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Robust adaptive control using multiple models, switching and tuning

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Abstract: The supervisory control problem is analysed as an online robust design problem using switching to select the relevant models for designing the control law. The proposed supervisory control algorithm is based on the integration of concepts used in supervisory adaptive control, robust control and receding horizon control. It involves a two-stage adaptive control algorithm: (i) the identification of a time-varying set of models $\mathcal{P}_{\mathcal{L}(k)}$, from the set of admissible models $\mathcal{P}_{\mathcal{L}}$, that explains the input–output behaviour of the system, followed by (ii) the design of the control law using a parametric linear optimisation problem. The authors show that under the proposed supervisory control algorithm, the system output remains bounded for any bounded disturbance. The use of superstability concepts, together with certain assumptions on $\mathcal{P}_{\mathcal{L}}$, allows us to establish overall performance and robust stability guarantees for the supervisory scheme and to include constrains in the closed-loop variables as well as in the controller structure. The relevant features of the proposed control algorithm are demonstrated in a numerical simulation.

1 Introduction

Adaptive systems control has been investigated for over four decades. Since the beginning, for the sake of mathematical tractability, adaptive control theorists confined their attention to time-invariant systems with unknown parameters or slow drifts in the parameters [1, 2]. The accepted philosophy was that if an adaptive system was faster than the system parameters variation and accurate enough, the convergence to real parameters and the closed-loop stability can be guaranteed [3]. Based on this general principle, adaptive control was extensively studied and numerous robust adaptive control algorithms were derived [4]. In this framework, the problem of selecting the best controller according to a given performance index can be addressed, along with a dual control approach [5]. However, such approach is generally difficult to implement because of its computational burden. Besides, the closed-loop system shows a poor performance during the transient phase, exhibiting an oscillatory behaviour, when there are large errors in the initial parameters estimates [3]. A computationally feasible, even though sub-optimal, approach to design adaptive controllers is the so-called supervisory control, also known as multiple model adaptive control (MMAC), originally introduced in [6] and further developed in the last two decades.

There is a great amount of work on supervisory control, see [7] and references therein. Early works on the subject used a strategy of sequentially stepping through controllers until we find one that stabilises the plant [6]. The selection of the candidate controller is based on monitoring the output of the system over a moving window of time. The next controller is switched into the loop if the value of the monitoring function for the second half of the observation window is higher than its value for the first half, assuming that the length of the window is sufficiently large. A model-based approach to supervisory control was introduced in [8, 9]. In this case, the supervisor evaluates a set of performance signals $\mu_l(k)$, which are estimates of the output error with respect to each candidate model $e_l(k) = y(k) - y_l(k)$, then the supervisor switches into the loop the controller with the best performance signal. Others approaches to supervisory control use calibrate forecasts [10] and a bank of observers for the system identification combined with a hypothesis testing strategy to select the controller [11, 12]. Other model-based approaches to supervisory control can be found in [13–15].

An alternative approach to supervisory control is the cost-based unfalsified control approach [16, 17], which is a non-identifier-based deterministic approach. It is a model-free approach that employs the closed-loop data to select the right controller. Based on the performance assessments, the controllers that do not meet a pre-specified desired performance condition are rejected from the set of candidate controllers. For a cost-based supervisory scheme to be effective, the cost should be representative of the objectives of the problem [18]. Simulation studies [19, 20] report unacceptable transients since closed-loop data may reflect the effect of initial conditions, resulting in the selection of poorly performing controllers until measurements quality improves. Therefore to achieve a good closed-loop performance, it is necessary to include information about the plant model [20]. To deal with this problem, an adaptive control scheme that combines both the approaches, that is, cost-based falsification and model-based switching, has been proposed recently [21, 22].

In most of the works that deal with these approaches to adaptive control, the switching is based on the certainty equivalence principle. The implementation and the analysis of a supervisory control are simplified by considering a finite number of candidate controllers (finite cover set) [23]. The compromise between robustness and performance is made offline when the cover set is designed. If the cover set comprises a small number of controllers, each one stabilising a wide set of models, then stability is rapidly achieved, even before a large amount of information has been accrued, but in the long run the resulting performance is typically poor. In contrast, if the cover set comprises a large number of controllers, each one tailored to a narrow set of models, high performance is potentially achieved, but poor performance will possibly arise during the transient phase until there is enough data to obtain an accurate estimate of the system model.

Supervisory control schemes provide an attractive framework for combining adaptive and robust control tools. They are capable of overcoming the loss of stabilisation problem and respond rapidly to abrupt parameters changes. However, switching may introduce undesirable behaviours that could affect closed-loop performance. One of these behaviours is the intermittent switching among similar models, which is caused by the algorithms hysteresis when the plant is near the boundary of several models. This fact may lead the supervisor to select the controller that does not achieve a desirable closed-loop behaviour, despite that the observed data indicate an acceptable candidate controller (persistent selection of poorly performing controllers). To encourage switching, hysteresis constant can be reduced or replaced with dwell-time logic but at the increased risk of long-term intermittent switching, resulting in transients from initialisation of the new controller. Finally, robust performance is only recovered in the steady-state phase.

The aim of this work is to develop a new robust supervisory adaptive control approach, called multiple models, switching and tuning, capable of overcoming the problems described in previous paragraphs, offering an alternative to the existing supervisory adaptive control approaches. The proposed supervisory adaptive scheme differs from previous developments in three aspects:

- The controller is designed online using a convex optimisation problem and a time-varying polytopic linear model (PLM) $\mathcal{P}_{\mathcal{L}(k)}$ that is identified from a parameterised class of admissible system models $\mathcal{P}_{\mathcal{L}}$.
- The PLM $\mathcal{P}_{\mathcal{L}(k)}$ is built by excluding those models of $\mathcal{P}_{\mathcal{L}}$ that cannot explain the time evolution of the plant input–output trajectories, and
- The switching takes places in the cost function and constraints of the optimisation problem employed to design the controller.

The proposed algorithm exploits the advantages of superstable systems [24] in the derivation of the optimisation problem used to design the controller, which is convex in the controller parameters and allows the inclusion of robustness and performance specifications. The PLM $\mathcal{P}_{\mathcal{L}(k)}$ is built using set-valued observers (SVOs) that track the output estimates with a minimal size confidence ellipsoid consistent with the measurements, the uncertainty description and the effect of unknown but bounded noises [25]. This approach to supervisory control is equivalent to a standard supervisory scheme that has an infinite number of controllers with different degrees of robustness, plus the additional benefit of constraints handling.

This paper is structured as follows: the class of superstable system is recalled and some properties of this class of systems are analysed in Section 2. The class of linear time-varying (LTV) superstable systems is introduced and analysed for the special case of switching systems. Besides, a controller design procedure based on superstability [24] is recalled at the end of the section. In Section 3 the multiple models switching and tuning control approach is proposed, and the stability and performance of the resulting closed-loop system is analysed. Section 4 presents a numerical example. Finally, concluding remarks and possible extensions of the proposed adaptive control algorithm are presented in Section 5.

2 Preliminaries

2.1 Superstable systems

Consider a linear time-invariant (LTI) discrete-time closed-loop system described by a scalar equation

$$(1 + A(z))y(k) = B(z)w(k) \tag{1}$$

where $w(k) \in l_\infty$ is an exogenous input, $y(k)$ is the system output, z is the back-shift operator $zy(k) = y(k-1)$, $A(z)$ and $B(z)$ are given polynomials

$$A(z) = \sum_{i=1}^{n_A} a_i z^i, \quad B(z) = \sum_{i=0}^{n_B} b_i z^i, \quad n_A \geq n_B \tag{2}$$

In the sequel, we denote

$$||A||_1 = \sum_{i=1}^{n_A} |a_i|, \quad ||B||_1 = \sum_{i=0}^{n_B} |b_i| \tag{3}$$

and l_∞ stands for $\|w(k)\|_\infty \leq 1$, $\forall k \geq 0$.

Definition 1: The system (1) is superstable if $\|A\|_1 < 1$.

Note that superstability is formulated in terms of coefficients of the polynomial instead of its roots. Therefore a superstable system is stable, but the converse does not hold. Similar results can be found in a number of textbooks on linear algebra and system theory [26–29].

Discrete-time superstable system has a number of important properties. The main one is that superstable systems admit non-asymptotic estimates for all time steps and arbitrary initial conditions while stable systems only have asymptotic estimates.

Lemma 1: If the discrete system (1) is superstable and the initial conditions $|y(i)| \leq \mu$, $\forall i < 0$, then

- for $|w(k)| \leq 1$ $\forall k$, the norm of the system output is bounded by

$$|y(k)| \leq \eta + ||A||_1^{(k+1)/n_A} \max\{0, \mu - \eta\}, \quad \forall k \geq 0 \tag{4}$$

where

$$\eta = ||B||_1/(1 - ||A||_1) \tag{5}$$

is the equalised performance of the system [30].

- for $w(k) = 0\ \forall k$, the norm of the system output is bounded by

$$|y(k)| \leq ||A||_1^{(k+1)/n_A}\mu, \quad \forall k \geq 0 \tag{6}$$

Proof: See [24], p. 135. □

These results indicate that the norm of a superstable system output decreases monotonically, while this property cannot be guaranteed for stable systems whose estimate is

$$|y(k)| \leq C(\varepsilon)(\rho + \varepsilon)^k \mu, \quad \varepsilon > 0, \ \rho + \varepsilon < 1$$

The constant $C(\varepsilon)$ may be large and $|y(k)|$ can increase rather than decay monotonically at initial iterations.

Remark 1: If a system is superstable and $\mu \leq \eta$, then all trajectories verifies $|y(k)| \leq \eta,\ \forall k$ for all admissible perturbations.

These results can be extended to uncertain systems in a straightforward way. Suppose that we use an additive uncertainty description to represent the true system

$$A(z) = A^0(z) + \Delta A(z), \quad B(z) = B^0(z) + \Delta B(z),$$
$$||\Delta A||_1 \leq \varepsilon_A, \quad ||\Delta B||_1 \leq \varepsilon_B \tag{7}$$

where $A^0(z)$, $\Delta A(z)$, $B^0(z)$ and $\Delta B(z)$ are polynomials of the same form as in (2). We say that the uncertain system is robustly superstable if

$$||A||_1 \leq 1 - \varepsilon_A \tag{8}$$

and the robust equalised performance is given by

$$\eta = \frac{||B^0||_1 + \varepsilon_B}{1 - ||A||_1 - \varepsilon_A} \tag{9}$$

Note that perturbations $\Delta A(z)$ and $\Delta B(z)$ can be LTV such that they model a bounded non-linear system as well.

Another important property of a superstable system is its simple behaviour for LTV systems

$$(1 + A_k(z))y(k) = B_k(z)w(k) \tag{10}$$

where

$$A_k(z) = \sum_{i=1}^{n_A} a_i(k)z^i, \quad B_k(z) = \sum_{i=0}^{n_B} b_i(k) \tag{11}$$

The results obtained for LTI superstable systems can be extended to LTV systems by analysing the behaviour of the LTI frozen systems. Indeed, the superstability of frozen LTI systems ($\|A_k\|_1 < 1\ \forall k$) implies superstability of LTV system, guaranteeing an equalised performance

$$\bar{\eta} = \frac{\bar{\beta}}{1 - \bar{\gamma}} \tag{12}$$

and the norm of the system output is bounded by

$$|y(k)| \leq \bar{\eta} + \bar{\gamma}^{(k+1)/n_A} \max\{0, \mu - \bar{\eta}\} \tag{13}$$

where

$$\bar{\beta} = \max_{\forall k} ||B_k||_1, \quad \bar{\gamma} = \max_{\forall k} ||A_k||_1 \tag{14}$$

Now, let us consider the special case of a LTV system of the form

$$(1 + A_{S(k)}(z))y(k) = B_{S(k)}(z)w(k) \tag{15}$$

where

$$A_{S(k)}(z) = \sum_{l \in \mathcal{L}} s_l(k) \sum_{i=1}^{n_A} a_i^l z^i$$
$$B_{S(k)}(z) = \sum_{l \in \mathcal{L}} s_l(k) \sum_{i=0}^{n_B} b_i^l z^i \tag{16}$$

$S(k) = [s_l(k)] \in \mathcal{S}\ \forall l \in \mathcal{L}$ are the time-varying parameters, $\mathcal{L}$ is a finite index set, that is, $\mathcal{L} = \{1, 2, \ldots, m\}$ and $\mathcal{S}$ is a given compact set.

The results obtained for LTI systems can be extended to LTV systems as follows:

Definition 2: A LTV system is superstable if $\|A_{S(k)}\|_1 < 1$, $\forall S(k) \in \mathcal{S}$.

Lemma 2: If the LTV system (15) is superstable and the initial conditions $|y(i)| \leq \mu\ \forall i < 0$, then the following facts hold for all admissible trajectories of $S(k) \in \mathcal{S}$:

- for $|w(k)| \leq 1\ \forall k$, the norm of the LTV system output is bounded by

$$|y(k)| \leq \bar{\eta} + \bar{\gamma}^{(k+1)/n_A} \max\{0, \mu - \bar{\eta}\} \tag{17}$$

where

$$\bar{\eta} = \frac{\bar{\beta}}{1 - \bar{\gamma}}, \quad \bar{\beta} = \sup_{\forall k} \sup_{\forall S(k) \in \mathcal{S}} ||B_{S(k)}||_1,$$
$$\bar{\gamma} = \sup_{\forall k} \sup_{\forall S(k) \in \mathcal{S}} ||A_{S(k)}||_1 \tag{18}$$

- for $w(k) = 0\ \forall k$, the norm of the LTV system output is bounded by

$$|y(k)| \leq \bar{\gamma}^{(k+1)/n_A}\mu \tag{19}$$

Proof: See Appendix 1. □

The first property is related to input–output stability (BIBO stability) and disturbance rejection, whereas the second property is the system behaviour with respect to initial conditions. Thus, the switching between superstable systems leads to a superstable LTV system. It is well

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known that this does not hold for the more general class of stable systems [31].

2.2 Superstability-based controller design

Let us consider a standard unity feedback discrete-time single-input single-output (SISO) system (other controller structures, like a two-degree of freedom controller can be considered in the controller design), comprising a plant $P(z) = N(z)/D(z)$ and a controller $C(z) = F(z)/G(z)$, whose performance will be optimised for the rejection of bounded disturbances (see Fig. 1). Among such controllers, we are interested in the one that minimises the performance index

$$J = \sup_{w(k)\in l_\infty} \sup_{\forall k} |e(k)| \tag{20}$$

which provides the maximal reduction of the effect of disturbances. In a more general setting, a linear function of the error $e(t)$ and manipulated variable $u(k)$ can be considered rather than the error itself. [Other cost functions or combination of them can be employed, leading to different design problems. The structure of the resulting optimisation problem is the same like (21) and they only differ on the cost function minimised (see [24, 32]).]

Lemma 3: If there exist a controller $C(z)$ such that the resulting closed-loop system is superstable, then the minimisation of (20) is equivalent to solve the parametric linear problem

$$\begin{aligned} &\min_{\sigma\in[0,1)} \quad \min_{F,G} \frac{1}{1-\sigma} ||DG||_1 \\ &\text{such that} \\ &||DG + NF - 1||_1 \le \sigma \end{aligned} \tag{21}$$

For the optimal values σ^*, F^* and G^* the inequality $|e(k)| < \eta^*\ \forall k > 0$ is satisfied for the closed-loop system, provided that it is satisfied for $e(i)\ \forall i < 0$.

Proof: See Appendix 2. □

The feasibility of this optimisation problem is guaranteed for polynomials $F(z)$ and $G(z)$ of order $n_F = n_D$ and $n_G = n_N - 1$, respectively, due to the Bezout theorem. Using standard tools, problem (21) can be reformulated as an LP with respect to the parameters of the controller. Therefore it allows the inclusion of constraints in the structure and parameters of $C(z)$.

Combining the feasibility of (21) with Lemma 1, the resulting closed-loop system is superstable with an equalised performance η^*, an estimate of the maximum closed-loop error $\sup_{\forall k}|e(k)| \le r\eta^*$ and the norm of its output is bounded by (4) for any initial condition or disturbance.

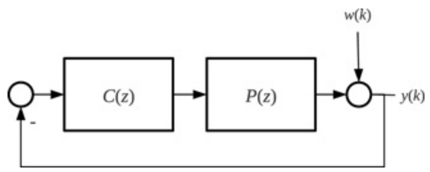

Fig. 1 *Closed-loop system considered in the design problem*

Robust design: The effect of uncertainty on the controller design can be easily included into problem (21). The structure of the resulting problem depends on the description of uncertainty adopted. If the additive uncertainty description is considered (7), the sensitivity function $\phi(z)$ is given by

$$\phi(z) = \frac{(D(z) + \Delta D(z))G(z)}{(D(z) + \Delta D(z))G(z) + (N(z) + \Delta N(z))F(z)} \tag{22}$$

and $\eta(\phi)$ becomes

$$\eta(\phi) = \frac{||DG||_1 + \varepsilon_D||G||_1}{1 - ||DG + NF - 1||_1 - \varepsilon_D||G||_1 - \varepsilon_N||F||_1} \tag{23}$$

This index leads to an LP optimisation problem similar to (21), with the inclusion of the uncertainty terms

$$\begin{aligned} &\min_{\sigma\in[0,1)} \min_{F,G} \frac{1}{1-\sigma} ||DG||_1 + \varepsilon_D||G||_1 \\ &\text{such that} \\ &||DG + NF - 1||_1 + \varepsilon_D||G||_1 + \varepsilon_N||F||_1 \le \sigma \end{aligned} \tag{24}$$

If a polytopic model $\mathcal{P}_\mathcal{L}$ is employed to represent the plant $P \in \mathcal{P}_\mathcal{L}$ such that

$$\begin{aligned} \mathcal{P}_\mathcal{L} &= \text{co}(P_l, \mathcal{L}) \\ &= \left\{ P/P = \sum_{l\in\mathcal{L}} \theta_l P_l,\ \sum_{l\in\mathcal{L}} \theta_l = 1,\ \theta_l \ge 0 \right\} \end{aligned} \tag{25}$$

the robust rejection of bounded disturbances problem becomes

$$\begin{aligned} &\min_{\sigma\in[0,1)} \min_{F,G} \frac{1}{1-\sigma} \sum_{l\in\mathcal{L}} \theta_l ||D_l G||_1 \\ &\text{such that} \\ &||D_l G + N_l F - 1||_1 \le \sigma, \quad l \in \mathcal{L} \end{aligned} \tag{26}$$

The coefficients θ_l allows the designer to assign different weights to each model P_l such that its influence on the controller design can be emphasised. This optimisation problem corresponds to a hybrid characterisation of the robust design problem where closed-loop performance is measured through a weighted-norm objective function

$$\frac{1}{1-\sigma} \sum_{l\in\mathcal{L}} \theta_l ||D_l G||_1$$

that includes the closed-loop performances of all models of $\mathcal{P}_\mathcal{L}$, while the superstability is guaranteed by the constraints by ensuring the superstability of each model of $\mathcal{P}_\mathcal{L}$

$$||D_l G + N_l F - 1||_1 \le \sigma, \quad l \in \mathcal{L}$$

Finally, these two approaches to robust design can be combined to tackle the uncertainty of the individual models $P_l \in \mathcal{P}_\mathcal{L}$ adding the extra terms of (24) in the objective function and constraints of (26).

3 Adaptive control using multiple models, switching and tuning

The objective of many supervisory adaptive control schemes is to control an unknown LTI plant P such that it is subject to an exogenous unknown disturbance $w(k) \in l_\infty$. Supervisory controllers use information obtained online to switch among a finite set of candidate robust non-adaptive controllers $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ such that the control law is given by $u(k) = C_{S(k)}(z)e(k)$, where the switching signal $S(k)$: $[0, \infty) \to \mathcal{L}$ is admissible if it is piecewise constant with a dwell-time $\tau > 0$ such that consecutive switching times $t_a < t_b$ satisfy $t_b - t_a \geq \tau$. The switching is based on the estimated performance of each controller $C_l(z)\ \forall l \in \mathcal{L}$, computed using the online measurements.

In this work we adopt a different approach: instead of switching among a set of non-adaptive controllers, we use a soft-variable controller $C_{\mathcal{L}(k)}(F_k(z), G_k(z))$ that is designed using superstability concepts and a time-varying PLM $\mathcal{P}_{\mathcal{L}(k)}$, whose structure is shown in Fig. 2*b*. At every sample, the monitoring block $\mathbf{M}$ builds $\mathcal{P}_{\mathcal{L}(k)} \subseteq \mathcal{P}_{\mathcal{L}}$ with only those models P_l that explain the input–output trajectory of the plant P, such that $P \in \mathcal{P}_{\mathcal{L}(k)}$, using information generated by the estimation block SVO, which is composed of one SVO for each model of $\mathcal{P}_{\mathcal{L}}$. $\mathcal{P}_{\mathcal{L}(k)}$ is characterised by the set of indexes $\mathcal{L}(k) \subseteq \mathcal{L}$, which is used by the supervisor block $\mathbf{S}$ to generate the switching signal $S(k) = [s_l(k)]$

$$s_l(k) = \begin{cases} 1, & \forall l \in \mathcal{L}(k) \\ 0, & \forall l \notin \mathcal{L}(k) \end{cases} \tag{27}$$

and the optimisation weights $\Theta(k) = [\theta_l(k)]$

$$\theta_l(k) = \begin{cases} 1 - \dfrac{\mathrm{tr}(\mathbf{P}_l(k|k))}{\sum_{l\in\mathcal{L}(k)} \mathrm{tr}(\mathbf{P}_l(k|k))}, & \forall l \in \mathcal{L}(k) \\ 0, & \forall l \notin \mathcal{L}(k) \end{cases} \tag{28}$$

where $\mathbf{P}_l(k|k)$ is the covariance matrix of each set-value observer. These signals are used to select the models, and the corresponding constraints, to design $C_{\mathcal{L}(k)}$ by computing the parameters of F_k and G_k through the optimiser. Then, the resulting controller polynomials $F_k(z)$ and $G_k(z)$ are updated at every sample to compute the control law $u(k)$.

Fig. 2 depicts the structure for both supervisory controllers, where we can see the differences between both schemes. They have a supervisor and a monitoring block with similar structure but different internal variables:

- the estimation errors $e_l(k)$, monitoring signals $\mu_l(k)$ and switching variables $S(k)$ in the supervisory adaptive controller;
- the covariance matrices $\mathbf{P}_l(k|k)$, set of index $\mathcal{L}(k)$, switching variables $S(k)$ and optimisation weights $\Theta(k)$ in the proposed controller.

The difference in the information employed by each scheme is due to the structure of the controller: a set of m robust non-adaptive controllers in the supervisory adaptive controllers and a soft-variable controller [33] in the proposed structure, which is equivalent to use an infinite cover set ($m \to \infty$). It should be obvious that employing an infinite number of controllers increase the flexibility in terms of the control objectives, robustness and performance, since the values of the eigenvalues can continuously change and the regulation rate will thus be greater, than would be in the case of using a switching controller.

3.1 Controller structure

The proposed adaptive control law has the format

$$u(k) = C_{\mathcal{L}(k)}(F_k(z), G_k(z))e(k) \tag{29}$$

where $F_k(z)$ and $G_k(z)$ are the polynomials of the controller

$$F_k(z) = \sum_{i=0}^{n_F} f_i(k)z^i, \quad G_k(z) = 1 + \sum_{i=1}^{n_B} g_i(k)z^i \tag{30}$$

which are obtained solving the following optimisation problem at every sample

$$\begin{aligned} &\min_{\sigma\in[0,1)} \min_{F_k, G_k} \frac{1}{1-\sigma} \sum_{l\in\mathcal{L}(k)} \theta_l(k) \|D_l G_k\|_1 \\ &\text{such that} \\ &s_l(k)\|D_l G_k + N_l F_k - 1\|_1 \leq \sigma \quad l \in \mathcal{L}(k) \end{aligned} \tag{31}$$

The main assumptions on the polytope $\mathcal{P}_{\mathcal{L}}$ and controller order are

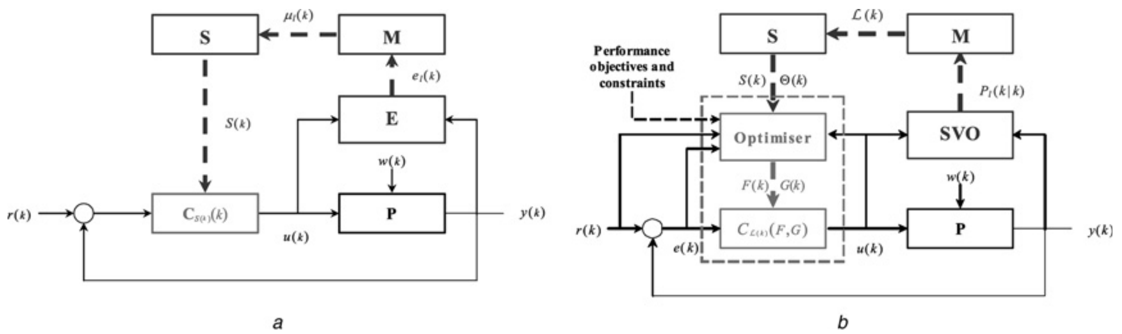

Fig. 2 *Structure of the supervisory controller*

a Supervisory adaptive
b Multiple, models, switching and tuning

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Assumption 1: The order of the polynomials $F_k(z)$ and $G_k(z)$ are

$$n_F = \max_{\forall l \in \mathcal{L}} (n_{Dl}), \quad n_G = \max_{\forall l \in \mathcal{L}} (n_{Nl}) - 1 \tag{32}$$

Assumption 2: The optimisation problem (31) is feasible for $\mathcal{L}(k) = \mathcal{L}$.

These assumptions ensure the existence of control law (29) for all samples by guaranteeing the feasibility of the optimisation problem (31). Assumption 1 guarantees that the control law has enough parameters to freely place the poles (a lower order controller may exist, but the feasibility of (31) needs to be checked), whereas Assumption 2 guarantees the feasibility of (31) for the worst case: the superstability of the entire polytope $\mathcal{P}_{\mathcal{L}(k)} = \mathcal{P}_{\mathcal{L}}$.

If the desired closed-loop performance is defined through constraints on the closed-loop variables, the feasibility of optimisation problem (31) will depend on this constraints. One way of ensuring the feasibility of the optimisation problem (31) is by softening performance constraints with slack variables and then penalised their deviation by including the slack variables in the objective function [34].

3.2 Supervisory algorithm

The role of the supervisory algorithm is to built $\mathcal{P}_{\mathcal{L}(k)}$ by excluding those models $P_l \in \mathcal{P}_{\mathcal{L}}$ that do not explain the time evolution of the input–output trajectories of plant P. $\mathcal{P}_{\mathcal{L}(k)}$ is build using the switching signal $S(k)$, which define the models P_l that will be employed by the optimisation problem (31). The problem of disqualifying models is addressed using SVOs [12]. This type of observers assume that the initial conditions of the plant is uncertain, there are disturbances acting upon the plant, the measurements are corrupted with noise and the plant is given by its state-space representation

$$\begin{aligned} x(k+1) &= (A + L_x\Delta A)x(k) + (B + L_x\Delta B)u(k) \\ y(k) &= (C + L_y\Delta C)x(k) + v(k) \end{aligned} \tag{33}$$

where L_x and L_y are problem-dependent scaling matrices and the uncertainties terms ΔA, ΔB and ΔC are bounded in norm. Therefore the estimate is a set (confidence set) instead of a single point. This set can be built using polytopes (differential inclusions) [35, 36] or being approximated through an ellipsoid [37–39]. Polytopes are accurate and non-conservative but they have a significant computational load due to the increasing number of constraints employed to approximate the set. On the other side, the use of ellipsoids to bound the confidence set reduce the computational burden at expenses of introducing a degree of conservativeness in the description of the confidence set. In this work we will use ellipsoids $\mathcal{E}_l(\hat{x}_l(k), \mathbf{E}_l(k))$ to bound the confidence set, defined by the central estimated $\hat{x}_l(k)$ and shape matrix $\mathbf{E}_l(k)$ such that $\{\xi(k): \xi(k) = \hat{x}_l(k|k) + \mathbf{E}_l(k|k)w(k)\}$. Its size is measured by means of the sum of squared semi-axes lengths given by $\mathrm{tr}(\mathbf{E}_l(k|k)\mathbf{E}_l^{\mathrm{T}}(k|k)) = \mathrm{tr}(\mathbf{P}_l(k|k))$.

Like in the supervisory adaptive control architecture proposed by Rosa *et al.* [12], we use a bank of SVOs each of which is tuned for a pre-specified model P_l, $\forall l \in \mathcal{L}$. In this work we use the robust SVO proposed by El Ghaoui and Calafiore [40] that provides the minimal size ellipsoid of confidence, computed recursively through two convex optimisation problems:

- one problem computes the minimal size ellipsoid of confidence for the estimate prediction

$$\min_{\mathbf{P}_l(k|k-1),\hat{x}(k|k-1),\tau_w,\tau_x,\tau_{\Delta_A}} \mathrm{tr}(\mathbf{P}_l(k|k-1)), \quad \forall l \in \mathcal{L}$$

such that

$$\tau_w,\ \tau_x,\ \tau_\Delta > 0$$

$$\begin{bmatrix} \mathbf{P}_l(k|k-1) & \Phi \\ \Phi^{\mathrm{T}} & \Omega \end{bmatrix} \succ 0 \tag{34}$$

where

$$\begin{aligned} \Phi &= [A\hat{x}(k-1|k-1) - \hat{x}(k|k-1)\ A_l\mathbf{E}_l(k)\ B\Delta A_l], \\ \Omega &= \mathrm{diag}[1-\tau_x,\ \tau_x I - \tau_{\Delta_A}P_l(k-1|k-1),\ \tau_w I,\ \tau_{\Delta_A} I] \end{aligned}$$

and τ_w, τ_x and τ_{Δ_A} are positive scalar.

- the other one that computes the minimal size ellipsoid of confidence for the measurement update is

$$\min_{\mathbf{P}_l(k|k-1),\hat{x}(k|k-1),\tau_v,\tau_x,\tau_{\Delta_A}} \mathrm{tr}(\mathbf{P}_l(k|k-1)), \quad \forall l \in \mathcal{L}$$

such that

Algorithm

Given $\mathcal{E}_l\left(\hat{x}_l(0), \mathbf{E}_l(0)\right), k = 1, \varepsilon > 0\ \forall l \in \mathcal{L}$

Step 1 Measure the current system output $y(k)$

Step 2 Compute $\mathcal{E}_l\left(\hat{x}_l(k), \mathbf{E}_l(k)\right)\ \forall l \in \mathcal{L}$

2.*a* Solve problem (34) to obtain $\mathbf{P}_l(k|k-1)$

2.*b* Solve problem (35) to obtain $\mathbf{P}_l(k|k)$

Step 3 Initialise the supervisory temporal variables

$\mathcal{L}(k) = \emptyset, S(k) = \mathbf{0}, \Theta(k) = \mathbf{0}$

, **Step 4** Select the models compatible with the input-output trajectory

4.*a* **for** $l \in \mathcal{L}$

$r_l = \mathrm{tr}(\mathbf{P}_l(k|k))$

if $r_l > \varepsilon$

$\mathcal{L}(k) = [\mathcal{L}(k)\ l]\,,\ s_l(k) = 1$

else

$s_l(k) = 0,\ \theta_l(k) = 0$

end

end

4.*b* Update the models' weights $\Theta(k)$

$\theta_l(k) = 1 - \dfrac{r_l(k)}{\sum_{l \in \mathcal{L}(k)} r_l(k)} \quad \forall l \in \mathcal{L}(k)$

Step 5 Solve LP problem (31) to design $C_{\mathcal{L}(k)}$

Step 6 $k = k + 1$ and goto Step 1

Fig. 3 *Supervisory algorithm*

$$\tau_x, \tau_w, \tau_{\Delta_C} > 0$$

$$\begin{bmatrix} \mathbf{P}_l(k|k) & \Phi_m \Psi \\ \Psi^{\mathrm{T}} \Phi_m^{\mathrm{T}} & \Psi^{\mathrm{T}} \Omega_m \Psi \end{bmatrix} \succ 0 \tag{35}$$

where

$$\begin{aligned} \Phi_m &= [A\hat{x}(k|k-1) - \hat{x}(k|k)\ \mathbf{E}_l(k|k)\ 0\ 0], \\ \Omega_m &= \mathrm{diag}(1 - \tau_x - \tau_v, \tau_x I - \tau_{\Delta_C} \mathbf{P}_l(k|k), \tau_v I, \tau_{\Delta_C} I) \end{aligned} \tag{36}$$

τ_x, τ_v and τ_{Δ_C} are positive scalars and Ψ is the orthogonal complement of

$$\Phi_y = [\hat{y}(k) - y(k)\ C\mathbf{E}_l(k|k)\ I\ \Delta_C]$$

If the ellipsoid of confidence $\mathcal{E}_l(\hat{x}_l(k|k), \mathbf{E}_l(k|k))$ provided by the second optimisation problem has a covariance matrix with very small trace ($\mathrm{tr}(\mathbf{P}_l(k|k)) \simeq 0$), then the intersection between the sets of predicted states and the measurements is void and the measurements are not compatible with the model. Therefore it can be discarded. Based upon this fact, the algorithm to construct $\mathcal{P}_{\mathcal{L}(k)}$ is proposed in Fig. 3.

4 Stability analysis

The proof of robust stability of the closed-loop is based upon the fact that the SVOs are non-conservative, that is, $\mathrm{tr}(\mathbf{P}_l(k|k)) > \varepsilon$ for some $l \in \mathcal{L}$, then $y(k)$ can be explained by input and output trajectories of those models P_l that $\mathrm{tr}(\mathbf{P}_l(k|k)) > \varepsilon$ and the existence of $C_{\mathcal{L}(k)}(F_k, G_k)$ can be guaranteed.

Assumption 3: The uncertainty scaling matrices L_x and L_y are chosen such that the regions of the different models of the polytope $\mathcal{P}_{\mathcal{L}}$ cover the entire uncertainty region.

Hereafter, the stability of the closed-loop system is discussed. The fact that the controller $C_{\mathcal{L}(k)}$ is computed such that all elements of $\mathcal{P}_{\mathcal{L}(k)}$ are superstable, leads to the first local stability result.

Theorem 1: Supposed Assumptions 1–3 are satisfied and using the algorithm described in Fig. 3, then the resulting closed-loop system at time k is stable.

Proof: To proof the closed-loop stability, we need to guarantee that the plant $P \in \mathcal{P}_{\mathcal{L}(k)}$ and the optimisation problem (31) is feasible at every sample.

The set $\mathcal{P}_{\mathcal{L}(k)}$ is non-empty due to Assumption 3 and the current output $y(k) \in \mathcal{E}_l(x_l(k|k), \mathbf{E}_l(k|k))$, $\forall l \in \mathcal{L}(k)$ [40], therefore the true plant $P \in \mathcal{P}_{\mathcal{L}(k)}$.

Assumptions 1 and 2 ensure the feasibility of (31) by ensuring: (*a*) $C_{\mathcal{L}(k)}$ has enough parameters to freely place the closed-loop poles and (*b*) the feasibility of (31) for $\mathcal{P}_{\mathcal{L}}$. Then, the controller $C_{\mathcal{L}(k)}$ superstabilises all models of $\mathcal{P}_{\mathcal{L}(k)}$ by satisfying

$$||D_l G_k + N_l F_k - 1||_1 \le \sigma, \quad \forall l \in \mathcal{L}(k) \tag{37}$$

which implies the uniform asymptotic stability of P. □

This result, valid for each individual polytope $\mathcal{P}_{\mathcal{L}(k)}$ $\forall k$, can be also applied to the switching between different polytopes $\mathcal{P}_{\mathcal{L}(k)} \to \mathcal{P}_{\mathcal{L}(k+1)}$, since it is a consequence of the robust superstability of each polytope. Furthermore, it is clear from Lemma 2 that the trajectories of the closed-loop system decrease in norm along all trajectories of $\mathcal{P}_{\mathcal{L}(k)}$ such that

$$\begin{aligned} |e(k)| &\le \prod_{j=0}^{k} \gamma(j) \max_{i \in [1, n_A]} (|e(k)|) \\ &\le \bar{\gamma}^{(k+1)/n_A} \max_{i \in [1, n_A]} (|e(k)|) \end{aligned} \tag{38}$$

and the closed-loop response admits the estimate

$$||y(k)|| \le \bar{\eta} + \bar{\gamma}^{(k+1)/n_A} \max(0, \mu - \bar{\eta}) \tag{39}$$

where

$$\begin{aligned} \gamma(j) &= \sum_{l=0}^{m} |s_l(j)|\, ||A_l||_1, \quad \bar{\gamma} = \sup_{j \in [0,k]} \gamma(j), \\ \beta(j) &= \sum_{l=0}^{m} |s_l(j)|\, ||B_l||_1, \quad \bar{\beta} = \sup_{j \in [0,k]} \beta(j), \\ \bar{\eta} &= \frac{\bar{\beta}}{1 - \bar{\gamma}} \end{aligned} \tag{40}$$

5 Simulations and results

In this section we provide an illustrative numerical example. Let us consider a stable plant P that results from the linearisation of a continuous stirred tank reactor (CSTR) at different operating points. It was originally used by Morningred *et al.* [41] for testing discrete control algorithms. The objective is to control the output concentration $y(k)$ using the coolant flow rate $u(k)$. The reactor has two disturbances: (a) the inlet coolant temperature, which is measurable and (b) the feed concentration which is non-measurable and acts on the reactor output. The PLM $\mathcal{P}_{\mathcal{L}}$ associated with the behaviour of CSTR within the operating space region given by the cube

$$\begin{aligned} -0.045 \le y(k) - 0.085 \le 0.045\ \mathrm{mol\,lt}^{-1}, \\ -20 \le u(k) - 100 \le 10 \quad \mathrm{lt\,min}^{-1} \end{aligned} \tag{41}$$

is defined by the discrete LTI models shown in Table 1 with $\mathcal{L} = \{1, 2, 3, 4\}$.

The performance of the closed-loop system is defined through the minimisation of the worst-case error ($\sup_{\forall k} \sup_{\forall l \in \mathcal{L}(k)} |e(k)|$) and a set of constraints on the closed-loop response:

Table 1 Vertices of $\mathcal{P}_{\mathcal{L}}$

$P_1(z) = \dfrac{0.186}{z^2 - 1.984z + 0.941}$
$P_2(z) = \dfrac{0.216}{z^2 - 1.727z + 0.779}$
$P_3(z) = \dfrac{0.115}{z^2 - 1.710z + 0.755}$
$P_4(z) = \dfrac{0.831}{z^2 - 1.792z + 0.824}$

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- An overshoot of 5%

$$y(k) \leq 1.05r_0, \quad \forall k \geq N_0 \tag{42}$$

- A settling time of 50 samples for an error of 5%

$$|e(k)| \leq 0.05r_0, \quad \forall k \geq N_0 + 50 \tag{43}$$

- A zero-offset steady-state response

$$\sum_{i=1}^{n_G} |g_i(k)| = 1, \quad \forall k \geq N_0 + 50 \tag{44}$$

where r_0 is the reference value and N_0 is the time instant when changes happen. These constraints were then included in the optimisation problem (31) for all models of $\mathcal{P}_{\mathcal{L}}$.

The proposed robust adaptive controller will be compared with (i) a supervisory adaptive controller with SVOs [12] and (ii) a robust model reference adaptive controller based on LPV systems [42]. Both adaptive controllers employ robust non-adaptive controllers designed for each model of $\mathcal{P}_{\mathcal{L}}$ using mixed-μ synthesis. Then, the supervisory adaptive controller [12] employs SVOs, built using polytopes, to select the controller that switches into the loop. On the other side, the robust model reference adaptive controller computes the control signal as a weighted sums of model-following control signals for each extreme model of $\mathcal{P}_{\mathcal{L}}$, and those weights are tuned adaptively [42]. When uncertain parameters are time-varying, stabilising signals are also introduced to stabilise plants and to regulate the effect of time-varying parameters. Those additional signals are derived as solutions of non-linear H_∞ control problems for certain virtual systems.

In a first case we will consider the situation that the reactor operates in the neighbourhood of the region corresponding to P_1

$$P(z) = \frac{0.186}{z^2 - 1.984z + 0.941} \tag{45}$$

This scenario leads to a potential intermittent switching between models associated with P_1 and P_2 in supervisory adaptive controllers since both models have the same steady-state and similar behaviours.

The plant input ($u(k)$) and output ($y(k)$) resulting from the simulations with the controllers described above are shown in Fig. 4. The proposed adaptive scheme exhibits a fast regulation and easily satisfied the performance requirements (constraints (42)–(44)). The closed-loop behaviour results from two facts: (a) all parameters of $C_{\mathcal{L}(k)}$ are allowed to vary continuously, due to online design of the controller, and (b) the supervisor quickly identify the relevant models (P_1 and P_2) that explain the plant behaviour (see Fig. 5). At this point it is necessary to highlight the lack of oscillations and abrupt changes in the control signal, generated by controller switches, which is due to the soft-variable nature of the controller.

Fig. 4 also shows the closed-loop responses of others adaptive controllers. The supervisory adaptive controller shows a quick response during the transient phase of the closed-loop response; however, it also exhibits a persistent oscillation during the steady-state phase. This behaviour is due to the characteristics of the plant dynamic (models with similar steady-state behaviour), which induce a persistent

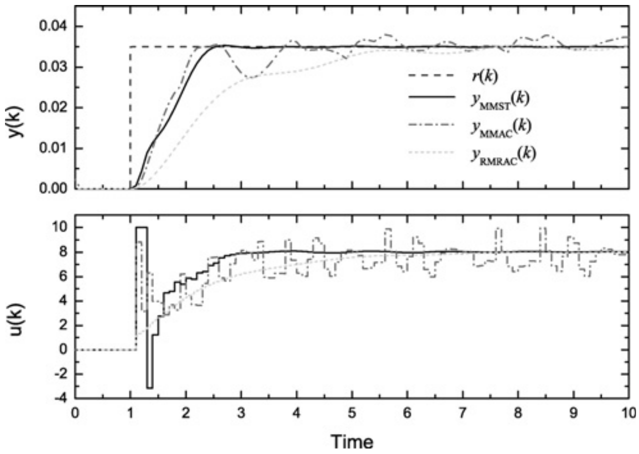

Fig. 4 *Simulation results for a multiple models switching and tuning controller (MMST), a supervisory adaptive controller (MMAC) and a robust model reference adaptive controller (RMRAC)*

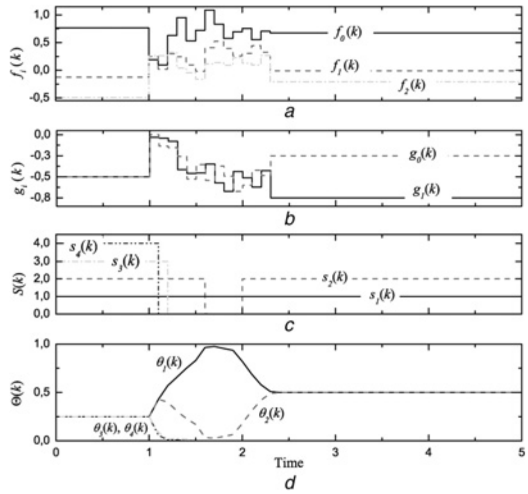

Fig. 5 *Time evolution of the controller's parameters*

a $f_i(k)$
b $g_i(k)$
c Switching variables ($s_l(k)$)
d Control weights ($\theta_l(k)$) of the proposed controller

switching between two candidate controllers (the ones corresponding to P_1 and P_2). On the other hand, the robust model reference controller avoid this problem, since the control input $u(k)$ is the result of a weighted combination, but exhibits a poor transient response.

The time evolution of the parameters of the proposed adaptive controller ($f_i(k)$ and $g_i(k)$), the switching variables ($s_l(k)$) and the weight of the models ($\theta_l(k)$) are shown in Fig. 5. An initial transient behaviour appears, after each change, before achieving their steady-state values. This fact can be appreciated in the behaviour of $S(k)$ and $\Theta(k)$ that show oscillations during the initial samples after the reference changes. This controller hesitation is owing to the fact that P_1 and P_2 has similar steady-state values and different dynamics.

In Section 2.2 it was pointed out that controllers with orders lower than the ones established by Bezout theorem ($n_F < n_D$, $n_G < n_N$) can superstabilise the system. In this example we consider the case of using a PID structure. The necessary conditions for guaranteeing the superstabilisation of $\mathcal{P}_{\mathcal{L}}$ were checked [24] and found that are satisfied. Then, simulations using a controller with a PID structure ($n_F = 2$ and $n_G = 1$) were done. The results are shown in Figs. 6 and 7. Fig. 6 shows the time evolution of plant input ($u(k)$) and output ($y(k)$) and its comparison with the responses resulting from the simulation of a full-order structure ($n_F = n_G = 2$). The time evolution of the parameters of the controller ($f_i(k)$ and $g_i(k)$), the switching variables ($s_l(k)$) and the weight of the models ($\theta_l(k)$) are shown in Fig. 7. In this case we can see that $S(k)$ and $\Theta(k)$ have similar behaviours to the full-order controller, but take longer time to achieve the steady-state values. Although in both controllers the supervisor quickly identified the relevant models (P_1 and P_2) that explained the plant behaviour (see Figs. 5 and 7) and (b), the structure of the PID imposes a severe limitation to control the plant due to its poorly dampened characteristics. This fact is reflected in the evolution of the control input $u(k)$ that shows softer changes in a longer period of time.

Now, let us consider the case in which the reactor operates in a region between the operating regions corresponding to P_2 and P_3

$$P(z) = \frac{0.201}{z^2 - 1.720z + 0.775} \qquad (46)$$

This scenario leads to the simultaneous use of a set of models (P_2 and P_3), but with different weights to explain the plant output behaviour. In this scenario, a supervisory adaptive control will choose only one robust controller.

The simulation results are shown in Figs. 8 and 9. Fig. 8 shows the time evolution of plant input ($u(k)$) and output ($y(k)$) and its comparison with the responses resulting from the simulation of the other adaptive controllers. In this case, the supervisory adaptive controller shows a quick response during the transient phase of the closed-loop response, and do not exhibit a persistent oscillation during the steady-state

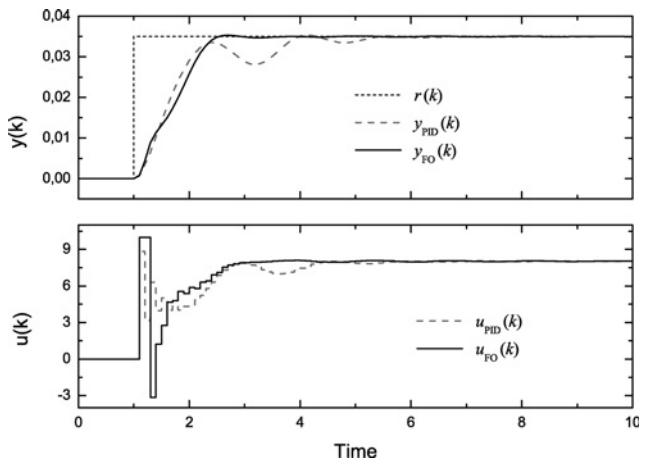

Fig. 6 *Simulation results for a full order and a PID structure of $C_{\mathcal{L}(k)}$*

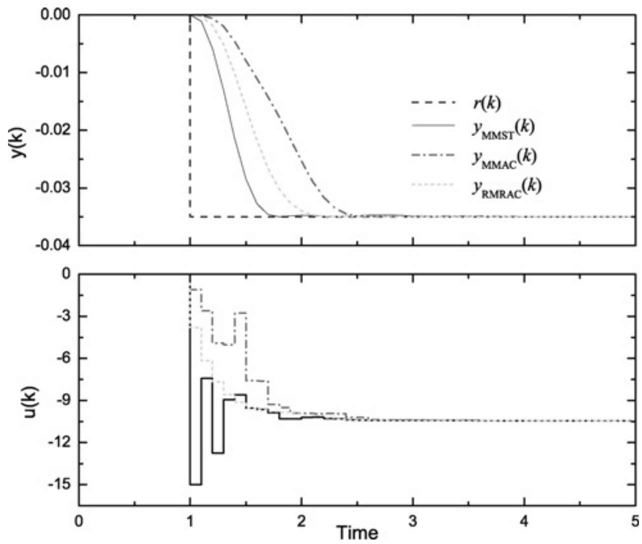

Fig. 8 *Simulation results for a multiple models switching and tuning controller (MMST), a supervisory adaptive controller (MMAC) and a robust model reference adaptive controller (RMRAC)*

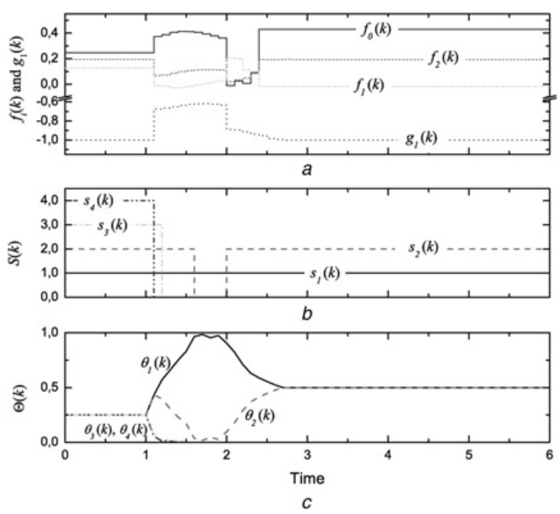

Fig. 7 *Time evolution of the controller's parameters*

a $f_i(k)$ and $g_i(k)$
b Switching variables ($s_l(k)$)
c Control weights ($\theta_l(k)$) for a PID controller

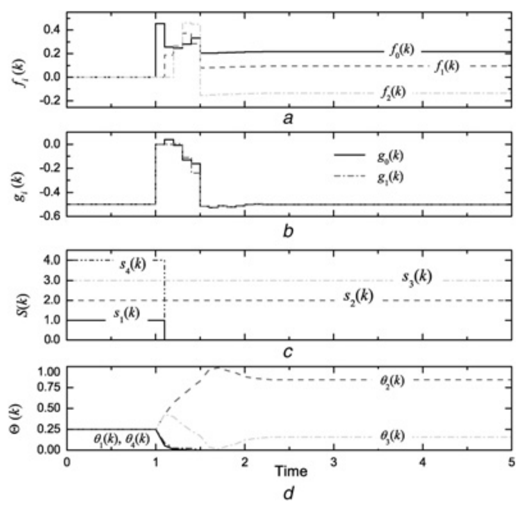

Fig. 9 *Time evolution of the controller's parameters*

a $f_i(k)$
b $g_i(k)$
c Switching variables ($s_l(k)$)
d Control weights ($\theta_l(k)$) of the proposed controller

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phase. This behaviour is due to the characteristics of the plant dynamic (models with a different dynamic and steady-state behaviour), which induce to select only one controller (the ones corresponding to P_2). Fig. 8 shows the time evolution of the parameters of the controller ($f_i(k)$ and $g_i(k)$), the switching variables ($s_l(k)$) and the weights of the models ($\theta_l(k)$). In this case the supervisory identify model P_2 is the closest model that explain the plant behaviour: $\theta_2(k)$ is bigger than the other weights ($\theta_2(k) \simeq 0.85$) and the other weight that is active is $\theta_3(k)$ ($\theta_3(k) \simeq 0.15$).

6 Conclusion

A simple scheme for designing robust supervisory adaptive controllers was presented. The motivation for multiple models switching and tuning adaptive control is to develop a deterministic approach that is capable of achieving high-performance by utilising robust LTI and switching-based adaptive tools, while avoiding issues of undesirable switching behaviours and uncertain disturbance models. The proposed approach is based on the receding horizon philosophy, switching-based adaptive control and parametric controller design techniques. In this way, the proposed approach combines the fast response of switching-based adaptive control schemes with the robustness of safe-adaptive and robust control design procedures. The resulting algorithms are able to handle constraints in the input and outputs of the system, as well as the controller structure and its parameters. The properties of the resulting closed-loop system and design guidelines have been discussed. Comparative results with others supervisory adaptive and robust adaptive controllers, based on simulations of a linear system, have been presented to illustrate the effectiveness of the proposed controller. These promising results warrant further evaluations.

The author is currently working towards extending this line of research to problems of adaptive estimation and control for MIMO systems and failure-robust design. Furthermore, the multiple models, switching and tuning framework is connected to some recent interesting works in robust adaptive control [42–44] where the robust adaptive controllers are derived using a polytopic representation of the uncertain system.

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9 Appendix 1

9.1 Proof of Lemma 2

At time k, the output of the closed-loop system is given by

$$y(k) = A_{S(k)}(z)y(k) + B_{S(k)}(z)w(k) \tag{47}$$

and its norm is bounded by

$$\begin{aligned}|y(k)| &\le |A_{S(k)}(z)y(k)| + |B_{S(k)}(z)w(k)| \\ &\le ||A_{S(k)}||_1 ||Y(k-1)||_\infty + ||B_{S(k)}||_1 ||W(k)||_\infty\end{aligned} \tag{48}$$

where

$$\begin{aligned}Y(k) &= [y(k)\cdots y(k-n_A-1)] \\ W(k) &= [w(k)\cdots w(k-n_B)]\end{aligned} \tag{49}$$

Using the recursion (42) the norm of the system output can be written in terms of the initial conditions

$$\begin{aligned}|y(k)| \le &\prod_{i=0}^{k} ||A_{S(k-i)}||_1 ||Y(-1)|| \\ &+ \sum_{i=1}^{k}\prod_{l=0}^{i-1} ||A_{S(k-l)}||_1 ||B_{S(k-i)}||_1 ||W(k-i)||_\infty\end{aligned} \tag{50}$$

If $w(k) = 0\ \forall k$, then the norm of the system output is

$$|y(k)| \le \prod_{i=0}^{k} ||A_{S(k-i)}||_1 ||Y(-1)|| \le \gamma^{(k+1)/n_A}\mu \tag{51}$$

where

$$\bar{\gamma} = \sup_{\forall k}\ \sup_{\forall S(k)\in\mathcal{S}} ||A_{S(k)}||_1 \tag{52}$$

If $\|w(k)\| \le 1\ \forall k$, then the norm of the system output when there is disturbance is given by

$$\begin{aligned}\|y(k)| &\le \bar{\gamma}^{(k+1)/n_A}\mu + \sum_{i=0}^{k}\bar{\gamma}^{j/n_A}\beta \\ &\le \eta + \bar{\gamma}^{(k+1)/n_A}\max(0, \mu-\eta)\end{aligned} \tag{53}$$

where

$$\bar{\beta} = \sup_{\forall k}\ \sup_{\forall S(k)\in\mathcal{S}} ||B_{S(k)}||_1 \tag{54}$$

This estimate shows that the PLM state trajectory is decreasing in norm until it reaches the invariant set $\{||e||_\infty \le \bar{\eta}\}$, that is, the trajectories originating in this set stay in it for all admissible perturbations. If a disturbance drives the state out of the invariant set, the control law will drive it again to the invariant set.

10 Appendix 2

10.1 Proof of Lemma 3

The goal is to design a controller $C(z)$ that ensures the superstability of the closed-loop system and minimises

$$J = \sup_{w(k)\in l_\infty}\ \sup_{\forall k} |e(k)| \tag{55}$$

The first step is to find an upper bound of J that includes the superstability of the closed-loop system. To this end, the error $e(k)$ is written in terms of the sensitivity function of the closed-loop system and the disturbance $w(k)$

$$e(k) = \frac{D(z)G(z)}{D(z)G(z) + N(z)F(z)} w(k) \tag{56}$$

Since the closed-loop must be superstable ($1 - \|DG + NF - 1\|_1 > 0$ or $\|DG + NF - 1\|_1 < 1$), Lemma 1 can be employed to obtain the estimate

$$|e(k)| \le \eta(\phi) + ||A_c||_1^{(k+1)/n_{A_c}} \max\{0, \mu_e - \eta(\phi)\}, \quad \forall k \ge 0 \tag{57}$$

where $A_c(z) = D(z)G(z) + N(z)F(z)$ and $|e(i)| \le \mu_e$, $\forall i < 0$. This is an upper bound of $|e(k)|$, $\forall k$ for any $w(k) \in l_\infty$. The effect of initial conditions is attenuated after enough steps, k_o, such that $|e(k)| \le \eta$, $\forall k > k_o$. This result provides motivation for the minimisation of η, since $J \le \eta(\phi)$, which leads to the following optimisation problem

$$\begin{aligned}&\min_{\sigma\in[0,1)}\ \min_{F,G} \frac{1}{1-\sigma} ||DG||_1 \\ &\text{such that} \\ &||DG + NF - 1||_1 \le \sigma\end{aligned} \tag{58}$$

The solution provides the maximal reduction of the effect of bounded disturbances. Such kinds of classical problems where stability (not superstability) of the closed-loop system is required are the subject of theory of l_1-optimisation. They are known to be extremely hard even for SISO systems, and reasonable solutions can be obtained only in particular cases.

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