A note on the use of variational methods for treatment of plate dynamics

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Abstract: The present note deals with a methodology for development of compact analytical schemes, for the derivation of boundary value problems, which describe the statical and dynamical behaviours of plates with complicating effects. A special set of multi-indices in differentiation symbols and two formulae of integration by parts are introduced to develop compact analytical expressions in the procedures for deriving the governing differential equations and natural boundary conditions.

Keywords: variational calculus, compact analytical procedures, vibrating plates, elastically restrained edges

1 INTRODUCTION

The calculus of variations is concerned with the determination of extremes of functionals, a generalization of the problem of finding extremes of real functions of several variables. It is well known that in a variational approach, the governing equations of a structural problem are obtained by seeking the minimum of total potential energy of the system. The variational principles are the only valid mathematical formulation of certain physical laws. Engineers and applied mathematicians increasingly use the techniques of calculus of variations to solve a large number of problems. Particularly in solid mechanics, the Hamiltonian principle constitutes a formidable tool for obtaining the analytical expressions of the equations of motion and their associated boundary conditions.

Substantial literature has been devoted to the formulation, by means of the calculus of variations, of boundary value problems in the statics and dynamics of isotropic plates [**1–8**]. In these works, the Kirchhoff classical plate theory or the first-order shear deformation theory is considered. Also, anisotropic plates have been treated [**9–11**]. More recently, as a consequence of the dramatic increase in the use of composite material in all types of engineering structures, new structural theories like the third-order laminate theory have been developed [**12**]. In all these texts, classical notations and algebraic manipulations are used. This standard procedure is based on the use of classical symbols to represent partial derivatives, involving tedious algebraic manipulations which inevitably lead to complicated analytical expressions whose details are commonly avoided.

The main aim of this note is to introduce a new analytical manipulation of the variational procedure to obtain the equations of motions and their associated boundary and transition conditions. This efficient approach is based on creating a compact analytical manipulation of the terms of the first variation of the functional involved in the application of Hamiltonian principle. The analytical condensation is performed in two stages: (a) by introducing four multi-indices in differentiation symbols, manipulated by some simple algebraic rules and (b) by developing two formulae of integration by parts, based on the well-known Green's theorem.

The efficiency and shorthand syntax of the approach is demonstrated by generating all the analytical manipulations needed to obtain the boundary value problems, which describe the statical and dynamical behaviours of anisotropic plates with several complicating effects. The use of the proposed condensed notation aids both intuition and mathematical manipulation since it avoids complicated and obscure formulae and allows including all the analytical details.

No claim of originality is made by the author since no new method is presented here. Nevertheless, it is felt that the suggested procedure has important advantages with respect to the traditional analytical manipulations.

2 CLASSICAL TREATMENT

Consider an isotropic plate that in the equilibrium position covers the two-dimensional domain G, with smooth boundary ∂G elastically restrained against rotation and translation, as shown in Fig. 1.

As usual, in order to analyse the transverse displacements of the system under study suppose that the vertical position of the plate at any time t, is described by the function $w = w(x, y, t)$, where $(x, y) \in \bar{G}$ and $\bar{G} = G \cup \partial G$. The rotational restraint is characterized by the function $c_r = c_r(s)$ and the translational restraint by the function $c_t = c_t(s)$, where s is the arc length along the boundary ∂G. At time t, the kinetic energy of the plate is given by

$$E_K(w) = \frac{1}{2}\int\int_G \rho h\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}y \tag{1}$$

where $h(x, y)$ is the plate thickness and ρ the mass density of the isotropic material.

On the other hand, at time t, the total potential energy due to the elastic deformation of the plate and to the elastic restraints on the boundary ∂G, is given by

$$E_D(w) = \frac{1}{2}\int\int_G D\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\partial G} c_r(s)\left(\frac{\partial w}{\partial n}\right)^2 ds + \frac{1}{2}\int_{\partial G} c_t(s) w^2 \mathrm{d}s \tag{2}$$

Hamiltonian principle requires that between times t_0 and t_1 at which the positions of the mechanical system are known, it should execute a motion which makes stationary the functional

$$I(w) = \int_{t_0}^{t_1} (E_K - E_D)\mathrm{d}t \tag{3}$$

on the space of admissible functions.

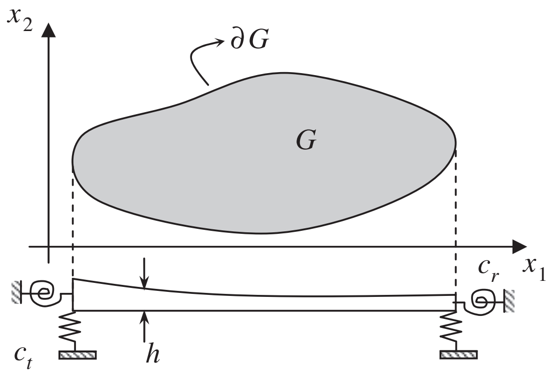

Fig. 1 Plate with elastically restrained boundary

In the classical variational calculus, it is a common practice to use δw to denote a variation of the function w. It is defined by $\delta w = \varepsilon v$, where ε is a small real number and v a function which satisfies determined conditions. Thus, δw is considered as an operator that changes from the function w into δw. The derivatives are changed in the same form. For instance, $\mathrm{d}w/\mathrm{d}x$ is changed into $\delta(\mathrm{d}w/\mathrm{d}x) = \varepsilon \mathrm{d}v/\mathrm{d}x$. The variational operator can be interchanged with derivatives and integrals. For instance

$$\delta \int_\Omega F dx = \int_\Omega \delta F \mathrm{d}x \tag{4}$$

This heuristic procedure requires now to define δF. Thus, if $F = F(x, w, w')$, then the variation δF is given by

$$\delta F = \frac{\partial F}{\partial w}\delta w + \frac{\partial F}{\partial w'}\delta w' \tag{5}$$

From (3), (4), and (5), the condition of stationary functional is given by

$$\delta I = \int_{t_0}^{t_1}\int\int_G \left\{\rho h \frac{\partial w}{\partial t}\frac{\partial(\delta w)}{\partial t} - D\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2(\delta w)}{\partial x^2} + \frac{\partial^2(\delta w)}{\partial y^2}\right) + 2(1-\mu)\left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2(\delta w)}{\partial x \partial y} - \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2(\delta w)}{\partial y^2} - \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\frac{\partial^2(\delta w)}{\partial x^2}\right)\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}t - \int_{t_0}^{t_1}\int_{\partial G} c_r(s)\left(\frac{\partial w}{\partial n}\right)\left(\frac{\partial(\delta w)}{\partial n}\right)\mathrm{d}s\mathrm{d}t - \int_{t_0}^{t_1}\int_{\partial G} c_t(s) w \delta w \mathrm{d}s\mathrm{d}t = 0 \tag{6}$$

Now, the application of the well-known Green's theorem of integration by parts results in a lengthy analytical procedure whose final product is the following boundary value problem

$$\rho h \frac{\partial^2 w}{\partial t^2} + D\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = 0, \quad \forall x \in G \tag{7}$$

$$c_r(s)\frac{\partial w}{\partial n} = -D\left[\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-(1-\mu)\right.$$
$$\left.\left(\frac{\partial^2 w}{\partial x^2}n_x^2+\frac{\partial^2 w}{\partial y^2}n_y^2+2n_x n_y\frac{\partial^2 w}{\partial x\partial y}\right)\right] \tag{8}$$

$$c_t(s)w = D\left\{\frac{\partial}{\partial n}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\left(1-\mu\right)\right.$$
$$\left.\frac{\partial}{\partial s}\left[\frac{\partial^2 w}{\partial x^2}n_x n_y+\frac{\partial^2 w}{\partial y^2}n_x n_y+\frac{\partial^2 w}{\partial x\partial y}\left(n_x^2-n_y^2\right)\right]\right\} \tag{9}$$

It is obvious that if for instance, the plate has anisotropic material, the corresponding variational treatment implies an extremely lengthy analytical manipulation process, revealing the necessity of a more efficient notation.

3 A NEW PROCEDURE FOR THE MANIPULATION OF DERIVATIVES

Consider the well-known notation

$$D^\alpha u(x) = \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}} \tag{10}$$

where $u : S \to \mathbb{R}$, $u \in C^{|\alpha|}(S)$, $S \subset \mathbb{R}^3$, and $x = (x_1, x_2, x_3)$. The vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index whose co-ordinates are non-negative integers and $|\alpha|$ the sum $|\alpha| = \sum_{i=1}^{3}\alpha_i$. Now, introduce the following multi-indices

$$\alpha^{(1)} = (2,0,0), \alpha^{(2)} = (0,2,0), \alpha^{(3)} = (1,1,0), \alpha^{(4)} = (0,0,2)$$
$$\mathbf{1}^{(i)} = (\delta_{1i}, \delta_{2i}, \delta_{3i}), i = 1,2,3 \tag{11}$$

where δ_{ji} is the Kronecker delta, $\delta_{ji} = 1$ if $j = i$ and $\delta_{ji} = 0$ if $j \neq i$.

Consider a function $v : S \to \mathbb{R}$, defined on $S = G \times [0, T]$ for some fixed time $T > 0$, with $x = (x_1, x_2) \in G$, $x_3 = t, G \subset \mathbb{R}^2$. The use of (10) and (11) leads to

$$D^{\mathbf{1}^{(i)}}v(x,t) = \frac{\partial v}{\partial x_i}(x,t),$$
$$D^{\alpha^{(i)}}v(x,t) = \frac{\partial^2 v}{\partial x_i^2}(x,t), i = 1,2,$$
$$D^{\alpha^{(3)}}v(x,t) = \frac{\partial^2 v}{\partial x_1\partial x_2}(x,t), \quad D^{\mathbf{1}^{(3)}}v(x,t) = \frac{\partial v}{\partial t}(x,t),$$
$$D^{\alpha^{(4)}}v(x,t) = \frac{\partial^2 v}{\partial t^2}(x,t) \tag{12}$$

Proposition 1

The multi-indices (11) verify the following algebraic rules

$$\mathbf{1}^{(i)} + \mathbf{1}^{(i)} = \alpha^{(i)}, \quad \forall i \in \{1,2\} \tag{13}$$
$$\mathbf{1}^{(3-i)} + \mathbf{1}^{(i)} = \alpha^{(3)}, \ \forall i \in \{1,2\} \tag{14}$$
$$\mathbf{1}^{(3)} + \mathbf{1}^{(3)} = \alpha^{(4)} \tag{15}$$

Proof

This follows from the sum operation of multi-indices.

Remark 1

If $v \in C^2(\bar{S})$, the order of differentiation is immaterial and by (14)

$$D^{\alpha^{(3)}}v = \frac{1}{2}\sum_{i=1}^{2}D^{\mathbf{1}^{(i)}}\left(D^{\mathbf{1}^{(3-i)}}v\right) \tag{16}$$

The decomposition (16) proves to be valuable in the analytical manipulations used in the next sections. The other essential step to compact analytical expressions, is the derivation of formulae needed to transform the terms which involves derivatives of variations.

Proposition 2

Suppose that $F : S \to \mathbb{R}, v : S \to \mathbb{R}$, $S = G \times [0,T], F(\bullet, t), v(\bullet, t) \in C^2(\bar{G}), G \subset \mathbb{R}^2$ and $i \in \{1,2\}$. Then, the following formula is valid

$$\int_G F(x,t)\left(D^{\alpha^{(i)}}v(x,t)\right)\mathrm{d}x$$
$$= \int_{\partial G}\left[F(x,t)\left(D^{\mathbf{1}^{(i)}}v(x,t)\right)n_i(x) - \left(D^{\mathbf{1}^{(i)}}F(x,t)\right)\right.$$
$$\left.v(x,t)n_i(x)\right]\mathrm{d}s + \int_G\left(D^{\alpha^{(i)}}F(x,t)\right)v(x,t)\mathrm{d}x \tag{17}$$

If $i = 3$, then

$$\int_G F(x,t)\left(D^{\alpha^{(3)}}v(x,t)\right)\mathrm{d}x$$
$$= \frac{1}{2}\sum_{i=1}^{2}\left\{\int_{\partial G}\left[F(x,t)\left(D^{\mathbf{1}^{(3-i)}}v(x,t)\right)n_i(x)\right.\right.$$
$$\left.\left.-\left(D^{\mathbf{1}^{(i)}}F(x,t)\right)v(x,t)n_{3-i}(x)\right]\mathrm{d}s\right\}$$
$$+\int_G\left(D^{\alpha^{(3)}}F(x,t)\right)v(x,t)\mathrm{d}x \tag{18}$$

Proof

The Green formula is given by

$$\iint_G u\frac{\partial v}{\partial x_i}\mathrm{d}x_1\mathrm{d}x_2 = \int_{\partial G}uvn_i\mathrm{d}s$$
$$-\iint_G v\frac{\partial u}{\partial x_i}\mathrm{d}x_1\mathrm{d}x_2, \ u, v \in C^1(\bar{G}) \tag{19}$$

where n_i denotes the ith component of the outward unit normal $\vec{n}$ to the boundary ∂G. Using this formula and the decomposition given by (13) yields

$$\int_G FD^{\alpha^{(i)}} v\mathrm{d}x = \int_G FD^{\mathbf{1}^{(i)}}\left(D^{\mathbf{1}^{(i)}} v\right)\mathrm{d}x$$
$$= \int_{\partial G} F\left(D^{\mathbf{1}^{(i)}} v\right)n_i \mathrm{d}s - \int_G \left(D^{\mathbf{1}^{(i)}} F\right)\left(D^{\mathbf{1}^{(i)}} v\right)\mathrm{d}x$$
$$= \int_{\partial G}\left[F\left(D^{\mathbf{1}^{(i)}} v\right)n_i - \left(D^{\mathbf{1}^{(i)}} F\right)v n_i\right]\mathrm{d}s$$
$$+ \int_G D^{\mathbf{1}^{(i)}}\left(D^{\mathbf{1}^{(i)}} F\right)v\mathrm{d}x$$

Hence, (17) is valid. On the other hand, by (16) and (19)

$$\int_G FD^{\alpha^{(3)}} v\mathrm{d}x = \frac{1}{2}\sum_{i=1}^{2}\int_G FD^{\mathbf{1}^{(i)}}\left(D^{\mathbf{1}^{(3-i)}} v\right)\mathrm{d}x$$
$$= \frac{1}{2}\sum_{i=1}^{2}\left[\int_{\partial G} F\left(D^{\mathbf{1}^{(3-i)}} v\right)n_i \mathrm{d}s - \int_G \left(D^{\mathbf{1}^{(i)}} F\right)\left(D^{\mathbf{1}^{(3-i)}} v\right)\mathrm{d}x\right]$$
$$= \frac{1}{2}\sum_{i=1}^{2}\left\{\int_{\partial G}\left[F\left(D^{\mathbf{1}^{(3-i)}} v\right)n_i - \left(D^{\mathbf{1}^{(i)}} F\right)v n_{3-i}\right]\mathrm{d}s\right.$$
$$\left. + \int_G D^{\mathbf{1}^{(3-i)}}\left(D^{\mathbf{1}^{(i)}} F\right)v\mathrm{d}x\right\}$$

from which (18) follows.

4 APPLICATION TO THE CLASSICAL ANISOTROPIC PLATE THEORY

Consider the plate described in section 2 when it has anisotropic material and is subjected to an external load $q = q(x, t)$. It is well known that the functional needed in the application of Hamiltonian principle is given by [**13**]

$$F(w) = \frac{1}{2}\int_{t_0}^{t_1}\left\{\int_G\left(\rho h\left(\frac{\partial w}{\partial t}\right)^2 - E_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2\right.\right.$$
$$- 2E_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - E_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2$$
$$- 4\frac{\partial^2 w}{\partial x_1 \partial x_2}\left(E_{16}\frac{\partial^2 w}{\partial x_1^2} + E_{26}\frac{\partial^2 w}{\partial x_2^2}\right) - 4E_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2$$
$$\left.\left. + 2qw\right)\mathrm{d}x - \int_{\partial G} c_r(s)\left(\frac{\partial w}{\partial n}\right)^2 \mathrm{d}s - \int_{\partial G} c_t(s) w^2 \mathrm{d}s\right\}\mathrm{d}t \tag{20}$$

where the coefficients $E_{ij} = E_{ij}(x)$ are the rigidities of the anisotropic material [**9**].

In order to avoid the vague mechanical δ method described briefly in section 2, consider the following. The condition of stationary functional which corresponds to (20) requires that

$$\delta F(w; v) = 0, \quad \forall v \in D_a \tag{21}$$

where $\delta F(w; v)$ is the first variation of F at w in the direction v and D_a the space of admissible directions at w for the domain D of this functional. The definition of the variation of F at w in the direction v is given as a generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$ [**14**]. Consequently, the definition of the first variation of F at w in the direction v is given by

$$\delta F(w; v) = \frac{\mathrm{d}F}{\mathrm{d}\varepsilon}(w + \varepsilon v)\bigg|_{\varepsilon=0} \tag{22}$$

It can be noted that v is simply an element of a vector space which generalizes the concept of direction and that a fundamental step is the determination of the spaces of admissible functions and directions. In order to make the mathematical developments required by the techniques of the calculus of variations assume that

$$\rho h \in C(\bar{G}),\ q(\bullet, t) \in C(\bar{G}),\ E_{ij} \in C^2(\bar{G}),$$
$$w(x, \bullet) \in C^2[t_0, t_1],\ w(\bullet, t) \in C^4(\bar{G})$$

From (22), it easily follows that the variation of the functional (20) is given by

$$\delta F(w; v) = \int_{t_0}^{t_1}\left\{\int_G\left[\rho h\frac{\partial w}{\partial t}\frac{\partial v}{\partial t} - E_{11}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2}\right.\right.$$
$$- E_{12}\left(\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right)$$
$$- E_{22}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} - 2E_{16}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)$$
$$- 2E_{26}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)$$
$$\left. - 4E_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2}\right) + qv\right]\mathrm{d}x$$
$$\left. - \int_{\partial G} c_r(s)\frac{\partial w}{\partial n}\frac{\partial v}{\partial n}\mathrm{d}s - \int_{\partial G} c_t(s) wv ds\right\}\mathrm{d}t \tag{23}$$

Introduce the coefficients A_{ij} as the elements of the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} E_{11} & E_{12} & 2E_{16} \\ E_{12} & E_{22} & 2E_{26} \\ 2E_{16} & 2E_{26} & 4E_{66} \end{bmatrix} \tag{24}$$

Using the notation introduced above, the variation (23) can be compacted to

$$\delta F(w; v) = \int_{t_0}^{t_1}\left\{\int_G\left[\rho h\left(D^{\mathbf{1}^{(3)}} w\right)\left(D^{\mathbf{1}^{(3)}} v\right)\right.\right.$$
$$\left. - \sum_{i=1}^{3}\sum_{j=1}^{3}\left(A_{ij} D^{\alpha^{(j)}} w\right)D^{\alpha^{(i)}} v + qv\right]\mathrm{d}x$$
$$\left. - \int_{\partial G} c_r\frac{\partial w}{\partial n}\frac{\partial v}{\partial n}ds - \int_{\partial G} c_t wv ds\right\}\mathrm{d}t \tag{25}$$

In view of all these observations and since Hamiltonian principle requires that at times t_0 and t_1 the positions are known, the space D where the functional (20) is defined is given by

$$D = \{w;\, w(x,\bullet) \in C^2[t_0,t_1],\, w(\bullet,t) \in C^4(\bar{G}),\\ w(x,t_0),\, w(x,t_1) \text{ prescribed}\} \tag{26}$$

The only admissible directions v at $w \in D$ are those for which $w + \varepsilon v \in D$ for sufficiently small ε and $\delta F(w;v)$ exists. In consequence, and in view of (26), v is an admissible direction at w for D if, and only if, $v \in D_a$ where

$$D_a = \{v;\, v(x,\bullet) \in C^2[t_0,t_1],\, v(\bullet,t) \in C^4(\bar{G}),\, v(x,t_0)\\ = v(x,t_1) = 0, \forall x \in \bar{G}\} \tag{27}$$

Consider the first term of (25). Since $w(x,\bullet), v(x,\bullet) \in C^2[t_0,t_1]$, integrating by parts with respect to t and applying the conditions $v(x,t_0) = v(x,t_1) = 0, \quad \forall x \in \bar{G}$, imposed in (27) yields

$$\begin{aligned}&\int_{t_0}^{t_1}\int_G \rho h\left(D^{1^{(3)}}w\right)\left(D^{1^{(3)}}v\right)\mathrm{d}x\mathrm{d}t\\ &= \int_G \rho h\left(D^{1^{(3)}}w\right)v\Big|_{t_0}^{t_1}\mathrm{d}x -\\ &- \int_{t_0}^{t_1}\int_G \rho h\left(D^{1^{(3)}}\left(D^{1^{(3)}}w\right)\right)v\mathrm{d}x\mathrm{d}t\\ &= -\int_{t_0}^{t_1}\int_G \rho h\left(D^{\alpha^{(4)}}w\right)v\mathrm{d}x\mathrm{d}t\end{aligned} \tag{28}$$

To transform the terms of (25) which are multiplied by a coefficient A_{ij}, the formulae (17) and (18) must be employed. Then, if $i \in \{1,2\}$ by (17)

$$\begin{aligned}\int_G S_i D^{\alpha^{(i)}}v\mathrm{d}x &= \int_{\partial G}\left[S_i\left(D^{\mathbf{1}^{(i)}}v\right)n_i - \left(D^{\mathbf{1}^{(i)}}S_i\right)vn_i\right]\mathrm{d}s +\\ &+ \int_G \left(D^{\alpha^{(i)}}S_i\right)v\mathrm{d}x\end{aligned} \tag{29}$$

where

$$S_i = \sum_{j=1}^{3} A_{ij}D^{\alpha^{(j)}}w \tag{30}$$

Finally, by (18)

$$\begin{aligned}&\int_G S_3\left(D^{\alpha^{(3)}}v\right)\mathrm{d}x\\ &= \frac{1}{2}\sum_{i=1}^{2}\int_{\partial G}\left[S_3\left(D^{\mathbf{1}^{(3-i)}}v\right)n_i - \left(D^{\mathbf{1}^{(i)}}S_3\right)vn_{3-i}\right]\mathrm{d}s\\ &+ \int_G\left(D^{\alpha^{(3)}}S_3\right)v\mathrm{d}x\end{aligned} \tag{31}$$

Substituting (28), (29), and (31) into (25) yields

$$\begin{aligned}&\delta F(w;v)\\ &= -\int_{t_0}^{t_1}\int_G\left(\rho h\left(D^{\alpha^{(4)}}w\right) + \sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i\right) - q\right)v\mathrm{d}x\mathrm{d}t\\ &- \int_{t_0}^{t_1}\int_{\partial G}\left\{\sum_{i=1}^{2}\left[S_i\left(D^{\mathbf{1}^{(i)}}v\right)n_i - \left(D^{\mathbf{1}^{(i)}}S_i\right)vn_i.\right.\right.\\ &\left.+\frac{1}{2}\left(S_3\left(D^{\mathbf{1}^{(3-i)}}v\right)n_i - \left(D^{\mathbf{1}^{(i)}}S_3\right)vn_{3-i}\right)\right]\\ &\left.+c_r(s)\frac{\partial w}{\partial n}\frac{\partial v}{\partial n} + c_t(s)wv\right\}\mathrm{d}s\mathrm{d}t\end{aligned} \tag{32}$$

The important properties that are shown in the following proposition, allow an adequate collection of terms.

Proposition 3

The sums in (32) verify the following properties

$$\sum_{i=1}^{2}\left(D^{\mathbf{1}^{(3-i)}}v\right)n_i = \sum_{i=1}^{2}\left(D^{\mathbf{1}^{(i)}}v\right)n_{3-i} \tag{33}$$

$$\sum_{i=1}^{2}\left(D^{\mathbf{1}^{(i)}}S_3\right)n_{3-i} = \sum_{i=1}^{2}\left(D^{\mathbf{1}^{(3-i)}}S_3\right)n_i \tag{34}$$

Proof

The development of the sums in both members leads to the corresponding demonstrations.

As usual, it is convenient from now on to introduce a change of variables in order to deal with the points which correspond to the boundary curve ∂G. The new variables are (n,s) where s is the arc length of the boundary curve ∂G and n a distance measured from ∂G along the unit normal $\vec{n}$ [**13**]. If ∂G is a smooth curve represented in the parametric form by the C^1 function

$$\beta : [0,l] \to \mathbb{R}^2;\ \beta = (\beta_1(s), \beta_2(s)),\, s \in [0,l]$$

where $l = l(\partial G)$ is the length of the curve ∂G, then the relation of the variables (x_1,x_2) with (n,s) leads to [**13**]

$$\begin{aligned}\frac{\partial v}{\partial x_1}(x,t)\Big|_{\partial G} &= \frac{\partial v}{\partial n}(\beta,t)n_1(\beta) - \frac{\partial v}{\partial s}(\beta,t)n_2(\beta)\\ \frac{\partial v}{\partial x_2}(x,t)\Big|_{\partial G} &= \frac{\partial v}{\partial n}(\beta,t)n_2(\beta) + \frac{\partial v}{\partial s}(\beta,t)n_1(\beta)\end{aligned}$$

which can be condensed by

$$D^{\mathbf{1}^{(i)}}v = \frac{\partial v}{\partial n}n_i + (-1)^i\frac{\partial v}{\partial s}n_{3-i},\ i = 1,2, \text{ in } \partial G \tag{35}$$

Substituting (35) into (32) and applying (33) and (34) yields

$$\delta F(w;\,v) = -\int_{t_0}^{t_1}\int_G \left[\rho h\left(D^{\alpha^{(4)}}w\right) + \sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i\right) - q\right] v\mathrm{d}x\mathrm{d}t - \int_{t_0}^{t_1}\int_{\partial G}\left\{\sum_{i=1}^{2}\left[-\left(D^{\mathbf{1}^{(i)}}S_i + 0.5D^{\mathbf{1}^{(3-i)}}S_3\right)vn_i + \frac{\partial v}{\partial n}\left(S_i n_i^2 + 0.5S_3 n_i n_{3-i}\right) + (-1)^i\frac{\partial v}{\partial s}\left(S_i n_i n_{3-i} + 0.5S_3 n_{3-i}^2\right)\right] + c_r(s)\frac{\partial w}{\partial n}\frac{\partial v}{\partial n} + c_t(s)wv\right\}\mathrm{d}s\mathrm{d}t \quad (36)$$

If $P_{i3-i} = (-1)^i\left(S_i n_i n_{3-i} + 0.5S_3 n_{3-i}^2\right)$, an integration by parts yields

$$\int_{\partial G}\frac{\partial v}{\partial s}P_{i3-i}\mathrm{d}s = -\int_{\partial G}\frac{\partial P_{i3-i}}{\partial s}v\mathrm{d}s \quad (37)$$

Replacing (37) into (36) yields

$$\delta F(w;\,v) = -\int_{t_0}^{t_1}\int_G \left[\rho h\left(D^{\alpha^{(4)}}w\right) + \sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i\right) - q\right] v\mathrm{d}x\mathrm{d}t + \int_{t_0}^{t_1}\int_{\partial G}\left\{\sum_{i=1}^{2}\left[\left(\left(D^{\mathbf{1}^{(i)}}S_i + 0.5D^{\mathbf{1}^{(3-i)}}S_3\right)n_i + \frac{\partial P_{i3-i}}{\partial s} - 0.5c_t(s)w\right)v - \frac{\partial v}{\partial n}\left(S_i n_i^2 + 0.5S_3 n_i n_{3-i} + 0.5c_r(s)\frac{\partial w}{\partial n}\right)\right]\right\}\mathrm{d}s\mathrm{d}t \quad (38)$$

According to the condition of stationary functional (21), the expression (38) must vanish for the function w corresponding to the actual motion of the plate for all admissible directions v, and in particular for those admissible v, for which the curvilinear integrals in (38) vanish. Then, the variation (38) reduces to

$$\delta F(w;\,v) = -\int_{t_0}^{t_1}\int_G \left(\rho h\left(D^{\alpha^{(4)}}w\right) + \sum_{i=1}^{3}\left(D^{\alpha^{(i)}}S_i\right) - q\right) v\mathrm{d}x\mathrm{d}t \quad (39)$$

If the fundamental lemma of the calculus of variations is applied, it is concluded that the function w must satisfy the following differential equation

$$\rho h(x)D^{\alpha^{(4)}}w(x,t) + \sum_{i=1}^{3}\left(D^{\alpha^{(i)}}\left(\sum_{j=1}^{3}A_{ij}(x)D^{\alpha^{(j)}}w(x,t)\right)\right) = q(x,t), \quad \forall x \in G,\ t \geq 0 \quad (40)$$

Finally, from (38) and (40), the condition (21) leads to the following natural boundary conditions

$$c_r(s)\frac{\partial w}{\partial n} = -\sum_{i=1}^{2}\left(S_i n_i^2 + 0.5S_3 n_i n_{3-i}\right) \quad (41)$$

$$c_t(s)w = \sum_{i=1}^{2}\left[\left(D^{\mathbf{1}^{(i)}}S_i + 0.5D^{\mathbf{1}^{(3-i)}}S_3\right)n_i + (-1)^i\frac{\partial}{\partial s}\left(S_i n_i n_{3-i} + 0.5S_3 n_{3-i}^2\right)\right] \quad (42)$$

where S_i is given by (30).

5 RECTANGULAR PLATE WITH AN INTERNAL LINE HINGE

Consider an isotropic rectangular plate that in the equilibrium position covers the two-dimensional domain G, with piecewise smooth boundary ∂G elastically restrained against rotation and translation. The plate has also an intermediate line hinge, as shown in Fig. 2. It is assumed that the domain G is divided into two parts $G^{(1)}$ and $G^{(2)}$(with boundaries $\partial G^{(1)}$ and $\partial G^{(2)}$, respectively) by the line $\Gamma^{(c)}$. In this case, consider the expression (25) with

$$\mathbf{A} = \begin{bmatrix} E^{(k)} & \mu E^{(k)} & 0 \\ \mu E^{(k)} & E^{(k)} & 0 \\ 0 & 0 & 2(1-\mu)E^{(k)} \end{bmatrix}$$

where $E^{(k)}$ denotes the flexural rigidity of the plate material which corresponds to the subdomain $G^{(k)}$. In the manner of achieving the boundary value problem in section 4, to the side

$$\Gamma^{(1),1} = \left\{(x_1, b),\, x_1 \in [0, c]\right\}$$

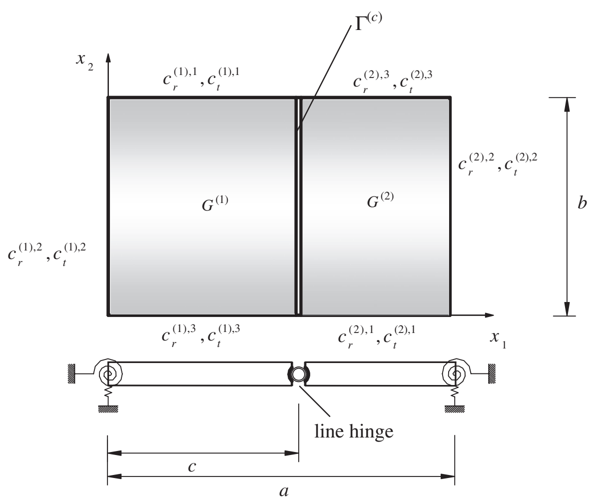

Fig. 2 A rectangular plate with an internal line hinge in a variable position

corresponds the following boundary conditions

$$c_r^{(1),1}(x)D^{\mathbf{1}^{(2)}}w(x,t)\Big|_{(x_1,b)} = -E^{(1)}(x)\Big(D^{\alpha^{(2)}}w(x,t)+\mu D^{\alpha^{(1)}}w(x,t)\Big)\Big|_{(x_1,b)}, \quad \forall x_1\in[0,c]$$

$$c_t^{(1),1}(x)w\Big|_{(x_1,b)} = D^{\mathbf{1}^{(2)}}\Big[E^{(1)}(x)\Big(D^{\alpha^{(2)}}w(x,t)+\mu D^{\alpha^{(1)}}w(x,t)\Big)\Big] +2(1-\mu)D^{\mathbf{1}^{(1)}}\Big(E^{(1)}(x)D^{\alpha^{(3)}}w(x,t)\Big)|_{(x_1,b)},\ \forall x_1\in[0,c]$$

In an analogue form, the remaining boundary conditions can be obtained. Moreover, from the curvilinear integral which corresponds to the line hinge when is considered as part of $A^{(1)}$, i.e.

$$\int_{t_0}^{t_1}\int_{\Gamma^{(c)}}\Big(S_1^{(1)}D^{\mathbf{1}^{(1)}}v-\Big(D^{\mathbf{1}^{(1)}}S_1^{(1)}+D^{\mathbf{1}^{(2)}}S_3\Big)v\Big)\mathrm{d}s\mathrm{d}t \tag{43}$$

and the corresponding to $A^{(2)}$, the transition conditions are given by

$$S_1^{(1)}\Big|_{(c,x_2)}=S_1^{(2)}\Big|_{(c,x_2)}=0,\ \forall x_2\in[0,b] \tag{44}$$

$$D^{\mathbf{1}^{(1)}}S_1^{(1)}+D^{\mathbf{1}^{(2)}}S_3^{(1)}\Big|_{(c,x_2)} = D^{\mathbf{1}^{(1)}}S_1^{(2)}+D^{\mathbf{1}^{(2)}}S_3^{(2)}\Big|_{(c,x_2)},\ \forall x_2\in[0,b] \tag{45}$$

In analogue form, the equations can be obtained which correspond to the subdomain $G^{(2)}$.

The proposed mathematical model is tested to find the non-dimensional frequency coefficient $\Omega=\omega b^2\sqrt{\rho h/E}$ for a rectangular plate elastically restrained against rotation and translation with an internal line hinge. A combination of the Ritz method and the Lagrange multipliers method [**15**] is used to determinate the values of the mentioned frequency coefficient. For the sake of simplicity, assume $E^{(1)}=E^{(2)}=E$ and $h^{(1)}=h^{(2)}=h$. Table 1 depicts the first five values of the frequency parameter $\Omega=\omega b^2\sqrt{\rho h/E}$ for a square plate with the following restraint parameters

$$R_1^{(k)}=c_r^{(k),1}b/E,\ R_2^{(k)}=c_r^{(k),2}a/E,\ R_3^{(k)}=c_r^{(k),3}b/E,$$
$$T_1^{(k)}=c_t^{(k),1}b^3/E,\ T_2^{(k)}=c_t^{(k),2}a^3/E,\ T_3^{(k)}=c_t^{(k),3}b^3/E,\ k=1,2$$

and the internal line hinge is located at $c/a=0.3$.

6 CONCLUDING REMARKS

The compact procedure presented is particularly adequate for the variational derivation of boundary and eigenvalue problems which correspond to different plate theories. The compact analytical expressions generated diminish the analytical effort and the amount of information. This method greatly broadens the applicability and simplifies analytical procedures and particularly, allows the determination of the properties (33) and (34), which are essential in the compactness of analytical expressions. These properties are not evidenced when using the classical analytical developments.

The compact procedure presented is also adequate in the determination of weak solutions. In effect, it is well known that the statical behaviour of the anisotropic plate described in section 4 when a load $q=q(x)$ is applied, is governed by the equation

$$\sum_{i=1}^{3}\left(D^{\alpha^{(i)}}\left(\sum_{j=1}^{3}A_{ij}(x)D^{\alpha^{(j)}}w(x)\right)\right)=q(x),\quad \forall x\in G \tag{46}$$

and the boundary conditions (41) and (42) when the variable t is deleted. Considering an arbitrary function $v\in H^2(G)$, where $H^2(G)$ is a Sobolev space [**16**, **17**], multiplying equation (46) by this function and integrating the result over the domain G yields

$$\int_G\sum_{i=1}^{3}\left[D^{\alpha^{(i)}}\left(\sum_{j=1}^{3}A_{ij}(x)D^{\alpha^{(j)}}w(x)\right)\right]v(x)\mathrm{d}x = \int_G q(x)v(x)\mathrm{d}x \tag{47}$$

Table 1 The first five values of the frequency parameter $\Omega=\omega b^2\sqrt{\rho h/E}$ for a square plate with four edges elastically restrained against rotation $\left(R=R_i^{(1)}=R_i^{(2)},\,i=1,2,3\right)$ and translation $\left(T=T_i^{(1)}=T_i^{(2)},\,i=1,2,3\right)$. The line hinge is located at $c/a=0.3$

		Mode sequence				
$R_i^{(k)}=R$	$T_i^{(k)}=T$	1	2	3	4	5
∞	1	1.997	7.904	9.940	18.645	36.881
	10	6.223	10.535	12.299	20.382	37.666
	100	17.208	23.146	24.946	32.413	44.892
	1000	29.526	46.627	53.785	67.845	79.881
1000	1	1.997	7.892	9.923	18.620	36.773
	10	6.221	10.528	12.287	20.362	37.564
	100	17.183	23.138	24.941	32.411	44.836
	1000	29.424	46.514	53.669	67.737	79.852
100	1	1.996	7.790	9.768	18.407	35.840
	10	6.211	10.470	12.187	20.196	36.683
	100	16.967	23.074	24.901	32.401	44.360
	1000	28.573	45.585	52.723	66.864	79.615
10	1	1.993	6.977	8.570	16.983	29.210
	10	6.117	10.024	11.435	19.105	30.482
	100	15.345	22.652	24.634	32.334	41.219
	1000	23.602	40.682	47.739	62.602	78.411

Now, upon applying the formulae (17) and (18) and the boundary conditions (41) and (42), the corresponding bilinear form is given by

$$B(w, v) = \int_G \sum_{i=1}^{3} \left(\left(\sum_{j=1}^{3} A_{ij}(x) D^{\alpha^{(j)}} w(x) \right) D^{\alpha^{(i)}} v \right) \mathrm{d}x + \int_{\partial G} \left(c_t w v + c_r \frac{\partial w}{\partial n} \frac{\partial v}{\partial n} \right) \mathrm{d}s \tag{48}$$

As usual, a function $w \in H^2(G)$ is called a weak solution of the boundary value problem if (*i*) $w \in H^2(G)$, (*ii*) $B(w, v) = (q, v)_{L^2(G)}, \quad \forall v \in V,$ where V is the space of elements of $H^2(G)$, which satisfy the stable homogeneous boundary conditions.

The compact notation can also be implemented in the variational treatment of the first-order laminate plate theory [**12**] by introducing

$$u_1 = u, u_2 = v, u_3 = w, v_1 = (u_1, 0, u_3), v_2 = (0, u_2, u_3), v_3 = (u_1, u_2, u_3)$$

$$Dv_i = D^{\mathbf{1}^{(i)}} u_i + 0.5 \left(D^{\mathbf{1}^{(i)}} u_3 \right)^2$$

$$Dv_3 = \sum_{k=1}^{2} \left(D^{\mathbf{1}^{(3-k)}} u_k \right) + D^{\mathbf{1}^{(1)}} u_3 D^{\mathbf{1}^{(2)}} u_3$$

Then, the classical terms N_{xx}, N_{yy}, and N_{xy} [**12**] can be compacted as

$$N_i = \sum_{j=1}^{3} \left(E_{ij} Dv_j - F_{ij} D^{\alpha^{(j)}} u_3 \right),$$

$$M_i = \sum_{j=1}^{3} \left(G_{ij} Dv_j - H_{ij} D^{\alpha^{(j)}} u_3 \right), \; i = 1, 2, 3$$

The usual variational procedure leads to the differential equations

$$D^{\mathbf{1}^{(i)}}(N_i) + \left(D^{\mathbf{1}^{(3-i)}} N_3 \right) = I_0 D^{\alpha^{(4)}} u_i - I_1 \left(D^{\mathbf{1}^{(i)}} D^{\alpha^{(4)}} u_3 \right), \quad i = 1, 2,$$

$$\sum_{i=1}^{2} \left(D^{\mathbf{1}^{(i)}} \left(N_i D^{\mathbf{1}^{(i)}} u_3 \right) + D^{\mathbf{1}^{(3-i)}} \left(N_3 D^{\mathbf{1}^{(i)}} u_3 \right) + D^{\alpha^{(i)}} M_i + D^{\alpha^{(3)}} M_3 - I_1 D^{\mathbf{1}^{(i)}} D^{\alpha^{(4)}} u_i + I_2 \left(D^{\alpha^{(i)}} D^{\alpha^{(4)}} u_3 \right) \right) + q - I_0 D^{\alpha^{(4)}} u_3 = 0$$

It is obvious from the presentation that the compact notation has many advantages. It must be remarked that the derived analytical expressions can be recognized at first sight in the standard notation. For instance, equations (44) and (45) can be immediately written as

$$E^{(1)} \left(\frac{\partial^2 w^{(1)}}{\partial x_1^2}(x, t) + \mu \frac{\partial^2 w^{(1)}}{\partial x_2^2}(x, t) \right) \Bigg|_{(c, x_2)} = E^{(2)} \left(\frac{\partial^2 w^{(2)}}{\partial x_1^2}(x, t) + \mu \frac{\partial^2 w^{(2)}}{\partial x_2^2}(x, t) \right) \Bigg|_{(c, x_2)} = 0, \quad \forall x_2 \in [0, b]$$

$$\frac{\partial}{\partial x_1} \left(E^{(1)}(x) \left(\frac{\partial^2 w^{(1)}}{\partial x_1^2}(x, t) + \mu \frac{\partial^2 w^{(1)}}{\partial x_2^2}(x, t) \right) \right) + 2 \frac{\partial}{\partial x_2} \left((1 - \mu) E^{(1)}(x) \frac{\partial^2 w^{(1)}}{\partial x_1 \partial x_2}(x, t) \right) \Bigg|_{(c, x_2)}$$
$$= \frac{\partial}{\partial x_1} \left(E^{(2)}(x) \left(\frac{\partial^2 w^{(2)}}{\partial x_1^2}(x, t) + \mu \frac{\partial^2 w^{(2)}}{\partial x_2^2}(x, t) \right) \right) + 2 \frac{\partial}{\partial x_2} \left((1 - \mu) E^{(2)}(x) \frac{\partial^2 w^{(2)}}{\partial x_1 \partial x_2}(x, t) \right) \Bigg|_{(c, x_2)}, \quad \forall x_2 \in [0, b]$$

Admittedly, it is not a new method but nevertheless it is an important tool for the analytical manipulation which exhibits advantages even from a pedagogical point of view.

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APPENDIX

Notation

a, b	rectangular plate dimensions (Fig. 2)
$B(u, v)$	bilinear form
$c_r(s)$	rotational stiffness
$c_t(s)$	translational stiffness
$C^n(G)$	space of all functions which, together with all their partial derivatives of orders $\leq n$, are continuous on G
$E, E^{(k)}$	flexural rigidities of isotropic plate
E_D	strain energy
E_{ij}	bending, twisting, and coupling rigidities of anisotropic plate
E_K	kinetic energy
G	plate domain
$\bar{G}$	closure of G, $\bar{G} = G \cup \partial G$
$G^{(i)}$	plate subdomain
$G \times [0, T]$	Cartesian product of G an $[0, T]$
h	plate thickness
$H^2(G)$	Sobolev space of order two
n_x, n_y	direction cosines of the outward unit normal
q	transversal load
$R_i^{(k)}, T_i^{(k)}$	non-dimensional rotational and translational coefficients
s	arc length
t	time
v	admissible direction
w	deflection function
x_1, x_2	Cartesian coordinates
$\alpha, \alpha^{(i)}, \mathbf{1}^{(i)}$	multi-index vectors
$\delta I(u; v)$	first variation of functional I
∂G	plate boundary
μ	Poisson's ratio
ρ	mass density
ω	circular natural frequency of plate vibration
Ω	non-dimensional frequency parameter