \in \mathbb{R}$, $f^{\varepsilon_j} \to 0$, $p_j \to p_0$ with p_0 constant and $\varepsilon_j \to 0$, then $\alpha = 0$ or $\alpha = \left(\frac{p_0}{p_0-1} M\right)^{1/p_0}$. We refer to [13] for the complete proof of this theorem. □

4 Final Remarks

We point out that in [12] it is shown that, in case $p(x) \equiv 2$, the assumptions required in the proof of Theorem 2 above are fulfilled in case u^{ε_j} are minimizers of a certain energy functional.

On the other hand, in the work [14], which applies to our problem in the particular case in which $p(x) \equiv p$ and $f^\varepsilon \equiv 0$, it is studied the smoothness of the free boundary for weak solutions in the sense of Definition 1 above. There it is shown that the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of every free boundary point where there is a normal in the measure theoretic sense.

We finally want to remark that, when $f^\varepsilon \not\equiv 0$ and $p(x) \not\equiv 2$, our results are new even in the case $p(x) \equiv p$. Moreover, when $p(x)$ is not constant, our results are new even if $f^\varepsilon \equiv 0$.

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