Research Article

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$L^{p(\cdot)}$–$L^{q(\cdot)}$ boundedness of some integral operators obtained by extrapolation techniques

https://doi.org/10.1515/gmj-2018-0066
Received October 10, 2016; revised January 27, 2017; accepted August 24, 2017

Abstract: Given a matrix A such that $A^M = I$ and $0 \leq \alpha < n$, for an exponent p satisfying $p(Ax) = p(x)$ for a.e. $x \in \mathbb{R}^n$, using extrapolation techniques, we obtain $L^{p(\cdot)} \to L^{q(\cdot)}$ boundedness, $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and weak type estimates for integral operators of the form

$$T_\alpha f(x) = \int \frac{f(y)}{|x - A_1 y|^{\alpha_1} \cdots |x - A_m y|^{\alpha_m}}\, dy,$$

where $A_1, \dots, A_m$ are different powers of A such that $A_i - A_j$ is invertible for $i \neq j$, $\alpha_1 + \cdots + \alpha_m = n - \alpha$. We give some generalizations of these results.

Keywords: Variable exponents, fractional integrals

MSC 2010: 42B25, 42B35

1 Introduction

Given a measurable set $\Omega \subset \mathbb{R}^n$ and a measurable function $p(\cdot)\colon \Omega \to [1, \infty)$, let $L^{p(\cdot)}(\Omega)$ denote the Banach space of measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx < \infty,$$

with norm

$$\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx \leq 1\Big\}.$$

These spaces are known as *variable exponent spaces* and are generalizations of the classical Lebesgue spaces $L^p(\Omega)$. In the last years many authors have extended the machinery of classical harmonic analysis to these spaces, see [1, 2, 4]. The first step was to determine sufficient conditions on $p(\cdot)$ for the boundedness on $L^{p(\cdot)}$ of the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_B \frac{1}{|B|} \int_{B\cap\Omega} |f(y)|\, dy,$$

where the supremun is taken over all balls B containing x. Let $p_- = \operatorname{ess\,inf} p(x)$ and $p_+ = \operatorname{ess\,sup} p(x)$. In [2], Cruz-Uribe, Fiorenza and Neugebauer proved the following result.

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Theorem. *Given an open set $\Omega \subset \mathbb{R}^n$, let $p(\cdot)\colon \Omega \to [1,\infty)$ be such that $1 < p_- \le p_+ < \infty$. Suppose further that $p(\cdot)$ satisfies*

$$|p(x) - p(y)| \le \frac{c}{-\log|x-y|}, \quad x, y \in \Omega,\ |x-y| < \frac{1}{2}, \tag{1}$$

and

$$|p(x) - p(y)| \le \frac{c}{\log(e+|x|)}, \quad x, y \in \Omega,\ |y| \ge |x|. \tag{2}$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

We recall that a weight ω is a locally integrable and non negative function. The Muckenhoupt class A_p, $1 < p < \infty$, is defined as the class of weights ω such that

$$\sup_Q \Big[\Big(\frac{1}{|Q|}\int_Q \omega\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{p-1}}\Big)^{p-1}\Big] < \infty,$$

where Q is a cube in $\mathbb{R}^n$. For $p = 1$, A_1 is the class of weights ω having the property that there exists $c > 0$ such that

$$\mathcal{M}\omega(x) \le c\omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

We denote by $[\omega]_{A_1}$ the infimum of the constant c such that ω satisfies the above inequality.

In [5], Muckenhoupt and Wheeden define $A(p,q)$, with $1 < p < \infty$ and $1 < q < \infty$, as the class of weights ω such that

$$\sup_Q \Big[\Big(\frac{1}{|Q|}\int_Q \omega(x)^q\,dx\Big)^{\frac{1}{q}}\Big(\frac{1}{|Q|}\int_Q \omega(x)^{-p'}\,dx\Big)^{\frac{1}{p'}}\Big] < \infty.$$

When $p = 1$, $\omega \in A(1,q)$ if only if

$$\sup_Q \Big[\|\omega^{-1}\chi_Q\|_\infty \Big(\frac{1}{|Q|}\int_Q \omega(x)^q\,dx\Big)^{\frac{1}{q}}\Big] < \infty.$$

Let $M \in \mathbb{N}$, $M > 1$. Let A be an invertible $n \times n$ matrix such that $A^M = I$, and also suppose that M is such that if $A^N = I$ for some $N \in \mathbb{N}$, then $M \le N$. Let $m \in \mathbb{N}$, $1 < m \le M$. Let $0 \le \alpha < n$. Let $\alpha_1, \dots, \alpha_m$ be real numbers such that

$$\alpha_1 + \dots + \alpha_m = n - \alpha.$$

Let T_α be the integral operator given by

$$T_\alpha f(x) = \int k(x,y) f(y)\,dy, \tag{3}$$

with

$$k(x,y) = \frac{1}{|x - A_1 y|^{\alpha_1}} \cdots \frac{1}{|x - A_m y|^{\alpha_m}},$$

where, for $1 \le i \le m$, the matrices A_i are certain power of A, $A_i = A^{k_i}$, $k_i \in \mathbb{N}$, $1 \le k_i \le M$.

In [6], Riveros and Urciuolo studied integral operators with kernels given by

$$k(x,y) = \frac{1}{|x - A_1 y|^{\alpha_1} \cdots |x - A_m y|^{\alpha_m}}, \tag{4}$$

where $A_1, \dots, A_m$ are invertible matrices such that $A_i - A_j$ is invertible for $i \ne j$, $1 \le i, j \le m$. They obtained weighted (p,q) estimates, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, for weights $w \in A(p,q)$ such that $w(A_i x) \le c w(x)$. We want to use extrapolation techniques to obtain $p(\cdot)$–$q(\cdot)$ and weak type estimates. In [7], Rocha and Urciuolo proved the following theorem that involves more general matrices A_i, with the additional hypothesis $p(A_i x) = p(x)$ for a.e. $x \in R^n$.

Theorem (Strong type). *Let $0 \le \alpha < n$ and let T_α be the integral operator with kernel given by (4), with A_i orthogonal matrices such that $A_i - A_j$ is invertible for $i \ne j$, $1 \le i, j \le m$. Let $h\colon \mathbb{R} \to [1,\infty)$ be such that $1 < h_- \le h_+ < \frac{n}{\alpha}$ and satisfying (1) and (2). Let $p\colon \mathbb{R}^n \to [1,\infty)$ given by $p(x) = h(|x|)$. Then T_α is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$.*

In this paper we prove a similar result using extrapolation techniques that allow us to replace the log-Hölder conditions about the exponent $p(\cdot)$ by a more general hypothesis concerning the boundeness of the maximal function $\mathcal{M}$. We obtain the following result.

Theorem 1. *let T_α be the integral operator given by* (3) *such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p\colon \mathbb{R}^n \to [1,\infty)$ be such that $1 < p_- \leq p_+ < \frac{n}{\alpha}$ and $p(Ax) = p(x)$ for a.e. $x \in \mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator $\mathcal{M}$ is bounded on $L^{(\frac{n-\alpha p_-}{np_-}q(\cdot))'}$, then T_α is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$.*

In [7], Rocha and Urciuolo obtained weak type estimates with the additional hypothesis $p(0) = 1$.

Theorem (Weak type). *Let $0 \leq \alpha < n$, and let $h\colon \mathbb{R} \to [1,\infty)$ be a function satisfying* (1) *and* (2), *with $h(0) = 1$ and $h_+ < \infty$. Let $p\colon \mathbb{R}^n \to [1,\infty)$ given by $p(x) = h(|x|)$. Let T_α be the integral operator with kernel given by* (4), *with A_i orthogonal matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. If $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$, then there exists $C > 0$ such that*

$$\sup_{\lambda>0} \lambda \|\chi_{\{x : T_\alpha f(x) > \lambda\}}\|_{q(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

We obtain a weak type estimate for the operator given by (3), without that additional hypothesis. Our result is the following.

Theorem 2. *Let T_α be the integral operator given by* (3) *such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $p\colon \mathbb{R}^n \to [1,\infty)$ be such that $1 \leq p_- \leq p_+ < \frac{n}{\alpha}$ and $p(Ax) = p(x)$ a.e. $x \in \mathbb{R}^n$. Let $q(\cdot)$ be defined by $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. If the maximal operator $\mathcal{M}$ is bounded on $L^{(\frac{n-\alpha p_-}{np_-}q(\cdot))'}$, then there exists $c > 0$ such that*

$$\|t\chi_{\{x : T_\alpha f(x) > t\}}\|_{q(\cdot)} \leq c\|f\|_{p(\cdot)}.$$

We will also show that this technique applies in the case when each of the matrices A_i is either a power of an orthogonal matrix A or a power of A^{-1}.

2 Proofs of the results

Proof of Theorem 1. We denote $q_0 = \frac{np_-}{n-\alpha p_-}$. In [6], Riveros and Urciuolo obtained a weighted (p_-, q_0) estimate for weights $w \in A(p,q)$ such that $w(A_i x) \leq cw(x)$. We let $\widetilde{q}(x) = \frac{q(\cdot)}{q_0}$ and define

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h(Ax)}{2^k \|\mathcal{M}\|^k_{\widetilde{q}(\cdot)'}} + \cdots + \sum_{k=0}^{\infty} \frac{\mathcal{M}^k h(A^M x)}{2^k \|\mathcal{M}\|^k_{\widetilde{q}(\cdot)'}}. \tag{5}$$

It is easy to check that
(1) for all $x \in \mathbb{R}^n$, $|h(x)| \leq \mathcal{R}h(x)$,
(2) $\mathcal{R}$ is bounded on $L^{\widetilde{q}(\cdot)'}(\mathbb{R}^n)$ and $\|\mathcal{R}h\|_{\widetilde{q}(\cdot)'} \leq 2M\|h\|_{\widetilde{q}(\cdot)'}$,
(3) $\mathcal{R}h \in A_1$ and $[\mathcal{R}h]_{A_1} \leq 2CM\|\mathcal{M}\|_{\widetilde{q}(\cdot)'}$
(4) $\mathcal{R}h(A^i x) \leq \mathcal{R}h(x)$, $x \in \mathbb{R}^n$.
Indeed, (1) is evident; (2) is verified as follows. Let $l \in \mathbb{N}$, $l \leq M$. Then

$$\|\mathcal{M}^k h(A^l \cdot)\|_{\widetilde{q}(\cdot)'} = \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{\mathcal{M}^k h(A^l x)}{\lambda}\Big)^{\widetilde{q}(x)'} dx \leq 1\Big\}.$$

But

$$\int_{\mathbb{R}^n} \Big(\frac{\mathcal{M}^k h(A^l x)}{\lambda}\Big)^{\widetilde{q}(x)'} dx = \int_{\mathbb{R}^n} \Big(\frac{\mathcal{M}^k h(y)}{\lambda}\Big)^{\widetilde{q}(A^{-l}y)'} dy = \int_{\mathbb{R}^n} \Big(\frac{\mathcal{M}^k h(y)}{\lambda}\Big)^{\widetilde{q}(y)'} dy,$$

where the first equality follows from a change of variables, using that $|\det A| = 1$. The second equality holds because $q(A^l x) = q(x)$ for a.e. $x \in \mathbb{R}^n$. Then we conclude that

$$\|\mathcal{M}^k h(A^l \cdot)\|_{\widetilde{q}(\cdot)'} = \|\mathcal{M}^k h\|_{\widetilde{q}(\cdot)'}.$$

Thus, we obtain (2) by subadditivity of the norm:

$$\|\mathcal{R}h\|_{\tilde{q}(\cdot)'} \le \sum_{k=0}^{\infty} \frac{\|\mathcal{M}^k h(A(\cdot))\|_{\tilde{q}(\cdot)'}}{2^k \|\mathcal{M}\|^k_{\tilde{q}(\cdot)'}} + \cdots + \sum_{k=0}^{\infty} \frac{\|\mathcal{M}^k h(A^M(\cdot))\|_{\tilde{q}(\cdot)'}}{2^k \|\mathcal{M}\|^k_{\tilde{q}(\cdot)'}} \le \|h\|_{\tilde{q}(\cdot)'} M \sum_{k=0}^{\infty} 2^{-k} = 2M\|h\|_{\tilde{q}(\cdot)'}.$$

Now, it is easy to check that there exists $C > 0$ such that for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\mathcal{M}(f \circ A)(x) \le C\mathcal{M}f(Ax)$. So, (3) follows as in [3, p. 157]:

$$\begin{aligned}\mathcal{M}(\mathcal{R}h)(x) &\le C\bigg(\sum_{k=0}^{\infty} \frac{\mathcal{M}^{k+1}h(Ax)}{2^k \|\mathcal{M}\|^k_{\tilde{q}(\cdot)'}} + \cdots + \sum_{k=0}^{\infty} \frac{\mathcal{M}^{k+1}h(A^M x)}{2^k \|\mathcal{M}\|^k_{\tilde{q}(\cdot)'}}\bigg)\\ &\le 2C\|\mathcal{M}\|_{\tilde{q}(\cdot)'}\bigg(\sum_{k=0}^{\infty} \frac{\mathcal{M}^{k+1}h(Ax)}{2^{k+1} \|\mathcal{M}\|^{k+1}_{\tilde{q}(\cdot)'}} + \cdots + \sum_{k=0}^{\infty} \frac{\mathcal{M}^{k+1}h(A^M x)}{2^{k+1} \|\mathcal{M}\|^{k+1}_{\tilde{q}(\cdot)'}}\bigg)\\ &\le 2\|\mathcal{M}\|_{\tilde{q}(\cdot)'}\mathcal{R}h(x),\end{aligned}$$

and (4) follows by definition. So, $\mathcal{R}h$ is a weight in A_1 such that $\mathcal{R}h(A_i x) \le \mathcal{R}h(x)$, $x \in \mathbb{R}^n$.

We now take a bounded f with compact support. We will check later that $\|T_\alpha f\|_{q(\cdot)} < \infty$, so, as in [3, Theorem 5.24],

$$\begin{aligned}\|T_\alpha f\|^{q_0}_{q(\cdot)} = \|(T_\alpha f)^{q_0}\|_{\tilde{q}(\cdot)} &= c \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int_{\mathbb{R}^n} (T_\alpha f)^{q_0}(x)h(x)\,dx\\ &\le c \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int_{\mathbb{R}^n} (T_\alpha f)^{q_0}(x)\mathcal{R}h(x)\,dx\\ &\le c \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \bigg(\int_{\mathbb{R}^n} |f(x)|^{p_-}\mathcal{R}h(x)^{\frac{p_-}{q_0}}\,dx\bigg)^{\frac{q_0}{p_-}},\end{aligned}$$

since $Rh^{\frac{1}{q_0}} \in A(p_-, q_0)$. Hölder's inequality gives

$$\|(T_\alpha f)^{q_0}\|_{\tilde{q}(\cdot)} \le c\|f^{p_-}\|^{\frac{q_0}{p_-}}_{\tilde{p}(\cdot)} \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \|\mathcal{R}h^{\frac{p_-}{q_0}}\|^{\frac{q_0}{p_-}}_{\tilde{p}(\cdot)'} \le c\|f\|^{q_0}_{p(\cdot)}\|\mathcal{R}h\|_{\tilde{q}(\cdot)'} \le 2Mc\|f\|^{q_0}_{p(\cdot)}\|h\|_{\tilde{q}(\cdot)'},$$

where the last inequality follows as in [3, p. 211].

Now we show that $\|T_\alpha f\|_{q(\cdot)} < \infty$. By [3, Proposition 2.12, p. 19], it is enough to check that $\int_{\mathbb{R}^n} T_\alpha f < \infty$. We have

$$|Tf(x)|^{q(x)} \le |Tf(x)|^{q_+}\chi_{\{x:T_\alpha f(x)>1\}} + |Tf(x)|^{q_-}\chi_{\{x:T_\alpha f(x)\le 1\}},$$

and now f is bounded and with compact support, so $T_\alpha f \in L^s(\mathbb{R}^n)$ for $\frac{n}{n-\alpha} < s < \infty$, thus $\int_{\mathbb{R}^n} |Tf(x)|^{q(x)}dx < \infty$.

The theorem follows since bounded functions with compact support are dense in $L^{p(\cdot)}(\mathbb{R}^n)$. □

Proof of Theorem 2. We consider first the case $p_- = 1$. We denote $q_0 = \frac{n}{n-\alpha}$ and $\tilde{q}(\cdot) = \frac{q(\cdot)}{q_0}$. Theorem 3.2 of [6] implies that if $\omega \in A(1, q_0)$ is such that $\omega(Ax) \le c\omega(x)$, then

$$\sup_\lambda \lambda^{q_0}\omega^{q_0}(\chi_{\{x:|T_\alpha f(x)|>\lambda\}}) \le C\bigg(\int_{\mathbb{R}^n} |f(x)|\omega(x)\,dx\bigg)^{q_0}.$$

Now, let $F_\lambda = \lambda^{q_0}\chi_{\{x:|T_\alpha f(x)|>\lambda\}}$. Then

$$\|\lambda\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|^{q_0}_{q(\cdot)} \le \|\lambda^{q_0}\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{\tilde{q}(\cdot)} = \|F_\lambda\|_{\tilde{q}(\cdot)} = C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int_{\mathbb{R}^n} F_\lambda(x)h(x)\,dx.$$

As in the previous theorem, we define $\mathcal{R}h$ by (5). Since $\mathcal{R}h \in A_1$, $\mathcal{R}h^{\frac{1}{q_0}} \in A(1, q_0)$. So,

$$\begin{aligned}\|\lambda\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|^{q_0}_{q(\cdot)} &\le C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int_{\mathbb{R}^n} F_\lambda(x)\mathcal{R}h(x)\,dx\\ &\le C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \int_{\mathbb{R}^n} F_\lambda(x)(\mathcal{R}h(x)^{\frac{1}{q_0}})^{q_0}\,dx\\ &\le C \sup_{\|h\|_{\tilde{q}(\cdot)'}=1} \bigg(\int_{\mathbb{R}^n} |f(x)|\mathcal{R}h(x)^{\frac{1}{q_0}}\,dx\bigg)^{q_0},\end{aligned}$$

and, as in the previous theorem, we get

$$\begin{aligned}\|\lambda\chi_{\{x:|T_\alpha f(x)|>\lambda\}}\|_{q(\cdot)}^{q_0} &\leq C\|f\|_{p(\cdot)}^{q_0} \sup_{\|h\|_{\widetilde{q}(\cdot)'}=1} \|\mathcal{R}h(x)^{\frac{1}{q_0}}\|_{p(\cdot)'}^{q_0} \\ &\leq C\|f\|_{p(\cdot)}^{q_0} \sup_{\|h\|_{\widetilde{q}(\cdot)'}=1} \|\mathcal{R}h\|_{\widetilde{q}(\cdot)'} \\ &\leq 2M\|f\|_{p(\cdot)}^{q_0} \sup_{\|h\|_{\widetilde{q}(\cdot)'}=1} \|h\|_{\widetilde{q}(\cdot)'} = 2M\|f\|_{p(\cdot)}^{q_0}.\end{aligned}$$

If $p_- > 1$, then we use that T_α is of weak type (p_-, q_0) and we proceed as before to get the statement of the theorem. □

Remark 3. Theorems 1 and 2 still hold if $m = 1$ and $\alpha > 0$. In this case, if $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$ and $\omega \in A(p, q)$ is such that $\omega(Ax) \leq \omega(x)$ for a.e. $x \in \mathbb{R}^n$, then

$$T_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - A^{k_1}y|^{n-\alpha}}\, dy = I_\alpha(f \circ A^{-k_1})(x),$$

where I_α is the classical fractional integral operator. Thus,

$$\begin{aligned}\int_{\mathbb{R}^n} (T_\alpha f(x))^q \omega^q(x)\, dx &= \int_{\mathbb{R}^n} (I_\alpha(f \circ A^{-k_1})(x))^q \omega^q(x)\, dx \\ &\leq C\Big(\int_{\mathbb{R}^n} (f \circ A^{-k_1}(x))^p \omega^p(x)\, dx\Big)^{\frac{q}{p}} \\ &= C\Big(\int_{\mathbb{R}^n} (f(x))^p \omega^p(A^{k_1}x)\, dx\Big)^{\frac{q}{p}} \\ &\leq C\Big(\int_{\mathbb{R}^n} (f(x))^p \omega^p(x)\, dx\Big)^{\frac{q}{p}}.\end{aligned}$$

So,

$$\|T_\alpha f\|_{q,\omega^q} \leq C\|f\|_{p,\omega^p},$$

In a similar way we obtain the corresponding weak type estimate and we proceed as in the previous theorems.

Remark 4. Let A be a orthogonal matrix and let T_α be as in (3), where the matrix A_i is either a power of A or a power of A^{-1}. If $A_i - A_j$ is invertible and $p(\cdot)$ is as in Theorem 2, we also obtain strong and weak type estimates. We simply define $\mathcal{R}$ as follows:

$$\mathcal{R}h(x) = \sum_{j=0}^{\infty} \frac{1}{2^j}\bigg(\sum_{k=0}^{\infty} \frac{M^k h(A^j x)}{2^k\|M\|_{\widetilde{q}(\cdot)'}^k} + \sum_{k=0}^{\infty} \frac{M^k h((A^{-1})^j x)}{2^k\|M\|_{\widetilde{q}(\cdot)'}^k}\bigg),$$

and the proof follows as in the proofs of Theorems 1 and 2.

Example 5. We take r satisfying (1) and (2), with $1 < r_- \leq r_+ < \frac{n}{\alpha}$ and

$$A = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}.$$

So, $A^4 = I$ and $A^i - A^j$ is invertible for $1 \leq i, j \leq 4$, $i \neq j$. We let $p(x) = \frac{1}{4}(r(Ax) + r(A^2x) + r(A^3x) + r(A^4x))$.

Example 6. We take an even function p satisfying (1) and (2), with $1 < p_- \leq p_+ < \frac{n}{\alpha}$ and $A = -I$.

Funding: Partially supported by CONICET and SECYTUNC.

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