Bisimilarity is not Borel

Pedro Sánchez Terraf*

February 15, 2013

Abstract

We prove that the relation of bisimilarity between countable labelled transition systems is $\mathbf{\Sigma}_1^1$-complete (hence not Borel), by reducing the set of non-wellorders over the natural numbers continuously to it.

This has an impact on the theory of probabilistic and nondeterministic processes over uncountable spaces, since the proofs of logical characterizations of bisimilarity based on the unique structure theorem for analytic spaces require a countable logic whose formulas have measurable semantics. Our reduction shows that such a logic does not exist in the case of image-infinite processes.

1 Introduction

Markov decision processes over continuous state spaces are an appropriate framework to study and formalize systems that involve continuously valued variables such as those arising in physics, biology, and economics; and where some of those variables are known only in a probabilistic way.

In this direction, *labelled Markov processes (LMP)* were developed in [6, 7] by Desharnais et alter. A LMP has a labelled set of *actions* that encode interaction with the environment; thus LMP are a reactive model in which there are different transition subprobabilities for each action. In this model, uncertainty is (only) considered to be probabilistic; therefore, LMP can be regarded as generalization of deterministic processes.

Models that include both probabilism and internal nondeterminism arise naturally, e.g., by abstraction of LMP. In the discrete case, *probabilistic automata* are an example. Over uncountable state spaces, the common generalization of LMP and probabilistic automata are *nondeterministic labelled Markov processes (NLMP)* [4, 15]. NLMP allow, for each state and each action, a (possibly infinite) set of probabilistic behaviors. A different approach, using super-additive functions, is proposed in [8].

*The author was partially supported by CONICET, SECYT-UNC and the MEALS Project (`http://meals-project.eu/`). Part of this work was presented at Dagstuhl Seminar 12411 on *Coalgebraic Logics.*

Keywords: measurable labelled transition system, non-deterministic labelled Markov process, modal logic, Borel hierarchy. *MSC 2010:* 03B70; 03E15; 28A05. *ACM class:* F.4.1; F.1.2.

In [4, 5] the problem of defining appropriate notions of bisimulation and finding logical characterizations for NLMP was addressed. It turns out that there are three different notions of bisimilarity: traditional, state-based and event-based. Two of these notions collapse to the state bisimilarity when one drops non-determinism (i.e., for LMP), and the third one is analogous to the concept for LMP bearing the same name. Furthermore, when the state space is analytic, state-based and event-based bisimilarity coincide for LMP [3]. This was proved alternatively, over Polish spaces, by Doberkat [9].

In both LMP and NLMP, event bisimilarity is a relation naturally characterized by a modal logic, in the sense that two states satisfy exactly the same formulas if and only if they are bisimilar eventwise. When event and traditional bisimilarities coincide, we speak of a *logical characterization of bisimilarity*, emphasizing the role of traditional (state) bisimilarity as a the most natural generalization of *probabilistic* bisimilarity by Larsen and Skou [13] to NLMP (LMP).

The *unique structure theorem* provides a very clear way to prove the logical characterization of bisimilarity for LMP with an analytic state space, as in the work of Danos et al. [3]. The argument there can be generalized to encompass image-finite NLMP (i.e., having a finite number of probabilistic behaviors for each pair $\langle$state,action$\rangle$) over analytic state spaces. Actually, a proof strategy can be found in [4, 2]: every countable 'measurable' logic $\mathscr{L}$ satisfying certain local restrictions must characterize bisimilarity. (Here we call a logic *measurable* if the extension of each formula is a measurable subset of the state space.) Both countability and measurability requirements are necessary for the proof to work.

In this paper we show that the relation of bisimilarity is not Borel in an appropriate process having a Polish state space, and therefore we prove that there is no countable measurable logic that characterizes it. In the next section we review some of the known results on NLMP, describing the available notions of bisimulation. Most calculations in the paper will be carried on a simpler model also described in Section 2, which is essentially a labelled transition system (LTS) over a measurable space. In Section 3 we use some machinery of sequence spaces and unwinding of labelled transition systems to assess the complexity of the relation of bisimilarity. The final section contains some concluding remarks.

2 Review of NLMP

2.1 Basic definitions

All of the material of this section appear in [5]. Let (S, Σ) be a measurable space. The set $\Delta(S)$ of probability measures on (S, Σ) has a natural σ-algebra $\Delta(\Sigma) = \sigma(\{\Delta^{\geq q}(Q) : q \in \mathbb{Q}, Q \in \Sigma\})$, where $\Delta^{\geq q}(Q) = \{\mu \in \Delta(S) : \mu(Q) \geq q\}$. This is the least σ-algebra making evaluation $\mu \mapsto \mu(Q)$ measurable.

Recall that a Markov kernel on (S, Σ) is a measurable map $T : (S, \Sigma) \to (\Delta(S), \Delta(\Sigma))$. The following definitions generalize this concept by enlarging the codomain to the family of all measurable sets of probability measures $(\Delta(\Sigma))$, and constructing a σ-algebra for this family in order to be able to say that T is measurable.

Definition 1. $H(\Delta(\Sigma))$ is the least σ-algebra containing all sets $H_\xi \doteq \{\zeta \in \Delta(\Sigma) : \zeta \cap \xi \neq \varnothing\}$ with $\xi \in \Delta(\Sigma)$.

Definition 2. A *nondeterministic labelled Markov process* (NLMP) is a tuple $(S, \Sigma, \{T_a : a \in L\})$ where Σ is a σ-algebra on the set of states S, and for each label $a \in L$, $T_a : (S, \Sigma) \to (\Delta(\Sigma), H(\Delta(\Sigma)))$ is measurable.

The motivation for the previous definitions is that we want the event "there exists a probabilistic behavior from s such that …" to be measurable; to be able to calculate the probability of such event (cf. the semantics of the logic below).

Some notation concerning binary relations will be needed to define bisimulations. Let R a binary relation over S. A set Q is *R-closed* if $Q \ni x \; R \; y$ implies $y \in Q$. $\Sigma(R)$ is the σ-algebra of R-closed sets in Σ. If μ, μ' are measures defined on Σ, we write $\mu \; R \; \mu'$ if they coincide in $\Sigma(R)$. Lastly, let Ξ be a subset of $\mathsf{Pow}(S)$, the powerset of S. The relation $\mathcal{R}(\Xi)$ is given by:

$$(s, t) \in \mathcal{R}(\Xi) \quad \iff \quad \forall Q \in \Xi : s \in Q \Leftrightarrow t \in Q.$$

Definition 3.

1. An *event bisimulation* on a NLMP $(S, \Sigma, \{T_a : a \in L\})$ is a sub-σ-algebra Λ of Σ such that $T_a : (S, \Lambda) \to (\Delta(\Sigma), H(\Delta(\Lambda)))$ is measurable for each $a \in L$. We also say that a relation R is an event bisimulation if there is an event bisimulation Ξ such that $R = \mathcal{R}(\Xi)$.

2. A relation $R \subseteq S \times S$ is a *state bisimulation* if it is symmetric and for all $a \in L$, $s \; R \; t$ implies $\forall \xi \in \Delta(\Sigma(R)) : T_a(s) \cap \xi \neq \varnothing \iff T_a(t) \cap \xi \neq \varnothing$.

3. A relation R is a *traditional bisimulation* if it is symmetric and for all $a \in L$, $s \; R \; t$ implies that for all $\mu \in T_a(s)$ there exists $\mu' \in T_a(t)$ such that $\mu \; R \; \mu'$.

We say that $s, t \in S$ are traditionally (resp. state-, event-) *bisimilar*, denoted by $s \sim_{\mathrm{t}} t$ ($s \sim_{\mathrm{s}} t$, $s \sim_{\mathrm{e}} t$), if there is a traditional (state, event) bisimulation R such that $s \; R \; t$.

We want to stress the fact that each notion of bisimulation/bisimilarity is defined relative to a particular NLMP. Event bisimulation is a straightforward generalization of the same concept for LMP, and it is the one that is more "compatible" with the measurable structure of the base space. For LMP, it is equivalent to the existence of a cospan of morphisms; see [3]. Traditional bisimulation is in a sense the most faithful generalization of both probabilistic bisimulation by Larsen and Skou and the standard notion of bisimulation for non deterministic processes, e.g., LTS. Finally, state bisimilarity is a good trade-off between the other two, since it is generally finer than event bisimilarity but it is a little more respectful to the measurable structure than the traditional version [5].

To close this section, we introduce the logic $\mathscr{L}$. There are two kinds of formulas: one that is interpreted on states, and another that is interpreted on measures.

$$\begin{array}{rcl} \varphi & \equiv & \top \;\mid\; \varphi_1 \wedge \varphi_2 \;\mid\; \langle a \rangle \psi \\ \psi & \equiv & \bigvee_{i \in I} \psi_i \;\mid\; \neg \psi \;\mid\; [\varphi]_{\geq q} \end{array}$$

where $a \in L$, I is a countable index set, and $q \in \mathbb{Q} \cap [0, 1]$. We denote by $\mathscr{L}$ the set of all formulas generated by the first production. Given a NLMP (S, Σ, T), the semantics is the following:

$$
\begin{aligned}
&[\![\top]\!] = S && [\![\textstyle\bigvee_{i \in I} \psi_i]\!] = \textstyle\bigcup_i [\![\psi_i]\!] \\
&[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \cap [\![\varphi_2]\!] && [\![\neg\psi]\!] = \Delta(S) \setminus [\![\psi]\!] \\
&[\![\langle a \rangle \psi]\!] = T_a^{-1}(H_{[\![\psi]\!]}) && [\![[\varphi]_{\geq q}]\!] = \Delta^{\geq q}([\![\varphi]\!])
\end{aligned}
$$

where $[\![\chi]\!]$ denotes the extension of χ. It can be proved by induction that all the sets $[\![\chi]\!]$ are measurable in the respective spaces. Let $[\![\mathscr{L}]\!] \doteq \{[\![\varphi]\!] : \varphi \in \mathscr{L}\}$.

Theorem 4. *The logic $\mathscr{L}$ completely characterizes event bisimulation. In other words, $\mathcal{R}([\![\mathscr{L}]\!]) = \sim_{\mathrm{e}}$.*

Theorem 5. $\sim_{\mathrm{t}} \subseteq \sim_{\mathrm{s}} \subseteq \sim_{\mathrm{e}} = \mathcal{R}([\![\mathscr{L}]\!])$.

Recall that a topological space is *Polish* if it is separable and completely metrizable and an *analytic* space is the image of a continuous map between Polish spaces. The following Lemma summarizes one strategy to prove completeness of a logic for traditional bisimilarity.

Lemma 6. *Let (S, Σ, T) be a NLMP with (S, Σ) being an analytic Borel space. Let $\mathfrak{L}$ be a logic such that (i) $\mathfrak{L}$ contains operators $\top$ and $\wedge$ with the usual semantics; (ii) for every formula $\varphi \in \mathfrak{L}$, $[\![\varphi]\!] \in \Sigma$; (iii) the set of all formulas in $\mathfrak{L}$ is countable; and (iv) for every $s \mathrel{\mathcal{R}(\mathfrak{L})} t$ and every $\mu \in T_a(s)$ there exists $\mu' \in T_a(t)$ such that $\forall\varphi \in \mathfrak{L}, \mu([\![\varphi]\!]) = \mu'([\![\varphi]\!])$. Then, two logically equivalent states s, t are traditionally bisimilar.*

The proof of this lemma is based on the *unique structure theorem* for analytic Borel spaces (S, Σ): every countably generated sub-σ-algebra of Σ that separates points must already be Σ.

By using this Lemma we were able to prove that a countable fragment of $\mathscr{L}$ was complete for traditional bisimilarity over image-finite NLMP. The next step would be to prove a similar result for image-countable processes, and the safest way to test this is in a more "discrete" setting. In the next section we consider a restricted class of processes.

2.2 Measurable LTS

Many interesting (counter)examples can be constructed by considering non-probabilistic NLMP, i.e., one $\mathbf{S} = (S, \Sigma, \{T_a : a \in L\})$ such that for all $a \in L$ and $s \in S$, $T_a(s)$ consists entirely of point-masses (i.e., Dirac measures). In this section, we will give a slightly simpler presentation of this kind of processes, that appears in Wolovick [15]. Actually, these processes are essentially a labelled transition system over a measurable space.

Definition 7. A *measurable labelled transition system* (MLTS) is a tuple $\mathbf{S} = (S, \Sigma, \{T_a : a \in L\})$ such that (S, Σ) is a measurable space and for each label $a \in L$, $T_a : (S, \Sigma) \to (\Sigma, H(\Sigma))$ is a measurable map.

If we write $\langle a \rangle Q$ ($\Diamond Q$ in case L is a singleton) for $\{s : T_a(s) \cap Q \neq \varnothing\}$, then the previous requirement on T_a amounts asking Σ to be *stable* under the map $\langle a \rangle$: for all $Q \in \Sigma$, $\langle a \rangle Q \in \Sigma$. This observation can be generalized to event bisimulations. The following lemma appears in [5].

Lemma 8. *1. A σ-algebra $\Lambda \subseteq \Sigma$ is an event bisimulation on $\mathbf{S}$ if and only if it is stable under the mapping $\langle a \rangle$.*

2. A symmetric relation R is a state bisimulation on $\mathbf{S}$ if and only if for all $s, t \in S$ such that $s\, R\, t$, it holds that for all $Q \in \Sigma(R)$, $s \in \langle a \rangle Q \Leftrightarrow t \in \langle a \rangle Q$.

3. A symmetric relation R is a traditional bisimulation on $\mathbf{S}$ if and only if for all $s, t \in S$ and $u \in T_a(s)$, if $s\, R\, t$ then there exists $v \in T_a(t)$ such that $u\, \mathcal{R}\,(\Sigma(R))\, v$.

Since for every relation one has $R \subseteq \mathcal{R}\,(\Sigma(R))$, standard bisimilarity for LTS is also a traditional bisimulation of MLTS. Observe also that the logic $\mathscr{L}$ is the same as Hennessy-Milner logic with countable conjunctions (and disjunctions) on this family of processes, and hence it characterizes standard bisimulation for image-countable MLTS. By appealing to Theorem 5 we can state

Proposition 9. *For image-countable MLTS, all kinds of bisimilarities (traditional, state, event and standard) coincide.*

We shall henceforth drop the subindexes and use simply $\sim$.

Let's return to the problem left open at the end of the previous section. To apply Lemma 6, the candidate logic should satisfy several requirements. So the first question is if there actually exists any countable logic that characterizes bisimilarity for countable LTS. The answer is given by the following

Example 10 (X. Caicedo)**.** Fix a countable set of labels L. There are at most $2^{\aleph_0}$ (bisimilarity classes of) countable LTS over $\mathbb{N}$. Hence there is an injective function f from bisimilarity classes to $\mathsf{Pow}(\mathbb{N})$. Our 'logic' will consist of countably many atomic formulas P_n ($n \in \mathbb{N}$) with the following semantics:

$$\mathbf{S}, s \models P_n \iff n \in f([(\mathbf{S}, s)]_\sim),$$

where $[\cdot]_\sim$ denotes $\sim$-classes of equivalence. The logic $\mathscr{L}_X := \{P_n : n \in \mathbb{N}\}$ is sound and complete for bisimilarity.

The logic $\mathscr{L}_X$ is devised in a non-constructive manner, and the main result in this work is to show that actually the extensions of formulas of such a countable logic cannot be Borel sets, confirming the intuition that formulas in $\mathscr{L}_X$ cannot be conceived as any reasonable kind of "test" on a process.

3 The main result

We will use some concepts from sequence (zero-dimensional) spaces. Let E be a set. The set of all finite sequences of elements of E will be denoted by E^*. The empty sequence will be denoted by ϵ. The ith element of a sequence $s \in E^*$ will be denoted by s^i; hence $s = (s^0, \dots, s^{|s|-1})$, where $|s|$ is the length of s. The concatenation of two sequences $s, t \in E^*$ will be denoted by $s^\frown t$; in case $t = (e)$ with $e \in E$, we will write $s^\frown e$ instead of $s^\frown(e)$. A *tree* on E is a subset of E^* closed by taking prefixes. We will be interested in the case where E is countable, and specially $E = \mathbb{N}$.

Let A be countable, and consider the discrete topology on it. The product space $A^{\mathbb{N}}$ of all infinite sequences of elements of A is Polish and has a (clopen) basis given by the sets $C_f = \{x \in A^{\mathbb{N}} : f \subset x\}$, where f a finite function. When $A = 2$, we obtain the Cantor space $\mathsf{Pow}(\mathbb{N})$. In general, for every countable set B, we regard $\mathsf{Pow}(B)$ as a compact Hausdorff space with basic open sets $\{X \subseteq B : P \subseteq X \;\&\; N \cap X = \varnothing\}$, where $P, N \subseteq B$ are finite. In case $B = \mathbb{N}$ we obtain exactly the basis given by $\{C_f\}$.

A binary structure $(\mathbb{N}, R)$ can be represented by a point in $\mathrm{Rel} \doteq 2^{\mathbb{N}\times\mathbb{N}}$. The set LO of strict linear order relations is a closed subset of Rel and hence a Polish space. The set $\mathrm{Tr}_{\mathbb{N}}$ of all trees on $\mathbb{N}$ is a closed subset of $2^{\mathbb{N}^*}$.

Let X be a Polish space. The family of Borel subsets of X is least σ-algebra $\mathcal{B}(X)$ containg the open sets of X. $\mathbf{\Sigma}^1_1(X)$ denotes the family

$$\{A \subseteq X : \exists Y \text{ Polish and } f : Y \to X \text{continuous with } f[Y] = A\}$$

of analytic subsets of X (see [11]). Both Borel and analytic sets are preserved by taking continuous preimages and countable unions and intersections. We will usually omit the reference to the space X.

3.1 Bisimilarity on denumerable trees

We first recall a standard construction of trees from linear orders. Given a strict linear order $\mathbf{E} = (E, R)$ over a countable set E we may define a new countable structure $(T_{\mathbf{E}}, \prec)$ as follows:

- $T_{\mathbf{E}} \doteq \{s \in E^* : s^{|s|-1} \; R \; s^{|s|-2} \; R \dots R \; s^0\} \cup \{\epsilon\}$.
- $s \prec s' \iff \exists e \in E : s^\frown e = s'$.

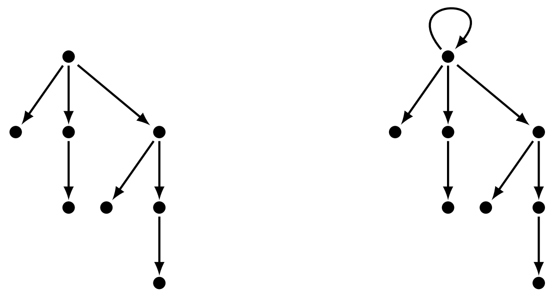
The tree $T_{\mathbf{E}}$ consists of all finite decreasing sequences in (E, R). We also use T_R to denote this tree, whenever E is clear from the context.

We'll now sketch the argument of the proof. Let WO (NWO) be the set of (non) wellorder relations on $\mathbb{N}$, regarded as subsets of LO. It is well known that the set NWO is an $\mathbf{\Sigma}^1_1$-*complete* set [11, Sect. 27.C], in the sense that it is as 'complicated' as any analytic subset of a (zero dimensional) Polish space; in particular, it is not Borel. We will be able to distinguish elements $R \in NWO$ among linear orderings essentially just by looking at the bisimilarity type of the tree T_R over $\mathbb{N}$, regarded as a processes with initial state ϵ. Thus we'll have succeeded *reducing* NWO to the relation of bisimilarity, thus showing that the latter is not Borel.

Definition 11. Let X, Y be Polish spaces and $A \subseteq X$ and $B \subseteq Y$. A *continuous reduction* of A to B is a continuous map $f : X \to Y$ such that $f^{-1}[B] = A$; in this case we say that A is *Wadge reducible* to B. B is $\mathbf{\Sigma}^1_1$-*hard* if for every zero dimensional Polish space X and every $A \in \mathbf{\Sigma}^1_1(X)$, A is Wadge reducible to B, and B is $\mathbf{\Sigma}^1_1$-*complete* if moreover $B \in \mathbf{\Sigma}^1_1(X)$.

Theorem 12. *NWO is a $\mathbf{\Sigma}^1_1$-complete and, in particular, non Borel subset of LO.*

A bit more precisely, the trees T_R corresponding to $R \in WO$ are well founded: regarded as processes, they're terminating. The depth of such a tree equals the length of the wellorder R and this can be 'measured' by using modal formulas. In the case of a non-wellorder relation S on $\mathbb{N}$, there must be an infinite branch, but we can say more: the tree T_S is bisimilar to the process that results after attaching a loop to the initial state of the tree T' associated to the maximal well-ordered initial segment of $(\mathbb{N}, S)$. For the purpose of having a manageable example, consider the linear orders $\mathbf{L} \doteq (\{0,1,2\}, <_3)$ and $\mathbf{L}' \doteq (\{0,1,2\} \cup \{\dots,-3,-2,-1\}, <')$ where $a <' b$ if b is negative and a is not, and otherwise $<'$ behaves as the ordering of the integers. Then the trees corresponding to each of them are bisimilar to the processes in the following picture:



$\sim\ T_{<_3}$ $\qquad$ $\sim\ T_{<'}$

Observe that the poset $\mathbf{L}'$ is the ordered sum of the poset $\mathbf{L}$ and that of the negative integers; it can be seen that the process $T_{\mathbf{A}+\mathbf{B}}$ corresponding to the ordered sum of posets $\mathbf{A}$ and $\mathbf{B}$ starts behaving like $T_{\mathbf{B}}$ and may be *disrupted* in any moment and proceed aftewards as $T_{\mathbf{A}}$. We call this process $T_{\mathbf{B}} \triangleright T_{\mathbf{A}}$; the operator $\triangleright$ is very much like LOTOS disabling operator [1].

Now we may apply the following argument by Dougherty (cited in [12]). If $\mathbf{L}$ is a wellorder, then $\mathbf{L}+\mathbf{L}$ is itself a wellorder and $T_{\mathbf{L}} \nsim T_{\mathbf{L}+\mathbf{L}}$ since these trees have different depth. But in case $\mathbf{L}$ is not, the trees $T_{\mathbf{L}}$ and $T_{\mathbf{L}+\mathbf{L}}$ are indeed bisimilar since $\mathbf{L}$ and $\mathbf{L}+\mathbf{L}$ have the same maximal well-ordered initial segment.

Then the reduction we are looking for is given by the map $R \mapsto (T_R, T_{R+R})$, where $R \in LO$.

We'll begin by proving that this map is indeed continuous.

Lemma 13. *The map $R \mapsto T_R$ from LO to $2^{\mathbb{N}^*}$ is continuous.*

Proof. We will show that the function from $2^{\mathbb{N}\times\mathbb{N}}$ to $2^{\mathbb{N}^*}$ with the same definition is continuous. Assume $s_1, \dots, s_n, t_1, \dots, t_m \in \mathbb{N}^*$. Then

$$\left.\begin{array}{r} s_1, \dots s_n \in T_R \\ t_1, \dots t_m \notin T_R, \end{array}\right] \iff \left[\begin{array}{l} \forall i < n, \forall j : 0 < j < |s_i| \Rightarrow s_i^j \mathrel{R} s_i^{j-1} \\ \forall i < m, \exists j : 0 < j < |t_i| \;\&\; t_i^j \not\mathrel{R} t_i^{j-1}. \end{array}\right.$$

The last condition involve finite intersections and unions of clopen sets, and hence it is open. For a little more detail, let $P = \{s_1, \dots, s_n\}$, $N = \{t_1, \dots, t_m\}$. Hence

$$(T_\bullet)^{-1}(\{T \subseteq \mathbb{N}^* : P \subseteq T \;\&\; N \cap T = \varnothing\}) =$$
$$= \{R \subseteq \mathbb{N}\times\mathbb{N} : U \subseteq R\} \cap \bigcap_{i<m} \bigcup_{0<j<|t_i|} \{R \subseteq \mathbb{N}\times\mathbb{N} : V_{i,j} \cap R = \varnothing\},$$

where $U = \{(s_i^j, s_i^{j-1}) : i < n,\ 0 < j < |s_i|\}$ and $V_{i,j} = \{(t_i^j, t_i^{j-1})\}$. $\square$

We'll need a definition of sum of linear orders such that LO is closed under this operation. Given $R, R' \in LO$, let

$$(R + R')(n,m) \doteq \begin{cases} 1 & 2 \mid n \ \&\ 2 \nmid m \\ R(\frac{n}{2}, \frac{m}{2}) & 2 \mid n \ \&\ 2 \mid m \\ R'(\frac{n-1}{2}, \frac{m-1}{2}) & 2 \nmid n \ \&\ 2 \nmid m \\ 0 & 2 \nmid n \ \&\ 2 \mid m. \end{cases}$$

Hence we have the following straightforward lemma.

Lemma 14. *$(\mathbb{N}, R + R')$ is isomorphic to the ordered sum of $(\mathbb{N}, R)$ and $(\mathbb{N}, R')$, and $(R, R') \mapsto R + R'$ is continuous from $LO \times LO$ to LO.*

Corollary 15. *The map $R \mapsto (T_R, T_{R+R})$ is continuous from LO to $\mathrm{Tr}_{\mathbb{N}} \times \mathrm{Tr}_{\mathbb{N}}$.*

Define by recursion on $\alpha < \omega_1$ the following modal formulas:

- $\varphi_0 \doteq \top$.
- $\varphi_{\alpha+1} \doteq \Diamond \varphi_\alpha$.
- $\varphi_\lambda \doteq \bigwedge_{\beta<\lambda} \varphi_\beta$, for limit λ.

Proposition 16. *For a wellorder $(\mathbb{N}, R)$ of type α, $T_R, \epsilon \models \varphi_\beta$ if and only if $\beta \leq \alpha$. If $(\mathbb{N}, R)$ is not well founded, $T_R, \epsilon \models \varphi_\beta$ for all $\beta < \omega_1$.*

Corollary 17. *If $(\mathbb{N}, R)$ is a wellorder and $R' \in LO$, $T_R, \epsilon \sim T_{R'}, \epsilon \iff (\mathbb{N}, R) \cong (\mathbb{N}, R')$. In particular, $T_R, \epsilon \nsim T_{R+R}, \epsilon$.*

Given two binary structures $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, T)$ where A and B disjoint, define $\mathbf{A} \triangleright \mathbf{B}$ to be structure with universe $(A \times B) \cup B$ and the least binary relation $\to$ such that:

$$\begin{aligned} (a,b) \to (a',b') &\iff a\ R\ a' \\ (a,b) \to b' &\iff b\ T\ b' \\ b \to b' &\iff b\ T\ b', \end{aligned}$$

for all $a, a' \in A$ and $b, b' \in B$.

Lemma 18. *Let $\mathbf{A} = (A, R)$ and $\mathbf{B} = (B, T)$ be strict linear orders with A and B disjoint. Then $T_{\mathbf{A}+\mathbf{B}}, \epsilon \ \sim\ T_{\mathbf{B}} \triangleright T_{\mathbf{A}}, \epsilon \triangleright \epsilon$.*

Proof. Observe that any decreasing sequence in $\mathbf{A} + \mathbf{B}$ decomposes uniquely as the concatenation of a sequence in $\mathbf{A}$ after a sequence in $\mathbf{B}$. Then it is easy to check that the relation

$$\theta \doteq \{(t, t \triangleright \epsilon) : t \in T_{\mathbf{B}}\} \cup \{(t^\frown s, s) : t \in T_{\mathbf{B}}, s \in T_{\mathbf{A}} \setminus \{\epsilon\}\}$$

is a bisimulation. $\square$

Lemma 19. *If $\mathbf{A} = (A, R)$ is a countable strict linear order without first element, then $(T_{\mathbf{A}}, \prec, \epsilon) \cong (\mathbb{N}^*, \prec, \epsilon)$. In particular, $T_{\mathbf{A}}, \epsilon \sim \mathbb{N}^*, \epsilon$.*

Proof. It is enough to note that under the hypothesis, $T_{\mathbf{A}}$ is a (countable) tree in which every node has infinitely many succesors. $\square$

Corollary 20. *If $R \in NWO$, $T_R, \epsilon \sim T_{R+R}, \epsilon$.*

Proof. Decompose $\mathbf{N} \doteq (\mathbb{N}, R)$ as $\mathbf{A} + \mathbf{B}$, where $\mathbf{A}$ is the maximal well-ordered initial segment of $\mathbf{N}$ and hence $\mathbf{B}$ has no first element. Then $\mathbf{N}+\mathbf{N} \cong \mathbf{A}+\mathbf{B}+\mathbf{A}+\mathbf{B} \cong \mathbf{A}+\mathbf{C}$, where $\mathbf{C}$ has no first element. Hence

$$
\begin{aligned}
T_{R+R}, \epsilon &\sim T_{\mathbf{A}+\mathbf{C}}, \epsilon && \text{since } T_{R+R} = T_{\mathbf{N}+\mathbf{N}} \cong T_{\mathbf{A}+\mathbf{C}}, \\
&\sim T_{\mathbf{C}} \triangleright T_{\mathbf{A}}, \epsilon \triangleright \epsilon && \text{by Lemma 18,} \\
&\sim T_{\mathbf{B}} \triangleright T_{\mathbf{A}}, \epsilon \triangleright \epsilon && \text{by Lemma 19, } T_{\mathbf{C}} \cong T_{\mathbf{B}} \\
&\sim T_{\mathbf{A}+\mathbf{B}}, \epsilon && \text{by Lemma 18,} \\
&\sim T_R, \epsilon.
\end{aligned}
$$

$\square$

Theorem 21. *Bisimilarity on $\mathrm{Tr}_{\mathbb{N}}$ is $\mathbf{\Sigma}^1_1$-hard, and hence not Borel.*

Proof. We have seen that $R \mapsto f(R) \doteq (T_R, T_{R+R})$ is continuous (by Lemmas 13 and 14). By Corollary 17 and Corollary 20, $f^{-1}[\sim] = NWO$ and hence NWO is Wadge reducible to $\sim$. This makes $\sim$ $\mathbf{\Sigma}^1_1$-hard, and not Borel (since Borel sets are preserved by continuous preimages). $\square$

3.2 Bisimilarity on MLTS is not Borel

Now we will merge all the tree processes T_R into a single MLTS. Let $\mathbf{F} = (F, \mathcal{B}(F), \overline{\prec})$, where:

- $(F, \mathcal{B}(F)) \doteq (\mathrm{Tr}_{\mathbb{N}} \times \mathbb{N}^*, \mathcal{B}(\mathrm{Tr}_{\mathbb{N}} \times \mathbb{N}^*))$ (where $\mathbb{N}^*$ is considered discrete). Note that this is a Polish space.
- $\overline{\prec}(T, s) = \{(T, s') : s, s' \in T \;\&\; s \prec s'\}$.

We prove that $\mathbf{F}$ is a MLTS. First note that sets $\overline{\prec}((T, s))$, being countable, are Borel. We will also use the symbol $\overline{\prec}$ also as a binary relation, defined in the obvious way: $(T, s) \mathrel{\overline{\prec}} (T', s')$ iff $T = T'$, $s, s' \in T$ and $s \prec s'$.

For a subset $A \subseteq X \times Y$ of a product and $c \in Y$ the *section* $A|_c$ is the set $\{x \in X : (x, c) \in A\}$, the preimage of the injection $x \mapsto (x, c)$.

Lemma 22. *$\Diamond Q$ is Borel for each $Q \in \mathcal{B}(\mathrm{Tr}_{\mathbb{N}} \times \mathbb{N}^*)$.*

Proof.

$$
\begin{aligned}
\Diamond Q &= \{(T, s) \in F : \exists (T', s') \in Q \; ((T, s) \mathrel{\overline{\prec}} (T', s'))\} \\
&= \{(T, s) \in F : \exists s' \; ((T, s') \in Q \;\&\; (T, s) \mathrel{\overline{\prec}} (T, s'))\} \\
&= \{(T, s) \in F : \exists n \in \mathbb{N} (s^\frown n \in T \;\&\; (T, s^\frown n) \in Q)\} \\
&= \bigcup_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^*} \{(T, s) \in F : s^\frown n \in T \;\&\; (T, s^\frown n) \in Q\}
\end{aligned}
$$

Now we may write the set inside the unions (now for fixed s, n) as

$$\begin{aligned}\{(T,s) \in F : s^\frown n \in T \;\&\; (T, s^\frown n) \in Q\} &= \Big(Q \cap \{(T, s^\frown n) : s^\frown n \in T\}\Big)|_{s^\frown n} \times \{s\} \\ &= \Big(Q \cap \big(\{T : s^\frown n \in T\} \times \{s^\frown n\}\big)\Big)|_{s^\frown n} \times \{s\}\end{aligned}$$

The inner rectangle is clopen, and since Q is Borel, the set between the big parentheses is Borel. The whole set is easily Borel, too. □

Theorem 23. *$\sim$ is a $\mathbf{\Sigma}^1_1$-hard subset of $F \times F$, and hence not Borel.*

Proof. It is clear that the injection $T \mapsto (T, \epsilon)$ is continuous from $\mathrm{Tr}_{\mathbb{N}}$ to $\mathrm{Tr}_{\mathbb{N}} \times \mathbb{N}^*$; hence the composition $R \mapsto f(R) \doteq ((T_R, \epsilon), (T_{R+R}, \epsilon))$ also is (by Lemmas 13 and 14). This f is a suitable reduction, since $f^{-1}[\sim] = NWO$. □

We arrive at the main result of this work.

Theorem 24. *There is no countable logic $\mathscr{L}$ that characterizes bisimulation on $\mathbf{F}$ such that $[\![\mathscr{L}]\!] \subseteq \mathcal{B}(F)$.*

Proof. Since $\mathcal{R}([\![\mathscr{L}]\!]) = \sim$,

$$s \sim t \iff (s,t) \in \bigcap \Big\{([\![\varphi]\!] \times [\![\varphi]\!]) \cup \big((F \setminus [\![\varphi]\!]) \times (F \setminus [\![\varphi]\!])\big) : \varphi \in \mathscr{L}\Big\}$$

This contradicts Theorem 23, since the right-hand side is a Borel definition of $\sim$. □

3.3 Bisimilarity is $\mathbf{\Sigma}^1_1$-complete

We finally show in this section that bisimilarity on $\mathbf{F}$ behaves similarly to the isomorphism relation: it is an analytic equivalence relation with Borel classes. Since we already proved it to be $\mathbf{\Sigma}^1_1$-hard, we would have seen it is a complete analytic set.

We will need a technical tool from Janin and Walukiewicz [10], adapted to our 'mono-modal' case. An *ω-indexed path from $s \in S$* on a LTS $\mathbf{S} = (S, R)$ is a sequence u of the form

$$u = s_0(s_1, a_1)(s_2, a_2)\ldots(s_n, a_n)$$

such that $s_0 = s$, $a_i \in \mathbb{N}$ for all i, and $(s_{i-1}, s_i) \in R$ for $i = 0, \ldots, n$. The *ω-expansion at s* of a LTS $\mathbf{S}$ is the LTS $\bar{\mathbf{\Omega}}_{\mathbf{S}}(s) = (\bar{\Omega}, \bar{R})$ such that $\bar{\Omega}$ is the set of all ω-indexed paths from s of $\mathbf{S}$ and the relation $\bar{R}$ is defined by $(u, v) \in \bar{R}$ iff v has the form $u(s, a)$ for some a and s.

Since we are dealing with trees on $\mathbb{N}$, the latter construction provides us with another tree that it is easily seen to be isomorphic to the one given by the following alternative description.

Definition 25. The *ω-expansion of* $\mathbf{T} = (T, \prec)$ *at* s is the LTS $\mathbf{\Omega}_{\mathbf{T}}(s) = (\Omega_{\mathbf{T}}(s), R_{\mathbf{T}}(s))$ such that $\Omega_{\mathbf{T}}(s) = \{(t, n) : s \subseteq t \in T \;\&\; n \in \omega\}$ and the relation $R_{\mathbf{T}}(s) \subseteq (T \times \mathbb{N})^2$ is given by

$$(u, n)\; R_{\mathbf{T}}(s)\; (t, m) \iff u \prec t.$$

If $\mathbf{T}$ is understood from the context will just write $\mathbf{\Omega}(s) = (\Omega(s), R(s))$.

It can be proved that two states in a tree are bisimilar if and only if have isomorphic ω-expansions.

Lemma 26. *Bisimilarity classes on* $\mathbf{F}$ *are Borel.*

Proof. By the previous observation we conclude that $[(T,s)]_\sim = R^{-1}([R((T,s))]_\cong)$. By Scott's Theorem [14], we know that $[R((T,s))]_\cong$ is a Borel subset of $2^{(\mathbb{N}^*\times\mathbb{N})^2}$. Then we just have to show that the map $R(\cdot) : \mathrm{Tr}_\mathbb{N} \times \mathbb{N}^* \to 2^{(\mathbb{N}^*\times\mathbb{N})^2}$ given by

$$(u,n)\ R((T,s))\ (t,m) \iff s \subseteq u \ \&\ u \prec t \ \&\ t \in T$$

is Borel measurable.

For a finite set $P = \{((u_i,n_i),(t_i,m_i)) : i = 1,\dots,k\}$ of elements of $(\mathbb{N}^*\times\mathbb{N})^2$, we have

$$\begin{aligned} R^{-1}(\{R : P \subseteq R\}) &= \{(T,s) : t_i \in T \ \&\ s \subseteq u_i \ i = 1,\dots,k\} \\ &= \{T : \forall i\ (t_i \in T)\} \times \{s : \forall i\ (s \subseteq u_i)\}, \end{aligned}$$

if $\forall i : u_i \prec t_i$, and it is empty otherwise.

This proves that the map is continuous and, *a fortiori*, Borel measurable. □

By using the previous reduction, one can prove that bisimilarity is $\mathbf{\Sigma}^1_1$, since isomorphism is. We also give a direct proof of this fact.

Lemma 27. *Let A be countable with the discrete topology, Y Polish and $B_k \subseteq Y$ Borel for all $k \in A$. Then*

$$C(R,y) \stackrel{\bullet}{\iff} \forall k \in R : (y \in B_k)$$

is Borel in $2^A \times Y$.

Proof. We have $(R,y) \in C \iff \forall k \in A : (k \in R \Rightarrow y \in B_k)$. Then

$$\begin{aligned} C &= \bigcap_{k\in A} \{(R,y) : k \in R \Rightarrow y \in B_k\} \\ &= \bigcap_{k\in A} \{(R,y) : k \notin R\} \cup \{(R,y) : y \in B_k\} \\ &= \bigcap_{k\in A} (\{R : k \notin R\} \times Y) \cup (2^A \times \{y : y \in B_k\}) \\ &= \bigcap_{k\in A} (\{R : \{k\} \cap R = \varnothing\} \times Y) \cup (2^A \times B_k) \end{aligned}$$

which is obviously Borel. □

Theorem 28. *Bisimilarity on* $\mathbf{F}$ *is* $\mathbf{\Sigma}^1_1$.

Proof. As usual, n, m denote non negative integers and s_i finite sequences. The definition of bisimilarity on $\mathbf{F}$ is as follows:

$$
\begin{aligned}
(T_1, s) \sim (T_2, s') \iff \exists R \in 2^{\mathbb{N}^* \times \mathbb{N}^*} : (s, s') \in R \, \& \\
\& \, \forall s_1 \forall s_2 \forall n. \Big(s_1{}^\frown n \in T_1 \, \& \, s_2 \in T_2 \, \& \, (s_1, s_2) \in R \implies \\
\exists m : s_2{}^\frown m \in T_2 \, \& \, (s_1{}^\frown n, s_2{}^\frown m) \in R \Big) \, \& \\
\& \, \forall s_1 \forall s_2 \forall n. \Big(s_1 \in T_1 \, \& \, s_2{}^\frown n \in T_2 \, \& \, (s_1, s_2) \in R \implies \\
\exists m : s_1{}^\frown m \in T_1 \, \& \, (s_1{}^\frown m, s_2{}^\frown n) \in R \Big).
\end{aligned}
$$

It suffices to prove that the set defined inside the outer existential quantifier is Borel in $2^{\mathbb{N}^* \times \mathbb{N}^*} \times F \times F$. We first consider the third line of the definition. The set defined by

$$
(R, (T_1, s), (T_2, s')) \in X_{(s_1, s_2), n, m} \overset{\bullet}{\iff} s_2{}^\frown m \in T_2 \, \& \, (s_1{}^\frown n, s_2{}^\frown m) \in R
$$

is easily Borel. Then the condition $\exists m : s_2{}^\frown m \in T_2 \, \& \, (s_1{}^\frown n, s_2{}^\frown m) \in R$ also is and

$$
\forall n. \Big(s_1{}^\frown n \in T_1 \, \& \, s_2 \in T_2 \implies \exists m : s_2{}^\frown m \in T_2 \, \& \, (s_1{}^\frown n, s_2{}^\frown m) \in R \Big)
$$

finally defines a Borel set of tuples $(R, (T_1, s), (T_2, s'))$ indexed by elements $(s_1, s_2) \in R$. We may apply now Lemma 27 and conclude that

$$
\forall (s_1, s_2) \in R : \forall n. \Big(s_1{}^\frown n \in T_1 \, \& \, s_2 \in T_2 \implies \exists m : s_2{}^\frown m \in T_2 \, \& \, (s_1{}^\frown n, s_2{}^\frown m) \in R \Big),
$$

which is equivalent to the second and third lines of our definition for bisimilarity, is Borel.

The rest of the formula is handled similarly. □

By using Theorem 23 we conclude

Corollary 29. *Bisimilarity on F is $\mathbf{\Sigma}_1^1$-complete.*

4 Conclusion

In the framework of nondeterministic labelled Markov process, we considered the problem of describing bisimilarity by using a modal logic. We studied this problem in a very restricted case, that of image-countable processes where the only kind of probabilities considered are Dirac measures (MLTS).

In this restricted setting, all proposed definitions of bisimilarity coincide (in particular, with the standard notion of bisimilarity for labelled transition systems). We proved that there is a model $\mathbf{F}$ consisting of trees whose base space F is Polish and the relation of bisimilarity on $\mathbf{F}$ is an analytic non Borel subset of F^2; hence there is no countable measurable logic that characterizes bisimilarity for this process.

Indeed, Hennessy-Milner logic already characterizes bisimilarity in this case, and it is a measurable logic (obviously with uncountably many formulas). But in order to cope with image-countable NLMP with no restrictions on the kind of probabilities we use, this is a serious limitation: to our knowledge, all the proofs for logical characterization of bisimilarities are based on the structure of analytic spaces, and evidence shows that one needs a countable measurable logic to be able to use these structural properties.

After obtaining these results and recalling the use of MLTS in other works [5, 15], we conclude that these models provide a simple framework that can be considered as a first test scenario for conjectures about nondeterministic and probabilistic processes over continuous state spaces.

Acknowlegdements. I want to thank Prof. Xavier Caicedo for a nice discussion concerning modal logics and for his Example 10. I would also like to thank José Mijares for some comments on descriptive-set-theoretical issues. Finally, I would like to express my gratitude to Prof. Pedro D'Argenio for a very instructive talk on process algebra.

References

[1] T. Bolognesi, E. Brinksma, Introduction to the ISO specification language LOTOS, *Computer Networks* **14**: 25–59 (1987).

[2] P. Celayes, "Procesos de Markov Etiquetados sobre Espacios de Borel Estándar", Master's thesis, FaMAF, Universidad Nacional de Córdoba (2006).

[3] V. Danos, J. Desharnais, F. Laviolette, P. Panangaden, Bisimulation and cocongruence for probabilistic systems, *Inf. Comput.* **204**: 503–523 (2006).

[4] P. D'Argenio, N. Wolovick, P. Sánchez Terraf, P. Celayes, Nondeterministic labeled Markov processes: Bisimulations and logical characterization, in: QEST, IEEE Computer Society: 11–20 (2009).

[5] P.R. D'Argenio, P. Sánchez Terraf, N. Wolovick, Bisimulations for nondeterministic labelled Markov processes, *Mathematical. Structures in Comp. Sci.* **22**: 43–68 (2012).

[6] J. Desharnais, "Labeled Markov Process", Ph.D. thesis, McGill University (1999).

[7] J. Desharnais, A. Edalat, P. Panangaden, Bisimulation for labelled Markov processes, *Inf. Comput.* **179**: 163–193 (2002).

[8] J. Desharnais, F. Laviolette, A. Turgeon, A logical duality for underspecified probabilistic systems, *Inf. Comput.* **209**: 850–871 (2011).

[9] E.E. Doberkat, Semi-pullbacks and bisimulations in categories of stochastic relations, in: ICALP'03: Proceedings of the 30th international conference on Automata, languages and programming, Springer-Verlag, Berlin, Heidelberg: 996–1007 (2003).

[10] D. Janin, I. Walukiewicz, On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic, in: U. Montanari, V. Sassone (Eds.), CONCUR, Lecture Notes in Computer Science **1119**, Springer: 263–277 (1996).

[11] A.S. Kechris, "Classical Descriptive Set Theory", Graduate Texts in Mathematics **156**, Springer-Verlag (1994).

[12] A.S. Kechris, Classical Descriptive Set Theory; corrections and updates, Webpage, (2011). `http://www.math.caltech.edu/papers/CDST-corrections.pdf`.

[13] K.G. Larsen, A. Skou, Bisimulation through probabilistic testing, *Inf. Comput.* **94**: 1–28 (1991).

[14] D. Scott, Invariant Borel sets, *Fund. Math.* **56**: 117–128 (1964).

[15] N. Wolovick, "Continuous Probability and Nondeterminism in Labeled Transition Systems", Ph.D. thesis, Universidad Nacional de Córdoba (2012).

CIEM — Facultad de Matemática, Astronomía y Física (Fa.M.A.F.)
Universidad Nacional de Córdoba — Ciudad Universitaria
Córdoba 5000. Argentina.

and

Algebraische und Logische Grundlagen der Informatik
Institut für Theoretische Informatik
Technische Universität Dresden — Fakultät Informatik
01062 Dresden