PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 8, August 2013, Pages 2629–2639
S 0002-9939(2013)11561-3
Article electronically published on April 25, 2013

# TENSOR PRODUCTS OF LEAVITT PATH ALGEBRAS

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(Communicated by Birge Huisgen-Zimmermann)

ABSTRACT. We compute the Hochschild homology of Leavitt path algebras over a field $k$. As an application, we show that $L_2$ and $L_2 \otimes L_2$ have different Hochschild homologies, and so they are not Morita equivalent; in particular, they are not isomorphic. Similarly, $L_\infty$ and $L_\infty \otimes L_\infty$ are distinguished by their Hochschild homologies, and so they are not Morita equivalent either. By contrast, we show that $K$-theory cannot distinguish these algebras; we have $K_*(L_2) = K_*(L_2 \otimes L_2) = 0$ and $K_*(L_\infty) = K_*(L_\infty \otimes L_\infty) = K_*(k)$.

## 1. INTRODUCTION

Elliott's theorem [21] states that $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ plays an important role in the proof of the celebrated classification theorem of Kirchberg algebras in the UCT class, due to Kirchberg [14] and Phillips [19]. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear and separable C*-algebra. The Kirchberg-Phillips theorem states that this class of simple C*-algebras is completely classified by its topological $K$-theory. The analogous question whether the algebras $L_2$ and $L_2 \otimes L_2$ are isomorphic has remained open for some time. Here $L_2$ is the Leavitt algebra of type $(1, 2)$ over a field $k$ (see [17]), that is, the $k$-algebra with generators $x_1, x_2, x_1^*, x_2^*$ and relations given by $x_i^* x_j = \delta_{i,j}$ and $\sum_{i=1}^2 x_i x_i^* = 1$.

In this paper we obtain a negative answer to this question. Indeed, we analyze a much larger class of algebras, namely the tensor products of Leavitt path algebras of finite quivers in terms of their Hochschild homology, and we prove that, for $1 \le n < m \le \infty$, the tensor products $E = \bigotimes_{i=1}^n L(E_i)$ and $F = \bigotimes_{j=1}^m L(F_j)$ of Leavitt path algebras of non-acyclic finite quivers $E_i$, $F_j$ are distinguished by their Hochschild homologies (Theorem 5.1). Because Hochschild homology is Morita invariant, we conclude that $E$ and $F$ are not Morita equivalent for $n < m$. Since $L_2$ is the Leavitt path algebra of the graph with one vertex and two arrows, we obtain that $L_2 \otimes L_2$ and $L_2$ are not Morita equivalent; in particular, they are not isomorphic.

Recall that, by a theorem of Kirchberg [15], a simple, nuclear and separable $C^*$-algebra $A$ is purely infinite if and only if $A \otimes \mathcal{O}_\infty \cong A$. We also show that the analogue of Kirchberg's result is not true for Leavitt algebras. We prove in

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Received by the editors August 1, 2011 and, in revised form, November 9, 2011.

2010 *Mathematics Subject Classification.* Primary 16E40, 16S99; Secondary 19D50.

The first author was partially supported by DGI MICIIN-FEDER MTM2008-06201-C02-01 and by the Comissionat per Universitats i Recerca de la Generalitat de Catalunya.

The second author was supported by CONICET and partially supported by grants PIP 112-200801-00900, UBACyTs X051 and 20020100100386, and MTM2007-64074.

Proposition 5.3 that if $E$ is a non-acyclic quiver, then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent, and also that $L_\infty \otimes L_\infty$ and $L_\infty$ are not Morita equivalent.

Using the results in [5] we prove that the algebras $L_2$ and $L_2 \otimes L(F)$, for $F$ an arbitrary finite quiver, have trivial $K$-theory: all algebraic $K$-theory groups $K_i$, $i \in \mathbb{Z}$, vanish on them (this follows from Lemma 6.1 and Proposition 6.2). We also compute $K_*(L(F)) = K_*(L_\infty \otimes L(F))$ and that $K_*(L_\infty) = K_*(L_\infty \otimes L_\infty) = K_*(k)$ is the $K$-theory of the ground field (see Proposition 6.3 and Corollary 6.4). This implies in particular that, in contrast with the analytic situation, no classification result, in terms solely of $K$-theory, can be expected for a class of central, simple $k$-algebras, containing all purely infinite simple unital Leavitt path algebras and closed under tensor products. It is worth mentioning that an important step towards a $K$-theoretic classification of purely infinite simple Leavitt path algebras of finite quivers has been achieved in [2].

We refer the reader to [3], [7] and [20] for the basics on Leavitt algebras, Leavitt path algebras and graph C*-algebras, and to [22] for a nice survey on the Kirchberg-Phillips Theorem.

*Notation.* We fix a field $k$; all vector spaces, tensor products and algebras are over $k$. If $R$ and $S$ are unital $k$-algebras, then by an $(R, S)$-bimodule we understand a left module over $R \otimes S^{op}$. By an $R$-bimodule we shall mean an $(R, R)$ bimodule, that is, a left module over the enveloping algebra $R^e = R \otimes R^{op}$. Hochschild homology of $k$-algebras is always taken over $k$. If $M$ is an $R$-bimodule, we write

$$HH_n(R, M) = \mathrm{Tor}_n^{R^e}(R, M)$$

for the Hochschild homology of $R$ with coefficients in $M$ and we abbreviate $HH_n(R) = HH_n(R, R)$.

## 2. Hochschild homology

Let $k$ be a field, $R$ a $k$-algebra and $M$ an $R$-bimodule. The Hochschild homology $HH_*(R, M)$ of $R$ with coefficients in $M$ was defined in the introduction. It is computed by the *Hochschild complex* $HH(R, M)$, which is given in degree $n$ by

$$HH(R, M)_n = M \otimes R^{\otimes n}.$$

It is equipped with the Hochschild boundary map $b$ defined by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

If $R$ and $M$ happen to be $\mathbb{Z}$-graded, then $HH(R, M)$ splits into a direct sum of subcomplexes

$$HH(R, M) = \bigoplus_{m \in \mathbb{Z}} {}_m HH(R, M).$$

The homogeneous component of degree $m$ of $HH(R, M)_n$ is the linear subspace of $HH(R, M)_n$ generated by all elementary tensors $a_0 \otimes \cdots \otimes a_n$ with $a_i$ homogeneous and $\sum_i |a_i| = m$. One of the first basic properties of the Hochschild complex is that it commutes with filtering colimits. Thus we have

**Lemma 2.1.** *Let $I$ be a filtered ordered set and let $\{(R_i, M_i) : i \in I\}$ be a directed system of pairs $(R_i, M_i)$ consisting of an algebra $R_i$ and an $R_i$-bimodule $M_i$, with algebra maps $R_i \to R_j$ and $R_i$-bimodule maps $M_i \to M_j$ for each $i \le j$. Let $(R, M) = \mathrm{colim}_i (R_i, M_i)$. Then $HH_n(R, M) = \mathrm{colim}_i\, HH_n(R_i, M_i)$ $(n \ge 0)$.*

Let $R_i$ be a $k$-algebra and $M_i$ an $R_i$-bimodule ($i = 1, 2$). The Künneth formula establishes a natural isomorphism ([23, 9.4.1])

$$HH_n(R_1 \otimes R_2, M_1 \otimes M_2) \cong \bigoplus_{p=0}^{n} HH_p(R_1, M_1) \otimes HH_{n-p}(R_2, M_2).$$

Another fundamental fact about Hochschild homology which we shall need is Morita invariance. Let $R$ and $S$ be Morita equivalent algebras, and let $P \in R \otimes S^{op}$-mod and $Q \in S \otimes R^{op}$-mod implement the Morita equivalence. Then ([23, Thm. 9.5.6])

$$HH_n(R, M) = HH_n(S, Q \otimes_R M \otimes_R P). \tag{2.2}$$

**Lemma 2.3.** *Let $R_1, \dots, R_n$ and $S_1, \dots, S_m, \dots$ be a finite and an infinite sequence of algebras, and let $R = \bigotimes_{i=1}^n R_i$, $S_{\le m} = \bigotimes_{j=1}^m S_j$ and $S = \bigotimes_{j=1}^\infty S_j$. Assume:*

1. *$HH_q(R_i) \neq 0 \neq HH_q(S_j)$ $\quad (q = 0, 1)$, $(1 \le i \le n)$, $(1 \le j)$.*
2. *$HH_p(R_i) = HH_p(S_j) = 0$ for $p \ge 2$, $1 \le i \le n$, $1 \le j$.*
3. *$n \neq m$.*

*Then no two of $R$, $S_{\le m}$ and $S$ are Morita equivalent.*

*Proof.* By the Künneth formula, we have

$$HH_n(R) = \bigotimes_{i=1}^{n} HH_1(R_i) \neq 0, \qquad HH_p(R) = 0, \qquad p > n.$$

By the same argument, $HH_p(S_{\le m})$ is non-zero for $p = m$ and zero for $p > m$. Hence if $n \neq m$, $R$ and $S_{\le m}$ do not have the same Hochschild homology, and therefore they cannot be Morita equivalent, by (2.2). Similarly, by Lemma 2.1, we have

$$HH_n(S) = \bigoplus_{J \subset \mathbb{N}, |J| = n} \Big(\bigotimes_{j \in J} HH_1(S_j)\Big) \otimes \Big(\bigotimes_{j \notin J} HH_0(S_j)\Big)$$

so that $HH_n(S)$ is non-zero for all $n \ge 1$, and thus it cannot be Morita equivalent to either $R$ or $S_{\le m}$. $\square$

## 3. Hochschild homology of crossed products

Let $R$ be a unital algebra and $G$ a group acting on $R$ by algebra automorphisms. Form the crossed-product algebra $S = R \rtimes G$, and consider the Hochschild complex $HH(S)$. For each conjugacy class $\xi$ of $G$, the graded submodule $HH^\xi(S) \subset HH(S)$ generated in degree $n$ by the elementary tensors $a_0 \rtimes g_0 \otimes \cdots \otimes a_n \rtimes g_n$ with $g_0 \cdots g_n \in \xi$ is a subcomplex, and we have a direct sum decomposition $HH(S) = \bigoplus_\xi HH^\xi(S)$. The following theorem of Lorenz describes the complex $HH^\xi(S)$ corresponding to the conjugacy class $\xi = [g]$ of an element $g \in G$ as hyperhomology over the centralizer subgroup $Z_g \subset G$.

**Theorem 3.1** ([16])**.** *Let $R$ be a unital $k$-algebra, $G$ a group acting on $R$ by automorphisms, $g \in G$ and $Z_g \subset G$ the centralizer subgoup. Let $S = R \rtimes G$ be the crossed product algebra and $HH^{\langle g \rangle}(S) \subset HH(S)$ be the subcomplex described above. Consider the $R$-submodule $S_g = R \rtimes g \subset S$. Then there is a quasi-isomorphism*

$$HH^{[g]}(S) \xrightarrow{\sim} \mathbb{H}(Z_g, HH(R, S_g)).$$

*In particular, we have a spectral sequence*

$$E^2_{p,q} = H_p(Z_g, HH_q(R, S_g)) \Rightarrow HH^{[g]}_{p+q}(S).$$

*Remark* 3.2. Lorenz formulates his result in terms of the spectral sequence alone, but his proof shows that there is a quasi-isomorphism as stated above. An explicit formula is given for example in the proof of [11, Lemma 7.2].

Let $A$ be a not necessarily unital $k$-algebra and write $\tilde{A}$ for its unitalization. Recall from [24] that $A$ is called *$H$-unital* if the groups $\mathrm{Tor}_n^{\tilde{A}}(k,A)$ vanish for all $n \geq 0$. Wodzicki proved in [24] that $A$ is $H$-unital if and only if for every embedding $A \triangleleft R$ of $A$ as a two-sided ideal of a unital ring $R$, the map

$$HH(A) \to HH(R:A) = \ker(HH(R) \to HH(R/A))$$

is a quasi-isomorphism.

**Lemma 3.3.** *Theorem* 3.1 *still holds if the condition that $R$ be unital is replaced by the condition that it be $H$-unital.*

*Proof.* Follows from Theorem 3.1 and the fact, proved in [11, Prop. A.6.5], that $R \rtimes G$ is $H$-unital if $R$ is as well. □

Let $R$ be a unital algebra and $\phi : R \to pRp$ a corner isomorphism. As in [6], we consider the skew Laurent polynomial algebra $R[t_+, t_-, \phi]$. This is the $R$-algebra generated by elements $t_+$ and $t_-$ subject to the following relations:

$$\begin{aligned} t_+a &= \phi(a)t_+ \\ at_- &= t_-\phi(a) \\ t_-t_+ &= 1 \\ t_+t_- &= p \end{aligned}$$

Observe that the algebra $S = R[t_+, t_-, \phi]$ is $\mathbb{Z}$-graded by $\deg(r) = 0$, $\deg(t_\pm) = \pm 1$. The homogeneous component of degree $n$ is given by

$$R[t_+, t_-, \phi]_n = \begin{cases} t_-^{-n}R & n < 0 \\ R & n = 0 \\ Rt_+^n & n > 0 \end{cases}$$

**Proposition 3.4.** *Let $R$ be a unital ring, $\phi : R \to pRp$ a corner isomorphism, and $S = R[t_+, t_-, \phi]$. Consider the weight decomposition $HH(S) = \bigoplus_{m\in\mathbb{Z}} {}_mHH(S)$. There is a quasi-isomorphism*

$$ {}_mHH(S) \overset{\sim}{\to} \mathrm{Cone}(1 - \phi : HH(R, S_m) \to HH(R, S_m)). \tag{3.5}$$

*Proof.* If $\phi$ is an automorphism, then $S = R \rtimes_\phi \mathbb{Z}$, the right hand side of (3.5) computes $\mathbb{H}(\mathbb{Z}, HH(R, S_m))$, and the proposition becomes the particular case $G = \mathbb{Z}$ of Theorem 3.1. In the general case, let $A$ be the colimit of the inductive system

$$R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \dots.$$

Note that $\phi$ induces an automorphism $\hat{\phi} : A \to A$. Now $A$ is $H$-unital, since it is a filtering colimit of unital algebras, and thus the assertion of the proposition is true for the pair $(A, \hat{\phi})$, by Lemma 3.3. Hence it suffices to show that for $B = A \rtimes_{\hat{\phi}} \mathbb{Z}$ the maps $HH(S) \to HH(B)$ and $\mathrm{Cone}(1 - \phi : HH(R, S_m) \to HH(R, S_m)) \to \mathrm{Cone}(1 - \phi : HH(A, B_m) \to HH(A, B_m))$ ($m \in \mathbb{Z}$) are quasi-isomorphisms. The analogous property for $K$-theory is shown in the course of the third step of the proof of [5, Thm. 3.6]. Since the proof in [5] uses only that $K$-theory commutes with filtering colimits and is matrix invariant on those rings for which it satisfies excision, it applies verbatim to Hochschild homology. This concludes the proof. □

## 4. Hochschild homology of the Leavitt path algebra

Let $E = (E_0, E_1, r, s)$ be a finite quiver and let $\hat{E} = (E_0, E_1 \sqcup E_1^*, r, s)$ be the double of $E$, which is the quiver obtained from $E$ by adding an arrow $\alpha^*$ for each arrow $\alpha \in E_1$, going in the opposite direction. The *Leavitt path algebra* of $E$ is the algebra $L(E)$ with one generator for each arrow $\alpha \in \hat{E}_1$ and one generator $p_i$ for each vertex $i \in E_0$, subject to the following relations:

$$\begin{aligned} p_i p_j &= \delta_{i,j} p_i \qquad (i, j \in E_0) \\ p_{s(\alpha)}\alpha &= \alpha = \alpha p_{r(\alpha)} \qquad (\alpha \in \hat{E}_1) \\ \alpha^*\beta &= \delta_{\alpha,\beta} p_{r(\alpha)} \qquad (\alpha, \beta \in E_1) \\ p_i &= \sum_{\alpha \in E_1, s(\alpha)=i} \alpha\alpha^* \qquad (i \in E_0 \setminus \mathrm{Sink}(E)) \end{aligned}$$

The algebra $L = L(E)$ is equipped with a $\mathbb{Z}$-grading. The grading is determined by $|\alpha| = 1$, $|\alpha^*| = -1$, for $\alpha \in E_1$. Let $L_{0,n}$ be the linear span of all elements of the form $\gamma\nu^*$, where $\gamma$ and $\nu$ are paths with $r(\gamma) = r(\nu)$ and $|\gamma| = |\nu| = n$. By [7, proof of Theorem 5.3], we have $L_0 = \bigcup_{n=0}^{\infty} L_{0,n}$. For each $i$ in $E_0$ and each $n \in \mathbb{Z}^+$, let us denote by $P(n, i)$ the set of paths $\gamma$ in $E$ such that $|\gamma| = n$ and $r(\gamma) = i$. The algebra $L_{0,0}$ is isomorphic to $\prod_{i \in E_0} k$. In general, the algebra $L_{0,n}$ is isomorphic to

$$\Big[\prod_{m=0}^{n-1}\Big(\prod_{i \in \mathrm{Sink}(E)} M_{|P(m,i)|}(k)\Big)\Big] \times \Big[\prod_{i \in E_0} M_{|P(n,i)|}(k)\Big]. \tag{4.1}$$

The transition homomorphism $L_{0,n} \to L_{0,n+1}$ is the identity on the factors

$$\prod_{i \in \mathrm{Sink}(E)} M_{|P(m,i)|}(k),$$

for $0 \le m \le n-1$, and also on the factor

$$\prod_{i \in \mathrm{Sink}(E)} M_{|P(n,i)|}(k)$$

of the last term of the displayed formula. The transition homomorphism

$$\prod_{i \in E_0 \setminus \mathrm{Sink}(E)} M_{|P(n,i)|}(k) \to \prod_{i \in E_0} M_{|P(n+1,i)|}(k)$$

is a block diagonal map induced by the following identification in $L(E)_0$: A matrix unit in a factor $M_{|P(n,i)|}(k)$, where $i \in E_0 \setminus \mathrm{Sink}(E)$, is a monomial of the form $\gamma\nu^*$, where $\gamma$ and $\nu$ are paths of length $n$ with $r(\gamma) = r(\nu) = i$. Since $i$ is not a sink, we can enlarge the paths $\gamma$ and $\nu$ using the edges that $i$ emits, obtaining paths of length $n+1$, and the last relation in the definition of $L(E)$ gives

$$\gamma\nu^* = \sum_{\{\alpha \in E_1 | s(\alpha) = i\}} (\gamma\alpha)(\nu\alpha)^*.$$

Assume $E$ has no sources. For each $i \in E_0$, choose an arrow $\alpha_i$ such that $r(\alpha_i) = i$. Consider the elements

$$t_+ = \sum_{i \in E_0} \alpha_i, \qquad t_- = t_+^*.$$

One checks that $t_-t_+ = 1$. Thus, since $|t_\pm| = \pm 1$, the endomorphism

$$\phi : L \to L, \qquad \phi(x) = t_+ x t_- \tag{4.2}$$

is homogeneous of degree 0 with respect to the $\mathbb{Z}$-grading. In particular, it restricts to an endomorphism of $L_0$. By [6, Lemma 2.4], we have

$$L = L_0[t_+, t_-, \phi]. \tag{4.3}$$

Consider the matrix $N'_E = [n_{i,j}] \in M_{e_0}\mathbb{Z}$ given by

$$n_{i,j} = \#\{\alpha \in E_1 : s(\alpha) = i, \quad r(\alpha) = j\}.$$

Let $e'_0 = |\operatorname{Sink}(E)|$. We assume that $E_0$ is ordered so that the first $e'_0$ elements of $E_0$ correspond to its sinks. Accordingly, the first $e'_0$ rows of the matrix $N'_E$ are 0. Let $N_E$ be the matrix obtained by deleting these $e'_0$ rows. The matrix that enters the computation of the Hochschild homology of the Leavitt path algebra is

$$\begin{pmatrix} 0 \\ 1_{e_0 - e'_0} \end{pmatrix} - N_E^t \colon \mathbb{Z}^{e_0 - e'_0} \longrightarrow \mathbb{Z}^{e_0}.$$

By a slight abuse of notation, we will write $1 - N_E^t$ for this matrix. Note that $1 - N_E^t \in M_{e_0 \times (e_0 - e'_0)}(\mathbb{Z})$. Of course, $N_E = N'_E$ in case $E$ has no sinks.

**Theorem 4.4.** *Let $E$ be a finite quiver without sources, and let $N = N_E$. For each $i \in E_0 \setminus \operatorname{Sink}(E)$ and $m \geq 1$, let $V_{i,m}$ be the vector space generated by all closed paths $c$ of length $m$ with $s(c) = r(c) = i$. Let $\mathbb{Z} = \langle \sigma \rangle$ act on*

$$V_m = \bigoplus_{i \in E_0 \setminus \operatorname{Sink}(E)} V_{i,m}$$

*by rotation of closed paths. We have*

$$_mHH_n(L(E)) = \begin{cases} \operatorname{coker}(1 - \sigma : V_{|m|} \to V_{|m|}) & n = 0, m \neq 0 \\ \operatorname{coker}(1 - N^t) & n = m = 0 \\ \ker(1 - \sigma : V_{|m|} \to V_{|m|}) & n = 1, m \neq 0 \\ \ker(1 - N^t) & n = 1, m = 0 \\ 0 & n \notin \{0, 1\} \end{cases}$$

*Proof.* Let $L = L(E)$, $P = P(E) \subset L$ be the path algebras of $E$ and $W_m \subset P$ be the subspace generated by all paths of length $m$. For each fixed $n \geq 1$ and $m \in \mathbb{Z}$, consider the following $L_{0,n}$-bimodule:

$$L_{m,n} = \begin{cases} L_{0,n} W_m L_{0,n} & m > 0 \\ L_{0,n} W_{-m}^* L_{0,n} & m < 0 \end{cases}$$

Write $L = L(E)$, and let $_mL$ be the homogeneous part of degree $m$; we have

$$_mL = \bigcup_{n \geq 1} L_{m,n}.$$

If $m$ is positive, then there is a basis of $L_{m,n}$ consisting of the products $\alpha\theta\beta^*$ where each of $\alpha$, $\beta$ and $\theta$ is a path in $E$, $r(\alpha) = s(\theta)$, $r(\beta) = r(\theta)$, $|\alpha| = |\beta| = n$ and $|\theta| = m$. Hence the formula

$$\pi(\alpha\theta\beta^*) = \begin{cases} \theta & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases}$$

defines a surjective linear map $L_{m,n} \to V_m$. One checks that $\pi$ induces an isomorphism

$$HH_0(L_{0,n}, L_{m,n}) \cong V_m \quad (m > 0).$$

Similarly, if $m < 0$, then

$$HH_0(L_{0,n}, L_{m,n}) = V^*_{|m|} \cong V_{-m}.$$

Next, by (4.1), we have

$$HH_0(L_{0,n}) = k[E \setminus \mathrm{Sink}(E)] \oplus \bigoplus_{i \in \mathrm{Sink}(E)} k^{r(i,n)}.$$

Here

$$r(i,n) = \max\{r \leq n : P(r,i) \neq \emptyset\}.$$

Now note that because $L_{0,n}$ is a product of matrix algebras, it is separable, and thus $HH_1(L_{0,n}, M) = 0$ for any bimodule $M$. As observed in (4.3), for the automorphism (4.2), we have $L = L_0[t_+, t_-, \phi]$. Hence in view of Proposition 3.4 and Lemma 2.1, it only remains to identify the maps $HH_0(L_{0,n}, L_{m,n}) \to HH_0(L_{0,n+1}, L_{m,n+1})$ induced by inclusion and by the homomorphism $\phi$. One checks that for $m \neq 0$, these are respectively the cyclic permutation and the identity $V_{|m|} \to V_{|m|}$. The case $m = 0$ is dealt with in the same way as in [5, Proof of Theorem 5.10]. □

**Corollary 4.5.** *Let $E$ be a finite quiver with at least one non-trivial closed path.*

i) *$HH_n(L(E)) = 0$ for $n \notin \{0, 1\}$.*
ii) *${}_mHH_*(L(E)) \cong {}_{-m}HH_*(L(E))$ $(m \in \mathbb{Z})$.*
iii) *There exist $m > 0$ such that ${}_mHH_0(L(E))$ and ${}_mHH_1(L(E))$ are both non-zero.*

*Proof.* We first reduce to the case where the graph does not have sources. By the proof of [5, Theorem 6.3], there is a finite complete subgraph $F$ of $E$ such that $F$ has no sources, $F$ contains all the non-trivial closed paths of $E$, $\mathrm{Sink}(F) = \mathrm{Sink}(E)$, and $L(F)$ is a full corner in $L(E)$ with respect to the homogeneous idempotent $\sum_{v \in F^0} p_v$. It follows that $HH_*(L(E))$ and $HH_*(L(F))$ are graded-isomorphic. Therefore we can assume that $E$ has no sources.

The first two assertions are already part of Theorem 4.4. For the last assertion, let $\alpha$ be a primitive closed path in $E$, and let $m = |\alpha|$. Let $\sigma$ be the cyclic permutation; then $\{\sigma^i \alpha : i = 0, \ldots, m-1\}$ is a linearly independent set. Hence $N(\alpha) = \sum_{i=0}^{m-1} \sigma^i \alpha$ is a non-zero element of $V_m^\sigma = {}_mHH_1(L(E))$. Since on the other hand $N$ vanishes on the image of $1 - \sigma : V_m \to V_m$, it also follows that the class of $\alpha$ in ${}_mHH_0(L(E))$ is non-zero. □

## 5. Applications

**Theorem 5.1.** *Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ be finite quivers. Assume that $n \neq m$ and that each of the $E_i$ and the $F_j$ has at least one non-trivial closed path. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $L(F_1) \otimes \cdots \otimes L(F_m)$ are not Morita equivalent.*

*Proof.* Immediate from Lemma 2.3 and Corollary 4.5(iii). □

**Example 5.2.** It follows from Theorem 5.1 that $L_2$ and $L_2 \otimes_k L_2$ are not Morita equivalent. There is another way of proving this, due to Jason Bell and George Bergman [8]. By Theorem 3.3 of [9], $\mathrm{l.gl.dim}\, L_2 \leq 1$. Using a module-theoretic construction, Bell and Bergman show that $\mathrm{l.gl.dim}(L_2 \otimes_k L_2) \geq 2$, which forces $L_2$ and $L_2 \otimes_k L_2$ to not be Morita equivalent. Bergman then asked Warren Dicks whether general results were known about global dimensions of tensor products and was pointed to Proposition 10(2) of [12], which is an immediate consequence of

Theorem XI.3.1 of [10] and says that if $k$ is a field and $R$ and $S$ are $k$-algebras, then $\operatorname{l.gl.dim} R + \operatorname{w.gl.dim} S \leq \operatorname{l.gl.dim}(R \otimes_k S)$. Consequently, if $\operatorname{l.gl.dim} R < \infty$ and $\operatorname{w.gl.dim} S > 0$, then $\operatorname{l.gl.dim} R < \operatorname{l.gl.dim}(R \otimes_k S)$; in particular, $R$ and $R \otimes_k S$ are then not Morita equivalent. To see that $\operatorname{w.gl.dim} L_2 > 0$, write $x_1$, $x_2$, $x_1^*$, $x_2^*$ for the usual generators of $L_2$ and use normal-form arguments to show that $\{a \in L_2 \mid ax_1 = a+1\} = \emptyset$ and $\{b \in L_2 \mid x_1 b = b\} = \{0\}$. Hence, in $L_2$, $x_1 - 1$ does not have a left inverse and is not a left zerodivisor (or see [4]); thus, $\operatorname{w.gl.dim} L_2 > 0$.

We denote by $L_\infty$ the unital algebra presented by generators $x_1, x_1^*, x_2, x_2^*, \ldots$ and relations $x_i^* x_j = \delta_{i,j} 1$.

**Proposition 5.3.** *Let $E$ be any finite quiver having at least one non-trivial closed path. Then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent. Similarly, $L_\infty \otimes L_\infty$ and $L_\infty$ are not Morita equivalent.*

*Proof.* Let $C_n$ be the algebra presented by generators $x_1, x_1^*, \ldots, x_n, x_n^*$ and relations $x_i^* x_j = \delta_{i,j} 1$, for $1 \leq i, j \leq n$. Then

$$L_\infty = \varinjlim C_n \tag{5.4}$$

and $C_n \cong L(E_n)$, where $E_n$ is the graph having two vertices $v, w$ and $2n$ arrows $e_1, \ldots, e_n, f_1, \ldots, f_n$, with $s(e_i) = r(e_i) = v = s(f_i)$ and $r(f_i) = w$ for $1 \leq i \leq n$. (The isomorphism $C_n \to L(E_n)$ is obtained by sending $x_i$ to $e_i + f_i$ and $x_i^*$ to $e_i^* + f_i^*$.) It follows from Theorem 4.4 and (5.4) that the formulas in Theorem 4.4 for ${}_mHH_n(L_\infty)$, $m \neq 0$, hold, taking as $V_{i,m}$ the vector space generated by all the words in $x_1, x_2, \ldots$ of length $m$, and that ${}_0HH_0(L_\infty) = k$ and ${}_0HH_n(L_\infty) = 0$ for $n \geq 1$. As before, Lemma 2.3 gives the result. $\square$

**Theorem 5.5.** *Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m, \ldots$ be a finite and an infinite sequence of quivers. Assume that the number of indices $i$ such that $F_i$ has at least one non-trivial closed path is infinite. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $\bigotimes_{i=1}^\infty L(F_i)$ are not Morita equivalent.*

*Proof.* Immediate from Lemma 2.3 and Corollary 4.5(iii). $\square$

**Example 5.6.** Let $L^{(\infty)} = \bigotimes_{i=1}^\infty L_2$, and let $E$ be any quiver having at least one non-trivial closed path. Then $L^{(\infty)} \otimes L(E)$ and $L(E)$ are not Morita equivalent.

It would be interesting to know the answer to the following question:

**Question 5.7.** Is there a unital homomorphism $\phi \colon L_2 \otimes L_2 \to L_2$?

Observe that to build a unital homomorphism $\phi \colon L_2 \otimes L_2 \to L_2$, it is enough to exhibit a *non-zero* homomorphism $\psi \colon L_2 \otimes L_2 \to L_2$ because $eL_2e \cong L_2$ for every non-zero idempotent $e$ in $L_2$.

## 6. $K$-theory

To conclude the paper we note that algebraic $K$-theory cannot distinguish between $L_2$ and $L_2 \otimes L_2$ or between $L_\infty$ and $L_\infty \otimes L_\infty$. For this we need a lemma which might be of independent interest. Recall that a unital ring $R$ is said to be *regular supercoherent* in case all the polynomial rings $R[t_1, \ldots, t_n]$ are regular coherent in the sense of [13].

**Lemma 6.1.** *Let $E$ be a finite graph. Then $L(E)$ is regular supercoherent.*

*Proof.* Let $P(E)$ be the usual path algebra of $E$. It was observed in the proof of [3, Lemma 7.4] that the algebra $P(E)[t]$ is regular coherent. The same proof gives that all the polynomial algebras $P(E)[t_1, \dots, t_n]$ are regular coherent. This shows that $P(E)$ is regular supercoherent. By [3, Proposition 4.1], the universal localization $P(E) \to L(E) = \Sigma^{-1}P(E)$ is flat on the left. It follows that $L(E)$ is left regular supercoherent (see [5, page 23]). Since $L(E) \otimes k[t_1, \dots, t_n]$ admits an involution, it follows that $L(E)$ is regular supercoherent. □

**Proposition 6.2.** *Let $R$ be regular supercoherent. Then the algebraic $K$-theories of $L_2$ and of $L_2 \otimes R$ are both trivial.*

*Proof.* Let $E$ be the quiver with one vertex and two arrows. Then $L_2 \cong L(E)$, and we have

$$L_2 \otimes R = L_R(E).$$

Applying [5, Theorem 7.6] we obtain that $K_*(L_R(E)) = K_*(L(E)) = 0$. The result follows. □

We finally obtain a $K$-absorbing result for Leavitt path algebras of finite graphs, indeed for any regular supercoherent algebra.

**Proposition 6.3.** *Let $R$ be a regular supercoherent algebra. Then the natural inclusion $R \to R \otimes L_\infty$ induces an isomorphism $K_i(R) \to K_i(R \otimes L_\infty)$ for all $i \in \mathbb{Z}$.*

*Proof.* Adopting the notation used in the proof of Proposition 5.3, we see that it is enough to show that the natural map $R \to R \otimes L(E_n)$ induces isomorphisms $K_i(R) \to K_i(R \otimes L(E_n))$ for all $i \in \mathbb{Z}$ and all $n \geq 1$. Since $R$ is regular supercoherent, the $K$-theory of $R \otimes L(E_n) \cong L_R(E_n)$ can be computed by using [5, Theorem 7.6]. By the explicit form of the quiver $E_n$, we thus obtain that

$$K_i(R \otimes L(E_n)) \cong (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

The natural map $R \to L_R(E_n)$ factors as

$$R \to Rv \oplus Rw \to L_R(E_n)\,.$$

The first map induces the diagonal homomorphism $K_i(R) \to K_i(R) \oplus K_i(R)$, sending $x$ to $(x, x)$. The second map induces the natural surjection

$$K_i(R) \oplus K_i(R) \to (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R).$$

Therefore the natural homomorphism $R \to L_R(E_n)$ induces an isomorphism

$$K_i(R) \xrightarrow{\sim} K_i(L_R(E_n)).$$

This concludes the proof. □

**Corollary 6.4.** *The natural maps $k \to L_\infty \to L_\infty \otimes L_\infty$ induce $K$-theory isomorphisms $K_*(k) = K_*(L_\infty) = K_*(L_\infty \otimes L_\infty)$.*

*Proof.* A first application of Proposition 6.3 gives $K_*(k) = K_*(L_\infty)$. A second application shows that for $E_n$ as in the proof above, the inclusion $L(E_n) \to L(E_n) \otimes L_\infty$ induces a $K$-theory isomorphism; passing to the limit, we obtain the corollary. $\square$


## Acknowledgement

Part of the research for this article was carried out during a visit of the second author to the Centre de Recerca Matemàtica. He is indebted to this institution for its hospitality.


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