Gaussian random permutation and the boson point process

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Abstract

We construct an infinite volume spatial random permutation (χ, σ), where $\chi \subset \mathbb{R}^d$ is a point process and $\sigma : \chi \to \chi$ is a permutation (bijection), associated to the formal Hamiltonian

$$H(\chi, \sigma) = \sum_{x \in \chi} \|x - \sigma(x)\|^2.$$

The measures are parametrized by the density ρ of points and the temperature α. Each finite cycle of σ induces a loop of points of χ. Spatial random permutations are naturally related to boson systems through a representation originally due to Feynman [10]. Bose-Einstein condensation occurs for dimension $d \geq 3$ and above a critical density $\rho_c = \rho_c(\alpha)$. For $\rho \leq \rho_c$ we define (χ, σ) as a Poisson process of finite unrooted loops that we call Gaussian loop soup after the Brownian loop soup of Lawler and Werner [14]. We also construct the Gaussian random interlacements, a Poisson process of trajectories of random walks with Gaussian increments analogous to the Brownian random interlacements introduced by Sznitman [21]. For $d \geq 3$ and $\rho > \rho_c$ we define (χ, σ) as the superposition of independent realizations of the Gaussian loop soup at density ρ_c and the Gaussian random interlacements at density $\rho - \rho_c$. In either case, we call the resulting (χ, σ) a Gaussian random permutation at density ρ and temperature α, and show that its χ-marginal has the same distribution as the boson point process introduced by Macchi [16] at the same density and temperature. This implies in particular that when Bose-Einstein condensation occurs the associated Gaussian random permutation exhibits infinite trajectories.

Keywords: Spatial random permutations, Bose gas, boson process, random interlacements, loop soup.

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1 Introduction

The free Bose gas has been intensively studied from different perspectives in mathematical physics. Araki-Woods [1] and Cannon [7] construct it as an infinite volume quantum model; in these works

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the free Bose gas at density ρ is given by a state φ_ρ of an appropriate C^*-algebra $\mathcal{A}$ of observables. In a finite volume box $\Lambda \subset \mathbb{R}^d$, the free Bose gas of N particles at temperature α is described as a quantum system on the Hilbert space of symmetric functions $L^2_s(\Lambda^N)$. The Hamiltonian is minus the Laplacian multiplied by a positive constant that we might choose as 1. The behavior of the grand-canonical ensemble as $\Lambda \to \mathbb{R}^d$, $N/|\Lambda| = \rho$, was studied by Einstein by diagonalizing the Hamiltonian, reaching the famous conclusion that when $d \geq 3$ there is a critical density above which the lowest energy level exhibits macroscopic occupation number. This critical density is

$$\rho_c(\alpha) = \left(\frac{\alpha}{\pi}\right)^{\frac{d}{2}} \sum_{k \geq 1} \frac{1}{k^{\frac{d}{2}}}. \tag{1.1}$$

Instead of diagonalizing the Hamiltonian, we might write the partition function as

$$\begin{aligned} \mathrm{Tr}_{L^2_s(\Lambda^N)}(e^{-\frac{1}{2\alpha}H}) &= \mathrm{Tr}_{L^2(\Lambda^N)}(P_+ e^{-\frac{1}{2\alpha}\Delta}) \\ &= \frac{(\alpha/\pi)^{Nd/2}}{N!} \sum_{\sigma \in \mathcal{S}_N} \int_{\Lambda^N} e^{-\alpha \sum_i \|x_i - x_{\sigma(i)}\|^2} \, d\underline{x}, \end{aligned}$$

where Δ is the Laplacian, P_+ is the symmetrization operator, $\underline{x} = (x_1, \ldots, x_N)$ and $\mathcal{S}_N$ denotes the set of permutations of $\{1, \ldots, N\}$. We abuse notation by writing $\sigma(x_i) = x_{\sigma(i)}$. The formula can be obtained by integrating the kernel of the integral operator $P_+ e^{-\frac{1}{2\alpha}\Delta}$ along the diagonal. This partition function coincides with a statistical mechanics model of spatial random permutations on the configuration space

$$\Lambda^N \times \mathcal{S}_N$$

and Radon-Nikodym density

$$f(\underline{x}, \sigma) = \frac{e^{-\alpha \sum_{i=1}^N \|\sigma(x_i) - x_i\|^2}}{\sum_{\tilde{\sigma} \in \mathcal{S}_N} \int_{\Lambda^N} d\underline{y} \; e^{-\alpha \sum_{i=1}^N \|\tilde{\sigma}(y_i) - y_i\|^2}} \tag{1.2}$$

with respect to the Lebesgue measure on Λ^N and uniform counting measure on $\mathcal{S}_N$. This representation was first proposed by Feynman [10] in 1953 in order to explain the transition of Helium-4 from fluid to superfluid. Feynman claimed that the transition to the superfluid phase coincides with the appearance of infinite cycles in a typical spatial random permutation. The approach was taken up by Sütő [19], [20], who proved Feynman's claim for Bose-Einstein condensation of the free Bose gas. As remarked by Feynman in [10], permutations in the model defined by (1.2) do not correspond to physical movement, even though the marginal distribution on the spatial variables does describe the physical point process; see for example [23].

The thermodynamic limit of the point marginal of the spatial random permutation associated to (1.2) with $N \approx \rho|\Lambda|$ has been considered by several authors. In the subcritical case when the point density is $\rho \leq \rho_c$, Tamura and Ito [23] identified the limit with the *boson point process* studied by Macchi [16] and Shirai and Takahashi [18], a process with n-point correlation functions

$$\varphi_n(x_1, \ldots, x_n) = \mathrm{perm}\left(K_\lambda(x_i, x_j)\right)_{i,j=1}^n, \tag{1.3}$$

$$K_\lambda(x, y) := \sum_{k \geq 1} \left(\frac{\alpha}{\pi k}\right)^{d/2} \lambda^k \, e^{-\frac{\alpha}{k}\|x - y\|^2}, \tag{1.4}$$

where the parameter $\lambda \in (0,1)$ is an increasing function of ρ and $\mathrm{perm}(A)$ denotes the permanent of the matrix $A \in \mathbb{R}^{n\times n}$. Since the series (1.4) diverges for $\lambda > 1$, the process with these correlations is well defined for densities $\rho \le \rho_c$, ρ_c the solution of $\lambda(\rho_c) = 1$. At the critical density the kernel K_1 determining the correlations is the Green function for the random walk with Gaussian increments. Tamura-Ito [22] consider the supercritical boson point process at density $\rho > \rho_c$ in dimension $d \ge 3$ and show that it consists of the convolution of the critical boson point process at density ρ_c with another point process at density $\rho - \rho_c$, see §3 for a precise description of the latter.

In this paper, to every positive density ρ and temperature α we associate an infinite volume spatial random permutation (χ, σ), where χ is a discrete subset of $\mathbb{R}^d$ and $\sigma : \chi \to \chi$ is a permutation, that is, a bijection. The law of (χ, σ) is translation-invariant and has point density ρ. Our construction stems from the observation that the density (1.2) can be written as a product of weights assigned to the loops γ induced by the cycles of the permutation σ. An unrooted *loop* of size k is described as $\gamma = [x_1, \dots, x_k]$ with $x_i \in \mathbb{R}^d$; the square brackets indicate that $[x_2, \dots, x_k, x_1] = [x_1, \dots, x_k]$. We denote by $\{\gamma\} = \{x_1, \dots, x_k\}$ the set of points in the loop, and write $\gamma(x_i) = x_{i+1}$. A spatial permutation $\Gamma = (\chi, \sigma)$ can be decomposed in loops, induced by the cycles of σ. We will say that $\gamma \in \Gamma$ if γ is a loop with $\{\gamma\} \subset \chi$ and $\gamma(x) = \sigma(x)$ for all $x \in \{\gamma\}$; with this notation we have $\bigcup_{\gamma\in\Gamma}\{\gamma\} = \chi$. The numerator of the density function (1.2) factorizes as follows

$$e^{-\alpha\sum_{i=1}^{N}\|\sigma(x_i)-x_i\|^2} = \prod_{\gamma\in\Gamma} e^{-\alpha\sum_{x\in\{\gamma\}}\|\gamma(x)-x\|^2}. \tag{1.5}$$

The independence of loops suggested by (1.5) was already present in Sütő [19].

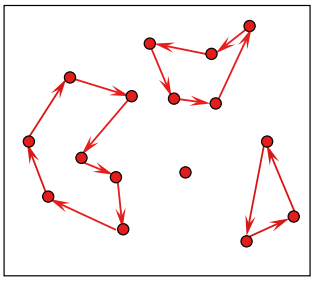

Figure 1: Loops induced by a spatial random permutation in a box. An arrow from x to y means $y = \sigma(x)$. Isolated dots correspond to points $x = \sigma(x)$, loops of length 1.

We define the *Gaussian loop soup* as a Poisson process of unrooted loops with Gaussian increments analogous to the Brownian loop soup introduced by Lawler and Werner in [14], see also Lawler and Trujillo Ferreras [13] and Le Jan [15]. We consider this process at density $\rho \le \rho_c$. When $d \ge 3$ and the point density ρ exceeds the critical density ρ_c, we allocate the leftover density $\rho - \rho_c$ to an independent realization of *Gaussian random interlacements*, a Poisson process of double infinite trajectories of a discrete-time random walk with Gaussian increments. This process is the discrete-time version of the Brownian random interlacements introduced by Sznitman [21].

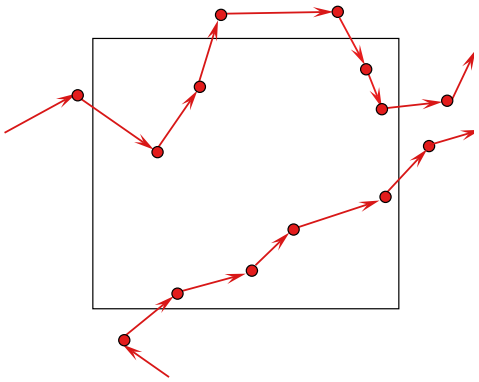

Figure 2: Random interlacements intersecting a box.

We define the *Gaussian random permutation in* $\mathbb{R}^d$*, with temperature* α *and density* ρ as the spatial random permutation obtained by superposing independent realizations of a Gaussian loop soup with increments of variance $\frac{1}{2\alpha}$ at point density $\rho \wedge \rho_c$, and a Gaussian random interlacements at point density $(\rho - \rho_c)^+$ with the same increments.

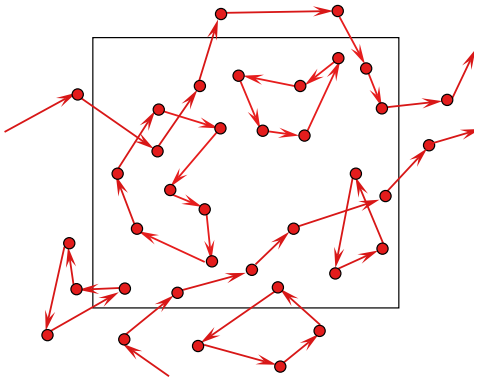

Figure 3: Loop soup plus random interlacements intersecting a box at supercritical density $\rho > \rho_c$.

Our main result can now be stated as follows.

Theorem 1.1. *For any dimension* d*, temperature* $\alpha > 0$ *and point density* $\rho > 0$*, the Gaussian random permutation with these parameters is a Gibbs measure for the specifications induced by the finite-volume distribution* (1.2)*. Furthermore, the point marginal of this measure is the boson point process.*

This result follow from Theorem 4.6 and Corollary 4.4.

The representation (1.2) and the results [19], [20] by Sütő generated further interest in spatial random permutations, and in particular in the problems of constructing infinite volume measures and describing the cycle sizes of typical infinite volume permutations. An early result due to Fichtner [11] in 1991, and more recently works by Gandolfo, Ruiz and Ueltschi[12], Betz and Ueltschi[4], Betz [3], Biskup and Ritchammer [6] and Armendáriz, Ferrari, Groisman and Leonardi [2], consider these issues when the set of points is fixed as either the regular lattice or a realization

of a locally finite point process. It is expected that for dimensions $d \geq 3$ there is a critical temperature below which, almost surely on the realization of the point process, a typical infinite volume permutation will contain infinite cycles. This is a challenging problem, at present all rigorous results pertain to the high temperature regime. In this article, on the other hand, we study the annealed model where configurations of points and permutations of these points are jointly sampled. Betz and Ueltschi [5] prove a phase transition for a family of annealed models that includes the one described by (1.2). Precisely, they prove that above a critical density that equals ρ_c for the model considered here, a typical permutation in a box of volume N/ρ contains macroscopic cycles, and the sequence of their rescaled lengths, when sorted in decreasing order, converges to a Poisson-Dirichlet distribution. These results were recently extended to a more general class of models by Elboim and Peled [9].

We expect that the Gaussian random permutation with temperature α and density ρ is the thermodynamic limit of the distribution in (1.2), and that infinite cycles evolve out of macroscopic cycles. In particular, this would provide an alternative proof that the Gaussian random permutation is a Gibbs measure for the specifications associated to (1.2).

The rest of the article is structured as follows. We construct the Gaussian loop soup in §2. In §2.1 we identify the Gaussian loop soup in finite volume conditioned to having a fixed number of points as the canonical measure of the spatial random permutation associated to (1.2), and the unconditioned Gaussian loop soup in finite volume as the grand-canonical measure. We then compute the Laplace functionals of the Gaussian loop soup point marginal and its n-point correlations in §2.2. In §3 we introduce Gaussian random interlacements, and derive the Laplace functionals and n-point correlations of the point marginal in §3.1. These Laplace functionals and n-point correlations are employed in §4.1 to prove that the boson point process, in any dimension and at any density, is in fact the point marginal of the Gaussian Random permutation with the same parameters; the latter is formally defined in §4. We finally show in §4.2 that the Gaussian random permutation has the Markov property, and moreover it is a Gibbs measure for the specifications determined by (1.2).

2 Gaussian loop soup

In this section we construct the Gaussian loop soup. We first introduce the Gaussian loop soup following the construction of the simple random walk loop soup by Lawler and Trujillo Ferreras [13]. In §2.1 we show that the loop soup restricted to A and conditioned to having N points is equal to the canonical distribution on permutations given by (1.2); an analogous result is obtained in the grand-canonical case. In §2.2 we show that the n-point correlations of the point marginal of the loop soup match those of the boson point process (1.3), a permanental process.

Consider the Brownian transition density

$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\Big\{-\frac{1}{2t}\|x-y\|^2\Big\}, \quad t>0, \quad x,y \in \mathbb{R}^d. \tag{2.1}$$

Given $k \geq 1$ and $\lambda > 0$, we consider the set of *rooted loops of length* k, or rooted k-loops for short,

$$D_k^{\mathrm{root}} = \big\{\gamma : \{0,\dots,k\} \to \mathbb{R}^d, \gamma(0) = \gamma(k)\big\} \tag{2.2}$$

with the Borel σ-algebra of $(\mathbb{R}^d)^{k+1}$, and define $Q^{\mathrm{root}}_{k,\lambda}$, the *rooted Gaussian k-loop measure with fugacity* λ, on D^{root}_k, as

$$Q^{\mathrm{root}}_{k,\lambda}(d\gamma) := \lambda^k \prod_{j=0}^{k-1} p_{\frac{1}{2\alpha}}(x_i, x_{i+1})\, dx_0 \dots dx_{k-1} \qquad \text{if } \gamma(i) = x_i, \quad 0 \le i \le k-1. \tag{2.3}$$

We can naturally define the periodic extension of a k-loop as a function $\gamma : \mathbb{Z} \to \mathbb{R}^d$ by setting $\gamma(j) := \gamma(mk + j)$, $j \in \mathbb{Z}$, where $m \in \mathbb{Z}$ is such that $0 \le mk + j < k$. Let θ^ℓ denote the time shift by ℓ, $\theta^\ell(\gamma)(j) := \gamma(j + \ell)$. An *unrooted loop of length* k, or unrooted k-loop, is an equivalence class of rooted k-loops under the equivalence $\gamma \sim \gamma'$ if there exists $\ell \in \mathbb{Z}$ such that $\gamma' = \theta^\ell \gamma$. Let

$$D_k := D^{\mathrm{root}}_k \,/\, \sim \tag{2.4}$$

denote the set of unrooted k-loops and

$$\pi_k : D^{\mathrm{root}}_k \to D_k \tag{2.5}$$

the projection. Consider the σ-algebra $\mathcal{D}_k$ on D_k given by the sets $\Upsilon \subset D_k$ such that $\pi_k^{-1}(\Upsilon) \subset D^{\mathrm{root}}_k$ is measurable, and denote $Q_{k,\lambda}$ the induced *unrooted Gaussian k-loop measure with fugacity* λ on D_k defined by

$$Q_{k,\lambda}(\Upsilon) := \frac{1}{k}\, Q^{\mathrm{root}}_{k,\lambda}\big(\pi_k^{-1}(\Upsilon)\big). \tag{2.6}$$

The factor $1/k$ in front of the push-forward measure compensates for the fact that a loop class is counted k times by $Q^{\mathrm{root}}_{k,\lambda}$, according to each possible root choice. For bounded measurable $g : D_k \to \mathbb{R}$ definitions (2.6) and (2.3) yield

$$\int_{D_k} g(\gamma)\, Q_{k,\lambda}(d\gamma) = \frac{\lambda^k}{k} \int_{(\mathbb{R}^d)^k} g([x_0, \dots, x_{k-1}]) \prod_{i=0}^{k-1} p_{\frac{1}{2\alpha}}(x_i, x_{i+1})\, dx_0 \dots dx_{k-1}. \tag{2.7}$$

We use the letter γ to denote an unrooted loop $[x_0, \dots, x_{k-1}]$, call *support* of γ the set $\{\gamma\} := \{x_0, \dots, x_{k-1}\} \subset \mathbb{R}^d$, and consider $\gamma : \{\gamma\} \to \{\gamma\}$ as a permutation of $\{\gamma\}$ satisfying $\gamma(x_i) = x_{i+1}$.

Given a Borel set $A \subset \mathbb{R}^d$ and an unrooted loop $\gamma \in D_k$ with support $\{\gamma\} = \{x_0, \dots, x_{k-1}\}$, consider the cardinality of the set $A \cap \{\gamma\}$

$$n_A(\gamma) := \sum_{j=0}^{k-1} \mathbf{1}_A(x_j). \tag{2.8}$$

If the set has finite Lebesgue measure $|A| < \infty$, the mean density of points belonging to k-loops in A is defined as

$$\rho_{k,\lambda}(A) := \frac{1}{|A|} \int_{D_k} n_A(\gamma)\, Q_{k,\lambda}(d\gamma)\,. \tag{2.9}$$

Proposition 2.1. *For any compact set $A \subset \mathbb{R}^d$,*

$$\rho_{k,\lambda} = \rho_{k,\lambda}(A) = \Big(\frac{\alpha}{\pi}\Big)^{d/2} \frac{\lambda^k}{k^{d/2}}\,. \tag{2.10}$$

Proof. By (2.7) we have

$$\begin{aligned}\rho_{k,\lambda}(A) &= \frac{\lambda^k}{k|A|}\int_{(\mathbb{R}^d)^k}\Big(\sum_{j=0}^{k-1}\mathbf{1}_A(x_j)\Big)\prod_{i=0}^{k-1}p_{\frac{1}{2\alpha}}(x_i,x_{i+1})\,dx_0\dots dx_{k-1}\\ &= \frac{\lambda^k}{k|A|}\sum_{j=0}^{k-1}\int_{(\mathbb{R}^d)^k}\mathbf{1}_A(x_j)\prod_{i=0}^{k-1}p_{\frac{1}{2\alpha}}(x_i,x_{i+1})\,dx_0\dots dx_{k-1}\\ &= \frac{\lambda^k}{k|A|}\sum_{j=0}^{k-1}\int_{\mathbb{R}^d}\mathbf{1}_A(x_j)\left(\frac{\alpha}{\pi k}\right)^{d/2}dx_j\\ &= \frac{\lambda^k}{k|A|}\left(\frac{\alpha}{\pi k}\right)^{d/2}k|A| = \left(\frac{\alpha}{\pi}\right)^{d/2}\frac{\lambda^k}{k^{d/2}}\,.\end{aligned}$$

□

We call $\rho_{k,\lambda}$ the point density of the measure $Q_{k,\lambda}$. Define

$$\rho(\lambda) := \sum_{k\geq 1}\rho_{k,\lambda}\,. \tag{2.11}$$

We have

$$\rho(\lambda) = \left(\frac{\alpha}{\pi}\right)^{d/2}\sum_{k\geq 1}\frac{\lambda^k}{k^{d/2}} < \infty \iff \begin{cases} d\leq 2 \text{ and } 0\leq\lambda<1, \text{ or}\\ d\geq 3 \text{ and } 0\leq\lambda\leq 1\,.\end{cases} \tag{2.12}$$

Define the critical density ρ_c by

$$\rho_c := \left(\frac{\alpha}{\pi}\right)^{d/2}\sum_{k\geq 1}\frac{1}{k^{d/2}}. \tag{2.13}$$

The function $\rho:[0,1]\to[0,\rho_c]$ is invertible; let $\lambda(\rho)$ be its inverse.

The space D of finite unrooted loops in $\mathbb{R}^d$ is

$$D := \bigcup_{k\geq 1} D_k, \tag{2.14}$$

with the minimal σ- algebra $\mathcal{D}$ that contains $\bigcup_{k\geq 1}\mathcal{D}_k$.

If λ and d satisfy (2.12), so that $\rho(\lambda)<\infty$, define the *Gaussian loop soup intensity measure* Q^{ls}_λ on D as

$$Q^{\mathrm{ls}}_\lambda := \sum_{k\geq 1} Q_{k,\lambda}\,. \tag{2.15}$$

This measure has point density $\rho(\lambda)$.

Proposition 2.2. *Let d and λ be as in* (2.12)*. Then Q^{ls}_λ is σ-finite.*

Proof. We start by showing that for $k\in\mathbb{N}$ and $\lambda>0$, the measure $Q_{k,\lambda}$ on D_k is σ-finite. Let $B_j\subset\mathbb{R}^d$ be the ball of radius j in $\mathbb{R}^d$, and define $\Upsilon_{k,j} := \big\{\gamma\in D_k : \{\gamma\}\cap B_j\neq\emptyset\big\}\subset D_k$. Then $\bigcup_j\Upsilon_{k,j} = D_k$, and

$$Q_{k,\lambda}(\Upsilon_{k,j}) \leq |B_j|\,\rho_{k,\lambda}(B_j) = |B_j|\rho_{k,\lambda} < \infty,$$

by (2.9) and Proposition 2.1. Letting $\Upsilon_j := \bigcup_{k\geq 1} \Upsilon_{k,j}$, we have $\bigcup_{j\geq 1} \Upsilon_j = D$. Then

$$Q^{\mathrm{ls}}_\lambda(\Upsilon_j) \leq \sum_{k\geq 1} Q_{k,\lambda}(\Upsilon_{k,j}) \leq \sum |B_j| \rho_{k,\lambda} = \rho |B_j| < \infty, \quad \text{by (2.12).} \qquad \square$$

Definition 2.3. *Let d and λ satisfy* (2.12). *We define the* Gaussian loop soup *at fugacity λ as a Poisson process on D with intensity Q^{ls}_λ. We will use $\Gamma^{\mathrm{ls}}_\lambda$ to denote a realization of the process, and $\mu^{\mathrm{ls}}_\lambda$ to denote its law.*

A sample Γ of a Gaussian loop soup is a countable collection of unrooted Gaussian loops in $\mathbb{R}^d$, with the property that any compact set contains finitely many points in the supports of these loops.

Let $\mathfrak{X}$ be the set of locally finite loop soup configurations,

$$\mathfrak{X} := \{\Gamma \subset D : \textstyle\sum_{\gamma\in\Gamma} |\{\gamma\} \cap A| < \infty, \text{ for all compact } A \subset \mathbb{R}^d\}. \tag{2.16}$$

Given $\Upsilon \subset D$ and $\Gamma \in \mathfrak{X}$ let

$$\mathfrak{X}_\Upsilon := \big\{\Gamma \in \mathfrak{X} : \Gamma \subseteq \Upsilon\big\}. \tag{2.17}$$

If $\Upsilon \subset D$ has finite Gaussian loop soup intensity, $Q^{\mathrm{ls}}_\lambda(\Upsilon) < \infty$, then, by definition of Poisson process, given a measurable, bounded function $g : \mathfrak{X}_\Upsilon \to \mathbb{R}$ we can explicitly write

$$\mu^{\mathrm{ls}}_\lambda g = e^{-Q^{\mathrm{ls}}_\lambda(\Upsilon)} \sum_{\ell\geq 0} \frac{1}{\ell!} \int_\Upsilon \cdots \int_\Upsilon g\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, Q^{\mathrm{ls}}_\lambda(d\gamma_1) \dots Q^{\mathrm{ls}}_\lambda(d\gamma_\ell) \tag{2.18}$$

$$= \sum_{\ell\geq 0} \frac{e^{-Q^{\mathrm{ls}}_\lambda(\Upsilon)}}{\ell!} \int_\Upsilon \cdots \int_\Upsilon g\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, f^{\mathrm{ls}}_\lambda\big(\{\gamma_1, \dots, \gamma_\ell\}\big)\, d\gamma_1 \dots d\gamma_\ell, \tag{2.19}$$

where

$$f^{\mathrm{ls}}_\lambda\big(\{\gamma_1, \dots, \gamma_\ell\}\big) := \textstyle\prod_{i=1}^{\ell} \omega_\lambda(\gamma_i), \tag{2.20}$$

$$\omega_\lambda\big([x_0, \dots, x_{k-1}]\big) := \lambda^k \textstyle\prod_{i=0}^{k-1} p_{\frac{1}{2\alpha}}(x_i, x_{i+1}) \quad \text{with } x_k = x_0, \tag{2.21}$$

and where for any bounded measurable $h : \Upsilon \to \mathbb{R}$,

$$\int_D h(\gamma)\, d\gamma := \sum_{k\geq 1} \frac{1}{k} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} h\big([x_0, \dots, x_{k-1}]\big)\, dx_0 \dots dx_{k-1}, \tag{2.22}$$

if the right hand side is well defined. Taking $h(\gamma) = h_1(\gamma)\, \omega_\lambda(\gamma) \mathbf{1}_{\{\gamma\in\Upsilon\}}$ for some bounded $h_1 : \Upsilon \to \mathbb{R}$ and recalling the assumption $Q^{\mathrm{ls}}_\lambda(\Upsilon) < \infty$, we conclude that (2.22) is well defined for this h and it is bounded by $\|h_1\|_\infty Q^{\mathrm{ls}}_\lambda(\Upsilon)$.

The function $e^{-Q^{\mathrm{ls}}_\lambda(\Upsilon)} f^{\mathrm{ls}}_\lambda$ is the *density* of the Gaussian loop soup at fugacity λ in the set Υ; in particular, if $g \equiv 1$ we have $\mu^{\mathrm{ls}}_\lambda g = 1$.

2.1 Finite-volume loop soup and spatial permutations

In this subsection we show that the Gaussian loop soup in a compact set $A \subset \mathbb{R}^d$ conditioned to have n points has the same law as the spatial random permutation with density (1.2). We prove a similar result in the grand-canonical case.

Note that there is a bijection between $\mathfrak{X}$ and the set

$$\{(\chi, \sigma) : \chi \text{ locally finite, for all } x \in \chi \text{ there is } k < \infty \text{ with } \sigma^k(x) = x\}, \tag{2.23}$$

the space of finite cycle permutations with locally finite supports. Indeed, for any $\Gamma \in \mathfrak{X}$, let (χ, σ) be given by $\chi = \cup_{\gamma\in\Gamma}\{\gamma\}$ and $\sigma(x) = \gamma(x)$, $x \in \{\gamma\}$, $\gamma \in \Gamma$. Conversely, given a spatial permutation (χ, σ) in the set (2.23), define the loop soup configuration

$$\Gamma := \cup_{x\in\chi}\{[x, \sigma(x), \dots, \sigma^{k(x)-1}(x)]\},$$

where $k(x) = \min\{j > 0 : \sigma^j(x) = x\}$ is the size of the cycle in σ containing x.

The set of loops with supports contained in A and the space of loop-soup configurations contained in A with exactly n points are denoted by

$$D_A := \{\gamma \in D : \{\gamma\} \subset A\}, \tag{2.24}$$

$$\mathfrak{X}_A := \mathfrak{X}_{D_A}, \tag{2.25}$$

$$\mathfrak{X}_{A,n} := \{\Gamma \in \mathfrak{X}_A : \textstyle\sum_{\gamma\in\Gamma}|\gamma| = n\}. \tag{2.26}$$

Canonical measures The Gaussian loop soup restricted to loops contained in A is defined by

$$\mu^{\mathrm{ls}}_{A,\lambda} := \text{Poisson process on } \mathfrak{X}_A \text{ with intensity } \mathbf{1}_{D_A}(\gamma)\, Q^{\mathrm{ls}}_\lambda(d\gamma). \tag{2.27}$$

We now show that $\mu^{\mathrm{ls}}_{A,\lambda}$ conditioned to have n points in A equals the spatial random permutation with density (1.2), that we denote $\mu_{A,n}$.

Proposition 2.4. *For any measurable, bounded test function $g : \mathfrak{X}_{A,n} \to \mathbb{R}$, we have*

$$\frac{1}{\mu^{\mathrm{ls}}_{A,\lambda}(\mathfrak{X}_{A,n})} \int_{\mathfrak{X}_{A,n}} g(\Gamma)\, \mu^{\mathrm{ls}}_{A,\lambda}(d\Gamma) = \mu_{A,n} g \tag{2.28}$$

Proof. Since there is a bijection between the supports of these probability measures, it suffices to verify that the weights assigned by their densities to any given configuration satisfy a fixed ratio. Let (χ, σ) be a spatial permutation such that $\chi \subset A$ and $|\chi| = n$. Let Γ be the cycle decomposition of (χ, σ); clearly $\Gamma \in \mathfrak{X}_{A,n}$. Then, by the definition of Poisson process, the loop soup conditioned density of $\Gamma \in \mathfrak{X}_{A,n}$ is

$$f^{\mathrm{ls}}_\lambda(\Gamma|\mathfrak{X}_{A,n}) = \frac{e^{-Q^{\mathrm{ls}}_\lambda(D_A)}}{\mu^{\mathrm{ls}}_\lambda(\mathfrak{X}_{A,n})} \prod_{\gamma\in\Gamma} \omega_\lambda(\gamma) = \frac{\lambda^n e^{-Q^{\mathrm{ls}}_\lambda(D_A)}}{\mu^{\mathrm{ls}}_\lambda(\mathfrak{X}_{A,n})} \prod_{\gamma\in\Gamma} \omega_1(\gamma), \tag{2.29}$$

where ω_λ was defined in (2.21).

On the other hand, the density of the canonical measure $\mu_{A,n}$ can be written as a function of the cycle decomposition of σ by

$$\frac{1}{Z_{A,n}} f_{A,n}(\chi, \sigma) = \frac{1}{Z_{A,n}}\, e^{-\alpha H(\chi,\sigma)} = \frac{1}{Z_{A,n}} \prod_{\gamma\in(\chi,\sigma)} \omega_1(\gamma). \qquad \square$$

Grand-canonical measures The grand-canonical spatial random permutation at fugacity $\lambda \leq 1$ associated to the canonical density (1.2) is defined by

$$\mu_{A,\lambda}\, g := \frac{1}{Z_{A,\lambda}} \sum_{n\geq 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} g(\underline{x},\sigma) e^{-\alpha H(\underline{x},\sigma)} d\underline{x}, \tag{2.30}$$

where $\underline{x} = (x_1,\dots,x_n)$, $H(\underline{x},\sigma) := \sum_{i=1}^n \|x_i - x_{\sigma(i)}\|^2$ and

$$Z_{A,\lambda} := \sum_{n\geq 0} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} e^{-\alpha H(\underline{x},\sigma)} d\underline{x}.$$

Proposition 2.5. *Let $\lambda \leq 1$. The Gaussian loop soup at fugacity λ restricted to A defined in* (2.27) *and the grand-canonical measure* (2.30) *at the same fugacity are equal,*

$$\mu^{\mathrm{ls}}_{A,\lambda} = \mu_{A,\lambda}. \tag{2.31}$$

Proof. It suffices to show that the measures have the same Laplace functionals. Given $\psi : D_A \to \mathbb{R}_+$, define $g : \mathfrak{X}_A \to \mathbb{R}$ as

$$g(\Gamma) := \exp\Big(-\textstyle\sum_{\gamma\in\Gamma}\psi(\gamma)\Big).$$

By Campbell's theorem,

$$\mu^{\mathrm{ls}}_{A,\lambda} g = \int_{\mathfrak{X}_A} \mu^{\mathrm{ls}}_\lambda(d\Gamma) e^{-\sum_{\gamma\in\Gamma}\psi(\gamma)} = \exp\Big(\int_{D_A} (e^{-\psi(\gamma)} - 1) Q^{\mathrm{ls}}_\lambda(d\gamma)\Big) = e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \exp\Big(\sum_{k\geq 1} \frac{1}{k!} a_k\Big),$$

where, using the definition of $Q_{k,\lambda}$ and denoting $\lambda(\alpha) := (\alpha/\pi)^{d/2}\lambda$,

$$\begin{aligned} a_k &:= \lambda(\alpha)^k (k-1)! \int_{A^k} dx_1 \dots dx_k\, e^{-\alpha H([x_1,\dots,x_k])}\, e^{-\psi([x_1,\dots,x_k])} \\ &= \lambda(\alpha)^k \sum_{\gamma\in\mathcal{C}_k} \int_{A^k} dx_1 \dots dx_k\, e^{-\alpha H(\underline{x},\gamma)}\, e^{-\psi(\underline{x},\gamma)} \end{aligned}$$

where $\mathcal{C}_k$ is the set of cycles of size k with elements $\{1,\dots,k\}$, $\underline{x} = (x_1,\dots,x_n)$, and $(\underline{x},\gamma) := [x_1, x_{\gamma(1)},\dots,x_{\gamma^{k-1}(1)}]$; notice that $\mathcal{C}_k$ has cardinality $(k-1)!$. By Lemma 2.8 below we have

$$\begin{aligned} \mu^{\mathrm{ls}}_{A,\lambda} g &= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|} \\ &= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0} \frac{\lambda(\alpha)^n}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} \sum_{\gamma\in\mathcal{C}_{|I|}} \int_{A^{|I|}} dx_1 \dots dx_{|I|} e^{-\alpha H([x_1,\dots,x_{|I|}])}\, e^{-\psi([x_1,\dots,x_{|I|}])} \\ &= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0} \frac{\lambda(\alpha)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \prod_{\gamma\in\sigma} \int_{A^{|\gamma|}} dx_1 \dots dx_{|\gamma|}\, e^{-\alpha H(\underline{x},\gamma)} e^{-\psi(\underline{x},\gamma)} \\ &= e^{-Q^{\mathrm{ls}}_\lambda(D_A)} \sum_{n\geq 0} \frac{\lambda(\alpha)^n}{n!} \sum_{\sigma\in\mathcal{S}_n} \int_{A^n} dx_1 \dots dx_n\, e^{-\alpha H(\underline{x},\sigma)} \prod_{\gamma\in\sigma} e^{-\psi((x_i : i\in\{\gamma\}),\gamma)} \\ &= \mu_{A,\lambda} g, \end{aligned}$$

where $(\underline{x},\sigma)$ is the spatial permutation that maps x_i to $x_{\sigma(i)}$ and $\{\gamma\}$ is the set of indices that appear in the cycle γ. □

2.2 Point marginal of the Gaussian loop soup

In this subsection we study the law of the point marginal of the Gaussian loop soup. Let $\Gamma_\lambda^{\mathrm{ls}}$ be the loop soup with distribution $\mu_\lambda^{\mathrm{ls}}$ and denote by $\chi_\lambda^{\mathrm{ls}}$ its point marginal:

$$\chi_\lambda^{\mathrm{ls}} := \bigcup_{\gamma\in\Gamma_\lambda^{\mathrm{ls}}} \{\gamma\}, \qquad \nu_\lambda^{\mathrm{ls}} := \text{ law of } \chi_\lambda^{\mathrm{ls}}. \tag{2.32}$$

For $k \geq 1$, $\lambda > 0$, define

$$J_\lambda^k(x,y) := \lambda^k \int_{(\mathbb{R}^d)^{k-1}} p_{\frac{1}{2\alpha}}(x,z_1)\dots p_{\frac{1}{2\alpha}}(z_{k-1},y)\, d\underline{z} = \lambda^k \Big(\frac{\alpha}{\pi k}\Big)^{d/2} \exp\Big\{-\frac{\alpha}{k}\|x-y\|^2\Big\}, \tag{2.33}$$

where $\underline{z} = (z1,\dots,z_{k-1})$. If d and λ satisfy (2.12), denote

$$K_\lambda(x,y) := \sum_{k\geq 1} J_\lambda^k(x,y)\,. \tag{2.34}$$

The choice of λ and d imply that the sum in (2.34) converges.

Let $\phi : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a non-negative measurable function with compact support and recall that $\mathcal{L}(\phi) := \int \nu(d\chi) \exp\big(-\sum_{x\in\chi}\phi(x)\big)$ is the Laplace functional of the point process with law ν.

Proposition 2.6 (Laplace functionals)**.** *Let d and λ be as in* (2.12)*. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a non-negative measurable function with compact support. Denote $\mathcal{L}_\lambda^{\mathrm{ls}}$ the Laplace functional of the point marginal of the Gaussian loop soup at fugacity λ. Then, if ϕ satisfies $K_\lambda(0,0)\|e^{-\phi}-1\|_1 < 1$ we have*

$$\mathcal{L}_\lambda^{\mathrm{ls}}(\phi) = \exp\Big\{\sum_{j\geq 1}\frac{1}{j}\int_{(\mathbb{R}^d)^j}\prod_{i=0}^{j-1}(e^{-\phi(y_i)}-1)\,K_\lambda(y_i,y_{i+1})\,dy_0\dots dy_{j-1}\Big\} \tag{2.35}$$

$$= 1 + \sum_{n\geq 1}\frac{1}{n!}\int_{(\mathbb{R}^d)^n}\sum_{\sigma\in\mathcal{S}_n}\prod_{i=0}^{n-1}\big(e^{-\phi(y_i)}-1\big)\prod_{i=0}^{n-1}K_\lambda(y_i,y_{\sigma(i)})\,dy_0\dots dy_{n-1}. \tag{2.36}$$

with the convention that $y_j := y_0$ in the integrals over $(\mathbb{R}^d)^j$ on the right of (2.35)*.*

Proof. Let $\Phi(\gamma) := \sum_{x\in\{\gamma\}}\phi(x)$, $\gamma \in D$, so that we can write

$$\mathcal{L}_\lambda^{\mathrm{ls}}(\phi) = \int \nu_\lambda^{\mathrm{ls}}(d\chi)\,\exp\Big\{-\sum_{x\in\chi}\phi(x)\Big\} = \int \mu_\lambda^{\mathrm{ls}}(d\Gamma)\,\exp\Big\{-\sum_{\gamma\in\Gamma}\Phi(\gamma)\Big\}$$
$$= \exp\Big\{\int_D (e^{-\Phi(\gamma)}-1)\,Q_\lambda(d\gamma)\Big\}, \tag{2.37}$$

by Campbell's theorem. Notice that $e^{-\Phi(\gamma)}-1$ is bounded and vanishing in the complement of Υ_j defined in Proposition 2.2, if $j \in \mathbb{N}$ is such that the support $\mathrm{supp}(\phi) \subseteq B_j$, the ball of radius

j in $\mathbb{R}^d$. We saw in the proof of Proposition 2.2 that $Q_\lambda^{\mathrm{ls}}(\Upsilon_j)$ is finite, hence the integral in the exponent of (2.37) converges.

If $\gamma = [x_0, \cdots, x_{k-1}] \in D_k$ then

$$e^{-\Phi(\gamma)} - 1 = \prod_{i=0}^{k-1} \left[\left(e^{-\phi(x_i)} - 1\right) + 1 \right] - 1 = \sum_{\emptyset \neq \gamma' \subset \gamma} \prod_{x \in \{\gamma'\}} \left(e^{-\phi(x)} - 1\right),$$

where $\gamma' \subset \gamma$ means that the trace of the loop γ' is a subset of the trace of γ: $\gamma' \in D_j$ with $j \le k$, and there is a subsequence $\{n_i\}_{0 \le i \le j-1}$ of $0, \ldots, k-1$ such that $\gamma'(i) = \gamma(n_i)$. Then

$$\begin{aligned}
&\int_D (e^{-\Phi(\gamma)} - 1)\, Q_\lambda(d\gamma) \\
&\quad = \sum_{k \ge 1} \int_{D_k} (e^{-\Phi(\gamma)} - 1)\, Q_{k,\lambda}(d\gamma) \\
&\quad = \sum_{k \ge 1} \int_{D_k} \sum_{\emptyset \neq \gamma' \subset \gamma} \prod_{x \in \{\gamma'\}} \left(e^{-\phi(x)} - 1\right)\, Q_{k,\lambda}(d\gamma) \\
&\quad = \sum_{k \ge 1} \frac{\lambda^k}{k} \int_{(\mathbb{R}^d)^k} \sum_{\substack{j \le k \\ \{n_0, \ldots, n_{j-1}\} \\ \subset \{0, \ldots, k-1\}}} \prod_{i=0}^{j-1} \left(e^{-\phi(x_{n_i})} - 1\right) \prod_{i=0}^{k-1} p_{\frac{1}{2\alpha}}(x_i, x_{i+1})\, dx_0 \ldots dx_{k-1} \qquad (2.38) \\
&\quad = \sum_{j \ge 1} \frac{1}{j} \sum_{\substack{r_0 \ge 1 \\ \cdots \\ r_{j-1} \ge 1}} \int_{(\mathbb{R}^d)^j} \prod_{i=0}^{j-1} \left(e^{-\phi(y_i)} - 1\right) \prod_{i=0}^{j-1} J_\lambda^{r_i}(y_i, y_{i+1})\, dy_0 \ldots dy_{j-1}\,. \qquad (2.39)
\end{aligned}$$

In order to derive (2.39) we first fix the points $y_0, \ldots, y_{j-1} \in \gamma'$, i.e. the arguments of the factors $e^{-\phi(\cdot)} - 1$ in (2.38), and integrate over the remaining points of those loops $\gamma \in D_k$, $k \ge j$, such that $\gamma' \subset \gamma$. This leads to the sum over the family of indices $(r_0, \ldots, r_{j-1}) \in \mathbb{N}^j$ and the product of transition kernels $\prod_i J_\lambda^{r_i}(y_i, y_{i+1})$ in the integral over $\left(\mathbb{R}^d\right)^j$. Note that the integral in (2.39) assigns density

$$j \prod_{i=0}^{j-1} \left(e^{-\phi(y_i)} - 1\right) \prod_{i=0}^{j-1} J_\lambda^{r_i}(y_i, y_{i+1})$$

to an unrooted loop $\gamma' = [y_0, \ldots, y_{j-1}]$ and a choice of indices $(r_0, \ldots, r_{j-1})$, which is corrected by the factor $1/j$ in front to match the corresponding expression in (2.38). Now (2.35) follows by changing the order of the sum and the integral. This works if there is absolute convergence of the exponent in (2.35); using $K(x_i, x_{i+1}) \le K(0,0) = \sum_{k \ge 1} \left(\frac{\alpha}{\pi k}\right)^{\frac{d}{2}} \lambda^k$ and $K(0,0)\|e^{-\phi} - 1\|_1 < 1$, we have the absolute convergence.

In order to prove (2.36), write

$$\frac{1}{j}\int_{(\mathbb{R}^d)^j}\prod_{i=0}^{j-1}\big(e^{-\phi(y_i)}-1\big)\,K_\lambda(y_i,y_{i+1})\,dy_0\dots dy_{j-1}$$
$$=\frac{1}{j!}\int_{(\mathbb{R}^d)^j}\prod_{i=0}^{j-1}\big(e^{-\phi(y_i)}-1\big)\Big[\sum_{\sigma\in C_j}\prod_{i=0}^{j-1}K_\lambda(y_i,y_{\sigma(i)})\Big]\,dy_0\dots dy_{j-1}.$$

where C_j denotes the set of cycles of length j,

$$C_j\subset\mathcal{S}_j=\{\sigma:[0,j-1]\to[0,j-1],\sigma\text{ bijective}\},$$

the set of permutations of j elements. We replace this expression in (2.39) to get

$$\mathcal{L}^{\rm ls}_\lambda(\phi)=\exp\Big\{\sum_{j\geq1}\frac{1}{j!}\sum_{\sigma\in C_j}\int_{(\mathbb{R}^d)^j}\prod_{i=0}^{j-1}\big(e^{-\phi(y_i)}-1\big)\prod_{i=0}^{j-1}K_\lambda(y_i,y_{\sigma(i)})\,dy_0\dots dy_{j-1}\Big\}.$$
$$=1+\sum_{n\geq1}\frac{1}{n!}\sum_{P\in\mathcal{P}_n}\prod_{I\in P}\sum_{\sigma\in C_{|I|}}\int_{(\mathbb{R}^d)^{|I|}}\prod_{i=0}^{|I|-1}\big(e^{-\phi(y_i)}-1\big)\prod_{i=0}^{|I|-1}K_\lambda(y_i,y_{\sigma(i)})\,dy_0\dots dy_{|I|-1}\quad(2.40)$$
$$=1+\sum_{n\geq1}\frac{1}{n!}\int_{(\mathbb{R}^d)^n}\sum_{\pi\in\mathcal{S}_n}\prod_{i=0}^{n}\big(e^{-\phi(y_i)}-1\big)\prod_{i=0}^{n}K_\lambda(y_i,y_{\pi(i)})\,dy_0\dots dy_n.$$

where in identity (2.40) we have used Lemma 2.8 below. This shows (2.36). □

Proposition 2.7 (Point correlations)**.** *The n-point correlation density of $\nu^{\rm ls}_\lambda$ is given by*

$$\varphi^{\rm ls}_\lambda(x_1,\dots,x_n)=\operatorname{perm}\big(K_\lambda(x_i,x_j)\big)_{i,j=1}^n,\qquad(2.41)$$

where $\operatorname{perm}(A)$ *is the permanent of the matrix $A\in\mathbb{R}^{n\times n}$.*

Sketch of the proof. We compute the 3-point correlation density. To simplify notation, in this proof we will denote $\mu=\mu^{\rm ls}_\lambda$, $Q=Q^{\rm ls}_\lambda$ and $K_{xy}=K_\lambda(x,y)$. Given pairwise disjoint bounded Borel sets $A,B,C\subset\mathbb{R}^d$, the third moment measure for the point marginal $\nu^{\rm ls}_\lambda$ over $A\times B\times C$ is given by

$$\int n_A(\Gamma)\,n_B(\Gamma)\,n_C(\Gamma)\,\mu(d\Gamma)\qquad(2.42)$$
$$=\int\textstyle\sum_{\gamma\in\Gamma}n_A(\gamma)\sum_{\gamma'\in\Gamma}n_B(\gamma')\sum_{\gamma''\in\Gamma}n_C(\gamma'')\,\mu(d\Gamma)$$
$$=\int\Big(\textstyle\sum_{\gamma\in\Gamma}n_A(\gamma)\,n_B(\gamma)\,n_C(\gamma)\;+\sum_{\gamma\in\Gamma}n_A(\gamma)\sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma}n_B(\gamma')\,n_C(\gamma')\qquad(2.43)$$
$$+\;\textstyle\sum_{\gamma\in\Gamma}n_B(\gamma)\sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma}n_A(\gamma')\,n_C(\gamma')\;+\;\sum_{\gamma\in\Gamma}n_C(\gamma)\sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma}n_A(\gamma')\,n_B(\gamma')$$
$$+\;\textstyle\sum_{\gamma\in\Gamma}n_A(\gamma)\sum_{\gamma'\in\Gamma,\,\gamma'\neq\gamma}n_B(\gamma')\sum_{\gamma''\in\Gamma\setminus\{\gamma,\gamma'\}}n_C(\gamma'')\Big)\mu(d\Gamma)$$
$$=Q(n_A\,n_B\,n_C)\;+Q(n_A)\,Q(n_B\,n_C)\qquad(2.44)$$
$$+\,Q(n_B)\,Q(n_A\,n_B)+Q(n_C)\,Q(n_A\,n_B)+Q(n_A)\,Q(n_B)\,Q(n_C).$$

To go from (2.43) to (2.44) we use that μ is a Poisson process of loops, then (a) the expectation of the product of functions of different loops factorize (Theorem 3.2 in [17]), and (b) $\int \sum_{\gamma\in\Gamma} g(\gamma)\mu(d\Gamma) = \int_D g(\gamma)Q(d\gamma)$, denoted $Q(g)$, by Campbell's theorem.

Define

$$\big\{\langle a_1 \dots a_k\rangle\big\} := \big\{\gamma \in D : \gamma \text{ goes through } a_1, \dots, a_k \text{ in this order}\big\} \tag{2.45}$$

and compute

$$Q(n_A\, n_B\, n_C) = \int \textstyle\sum_{a\in\gamma} \mathbf{1}_A(a) \sum_{b\in\gamma} \mathbf{1}_B(b) \sum_{c\in\gamma} \mathbf{1}_C(c)\, Q(d\gamma) \tag{2.46}$$

$$= \int \sum_{\substack{\{a,b,c\}\subset\{\gamma\}\\(a,b,c)\in A\times B\times C}} \big(\mathbf{1}_{\langle abc\rangle}(\gamma) + \mathbf{1}_{\langle acb\rangle}(\gamma)\big)\, Q(d\gamma) \tag{2.47}$$

$$= \int_A \int_B \int_C (K_{ab}K_{bc}K_{ca} + K_{ac}K_{cb}K_{ba})\, dc\, db\, da, \tag{2.48}$$

where (2.47) follows from partitioning the set of cycles that go through a, b, c according to the order in which they visit the points, and (2.48) can be proved using the argument applied to derive (2.38)-(2.39). Using the same argument to compute the other terms in (2.44), we conclude that the third moment measure (2.42) is absolutely continuous with respect to Lebesgue measure in $(\mathbb{R}^d)^3$ with Radon-Nikodym derivative

$$\begin{aligned}\varphi^{\mathrm{ls}}_\lambda(x,y,z) = {} & K_{xx}K_{yy}K_{zz} + K_{xx}K_{yz}K_{zy} + K_{xy}K_{yx}K_{zz} \\ & + K_{xy}K_{yz}K_{zx} + K_{xz}K_{yx}K_{zy} + K_{xz}K_{yy}K_{zx},\end{aligned}$$

which proves (2.41) for $n = 3$; see Fig. 4. We leave the proof of the general case to the reader. □

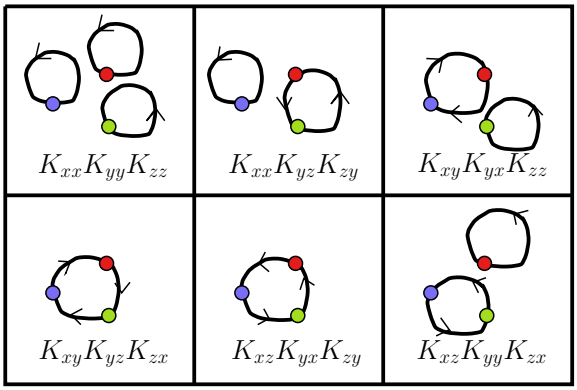

Figure 4: Gaussian loop soup 3-point correlations. A directed lace between two points means that the loop goes through the points in the indicated order. Point x is blue, y is red and z is green.

2.3 Combinatorial lemma

We now prove the lemma used to show Proposition 2.5 and identity (2.40).

Lemma 2.8. *Let $(a_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}}$ be such that $\sum_{n\geq 1} \frac{1}{n!}|a_n| < \infty$. Then*

$$\exp\Big(\sum_{n\geq 1} \frac{1}{n!}\, a_n\Big) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|}, \tag{2.49}$$

where $\mathcal{P}_n$ is the set of partitions of $\{1,\dots,n\}$ into non-empty sets,

$$\mathcal{P}_n = \Big\{P = \{I_j\}_{1\leq j\leq k},\, k\in\mathbb{N} : \emptyset \neq I_j \subset [1,n],\, I_i\cap I_j = \emptyset \text{ if } i\neq j \text{ and } \bigcup_{j=1}^{k} I_j = [1,n]\Big\}, \tag{2.50}$$

and $|I|$ stands for the cardinality of the set I.

Proof. By the series expansion of the exponential function

$$\begin{aligned}
\exp\Big(\sum_{n\geq 1} \frac{1}{n!}\, a_n\Big) &= \sum_{j\geq 0} \frac{1}{j!}\Big(\sum_{\ell\geq 1} \frac{1}{\ell!} a_l\Big)^j \\
&= 1 + \sum_{j\geq 1} \frac{1}{j!} \sum_{\substack{i_1,\dots,i_j \\ i_\ell\geq 1}} \frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!} \\
&= 1 + \sum_{j\geq 1} \frac{1}{j!} \sum_{n\geq 1} \sum_{\substack{i_1+\cdots+i_j=n \\ i_\ell\geq 1}} \frac{a_{i_1}}{i_1!}\cdots\frac{a_{i_j}}{i_j!} \\
&= 1 + \sum_{j\geq 1} \frac{1}{j!} \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{i_1+\cdots+i_j=n \\ i_\ell\geq 1}} \binom{n}{i_1\dots i_j}\, a_{i_1}\dots a_{i_j} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{j\geq 1} \frac{1}{j!} \sum_{\substack{i_1+\cdots+i_j=n \\ i_\ell\geq 1}} \binom{n}{i_1\dots i_j}\, a_{i_1}\dots a_{i_j} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{j\geq 1} \sum_{P=\{I_1,\dots,I_j\}\in\mathcal{P}_n} a_{|I_1|}\dots a_{|I_j|} \\
&= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{P\in\mathcal{P}_n} \prod_{I\in P} a_{|I|}.
\end{aligned}$$

The change of the order of summation in the fifth line above is justified by the absolute convergence of $\sum_{n\geq 1} \frac{1}{n!}\, a_n$. □

3 Gaussian random interlacements

The goal of this section is to construct a measure on the space of doubly infinite trajectories in $\mathbb{R}^d$, $d \geq 3$, analogous to the intensity of the Gaussian loop soup. The Gaussian random interlacements process with intensity β is defined as a Poisson process with an intensity proportional to this measure. We adapt the strategy developed by Sznitman [21] to introduce the model of random

interlacements in $\mathbb{Z}^d$. The intensity measure of set of paths that intersect a bounded set A is obtained as the projection of a measure supported on trajectories that visit A at time 0 for the first time. Precisely, the location of the first visit to A is distributed according to the equilibrium measure of the associated random walk, and from this location two independent random walk trajectories are drawn, with the one running up to time $-\infty$ being conditioned to never return to A. We here propose an alternative, equivalent measure that chooses the location x of the trajectory at time 0 uniformly in A, draws two independent random walk trajectories starting at x, up to $-\infty$ and $+\infty$ times respectively, and assigns weight to the resulting doubly infinite path that is inversely proportional to its total number of visits to A. In §3.1 we compute the density, Laplace functionals and point correlations of the point marginal of the Gaussian random interlacements.

We define the infinite volume Gaussian random permutation in §4. In §4.2 we show that the Gaussian random permutation is Gibbs for the specifications induced by (1.2).

For $d \geq 3$, define

$$W := \big\{ w : \mathbb{Z} \to \mathbb{R}^d, \lim_{n \to \pm\infty} \|w(n)\| = \infty \big\}, \tag{3.1}$$

the space of discrete $\mathbb{R}^d$-valued doubly infinite trajectories that spend finite time in compact subsets of $\mathbb{R}^d$. Endow W with the σ-algebra $\mathcal{W}$ generated by the canonical coordinates $X_n(w) := w(n)$, $n \in \mathbb{Z}$. Given a bounded set $A \subset \mathbb{R}^d$ and $w \in W$, let

$$T_A(w) := \inf \big\{ n \in \mathbb{Z},\, X_n(w) \in A \big\}, \quad \text{the entrance time in } A. \tag{3.2}$$

Notice that $T_A(w) > -\infty$ because by definition w intersects A finitely many times. Let W_A be the set of trajectories that enter A,

$$W_A = \{ w \in W : X_n(w) \in A \text{ for some } n \in \mathbb{Z} \}, \tag{3.3}$$

and for $w \in W_A$ denote by

$$n_A(w) = \sum_{n \in \mathbb{Z}} \mathbf{1}_A(X_n(w)) \tag{3.4}$$

the number of visits of the trajectory w to A.

Define the time shift $\theta : W \to W$ by $[\theta w](k) := w(k+1)$, and let θ^ℓ be the shift by ℓ time units, $[\theta^\ell w](k) := w(k+\ell)$, $\ell \in \mathbb{Z}$. Given a function $g : W \to \mathbb{R}$ let $(\theta g)(w) := g(\theta w)$.

Given $x \in \mathbb{R}^d$ let P^x denote the probability measure on $\big(W, \mathcal{W}\big)$ having finite dimensional distributions

$$\begin{aligned} &P^x \big[X_\ell \in dx_\ell, \dots, X_0 \in dx, \dots, X_k \in dx_k \big] \\ &\qquad = \delta_x(dx_0) \prod_{i=1}^{k} p_{\frac{1}{2\alpha}}(x_{i-1}, x_i)\, dx_i \prod_{j=1}^{\ell} p_{\frac{1}{2\alpha}}(x_{-j+1}, x_{-j})\, dx_{-j}, \end{aligned}$$

where δ_x is the Dirac distribution at x, and $p_{\frac{1}{2\alpha}}$ is defined in (2.1). That is, P^x is the law of a doubly infinite random walk in $\mathbb{R}^d$ that satisfies $X_0 = x$ and has independent, identically distributed

Gaussian increments $N(0, \frac{1}{2\alpha})$. The fact that the walk is transient implies $P^x(W) = 1$. Denote by E^x the expectation with respect to P^x. Since the Lebesgue measure is reversible for the random walk, we have

$$\int_{\mathbb{R}^d} E^x[g]\,dx = \int_{\mathbb{R}^d} E^x[\theta g]\,dx \tag{3.5}$$

for bounded measurable test functions $g : W \to \mathbb{R}$.

Consider the measure Q_A^{cap} on W_A, that integrates a test function $g : W \to \mathbb{R}$ as

$$Q_A^{\mathrm{cap}} g := \int_A E^x\big[g\,\mathbf{1}_{\{T_A=0\}}\big]\,dx. \tag{3.6}$$

The function $e_A(x) := P^x[T_A = 0]$ is the density of a measure on $\mathbb{R}^d$ supported on A called the equilibrium measure associated to the Gaussian random walk. The capacity of A is defined by $\mathrm{cap}(A) := \int_A e_A(x)\,dx$; see [21].

Define the measure Q_A^{unif} on W_A by

$$Q_A^{\mathrm{unif}} g := \int_A E^x\left[\frac{g}{n_A}\right]\,dx. \tag{3.7}$$

The weight assigned to a trajectory w is inversely proportional to the number of visits of w to the set A.

Proposition 3.1. *For any bounded set $A \subset \mathbb{R}^d$ and measurable bounded function $g : W \to \mathbb{R}$ invariant under time shifts, $g = \theta g$, we have*

$$Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{cap}} g. \tag{3.8}$$

Proof. Write

$$\begin{aligned}
Q_A^{\mathrm{unif}} g &= \int_A dx\, E^x\left[\frac{g}{n_A}\sum\nolimits_{i\le 0}\,\mathbf{1}_{\{T_A=i\}}\right] \\
&= \sum_{i\le 0}\int_{\mathbb{R}^d} dx\, E^x\left[\mathbf{1}_A(X_0)\mathbf{1}_{\{T_A=i\}}\,\frac{g}{n_A}\right] \qquad \text{by Fubini's theorem} \\
&= \sum_{i\ge 0}\int_{\mathbb{R}^d} dx\, E^x\left[\mathbf{1}_A(X_i)\mathbf{1}_{\{T_A=0\}}\,\theta^i\Big\{\frac{g}{n_A}\Big\}\right] \qquad \text{by (3.5)} \\
&= \int_A dx\, E^x\left[\mathbf{1}_{\{T_A=0\}}\,\frac{g}{n_A}\sum\nolimits_{i\ge 0}\,\mathbf{1}_A(X_i)\right] \qquad \text{since } \theta^i\Big\{\frac{g}{n_A}\Big\} = \frac{g}{n_A} \\
&= \int_A dx\, E^x\big[\mathbf{1}_{\{T_A=0\}}\,g\big] = Q_A^{\mathrm{cap}} g.
\end{aligned}$$

□

Lemma 3.2. *Let $A \subset B$ be bounded sets of $\mathbb{R}^d$, and let g be a test function that is invariant under time shifts, $g = \theta g$. Then*

$$Q_B^{\mathrm{cap}} g \mathbf{1}_{W_A} = Q_A^{\mathrm{cap}} g \qquad \textit{(compatibility)} \tag{3.9}$$

$$Q_B^{\mathrm{cap}} g = Q_A^{\mathrm{cap}} g + Q_{B\setminus A}^{\mathrm{cap}}\, g\mathbf{1}_{W_A^{\mathrm{c}}} \qquad \textit{(additivity)}, \tag{3.10}$$

where $W_A^{\mathrm{c}} = W \setminus W_A$. The same holds for Q^{unif}.

Proof. Write $\mathbf{1}_{W_A} = \sum_{i\in\mathbb{Z}} \mathbf{1}_{\{T_A=i\}}$. Then

$$\begin{aligned} Q_B^{\mathrm{cap}} g\mathbf{1}_{W_A} &= \sum_{i\geq 0} \int dx\, \mathbb{E}^x\left[\mathbf{1}_{\{T_B=0\}}\mathbf{1}_{\{T_A=i\}}\, g\right] \\ &= \sum_{i\geq 0} \int dx\, \mathbb{E}^x\left[\theta^i\left(\mathbf{1}_{\{T_B=-i\}}\mathbf{1}_{\{T_A=0\}}\, g\right)\right] \\ &= \sum_{j\leq 0} \int dx\, \mathbb{E}^x\left[\mathbf{1}_{\{T_B=j\}}\mathbf{1}_{\{T_A=0\}}\, g\right] \\ &= \int dx\, \mathbb{E}^x\left[\mathbf{1}_{\{T_A=0\}}\, g \sum_{i\leq 0}\, \mathbf{1}_{\{T_B=i\}}\right] = Q_A^{\mathrm{cap}} g, \end{aligned}$$

since $\mathbf{1}_{\{T_A=0\}} \sum_{i\leq 0} \mathbf{1}_{\{T_B=i\}} = \mathbf{1}_{\{T_A=0\}}$. This proves (3.9). To get (3.10) write

$$Q_B^{\mathrm{cap}} g = Q_B^{\mathrm{cap}} g(\mathbf{1}_{W_A} + \mathbf{1}_{W_A^c}) = Q_A^{\mathrm{cap}} g + Q_B^{\mathrm{cap}} g\mathbf{1}_{W_A^c} = Q_A^{\mathrm{cap}} g + Q_{B\setminus A}^{\mathrm{cap}} g\mathbf{1}_{W_A^c}.$$

Since $g\mathbf{1}_{W_A} = \theta(g\mathbf{1}_{W_A})$, $g = \theta(g)$ and $g\mathbf{1}_{W_A^c} = \theta(g\mathbf{1}_{W_A^c})$, Proposition 3.1 implies that (3.9) and (3.10) hold for Q^{unif} as well. □

Now let us identify trajectories that differ by time shift: given two doubly-infinite trajectories w, $w' \in W$, we say that $w \sim w'$ if there exists $k \in \mathbb{Z}$ such that $w' = \theta^k w$. Let

$$\widetilde{W} := W/\sim \tag{3.11}$$

be the space of trajectories modulo time shift, $\pi : W \to \widetilde{W}$ the projection, and $\widetilde{\mathcal{W}}$ the push-forward σ-algebra on $\widetilde{W}$. Given $A \subset \mathbb{R}^d$ let $\widetilde{W}_A = \pi(W_A)$. If $g : W \to \mathbb{R}$ is shift invariant then it can be extended to $\tilde{g} : \widetilde{W} \to \mathbb{R}$ by $g(\tilde{w}) = g(w)$, for any choice of representative $w \in \pi^{-1}(\tilde{w})$.

Theorem 3.3. *There exists a unique σ-finite measure Q^{ri} on $(\widetilde{W}, \widetilde{\mathcal{W}})$ such that for each bounded set $A \subset \mathbb{R}^d$*

$$\mathbf{1}_{\widetilde{W}_A} Q^{\mathrm{ri}} = \pi_* Q_A^{\mathrm{cap}} = \pi_* \circ Q_A^{\mathrm{unif}}, \tag{3.12}$$

where $\pi_* Q_A^{\mathrm{cap}}$ and $\pi_* Q_A^{\mathrm{unif}}$ denote the push-forward measures.

Proof. Let $\tilde{g} : \widetilde{W} \to \mathbb{R}$ and define $g : W \to \mathbb{R}$ by $g = \tilde{g} \circ \pi$. Then $\theta g = g$ and

$$\pi_* Q_A^{\mathrm{cap}} \tilde{g} = Q_A^{\mathrm{cap}} \tilde{g} \circ \pi = Q_A^{\mathrm{cap}} g = Q_A^{\mathrm{unif}} g = Q_A^{\mathrm{unif}} \tilde{g} \circ \pi = \pi_* Q_A^{\mathrm{unif}} \tilde{g}$$

by Proposition 3.1. This proves the second equality in (3.12).

Let $\{A_n\}_{n\geq 1}$ be an increasing sequence of bounded sets such that $A_n \nearrow_{n\to\infty} \mathbb{R}^d$. Then $\widetilde{W} = \bigcup_{n\geq 1} \widetilde{W}_{A_n}$ and uniqueness of the measure satisfying (3.12) follows. Define Q^{ri} on $\widetilde{W}_{A_n}$ by

$$\mathbf{1}_{\widetilde{W}_{A_n}} Q^{\mathrm{ri}} := \pi_* Q_{A_n}^{\mathrm{cap}}. \tag{3.13}$$

Let A be a bounded set and take n sufficiently large such that $A_n \supset A$. Then

$$\mathbf{1}_{\widetilde{W}_A}\mathbf{1}_{\widetilde{W}_{A_n}}\big(\pi_* Q^{\mathrm{cap}}_{A_n}\big) = \mathbf{1}_{\widetilde{W}_A}\big(\pi_* Q^{\mathrm{cap}}_{A_n}\big) = \pi_* \mathbf{1}_{W_A} Q^{\mathrm{cap}}_{A_n} = \pi_* Q^{\mathrm{cap}}_{A}, \tag{3.14}$$

where the last identity follows from (3.9). In the case when $A = A_m$ for some $m < n$, (3.14) proves that the definition (3.13) is consistent and that the measure Q^{ri} defined in (3.13) satisfies (3.12). By (3.12), $Q^{\mathrm{ri}}(\widetilde{W}_{A_n}) = Q^{\mathrm{cap}}_{A_n}(W) = \mathrm{cap}(A_n) < \infty$, which proves the σ-finite property. □

Definition 3.4. *Let $d \geq 3$ and $\beta > 0$. The Gaussian random interlacements process at level β is the Poisson point process on the space $\widetilde{W}$ of discrete-time trajectories in $\mathbb{R}^d$ modulo time shift defined in* (3.11)*, with intensity measure $\beta Q^{\mathrm{ri}}(d\tilde{w})$. We will use $\Gamma^{\mathrm{ri}}_\beta$ to denote a configuration of the random interlacements at level β, and μ^{ri}_β to denote its law.*

This definition is slightly simpler than the one proposed by Sznitman [21] in $\mathbb{Z}^d$. The original definition considers a point process in the product space $\mathbb{R}_{\geq 0} \times \widetilde{W}$ with intensity measure $du \otimes Q^{\mathrm{ri}}(d\tilde{w})$, so that each trajectory $\tilde{w}$ is sampled with an associated level $u > 0$ used to couple realizations of the model at different densities. For our purposes it is enough to a priori fix the level β.

3.1 Point marginal of the Gaussian random interlacements

We now study the point process composed of the points in the trajectories of the random interlacements. Define the random point process

$$\chi^{\mathrm{ri}}_\beta := \bigcup_{\tilde{w} \in \Gamma^{\mathrm{ri}}_\beta} \{\tilde{w}\}, \quad \text{and let} \quad \nu^{\mathrm{ri}}_\beta \quad \text{be its law.} \tag{3.15}$$

Lemma 3.5. *For any pair $A \subset B \subset \mathbb{R}^d$ of bounded sets we have*

$$Q^{\mathrm{cap}}_B n_A = Q^{\mathrm{unif}}_B n_A = |A|, \tag{3.16}$$

where $|A|$ is the Lebesgue measure of A. Furthermore, if $y \in B$ and $K = K_1$ is the kernel (2.34)*, we have*

$$e_B(y) + \int_B e_B(x) K(x,y)\, dx = 1, \qquad y \in A. \tag{3.17}$$

Proof. We have

$$\begin{aligned} Q^{\mathrm{unif}}_B n_A &= \int_B E^x\left[\frac{n_A}{n_B}\right] dx = \int_B E^x\left[\textstyle\sum_i \mathbf{1}_A(X_i)\mathbf{1}_B(X_0)\frac{1}{n_B}\right] dx \\ &= \int_B E^x\left[\textstyle\sum_i \mathbf{1}_A(X_0)\mathbf{1}_B(X_i)\frac{1}{n_B}\right] dx = \int_B E^x\left[\mathbf{1}_A(X_0)\frac{n_B}{n_B}\right] dx = |A|, \end{aligned} \tag{3.18}$$

where the first identity in (3.18) follows from (3.5). Now

$$Q^{\mathrm{unif}}_B n_A = \lim_{R\to\infty} Q^{\mathrm{unif}}_B (n_A \wedge R) = \lim_{R\to\infty} Q^{\mathrm{cap}}_B (n_A \wedge R) = Q^{\mathrm{cap}}_B n_A \tag{3.19}$$

by the monotone convergence theorem and Proposition 3.1. Identity (3.16) follows from (3.18) and (3.19).

In order to prove (3.17), notice that for any pair of sets $A \subset B$

$$\begin{aligned}
|A| = Q_B^{\mathrm{cap}} n_A &= \int_B E^x \Big[\mathbf{1}_{T_B=0} \sum_{j\in\mathbb{Z}} \mathbf{1}_A(X_j) \Big]\, dx \\
&= \int_B E^x \Big[\mathbf{1}_{T_B=0} \sum_{k\geq 0} \mathbf{1}_A(X_k) \Big]\, dx \qquad \text{as } A \subset B \\
&= \int_B \mathbb{E}^x \Big[\mathbf{1}_{T_B=0} \mathbf{1}_A(x) \Big]\, dx + \int_A \int_B \sum_{k\geq 1} \mathbb{P}^x(T_B = 0) J^k(x,y)\, dx\, dy \\
&= \int_B e_B(x) \mathbf{1}_A(x)\, dx + \int_A \int_B e_B(x) K(x,y)\, dx\, dy \\
&= \int_A \Big[e_B(y) + \int_B e_B(x) K(x,y)\, dx \Big] dy
\end{aligned}$$

Since this holds for any subset A of B, it follows that $e_B(y) + \int_B e_B(x) K(x,y)\, dx = 1$, $y \in A$. □

We next compute the point density of χ_β^{ri}. Given $A \subset \mathbb{R}^d$ and $\tilde{w} \in \widetilde{W}$, let $n_A(\tilde{w}) = n_A(w)$ for any $w \in \pi^{-1}(\tilde{w})$. Note that this definition does not depend on the choice of representative w. For a locally finite set of points $\chi \subset \mathbb{R}^d$, let $n_A(\chi) = \sum_{x\in\chi} \mathbf{1}_A(x)$. The mean density of χ_β^{ri} in A is defined by

$$\varrho_\beta(A) := \frac{1}{|A|} \nu_\beta^{\mathrm{ri}}\, n_A = \frac{1}{|A|} \int \sum_{\tilde{w}\in\Gamma} n_A(\tilde{w})\, \mu_\beta^{\mathrm{ri}}(d\Gamma). \tag{3.20}$$

Lemma 3.6 (Point density)**.** *For any bounded set $A \subset \mathbb{R}^d$ we have*

$$\varrho_\beta(A) = \beta. \tag{3.21}$$

In particular the mean density does not depend on the set A.

Proof. We compute the mean number of points of the process in A as

$$\begin{aligned}
|A|\, \varrho_\beta(A) &= \int \sum_{\tilde{w}\in\Gamma} n_A(\tilde{w})\, \mu_\beta^{\mathrm{ri}}(d\Gamma) \\
&= \beta \int_{\widetilde{W}} \tilde{n}_A(\tilde{w})\, Q^{\mathrm{ri}}(d\tilde{w}) \qquad \text{by Campbell's theorem} \\
&= \beta \int_{W_A} n_A(w)\, Q_A^{\mathrm{unif}}(dw) \qquad \text{by (3.12) and (3.8)} \\
&= \beta |A| \qquad \text{by (3.16).}
\end{aligned}$$

□

We next compute the Laplace functionals of χ_β^{ri}, denoted by

$$\mathcal{L}_\beta^{\mathrm{ri}}(\phi) := \int \nu_\beta^{\mathrm{ri}}(d\chi) \exp\Big\{ -\sum_{x\in\chi} \phi(x) \Big\}$$

Theorem 3.7 (Laplace functionals)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a measurable function with compact support and satisfying $K(0,0)\|e^{-\phi} - 1\|_1 < 1$. Then*

$$\mathcal{L}^{\mathrm{ri}}_\beta(\phi) = \exp\Big\{\beta \sum_{n\geq 1} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^{n-1} K(x_k, x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)} - 1\big)\, dx_1 \dots dx_n\Big\}, \tag{3.22}$$

with $\prod_{k=1}^{0} K(x_k, x_{k+1}) := 1$.

Proof. Define $\Phi : \widetilde{W} \to \mathbb{R}$ by

$$\Phi(\tilde{w}) = \sum_{n\in\mathbb{Z}} \phi(w(n)), \quad \text{for any } w \in \pi^{-1}(\tilde{w}). \tag{3.23}$$

The definition does not depend on the choice of representative $w \in \pi^{-1}(\tilde{w})$. Then,

$$\mathcal{L}^{\mathrm{ri}}_\beta(\phi) = \int_{\widetilde{W}} \exp\big\{ -\sum_{\tilde{w}\in\Gamma} \Phi(\tilde{w})\big\}\, \mu^{\mathrm{ri}}_\beta(d\Gamma) = \exp\Big\{\beta \int_{\widetilde{W}} \big(e^{-\Phi(\tilde{w})} - 1\big) Q^{\mathrm{ri}}(d\tilde{w})\Big\}, \tag{3.24}$$

by Campbell's theorem. Let $A = \mathrm{supp}(\phi) \subset \mathbb{R}^d$ which is compact by hypothesis. Then

$$\int_{\tilde{W}} \big(e^{-\Phi(\tilde{w})} - 1\big) Q^{\mathrm{ri}}(d\tilde{w}) = Q^{\mathrm{cap}}_A\Big(\prod_{i=0}^{\infty} e^{-\phi(w(i))} - 1\Big), \tag{3.25}$$

by Theorem 3.3. For $w \in W$, let us write

$$\begin{aligned}
\prod_{i=0}^{\infty} e^{-\phi(w(i))} - 1 &= \prod_{i=0}^{\infty} \Big[\big(e^{-\phi(w(i))} - 1\big) + 1\Big] - 1 \\
&= \sum_{\emptyset\neq S\subset\mathbb{N}_0} \prod_{i\in S} \big(e^{-\phi(w(i))} - 1\big) \\
&= \sum_{n\geq 1} \sum_{0\leq i_1<i_2<\dots<i_n} \prod_{j=1}^{n} \big(e^{-\phi(w(i_j))} - 1\big).
\end{aligned} \tag{3.26}$$

The trajectory $w \in W$ enters the support of ϕ finitely many times and the factor $e^{-\phi(x)} - 1$ vanishes outside this set, hence the sum on the second line above may be restricted to finite sets $S \subset \mathbb{N}_0$ to gt the third line.

Let $A^k = A \times \cdots \times A$ denote the product of k copies of A, $k \geq 1$. We replace the last expression from (3.26) in (3.25), and exchange the order of the sums and the integral, which is justified by

the monotone convergence theorem, to obtain

$$
\begin{aligned}
&\frac{1}{\beta}\log \mathcal{L}^{\mathrm{ri}}_{\beta}(\phi) = \\
&= \sum_{n\geq 1} \sum_{0\leq i_1<i_2<\ldots<i_n} \int_W \prod_{j=1}^{n} \big(e^{-\phi(w(i_j))}-1\big) Q^{\mathrm{cap}}_A(dw) \\
&= \sum_{n\geq 1} \sum_{\substack{1\leq \ell_1 \\ 1\leq \ddot{\ell}_{n-1}}} \int_{A^n} e_A(x_1) \prod_{k=1}^{n-1} J_1^{\ell_k}(x_k,x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)}-1\big)\, dx_1\ldots dx_n \\
&\quad + \sum_{n\geq 1}\sum_{\ell\geq 1} \sum_{\substack{1\leq \ell_1 \\ 1\leq \ddot{\ell}_{n-1}}} \int_{A^n+1} e_A(x_0) J_1^{\ell}(x,x_1) \prod_{k=1}^{n-1} J_1^{\ell_k}(x_k,x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)}-1\big)\, dx_0\ldots dx_n \\
&= \sum_{n\geq 1} \sum_{\substack{1\leq \ell_1 \\ 1\leq \ddot{\ell}_{n-1}}} \int_{A^n} \prod_{k=1}^{n-1} J_1^{\ell_k}(x_k,x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)}-1\big)\, dx_1\ldots dx_n \qquad \text{by (3.17)} \\
&= \sum_{n\geq 1} \int_{A^n} \prod_{k=1}^{n-1} K(x_k,x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)}-1\big)\, dx_1\ldots dx_n \qquad (3.27) \\
&= \sum_{n\geq 1} \int_{(\mathbb{R}^d)^n} \prod_{k=1}^{n-1} K(x_k,x_{k+1}) \prod_{k=1}^{n} \big(e^{-\phi(x_k)}-1\big)\, dx_1\ldots dx_n
\end{aligned}
$$

since $e^{-\phi(x)}-1\equiv 0$ for $x\notin A$; identity (3.27) holds because ϕ satisfies $K(0,0)\|e^{-g}-1\|_1<1$. The result follows by replacing the last expression in (3.24). □

We now describe the point correlations of the point marginal of the Gaussian random interlacements.

Proposition 3.8 (Point correlations)**.** *The n-point correlation density of the point process $\chi^{\mathrm{ri}}_{\beta}$ is given by*

$$
\varphi^{\mathrm{ri}}_{\beta}(x_1,\ldots,x_n) = \sum_{P\in\mathcal{P}_n} \prod_{I=\{i_1,\ldots,i_m\}\in P} \sum_{\sigma\in\mathcal{S}_m} \beta K(x_{i_{\sigma(1)}},x_{i_{\sigma(2)}})\ldots K(x_{i_{\sigma(m-1)}},x_{i_{\sigma(m)}}), \qquad (3.28)
$$

where $\mathcal{P}_n$ is the set of partitions of $\{1,\ldots,n\}$ and $\mathcal{S}_m$ is the set of permutations of $\{1,\ldots,m\}$.

A direct proof of (3.28) can be obtained as in the loop soup case, Proposition 2.7. For instance, for $n=3$, taking x,y,z distinct points in $\mathbb{R}^d$ and denoting $K_{xy}=K(x,y)$ we have

$$
\varphi^{\mathrm{ri}}_{\beta}(x,y,z) = \beta^3 + 2\beta^2K_{xy} + 2\beta^2K_{yz} + 2\beta^2K_{xz} + 2\beta K_{xy}K_{yz} + 2\beta K_{xz}K_{zy} + 2\beta K_{zx}K_{xy}. \qquad (3.29)
$$

Each term in the above sum corresponds to a set of a partition of the interlacements configurations containing $\{x,y,z\}$. For instance, the first term β^3 is the density of the event $\{\Gamma : x,\ y \text{ and } z \text{ belong to different trajectories in } \Gamma\}$, while $2\beta K_{xy}K_{yz}$ is the density of $\{\Gamma : x,\ y \text{ and } z \text{ belong to the same}$

trajectory of $\Gamma\}$. Notice that in the latter case the infinite path may go through x first and y later on, or visit the points in the opposite order; the factor 2 in front of the term associated to this partition accounts for these two cases. See an illustration of this partition in Fig. 5.

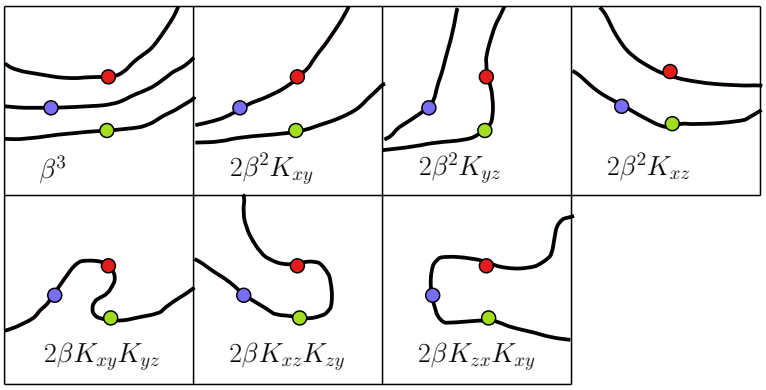

Figure 5: Gaussian random interlacements 3-point correlations. Each square represents one set of the trajectory partition; its density was computed after (3.29). The point x is blue, y is red and z is green.

4 Infinite volume Gaussian random permutations

In this section we define the infinite volume Gaussian random permutation, and show that it is a Gibbs measure for the specification induced by (1.2) in Section 4.2.

The superposition of a loop soup and an interlacements configuration $\Gamma \subset D \cup \widetilde{W}$ is in bijection with the spatial permutation (χ, σ) defined by

$$\chi := \cup_{\gamma\in\Gamma}\{\gamma\}, \quad \sigma(x) := \gamma(x), \quad \text{if } x \in \{\gamma\} \text{ and } \gamma \in \Gamma. \tag{4.1}$$

We will abuse notation and use indistinctly Γ and (χ, σ) to refer to the spatial permutation.

Fix a point density $\rho > 0$, recall $\rho_c = \left(\frac{\alpha}{\pi}\right)^{\frac{d}{2}} \sum_{k\geq 1} k^{-\frac{d}{2}}$ and for $\rho < \rho_c$ denote

$$\lambda(\rho) := \text{the solution to } \rho = \left(\frac{\alpha}{\pi}\right)^{\frac{d}{2}} \sum_{k\geq 1} \lambda^k \, k^{-\frac{d}{2}}. \tag{4.2}$$

Recall that when $d \leq 2$, $\rho_c = \infty$ and when $d \geq 3$, $\rho_c < \infty$ and $\lambda(\rho_c) = 1$.

Definition 4.1. *The* Gaussian random permutation *(GRP) in* $\mathbb{R}^d$ *at density* $\rho > 0$ *is the measure* $\mu_\rho := \mu^{\mathrm{ls}}_{\lambda(\rho)}$ *when* $d \leq 2$ *and, when* $d \geq 3$,

$$\mu_\rho := \begin{cases} \mu^{\mathrm{ls}}_{\lambda(\rho)}, & \text{if } \rho \leq \rho_c \\ \mu^{\mathrm{ls}}_1 * \mu^{\mathrm{ri}}_{\rho-\rho_c}, & \text{if } \rho > \rho_c \end{cases} \tag{4.3}$$

where the Gaussian loop soup $\mu^{\mathrm{ls}}_\lambda$ *and the Gaussian random interlacements* $\mu^{\mathrm{ri}}_{\rho-\rho_c}$ *are as in Definitions 2.3 and 3.4, respectively.*

4.1 The point marginal of the GRP is the boson point process

We show that the point marginal of the Gaussian random permutation coincides with the boson point process associated to the free Bose gas.

Subcritical case The point process for the subcritical free Bose gas in $\mathbb{R}^d$ at fugacity $\lambda \in (0,1)$ (also $\lambda = 1$ if $d \geq 3$) is identified by Shirai and Takahashi [18] (equation (1.1) and Theorem 1.2) as the probability measure $\nu_\lambda^{\mathrm{ST}}$ on locally finite point processes in $\mathbb{R}^d$ with Laplace functional given by

$$\mathcal{L}_\lambda^{\mathrm{ST}}(\phi) = \mathrm{Det}(I + \mathcal{K}_\phi^\lambda)^{-1}, \tag{4.4}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a non-negative function with compact support, $\mathcal{K}_\phi^\lambda$ is the linear operator on $L^2(\mathbb{R}^d)$ with kernel

$$\sqrt{1 - e^{-\phi(x)}}\, K_\lambda(x,y) \sqrt{1 - e^{-\phi(y)}}, \qquad K_\lambda(x,y) \text{ as in (2.34)},$$

and Det is the Fredholm determinant. They also show that the n-point correlation functions of $\nu_\lambda^{\mathrm{ST}}$ are given by

$$\varphi_\lambda^{\mathrm{ST}}(x_1, ..., x_n) = \mathrm{perm}\left(K_\lambda(x_i, x_j)\right)_{i,j=1}^n. \tag{4.5}$$

In [8], Eisenbaum proves that the process with law $\nu_\lambda^{\mathrm{ST}}$ is a Cox process.

Our next result shows that in the subcritical (or critical, for $d \geq 3$) phase, the boson point process is distributed as the point marginal of the Gaussian loop soup.

Proposition 4.2. *Let $0 < \lambda < 1$, or $\lambda = 1$ if $d \geq 3$. Then $\nu_\lambda^{\mathrm{ST}}$ is equal to the distribution $\nu_\lambda^{\mathrm{ls}}$ introduced in* (2.32)*.*

Proof. It suffices to show that these process have the same n-point correlations, for any $n \geq 1$. This follows from (4.5) and Proposition 2.7. Alternatively, one can show that the Laplace functional computed in Proposition 2.6 matches the expression in (4.4). □

Supercritical case Bose-Einstein condensation occurs for the free Bose gas when $d \geq 3$ and the density is supercritical, $\rho > \rho_c$. In this case Tamura and Ito describe the boson point process as the convolution of the process ν_1^{ST} defined above with a process that we will denote $\nu_{\rho-\rho_c}^{\mathrm{TI}}$.

The measure $\nu_{\rho-\rho_c}^{\mathrm{TI}}$ is the law of a point process with Laplace functional

$$\mathcal{L}_{\rho-\rho_c}^{\mathrm{TI}}(\phi) = \exp\left(-(\rho - \rho_c)\left\langle \sqrt{1 - e^{-\phi}}, (I + \mathcal{K}_\phi^1)^{-1} \sqrt{1 - e^{-\phi}} \right\rangle\right), \tag{4.6}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a non-negative function with compact support, $\mathcal{K}_\phi^1$ is the operator defined in (4.4), and $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(\mathbb{R}^d)$.

We show that the point process ν_β^{TI} is in fact the point marginal of the Gaussian random interlacements.

Proposition 4.3. *Let $d \geq 3$ and $\beta > 0$. The point processes ν_β^{TI} and ν_β^{ri}, introduced in* (3.15)*, are equal.*

Proof. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function with compact support. Expand $(I + \mathcal{K}_\phi^1)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\mathcal{K}_\phi^1)^n$ and replace in (4.6) to obtain the Laplace functional $\mathcal{L}_\beta^{\mathrm{ri}}(\phi)$ computed in (3.22). □

Corollary 4.4. *Let $d \geq 3$ and $\rho > \rho_c$. The boson point process $\nu_1^{\mathrm{ST}} * \nu_{\rho-\rho_c}^{\mathrm{TI}}$ is distributed as the point marginal of the Gaussian random permutation at density ρ.*

Proof. Straighforward from Propositions 4.2 and 4.3. □

The process $\nu_1^{\mathrm{ST}} * \nu_{\rho-\rho_c}^{\mathrm{TI}}$ is a Cox process, as proved by Eisenbaum in [8].

4.2 The Gaussian random permutation is Markov and Gibbs

We show first that for all $\rho > 0$ the Gaussian spatial permutation μ_ρ is Markov in the sense proposed in the statement of Theorem 4.5 below. We then define the specifications induced by (1.2) and show that μ_ρ is Gibbs with respect to these specifications.

Notation Given a set $A \subset \mathbb{R}^d$ and a spatial permutation $\Gamma = (\zeta, \kappa) \in \mathfrak{X}$, define the sets of points

$I_A\zeta := \zeta \cap A$,	points in A; in red in Fig. 6,
$O_A\zeta := \zeta \cap A^c$,	points in A^c; painted purple and yellow in Fig. 6,
$U_A\zeta := \{u \in \zeta \cap A^c : \kappa(u) \in A\}$,	points in A^c that are mapped to points in A, painted yellow and labelled u in Fig. 6,
$V_A\zeta := \{v \in \zeta \cap A^c : \kappa^{-1}(v) \in A\}$	points in A^c with pre-images in A, painted yellow and labelled v in Fig. 6,

and the maps

$$\begin{array}{llll} I_A\kappa : I_A\zeta \cup U_A\zeta \to I_A\zeta \cup V_A\zeta, & I_A\kappa(x) = \kappa(x) & \text{red arrows in Fig. 6,} \\ O_A\kappa : O_A\zeta \setminus U_A\zeta \to O_A\zeta \setminus V_A\zeta, & O_A\kappa(x) = \kappa(x) & \text{purple arrows in Fig. 6.} \end{array} \tag{4.7}$$

Note that $O_A\kappa$ and $I_A\kappa$ might not be bijective.

Define the inside and outside projections (with respect to A) by

$$I_A(\zeta, \kappa) := (I_A\zeta, I_A\kappa), \qquad O_A(\zeta, \kappa) := (O_A\zeta, O_A\kappa). \tag{4.8}$$

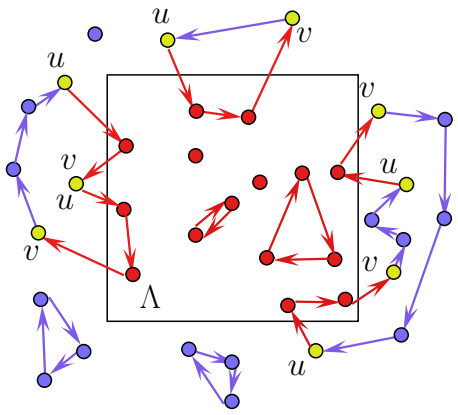

Figure 6: Decomposition of a loop soup spatial permutation $\Gamma = (\zeta, \kappa)$ intersecting a set A.

Markov property We now show that the law of the restriction of the spatial random permutation to points contained in a bounded set $A \subset \mathbb{R}^d$, conditioned on the configuration outside the set, is determined by the external boundary points U_A, V_A.

Theorem 4.5 (Markov property)**.** *The Gaussian random permutation μ_ρ is Markov in the following sense,*

$$\mu_\rho\big(dI_A(\Gamma)\big|\, O_A(\Gamma) \text{ occurs outside } A\big) = \mu_\rho\big(dI_A(\Gamma)\big|\, (U_A, V_A) \text{ occur outside } A\big). \tag{4.9}$$

Informally, the statement of the theorem is: if we condition on the purple and labelled yellow points and the purple arrows of Fig. 6, the law of the red points and red arrows is determined by the positions and labels of the yellow points.

Proof. We first prove the result for the Gaussian loop soup. Recall the notation $D_A := \{\gamma \in D : \{\gamma\} \subset A\}$, and denote $\partial D_A := \big(D_A \cup D_{A^c}\big)^c$. Note that the restricted Gaussian loop soups

$$\begin{aligned} &\Gamma_\lambda^{\mathrm{ls}} \cap D_A \quad \text{(loops contained in } A\text{)},\\ &\Gamma_\lambda^{\mathrm{ls}} \cap D_{A^c} \quad \text{(loops contained in } A^c\text{)},\\ &\Gamma_\lambda^{\mathrm{ls}} \cap \partial D_A \quad \text{(loops intersecting } A \text{ and } A^c\text{)} \end{aligned} \tag{4.10}$$

are independent Poisson processes with intensity measures $Q_\lambda^{\mathrm{ls}}\mathbf{1}_{D_A}$, $Q_\lambda^{\mathrm{ls}}\mathbf{1}_{D_{A^c}}$, $Q_\lambda^{\mathrm{ls}}\mathbf{1}_{\partial D_A}$, respectively, and form a partition of $\Gamma_\lambda^{\mathrm{ls}}$.

Due to the independence of the partition (4.10), the inside and outside components of $\Gamma_\lambda^{\mathrm{ls}}$ are partitioned into independent pieces as follows,

$$I_A(\Gamma_\lambda^{\mathrm{ls}}) = \big(\Gamma_\lambda^{\mathrm{ls}} \cap D_A\big) \dot\cup\, \partial I_A(\Gamma_\lambda^{\mathrm{ls}}), \tag{4.11}$$

$$O_A(\Gamma_\lambda^{\mathrm{ls}}) = \big(\Gamma_\lambda^{\mathrm{ls}} \cap D_{A^c}\big) \dot\cup\, \partial O_A(\Gamma_\lambda^{\mathrm{ls}}), \tag{4.12}$$

where

$$\partial I_A(\Gamma) := \big\{\eta = (u, x_1, \dots, x_{\ell(\eta)}, v) : u \in U_A(\Gamma),\, x_i = \kappa^i(u) \in A,\, v = \kappa^{\ell(\eta)+1}(u) \in V_A(\Gamma)\big\}, \tag{4.13}$$

$$\partial O_A(\Gamma) := \big\{\eta' = (v, y_1, \dots, y_{\ell(\eta')}, u) : v \in V_A(\Gamma),\, y_i = \kappa^i(v) \in A^c,\, u = \kappa^{\ell(\eta')+1}(v) \in U_A(\Gamma)\big\}, \tag{4.14}$$

where $\ell(\eta) = \min\{\ell \geq 1 : \kappa^{\ell+1}(u) \in V_A(\Gamma)\}$ and $\ell(\eta') = \min\{\ell \geq 0 : \kappa^{\ell+1}(v) \in U_A(\Gamma)\}$. These numbers count the number of points visited by the associated path, excluding the endpoints u and v. In Fig. 6 an element of $\partial I_A(\Gamma)$ is given by a red path linking two yellow points with labels u and v respectively, while an element of $\partial O_A(\Gamma)$ is a purple path that links two yellow points v and u. Each path in the inside boundary $\partial I_A(\Gamma)$ contains at least one point in A, so that $\ell(\eta) \geq 1$, while the outside boundary $\partial O_A(\Gamma)$ might contain a path (v, u) with $v \in V_A$, $u \in U_A$ (see Fig. 6); $\ell(\eta') = 0$ in this case. There is no path when $u = v \in U_A \cap V_A$.

Given $\eta = (u, x_1, \dots, x_\ell, v) \in \partial I_A(\Gamma)$ and $\eta' = (v, y_1, \dots, y_\ell, u) \in \partial O_A(\Gamma)$, consider the weights

$$\omega(\eta) := p(u, x_1)\, p(x_\ell, v) \prod_{i=1}^{\ell-1} p(x_i, x_{i+1}) \qquad \ell \geq 1, \tag{4.15}$$

$$\omega(\eta') := p(v, y_1)\, p(y_\ell, u) \prod_{i=1}^{\ell-1} p(y_i, y_{i+1}) \qquad \ell \geq 1, \tag{4.16}$$

$$\omega(v, u) := p(v, u) \qquad \ell = 0, \tag{4.17}$$

where $p = p_{1/2\alpha}$ the Brownian transition density in (2.1). By (2.21), the weight of a cycle $\gamma = [x_0, \dots, x_{n-1}]$ is given by

$$\omega_\lambda(\gamma) = \lambda^n \prod_{i=0}^{n-1} p(x_i, x_{i+1}), \quad \text{with } x_n = x_0. \tag{4.18}$$

If γ intersects both A and A^c, this weight factorizes as

$$\omega_\lambda(\gamma) = \lambda^n \prod_{\eta \in \partial I_A(\gamma)} \omega(\eta) \prod_{\eta' \in \partial O_A(\gamma)} \omega(\eta'). \tag{4.19}$$

Replacing (4.19) in (2.20) we obtain

$$\begin{aligned} f^{\mathrm{ls}}_\lambda(\Gamma \cap \partial D_A) &= e^{-Q^{\mathrm{ls}}_\lambda(\partial D_A)}\, \lambda^{|U_A(\Gamma) \cup V_A(\Gamma)|} \prod_{\gamma \in \Gamma \cap \partial D_A} \Big[\prod_{\eta \in \partial I_A(\gamma)} \omega(\eta)\, \lambda^{\ell(\eta)} \prod_{\eta' \in \partial O_A(\gamma)} \omega(\eta') \lambda^{\ell(\eta')} \Big] \\ &= e^{-Q^{\mathrm{ls}}_\lambda(\partial D_A)}\, \lambda^{|U_A(\Gamma) \cup V_A(\Gamma)|} \prod_{\eta \in \partial I_A(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta) \prod_{\eta' \in \partial O_A(\Gamma)} \lambda^{\ell(\eta')} \omega(\eta')\,. \end{aligned} \tag{4.20}$$

In view of the partition into independent processes (4.10) and the representation (4.20) above, we conclude that

$$\mu_\rho\big(dI_A(\Gamma) \big|\, O_A(\Gamma)\big) = \frac{1}{Z}\, f^{ls}_\lambda(\Gamma \cap D_A)\, dx_1 \dots dx_{|\zeta \cap A|} \prod_{\eta \in \partial I_A(\Gamma)} \lambda^{\ell(\eta)} \omega(\eta)\, dx^\eta_1 \dots dx^\eta_{\ell(\eta)}, \tag{4.21}$$

where Z is a normalizing constant $Z\big(\alpha, \lambda, A, U_A(\Gamma), V_A(\Gamma)\big)$. Identity (4.21) above implies that the conditioned measure on the left only depends on the sets $U_A(\Gamma)$ and $V_A(\Gamma)$, proving the Markov property (4.9) in the critical and subcritical cases $\rho \leq \rho_c$.

We now turn to the supercritical case. For $d \geq 3$ and $\rho > \rho_c$ the GRP μ_ρ is the superposition of independent realizations of the Gaussian loop soup Γ^{ls}_1 with law μ^{ls}_1 and the Gaussian random interlacements $\Gamma^{\mathrm{ri}}_{\rho-\rho_c}$ with law $\mu^{\mathrm{ri}}_{\rho-\rho_c}$.

An infinite trajectory $w \in \widetilde{W}_A$ enters A for the first time from a point $s_A(w)$ and visits A for the last time before jumping to a point $t_A(w)$, defined by

$$\begin{aligned} &s_A(w) := s \text{ if and only if } s \in U_A(w) \text{ and } w^{-\ell}(s) \in A^c, \text{ for all } \ell \geq 1, \\ &t_A(w) := t \text{ if and only if } t \in V_A(w) \text{ and } w^{\ell}(t) \in A^c, \text{ for all } \ell \geq 1. \end{aligned} \tag{4.22}$$

Given $w \in \widetilde{W}_A$, define w_A as the piece of trajectory between $s = s_A(w)$ and $t = t_A(w)$,

$$w_A = (s, z_1, \dots, z_{m(w)}, t), \text{ with } s = s_A(w), t = t_A(w) \text{ and } z_i = w^i(s), \tag{4.23}$$

where $m(w)$ is the number of points visited by w between the first and the last visits to A. Note that z_1 and $z_{m(w)}$ belong to A but intermediate points might not.

Since $\widetilde{W}$ is the disjoint union of $\widetilde{W}_A$ and $\widetilde{W}_A^c$, the infinite trajectories in the configuration Γ^{ri} that do not intersect A are independent of the configuration inside A. Let us then focus on $w \in \widetilde{W}_A$. In view of the definition of Q^{ri}, the weight $\omega(w_A)$ of the trajectory (4.23) under $\mu^{\mathrm{ri}}_{\rho-\rho_c}$ is

$$\begin{aligned} \omega^{\mathrm{ri}}(w_A) &= (\rho - \rho_c)\, P^s(X_n \notin A, n \leq -1)\, P^t(X_n \notin A, n \geq 1)\, p(s, z_1)\; p(z_1, z_2) \dots p(z_m, t) \\ &= (\rho - \rho_c)\, P^s(X_n \notin A, n \leq -1)\, P^t(X_n \notin A, n \geq 1) \prod_{\eta \in \partial I_A(w_A)} \omega(\eta) \prod_{\eta' \in \partial O_A(w_A)} \omega(\eta'), \end{aligned} \tag{4.24}$$

if we define, see Fig. 7,

$\partial I_A(w_A) := \big\{\eta = (a, x_1, \dots, x_{\ell(\eta)}, b) :\ a \in U_A(w),\ x_i = \kappa^i(a) \in A,\ b = \kappa^{\ell(\eta)+1}(a) \in V_A(w)\big\}$
(red paths joining two yellow points, or a yellow (green) point and a green (resp. yellow) point),

$\partial O_A(w_A) := \big\{\eta' = (v, y_1, \dots, y_{\ell(\eta')}, u) :\ v \in V_A(w_A) \setminus \{t_A(w)\},\ y_i = \kappa^i(v) \in A^c,$
$u = \kappa^{\ell(\eta')+1} \in U_A(w) \setminus \{s_A(w)\}\big\}$
(purple paths between yellow points),

and the weights

$$\omega(\eta) := p(u, x_1)\, p(x_\ell, v) \prod_{i=1}^{\ell-1} p(x_i, x_{i+1}), \quad \eta \in \partial I_A(w_A) \tag{4.25}$$

$$\omega(\eta') := p(v, y_1)\, p(y_\ell, u) \prod_{i=1}^{\ell-1} p(y_i, y_{i+1}), \quad \eta' \in \partial O_A(w_A),\ \ell(\eta') \geq 1, \tag{4.26}$$

$$\omega(v, u) := p(v, u), \quad \ell = 0. \tag{4.27}$$

Note that the weights (4.25), (4.26), (4.27) are equal to those in (4.15), (4.16) and (4.17), respectively, for the subcritical configuration. This remark implies that the distribution of the crossings of A by an infinite trajectory $w \in \widetilde{W}_A$, as well as the distribution of the paths in $O_A(w_A)$ between crossings, given the sets

$$U_A(w) := \big\{u \in \{w\} \cap A^c : \kappa(u) \in A\big\} \quad \text{and} \quad V_A(w) := \big\{v \in \{w\} \cap A^c : \kappa^{-1}(v) \in A\big\}, \tag{4.28}$$

is the same as the distribution of the crossings of A, and the paths between these crossings, for the finite cycles of the loop soup, conditioned on the boundary points. In other words, given the boundary points, the distribution of the crossings of the set A (and the connections between two consecutive crossings) is the same, regardless of whether they are part of finite cycles or infinite paths.

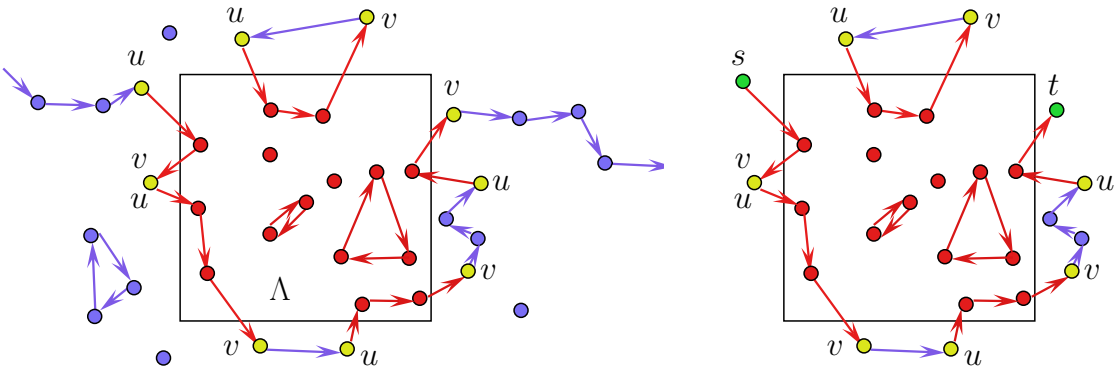

Figure 7: Decomposition of a spatial permutation Γ intersecting A in the supercritical case. Loops are decomposed as in Fig. 6. Only the finite trajectory between points labelled s and t of infinite paths is relevant for the intersection of the configuration with A, as shown in the picture on the right.

Let now $\Gamma = \Gamma^{\mathrm{ls}} \cup \Gamma^{\mathrm{ri}}$ be the full configuration, and split $\Gamma^{\mathrm{ls}} = \Gamma_1^{\mathrm{ls}} \cup \Gamma_2^{\mathrm{ls}} \cup \Gamma_3^{\mathrm{ls}}$, where $\Gamma_1^{\mathrm{ls}} \in D_A$, $\Gamma_2^{\mathrm{ls}} \in \partial D_{A^c}$ and $\Gamma_3^{\mathrm{ls}} \in D_{A^c}$ are independent (see (4.10)); and similarly, $\Gamma^{\mathrm{ri}} = \Gamma_1^{\mathrm{ri}} \cup \Gamma_2^{\mathrm{ri}}$, with independent configurations $\Gamma_1^{\mathrm{ri}} \in \widetilde{W}_A$, $\Gamma_2^{\mathrm{ri}} \in \widetilde{W}_A^c$.

We get

$$\mu_\rho(dI_A(\Gamma)\big|O_A(\Gamma)) = \frac{1}{Z}\, f_1^{\mathrm{ls}}(\Gamma_1^{\mathrm{ls}})\, dx_1 \dots dx_{|\Gamma_1^{\mathrm{ls}} \cap A|} \prod_{\eta \in \partial I_A(\Gamma)} \omega(\eta)\, dx_1^\eta \dots dx_{\ell(\eta)}^\eta, \tag{4.29}$$

where we note that $\partial I_A(\Gamma) = \partial I_A(\Gamma_2^{\mathrm{ls}}) \cup \partial I_A(\Gamma_1^{\mathrm{ri}})$. Here $Z = Z(\alpha, A, U_A(\Gamma), V_A(\Gamma))$ is a normalizing constant. The right hand side above is determined by the sets $U_A(\Gamma)$ and $V_A(\Gamma)$ of the outside configuration $O_A(\Gamma^{\mathrm{ri}})$, hence the Markov property of the supercritical GRP follows. □

Specifications and Gibbs measures We define the specifications related to the density (1.2). Let A be a bounded Borel set and let (χ, σ) and (ζ, κ) be two spatial permutations. We say that (χ, σ) is A-compatible with (ζ, κ), and write $(\chi, \sigma) \sim_A (\zeta, \kappa)$, if they match in A^c, that is, if

$$O_A(\chi, \sigma) = O_A(\zeta, \kappa). \tag{4.30}$$

Given a permutation (ζ, κ), let us define $H_A\big((\chi, \sigma)\big|(\zeta, \kappa)\big)$ on the set $(\chi, \sigma) \sim_A (\zeta, \kappa)$ by

$$H_A\big((\chi, \sigma)\big|(\zeta, \kappa)\big) := \sum_{x \in I_A(\chi)} \|x - \sigma(x)\|^2.$$

Let $\lambda \in (0,1)$, or $\lambda = 1$ if $d \geq 3$. Consider the measure $G_{A,\lambda}\big(\,\cdot\,\big|(\zeta,\kappa)\big)$ such that the integral of a bounded measurable function $g : \mathfrak{X} \to \mathbb{R}$ is given by

$$G_{A,\lambda}\big(\,g\,\big|(\zeta,\kappa)\big) := \sum_{n \geq |V_A(\zeta,\kappa)|} \frac{1}{n!} \int_{A^n} \sum_{\substack{\chi \cap A = \{x_1,\dots,x_n\},\\ (\chi,\sigma)\sim_A(\zeta,\kappa)}} g(\chi,\sigma)\, f_{A,\lambda}\big((\chi,\sigma)\big|(\zeta,\kappa)\big)\, dx_1, \dots dx_n, \tag{4.31}$$

with

$$f_{A,\lambda}\big((\chi,\sigma)\big|(\zeta,\kappa)\big) := \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^{|\chi\cap A|}}{Z_{A,\lambda}(\zeta,\kappa)} \exp\big(-\alpha H_A\big((\chi,\sigma)\big|(\zeta,\kappa)\big)\big), \tag{4.32}$$

and

$$Z_{A,\lambda}(\zeta,\kappa) := \sum_{n \geq |V_A(\zeta,\kappa)|} \frac{\big((\alpha/\pi)^{d/2}\lambda\big)^{n}}{n!} \int_{A^n} \sum_{\substack{\chi \cap A = \{x_1,\dots,x_n\},\\ (\chi,\sigma)\sim_A(\zeta,\kappa)}} \exp\big(-\alpha H_A\big((\chi,\sigma)\big|(\zeta,\kappa)\big)\big)\, dx_1 \dots dx_n.$$

An immediate corollary of Theorem 4.5 is that the Gaussian random permutation μ_ρ is Gibbs.

Theorem 4.6 (Gibbs measures)**.** *For $0 < \lambda < 1$, and $\lambda = 1$ if $d \geq 3$, the measure $\mu_\lambda^{\mathrm{ls}}$ given in Definition 2.3 is Gibbs for the specifications $(G_{A,\lambda} : A$ bounded$)$. That is, $\mu_\lambda^{\mathrm{ls}}$ satisfies the DLR equations*

$$\mu_\lambda^{\mathrm{ls}} g = \int d\mu_\lambda^{\mathrm{ls}}(\zeta,\kappa)\, G_{A,\lambda}\big(\,g\,\big|(\zeta,\kappa)\big). \tag{4.33}$$

Furthermore, if $d \geq 3$ and $\rho > \rho_c$, for all $\rho \geq \rho_c$ the measure $\mu_\rho = \mu_1^{\mathrm{ls}} * \mu_{\rho-\rho_c}^{\mathrm{ri}}$ is Gibbs for the specifications $(G_{A,1} : A$ bounded$)$.

Proof. It suffices to show that the specifications are equal to the conditioned probabilities in (4.9). This follows from identity (4.21) for $0 < \lambda < 1$ and $\lambda = 1$ if $d \geq 3$, and identity (4.29) for $d \geq 3$ and $\rho > \rho_c$. □

We point out the fact that all supercritical measures are Gibbs for the same specifications $G_{A,1}$.

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