The Aumann functional equation for general weighting procedures

Lucio R. Berrone*

Abstract

The functional equation of composite type

$$M(M(x, M(x, y)), M(M(x, y), y)) = M(x, y) \tag{1}$$

arose in the course of the studies on the problem of extension and restriction of the number of arguments of a mean M performed by G. Aumann at the third decade of the past century. A solution to (1) in the analytic case was ulteriorly obtained by Aumann himself and remained as a noteworthy characterization of analytic quasiarithmetic means. An ample generalization of equation (1) which involves general weighting operators is considered in this paper. Under mild conditions on the regularity of the involved means, the general solution to this generalized equation is obtained for a particularly tractable class of weighting operators.

AMS Mathematics Subject Classification: 39B22, 37E05.

1 Introduction

In his "habilitationsschrift" presented at the "Technixsche Hochshule", Munich, in 1933 ([1]), George Aumann discussed an iterative process to derive a mean in $n+1$ variables from a mean in n variables. The process, named by Aumann himself *augmentation (erhöhung) of a mean*, is as follows: for a real interval I and a symmetric mean M_n in the n (≥ 2) variables $x_1, x_2, \ldots, x_n \in I$, consider all the $n+1$ possible n-tuples $(x_1, x_2, \ldots, x_{n+1})^{\vee j} = (x_1, \ldots, x_{j-1}, x_{j+1} \ldots, x_{n+1})$ obtained from the $(n+1)$-tuple $(x_1, x_2, \ldots, x_{n+1})$ by omitting the j-th coordinate and then, define a transformation $\mathcal{A} : I^{n+1} \to I^{n+1}$ by $\mathcal{A}(x_1, x_2, \ldots, x_{n+1}) = (X_1, X_2, \ldots, X_{n+1})$, with

$$X_k = M_n((x_1, x_2, \ldots, x_{n+1})^{\vee(n+2-k)}), \ k = 1, 2, \ldots, n+1.$$

*Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Laboratorio de Acústica y Electroacústica, Facultad de Ciencias Exactas, Ing. y Agrim., Universidad Nacional de Rosario, Riobamba 245 bis, (2000) Rosario, Argentina; e-mail address: berrone@fceia.unr.edu.ar

Under appropriate hypotheses on the mean M, the iterations

$$\mathcal{A}^n = \overbrace{\mathcal{A} \circ \mathcal{A} \circ \cdots \circ \mathcal{A}}^{n \text{ times}}$$

of the so defined transformation $\mathcal{A}$ pointwise converge, when $n \uparrow \infty$, to a transformation $\mathcal{A}^\infty(x_1, x_2, \ldots, x_{n+1})$ with all its coordinate functions equal one each other; i.e.,

$$\mathcal{A}^\infty(x_1, x_2, \ldots, x_{n+1}) = (M_{n+1}, M_{n+1}, \ldots, M_{n+1}), \tag{2}$$

where $M_{n+1} = M_{n+1}(x_1, x_2, \ldots, x_{n+1})$ is the *upper mean (obermittel)* of M_n. As a consequence of (2), the mean M_{n+1} turns out to be invariant under the transformation $\mathcal{A}$, what is symbolically expressed by $M_{n+1} \circ \mathcal{A} = M_{n+1}$ or, writing in the variables,

$$M_{n+1}(M_n(x_1, x_2, \ldots, x_n), M_n(x_1, \ldots, x_{n-1}, x_{n+1}), \ldots, M_n(x_2, x_3, \ldots, x_{n+1})) = M_{n+1}(x_1, x_2, \ldots, x_{n+1}). \tag{3}$$

Later, a process opposite to the augmentation of a mean was considered by Aumann. In [2], the *lower mean (untermittel)* of the symmetric mean in n variables M_n is defined as the solution w of the equation

$$M_n(x_1, \ldots, x_{n-1}, w) = w \tag{4}$$

which turns out, again imposing suitable hypotheses on M_n, to be a unique mean in $n-1$ variables M_{n-1}. The process named *reduction (erniedrigung) of a mean* consists of the classical solution of the equation (4) by iteration. The functional relation among M_n and M_{n-1} expressed by

$$M_n(x_1, \ldots, x_{n-1}, M_{n-1}(x_1, \ldots, x_{n-1})) = M_{n-1}(x_1, \ldots, x_{n-1}) \tag{5}$$

is quickly derived from (4).

The question of establishing when the above defined processes are inverse one each other led him, in the most simple case in which $n = 2$, to the composite functional equation

$$M(M(x, M(x,y)), M(M(x,y), y)) = M(x,y), \ x, y \in I. \tag{6}$$

Indeed, setting $x_3 = M_2(x_1, x_2)$ in the case $n = 2$ of (3), it is deduced that

$$M_3(M_2(x_1, x_2), M_2(x_1, M_2(x_1, x_2)), M_2(x_2, M_2(x_1, x_2))) = M_3(x_1, x_2, M_2(x_1, x_2)),$$

or, taking into account the symmetry of the involved means,

$$M_3(M_2(x_1, M_2(x_1, x_2)), M_2(M_2(x_1, x_2), x_2), M_2(x_1, x_2)) = M_3(x_1, x_2, M_2(x_1, x_2)).$$

The right hand side of this equality coincides with $M_2(x_1, x_2)$ by the case $n = 3$ of (5); therefore,

$$M_3(M_2(x_1, M_2(x_1, x_2)), M_2(M_2(x_1, x_2), x_2), M_2(x_1, x_2)) = M_2(x_1, x_2)$$

and hence, by the uniqueness of solution of (the case $n = 3$ of) equation (4),

$$M_2(M_2(x_1, M_2(x_1, x_2)), M_2(M_2(x_1, x_2), x_2)) = M_2(x_1, x_2),$$

which is, saving the differences in the notation, identical to equation (6).

In recent years, the problem of constructing general augmentation or reduction algorithms for general means has received some attention. In this regard, the reader is referred to the articles [9] and [10]. The present paper is instead aimed to the study of an ample generalization of equation (6).

In [2], Aumann considered the equation (6) in the analytic case, showing that the *quasiarithmetic (analytic) mean*

$$M(z_1, z_2) = f^{-1}\left(\frac{f(z_1) + f(z_2)}{2}\right)$$

with f an arbitrary regular function, is its general solution in the class of *analytic means*; i.e., functions $M(z_1, z_2)$ holomorphic in a neighborhood V of a regular point $(a, a) \in \mathbb{C}^2$ which are symmetric and reflexive in V. In attempting to prove a similar result for real variables, one is faced with two obstacles at least. At a primary level, the usual order of the real line enters directly in the formulations of the internality, the characteristic property of real means. Moreover, the complex variable methods used by Aumann do not apply to real variables. It is an interesting example of the conservation of the functional forms, the fact that the quasiarithmetic mean

$$A_f(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right), \; x, y \in I, \tag{7}$$

with $f : I \to \mathbb{R}$ a strictly monotonic and continuous function, continues to be a solution to (6) in the real case. Indeed, a continuous symmetric strict solution of (6) which is not a quasiarithmetic mean is not known. The general solution of (6) in the class of *non-strict* continuous symmetric means includes means like $\min\{x, y\}$ and $\max\{x, y\}$. In its turn, the means given by

$$A_{f,g}(x, y) = \begin{cases} A_f(x, y), & x \leq y \\ A_g(x, y), & x \geq y \end{cases},$$

where f, $g : I \to \mathbb{R}$ are strictly monotonic and continuous functions satisfying $f \neq \alpha g + \beta$ for every α, $\beta \in \mathbb{R}$, constitute a family of continuous non-symmetric and strict solutions. Aumann equation also has a rich family of discontinuous solutions; among them, the following ones. For a nontrivial partition $\mathcal{P} = \{A, B\}$ of $\mathbb{R}^+$, the mean defined by

$$M_{\mathcal{P}}(x, y) = \begin{cases} y, & x/y \in A \\ x, & x/y \in B \end{cases}$$

is an homogeneous discontinuous solution in $I = \mathbb{R}^+$; while

$$M_d(x, y) = \begin{cases} y, & x, y \in \mathbb{Q} \\ \\ x, & \text{in other case} \end{cases}$$

is a solution discontinuous at every point of $\mathbb{R}^2$.

As already mentioned, a generalization of equation (6) is presented and discussed in this paper. After recalling a series of basic concepts, abstract weighting operators acting on general classes of means are defined in Section 2. The main properties of these operators named weighting procedures are there studied and some important examples are considered in detail. Even though weighting versions of means and inequalities among them play a capital role in the theory of means, only an occasional treatment of particular weightings seems to be find a place in the specific literature ([7], [8]), so that the length of Section 2 would be justified in part by this fact. Weighting procedures for means in n variables are studied in [6]. The useful weight representation of two variables means is also introduced in Section 2, while the Aumann equation is extended in the following Section 3 to general weighting procedures. In Section 4, a noticeable type of weighting procedures is isolated. The Aumann generalized equation turns out to be elementary tractable for these weighting procedures, a fact that ultimately derives in the statement and proof of some results furnishing its general solution in the last Section 4.

2 Means and weighting procedures

Let I be a real interval. A (two variables) mean M defined on I is a function $M : I \times I \to I$ which is *internal*; i.e., it satisfies the property

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \ x, y \in I. \tag{8}$$

The mean is said to be *strict* when the inequalities in (8) are strict if $x \neq y$ (*strict internality*). In view of the equality

$$M(x, x) = x, \ x \in I, \tag{9}$$

holds for every mean M, means are *reflexive* functions. A mean M is said to be *symmetric* when

$$M(x, y) = M(y, x), \ x, y \in I. \tag{10}$$

The *coordinate means* $X(x, y) \equiv x$ and $Y(x, y) \equiv y$ are the unique means depending on a sole variable. The functions at the leftmost and rightmost members of the inequalities (8) are, respectively, the *extremal means* $\min\{x, y\}$ and $\max\{x, y\}$. With few exceptions, the means considered throughout this paper will be *continuous means*; i.e., means that are continuous functions. The classes of all means and symmetric means defined on I will be respectively denoted by $\mathcal{GM}(I)$ and $\mathcal{SM}(I)$. If $\mathcal{M}(I)$ is a given class of means, then,

$\mathcal{M}^k(I)$, $k = 0, 1, \ldots$, and $\mathcal{M}_S(I)$ will denote the subclasses of $\mathcal{M}(I)$ respectively composed by the $\mathcal{C}^k$ means and the strict means. For instance, $\mathcal{SM}^2_S(I)$ stands for the class of symmetric strict $\mathcal{C}^2$ means defined on I.

Let $\Omega \subseteq \mathbb{R}^2$ a non-void set and $F : \Omega \to \mathbb{R}$ be a real function. Moreover, let $W \subseteq \mathbb{R}$ be a set (the *set of parameters*) containing two points at least. A real *parameterization* of F is a function $\Phi : \Omega \times W \to \mathbb{R}$ such that

(P1) for a certain $w_1 \in W$, $\Phi(\cdot, w_1) = F$;

(P2) for another $w_2 \in W$, $\Phi(\cdot, w_2) \neq F$.

By a weighting of a mean M is to be understood a set of means parameterized by the unitary interval $[0, 1]$ and fulfilling special properties. Namely, a parameterization $M(\cdot; w)$ of a mean M is said to be a *weighting* (of M), when

(W1) $M(\cdot; w)$ is a mean defined on I for every $w \in [0, 1]$,

(W2) $M(\cdot; 1/2) = M$,

(W3) $M(x, y; 0) = x$ and $M(x, y; 1) = y$, for all $x, y \in I$.

Note that, in view of **(W1),** $M(x, x; w) \equiv x$, $w \in [0, 1]$. A notation equivalent to $M(\cdot; w)$ in which the weight w is written as a superscript will be frequently used along this paper and it must not be confused with a power of the mean M; in symbols, $M(\cdot; w) = M^w(\cdot) \neq [M(\cdot)]^w$.

A weighting of a mean M is said to be *continuous* [$\mathcal{C}^k$*, analytic*] when

(CW) $w \mapsto M(x, y; w)$ is continuous [C^k, analytic] for every $x, y \in I$;

while it is said to be [*strictly*] *monotonic* when

(MW) $w \mapsto M(x, y; w)$ is [strictly] increasing if $x < y$ and [strictly] decreasing if $x > y$.

A weighting is said to be *exhaustive* when

(EW) $w \mapsto M(x, y; w)$ is onto $[x, y]$.

By the Darboux property and **(W3)**, a weighting is exhaustive provided that is continuous. Furthermore, for a continuous and strictly monotonic weighting $M(\cdot, w)$, the function $w \mapsto M(x, y; w)$ is an homeomorphism onto $[\min\{x, y\}, \max\{x, y\}]$ provided that $x \neq y$.

Several invariance properties possibly enjoyed by weightings are now defined. Let $f : I \to I$ a strictly monotonic and continuous function. A weighting $M(\cdot; w)$ is said to be *scale invariant* when

(IW) $N(x, y; w) = f^{-1}(M(f(x), f(y); w))$ for every $w \in [0, 1]$ provided that $N(x, y) = f^{-1}(M(f(x), f(y)))$.

A weighting is said to be *homogeneity preserving* provided that

(HW) $M(\cdot;w)$ is a homogeneous mean for every w when $M(\cdot)$ is an homogeneous mean;

while it is said to be *translation preserving* when

(TW) $M(x+\mathbf{c};w) = M(x;w)+\mathbf{c}$ holds for every $\mathbf{c}=(c,c)$ and for every $w\in[0,1]$ provided that $M(x+\mathbf{c}) = M(x)+\mathbf{c}$ for every $\mathbf{c}=(c,c)$.

Observe that property **(HW)** requires $M(\cdot;w)$ to be defined on a cone of $\mathbb{R}$ for every $w\in[0,1]$. In its turn, $M(\cdot;w)$, $w\in[0,1]$, must be defined on the whole $\mathbb{R}$ in property **(TW)**.

Now, let $\mathcal{M}(I)$ and $\mathcal{N}(I)$, $\mathcal{M}(I)\subseteq\mathcal{N}(I)$, be two classes of means defined on an interval I. A function $\mathcal{W}:\mathcal{M}(I)\times[0,1]\to\mathcal{N}(I)$ is to be named a *weighting procedure* (*w.p.*, for short) provided that, for every $M\in\mathcal{M}(I)$, $\mathcal{W}(M;w)$ is a weighting of the mean M. Intuitively, a *w.p.* is an algorithm assigning a weighted form $\mathcal{W}(M;\cdot)$ to a general mean M.

Several notions of continuity are naturally connected with weighting procedures. A *w.p.* $\mathcal{W}:\mathcal{M}(I)\times[0,1]\to\mathcal{N}(I)$ is said to be *weight-continuous* [*weight-$\mathcal{C}^k$, weight-analytic*] when

(WWP) $w\mapsto\mathcal{W}(M;w)$ is continuous [$\mathcal{C}^k$, analytic] for every $M\in\mathcal{M}(I)$.

If $\mathcal{M}(I)$ and $\mathcal{N}(I)$ are suitably topologized, the *w.p.* is said to be *mean-continuous* [*sequentially mean-continuous*] when

(MWP) $M\mapsto\mathcal{W}(M;w)$ is continuous [sequentially continuous] for every $w\in[0,1]$;

and it is simply said to be *continuous* when

(CWP) $(M,w)\mapsto\mathcal{W}(M;w)$ is continuous.

For many purposes $\mathcal{M}(I)$ will be endowed with the pointwise convergence topology.

A [*strictly*] *monotonic w.p.* $\mathcal{W}:\mathcal{M}(I)\times[0,1]\to\mathcal{N}(I)$ is one satisfying

(MMP) $\mathcal{W}(M;w)$ is a [strictly] monotonic weighting for every $M\in\mathcal{M}(I)$.

As illustrated by the examples at the end of this section, the monotonicity of a *w.p.* depends heavily on the class $\mathcal{M}(I)$. If $\mathcal{W}(\cdot;w)$ is, when defined on a certain class $\mathcal{M}(I)$, not a strictly monotonic *w.p.*, then, nontrivial solutions K to the equation

$$\mathcal{W}(K;w_1) = \mathcal{W}(K;w_2) \tag{11}$$

might exist in $\mathcal{M}(I)$ for weights $w_1<w_2$. In particular, nontrivial fixed points of the operator $\mathcal{W}(\cdot;w)$ might exist in $\mathcal{M}(I)$ for weights $w\neq 0,1/2,1$; i.e., nontrivial solutions K to the equation

$$\mathcal{W}(K;w) = K. \tag{12}$$

Let us denote by $\mathcal{K}_w(\mathcal{W})$ the set of such solutions; i.e.,

$$\mathcal{K}_w(\mathcal{W}) = \{K \in \mathcal{M}(I) : \mathcal{W}(K; w) = K\}, \tag{13}$$

and name *singular set* $\mathcal{K}(\mathcal{W})$ of the *w.p.* to the set

$$\mathcal{K}(\mathcal{W}) = \bigcup \{\mathcal{K}_w(\mathcal{W}) : w \neq 0, 1/2, 1\}. \tag{14}$$

In the case in which $\mathcal{K}(\mathcal{W}) = \{X, Y\}$ where X and Y are the coordinate means (trivial solutions to equation (12)), the *w.p.* is said to posses a *trivial singular set.*

A *w.p.* $\mathcal{W} : \mathcal{M}(I) \times [0,1] \to \mathcal{N}(I)$ is said to be *order preserving* when

(OWP) $\mathcal{W}(M; \cdot) \leq \mathcal{W}(N; \cdot)$ provided that $M \leq N$.

If the property **(OWP)** is satisfied with strict inequality signs, then, the w.p. is said to be *strictly order preserving.*

In general a *w.p.* $\mathcal{W}$ is said to be invariant in some sense if the weighting $\mathcal{W}(M; \cdot)$ is invariant in the same sense for every $M \in \mathcal{M}(I)$. Thus, a w.p. $\mathcal{W}$ will be said *scale invariant* when

(IWP) for every $f : I \to I$ continuous and strictly monotonic, $\mathcal{W}(N; \cdot)(x, y) = f^{-1}(\mathcal{W}(M; \cdot)(f(x), f(y)),\ x, y \in I$, provided that $N(x, y) \equiv f^{-1}(M(f(x), f(y))),\ x, y \in I$.

A comprehensive way of deriving a *w.p.* is based on a particular representation of two variables means. Concretely, let M be a mean defined on I and denote by $\Delta(I^2) = \{(x, x) : x \in I\}$ the set of points at the diagonal of the square I^2. The *functional weight* $\lambda_M : I^2 \setminus \Delta(I^2) \to [0, 1]$ of the mean M is defined by

$$\lambda_M(x, y) = \frac{M(x, y) - x}{y - x},\ y \neq x. \tag{15}$$

In terms of its functional weight λ_M, a mean M is uniquely representable in the form

$$M(x, y) = \begin{cases} (1 - \lambda_M(x, y))x + \lambda_M(x, y)y, & y \neq x \\ x, & y = x \end{cases}. \tag{16}$$

Observe in passing that the representation (16) can not be directly extended to means in $n > 2$ variables.

The main properties of the functional weight λ_M are gather together in the next proposition, whose simple proof will be omitted.

Proposition 1 *Let M be a mean defined on the real interval I.*

i) *The functional weight λ_M is a continuous function on $I^2 \setminus \Delta(I^2)$ provided that M is a continuous mean. Conversely, the continuous mean*

$$M(x, y) = (1 - \lambda(x, y))x + \lambda(x, y)y,\ x, y \in I,$$

is associated to an arbitrary bounded extension to I^2 of a continuous function $\lambda : I^2 \setminus \Delta(I^2) \to [0, 1]$; moreover,

ii) *λ_M can be continuously extended to the whole I^2 if and only if there exist the partial derivatives $\frac{\partial M}{\partial x}(t,t)$, $\frac{\partial M}{\partial y}(t,t)$ for every $t \in I$ and they are continuous on I (as functions of t).*

iii) *$0 < \lambda_M(x,y) < 1$ for every $(x,y) \in I^2 \setminus \Delta(I^2)$ if and only if M is a strict mean.*

iv) *M is a symmetric mean if and only if $\lambda_M(x,y) + \lambda_M(y,x) = 1,\ x,y \in I$.*

v) *$\lambda_M = \alpha$ with $\alpha \in [0,1]$ if and only if M is the weighted arithmetic mean $L_\alpha(x,y) = (1-\alpha)x + \alpha y$.*

vi) *If X, Y are the coordinate means, then, $\lambda_X = 0$ and $\lambda_Y = 1$.*

vii) *The functional weights corresponding to the extremal means* min *and* max *are respectively given by*

$$\lambda_{\min}(x,y) = \begin{cases} 0, & x<y \\ 1, & x>y \end{cases} \quad \textit{and } \lambda_{\max}(x,y) = \begin{cases} 1, & x<y \\ 0, & x>y \end{cases} .$$

The assert in the next proposition is exemplified by Proposition 1, **vii)**.

Proposition 2 *Let M, N be two means defined on the real interval I. Then, $M \le N$ if and only if*

$$\lambda_M \le \lambda_N \ \textit{if } x<y, \textit{ and } \lambda_M \ge \lambda_N \ \textit{if } x>y. \tag{17}$$

The inequalities in (17) are strict when $M < N,\ x \ne y$.

Proof. The proof is a straightforward consequence of (15). ∎

Setting $\mathcal{F}(I) = \{\lambda |\ \lambda : I^2 \setminus \Delta(I^2) \to [0,1]\}$ and taking into account that a mean M is completely characterized by the function λ_M, a *w.p.* $\mathcal{W}$ can be represented as a function $\Lambda : \mathcal{F}_1(I) \times [0,1] \to \mathcal{F}_2(I)$ with $\mathcal{F}_1(I) \subseteq \mathcal{F}_2(I) \subseteq \mathcal{F}(I)$ and satisfying the following conditions:

(Λ1) $\Lambda(\lambda;0) = 0$ and $\Lambda(\lambda;1) = 1$ for every $\lambda \in \mathcal{F}(I)$;

(Λ2) $\Lambda(\lambda;1/2) = \lambda$ for every $\lambda \in \mathcal{F}_1(I)$.

From now on, the representation Λ of a *w.p.* $\mathcal{W}$ in terms of functional weights will be named *weight representation* of $\mathcal{W}$. Clearly, a "dictionary" can be created to specify the correspondence among the standard representation $\mathcal{W}$ and the weight representation Λ of a *w.p.*. Regarding the classes of means, the family of functions $\mathcal{F}_S(I) = \{\lambda \in \mathcal{F}(I) : 0 < \lambda < 1\}$ corresponds to the class $\mathcal{GM}_S(I)$ of strict means and $\mathcal{F}_S^0(I) = \{\lambda \in \mathcal{F}_S(I) : \lambda \text{ is continuous}\}$ corresponds to the class $\mathcal{GM}_S^0(I)$ of continuous strict means. Of course, the above defined properties of a *w.p.* in the standard representation also find, in this dictionary, their counterparts expressed in terms of the weight representation. The sole case of weight-continuity and monotonicity is shown by the next result.

Proposition 3 *For every $\lambda \in \mathcal{F}(I)$, the function $w \to \Lambda(\lambda; w)$ turns out to be continuous or [strictly] increasing provided that the* w.p. *is weight-continuous or [strictly] monotonic, respectively.*

The proof of this result as well as of conditions (**Λ1**), (**Λ2**) is a straightforward consequence of the equality

$$\Lambda(\lambda; w) = \begin{cases} \frac{M^w(x,y)-x}{y-x}, & y \neq x \\ \\ M_y^w(x,x), & y = x \end{cases}, \tag{18}$$

which can be equivalently expressed in the form

$$M^w(x,y) = (1 - \Lambda(\lambda; w))x + \Lambda(\lambda; w)y. \tag{19}$$

To end this section, a series of examples of particular *w.p.* is exhibited and the main properties of every one of them are registered.

Example 4 *(The* sectionally linear w.p.*) A linear interpolation between the values $\Lambda(\lambda; 0) = 0$ and $\Lambda(\lambda; 1/2) = \lambda$ followed by another between the values $\Lambda(\lambda; 1/2) = \lambda$ and $\Lambda(\lambda; 1) = 1$, yields*

$$\Lambda(\lambda; w) = \begin{cases} 2w\lambda, & 0 \leq w \leq 1/2 \\ \lambda - (1-2w)(1-\lambda), & 1/2 \leq w \leq 1 \end{cases}. \tag{20}$$

Clearly, (20) is weight-continuous but it is not differentiable at $w = 1/2$ unless $\lambda(x,y) = 1/2$. Moreover, it is a strictly monotonic w.p. *in the class $\mathcal{F}_S(I) = \{\lambda \in \mathcal{F}(I) : 0 < \lambda < 1\}$ corresponding to strict means, but it is only monotonic when defined on the whole class $\mathcal{F}(I)$. However, the singular set of this* w.p. *is trivial even in that case; more precisely,*

$$\mathcal{K}_w(\mathcal{W}) = \begin{cases} \{X\}, & w \in (0, 1/2) \\ \{Y\}, & w \in (1/2, 1) \end{cases}.$$

Example 5 *(The homographic* w.p.*) When the interpolation is made by fitting a homographic function to the three points arising in conditions (**Λ1**), (**Λ2**), the continuous* w.p.

$$\Lambda(\lambda; w) = \frac{w\lambda(x,y)}{(1-w)(1-\lambda(x,y)) + w\lambda(x,y)} \tag{21}$$

is obtained. Clearly, the w.p. *Λ given by (21) is weight-analytic and, in view of*

$$\frac{\partial \Lambda}{\partial w}(\lambda; w) = \frac{\lambda(x,y)(1-\lambda(x,y))}{((1-w)(1-\lambda(x,y)) + w\lambda(x,y))^2} > 0,$$

it turns out to be strictly monotonic when defined on the class $\mathcal{F}_S(I)$. Like the sectionally linear w.p.*, the homographic* w.p. *has a trivial singular set even if defined on the class $\mathcal{F}(I)$; more precisely,*

$$\mathcal{K}_w(\mathcal{W}) = \{X, Y\}, \; w \in (0, 1/2) \cup (1/2, 1).$$

In [6], other properties of (21) are established.

The weighting procedures described in the examples above are particular cases of functional weighting procedures. Given a function $\Phi : [0,1]^2 \to [0,1]$, the *w.p.* $\Lambda : \mathcal{F}_1(I) \times [0,1] \to \mathcal{F}_2(I)$ defined by

$$\Lambda(\lambda; w) = \Phi(\lambda, w), \ \lambda \in \mathcal{F}_1(I),$$

is the *functional weighting procedure* associated to the function Φ. Clearly, there exists a correspondence among the properties of the *w.p.* on one hand and, on the other, the properties of the function Φ. The development of this point will not be pursued here and it is limited only to a straightforward observation: the weight-continuity and monotonicity of the *w.p.* respectively corresponds to the continuity and monotonicity of the function $w \mapsto \Phi(\lambda, w)$.

Example 6 *(The dyadic* w.p.*) A noteworthy not functional* w.p. *is defined by iteration. Namely, given a function $F : I \times I \to I$, a family $\{F^d(x,y) : d \in D([0,1])\}$ of* dyadic iterates *on $[x,y]$ of F is inductively defined as follows (cf. [4], [3], [5]): the first step consists in setting*

$$F^0(x,y) \equiv x, \ F^1(x,y) \equiv y; \tag{22}$$

then, assuming that $F^{\frac{j}{2^n}}(x,y)$ is known for $n \geq 0$ and for every $0 \leq j \leq 2^n$, the inductive step establishes that

$$F^{\frac{k}{2^{n+1}}}(x,y) = \begin{cases} F^{\frac{h}{2^n}}(x,y), & \text{if } k = 2h, \ 0 \leq h \leq 2^n \\ \\ F\left(F^{\frac{h}{2^n}}(x,y), F^{\frac{h+1}{2^n}}(x,y)\right), & \text{if } k = 2h+1, \ 0 \leq h \leq 2^n - 1 \end{cases} . \tag{23}$$

When M is a continuous strict mean, the dyadic iterates $\{M^d(x,y) : d \in D([0,1])\}$ are dense in the interval $[x,y]$ and it makes sense to consider the completation $\{M^\delta(x,y) : \delta \in [0,1]\}$. In fact, for a given $\delta \in (0,1)$, there exists an increasing sequence $\{d_n\} \subseteq D([0,1])$ such that $d_n \uparrow \delta$ when $n \uparrow \infty$, then, the sequence $\{M^{d_n}(x,y)\}$ is strictly monotonic (increasing when $x < y$, decreasing when $x > y$ and stationary when $x = y$) and bounded ($\min\{x,y\} \leq M^{d_n}(x,y) \leq \max\{x,y\}, \ n \in \mathbb{N}$); so that it makes sense to define $M^\delta(x,y)$ by

$$M^\delta(x,y) = \lim_{n \uparrow \infty} M^{d_n}(x,y). \tag{24}$$

The following result, whose proof is referred to [3] and [4], summarizes the above discussion.

Theorem 7 *For a strictly internal and reflexive function M, the function $d \mapsto M^d(x,y)$ defined on $D([0,1])$ is monotonically extended by (24) to the interval $[0,1]$. The extension $\delta \mapsto M^\delta(x,y)$ is a continuous function provided that M is a continuous mean. $\delta \mapsto M^\delta(x,y)$ is a monotonic function; increasing when $x < y$ and decreasing when $x > y$. Furthermore, M^δ is a continuous mean when $0 < \delta < 1$ and $M^0(x,y) = x, \ M^1(x,y) = y$.*

After Theorem 7, it turns out to be that $\mathcal{W} : \mathcal{GM}^0_S(I) \times [0,1] \to \mathcal{GM}^0_S(I)$ *defined by the dyadic iteration*

$$\mathcal{W}(M; w) = M^w$$

is a continuous and strictly monotonic w.p.. *Very noticeably, this* w.p. *is scale invariant; however, its monotonicity properties get worse if applied to non-strict means. For example, it is not difficult to see that, in the class* $\mathcal{GM}^0(I)$ *of continuous means, the general solution to the functional equation*

$$M(x, M(x, y)) = M(x, y),\ \ x, y \in [0, 1], \tag{25}$$

which is no other than the equation $M^{1/4}(x, y) = M(x, y)$*, includes the means expressed by*

$$M(x, y) = \left\{ \begin{array}{ll} \phi(x, y), & 0 \le x \le a(y) \\ x, & a(y) \le x \le b(y) \\ \psi(x, y), & b(y) \le x \le 1 \end{array} \right. , \tag{26}$$

where the functions ϕ, ψ, a *and* b *are continuous in their respective domains and satisfy the inequalities*

$$\begin{array}{rcl} 0 & \le & a(y) \le \phi(x, y) \le y,\ 0 \le x \le a(y),\ 0 \le y \le 1; \\ y & \le & \psi(x, y) \le b(y) \le 1,\ 0 \le x \le a(y),\ 0 \le y \le 1. \end{array}$$

Indeed, the general solution to (25) continues to be expressed by formula (26) provided that functions a *and* b *vary in the classes of upper and lower semicontinuous functions, respectively, and* M *is suitably defined at the points* (x, y_0) *with* y_0 *a discontinuity point of* a *or* b*. In any case, the singular set* $\mathcal{K}_{1/4} = \{K : K^{1/4} = K\}$ *turns out to be a family of means depending on arbitrary functions.*

A density argument (see [4], [3]) enable us to express A^w *in the form*

$$A^w(x, y) = (1 - w)x + wy = L_w(x, y); \tag{27}$$

in other words, the weighted arithmetic means L_w *arise as images of the arithmetic mean under the dyadic* w.p.. *Thus, the scale invariance of the* w.p. *gives*

$$A^w_f(x, y) = f^{-1}((1 - w)f(x) + wf(y)), \tag{28}$$

where A_f *is the quasiarithmetic mean (7).*

As shown in the next section, a direct generalization of the Aumann equation can be obtained by employing the dyadic w.p..

Example 8 *(A discontinuous* w.p.*) Perhaps, the simplest way of satisfying conditions* ($\Lambda 1$), ($\Lambda 2$) *is represented by the* w.p. $\Lambda_\infty : \mathcal{F}(I) \times [0,1] \to \mathcal{F}(I)$ *given by*

$$\Lambda_\infty(\lambda; w) = \left\{ \begin{array}{ll} 0, & w \in [0, 1/2) \\ \lambda, & w = 1/2 \\ 1, & w \in (1/2, 1] \end{array} \right. . \tag{29}$$

Now consider a w.p. Λ *such that:* ***i)*** Λ *is monotonic;* ***ii)*** *it is sequentially mean-continuous (in the topology of the pointwise convergence) and,* ***iii)*** *it has a trivial singular set; i.e.* $\mathcal{K}(\mathcal{W}) = \{X, Y\}$*. These conditions are satisfied by the weighting procedures presented in the previous examples when they are defined on proper classes of means. Let us compute the value of a limit which, in the superscript notation, is expressed by the "infinite tower"* $\left(\left(\left((M^w)^w\right)^{\cdots}\right)^w\right)^{\cdots}$*. To this end, define the iterates* $\Lambda_n,\ n \in \mathbb{N}$*, by*

$$\Lambda_n(\lambda; w) = \begin{cases} \Lambda(\lambda; w), & n = 1 \\ \\ \Lambda(\Lambda_{n-1}(\lambda; w); w), & n > 1 \end{cases}, \ w \in [0, 1].$$

Let us show that the discontinuous w.p. *(29) arises as the limit value of these iterates when* $n \uparrow \infty$*; i.e.,*

$$\lim_{n \uparrow \infty} \Lambda_n(\lambda; w) = \Lambda_\infty(\lambda; w),\ \lambda \in \mathcal{F}(I).$$

In fact, taking into account that $\Lambda_n(\lambda; 0) = 0,\ \Lambda_n(\lambda; 1/2) = \lambda$ *and* $\Lambda_n(\lambda; 1) = 1$ *for* $n \in \mathbb{N}$, $\lambda \in \mathcal{F}(I)$*, the equality (29) is true for the weights* $w = 0,\ 1/2,\ 1$*. Now assume that* $w \in (0, 1/2)$*. After Proposition 3, the inequalities* $\Lambda_n(\lambda; w) \leq \Lambda_{n-1}(\lambda; w),\ n \in \mathbb{N}$*, hold by the monotonicity of the* w.p.*; thence, the bounded below sequence* $\{\Lambda_n(\lambda; w)\}$ *is decreasing and there exists the limit*

$$\lim_{n \uparrow \infty} \Lambda_n(\lambda; w) = l_0 \geq 0.$$

By the sequential continuity of the w.p.*, this limit satisfies* $\Lambda(l_0; w) = l_0$ *and therefore, in view of the triviality of the singular set* $\mathcal{K}(\mathcal{W})$*, it turns out to be* $l_0 = 0$*. In the case* $w \in (1/2, 1)$*, it can be similarly proved that the increasing sequence* $\{\Lambda_n(\lambda; w)\}$ *converges to* $l_1 = 1$*.*

3 Aumann equation for weighting procedures

Let M^w be a *w.p.* defined on a class of continuous means $\mathcal{M}(I) \subseteq \mathcal{GM}^0(I)$ and let $p,\ q \in [0, 1]$. The functional equation

$$M(M^p(x, y), M^q(x, y)) = M(x, y),\ x, y \in I, \tag{30}$$

will be named an *Aumann generalized equation for the weighting procedure* M^w. To stress the role of the particular weights, the equation (30) will be also named *Aumann generalized equation with weights* p, q.

The Aumann equation (6) is the Aumann generalized equation corresponding to the dyadic *w.p.* with weights $p = 1/4$, $q = 3/4$. In fact, (22)-(23) yield

$$M^{1/4}(x, y) = M(x, M(x, y)),\ M^{3/4}(x, y) = M(M(x, y), y).$$

Observe that the Aumann generalized equation for the dyadic *w.p.* is an equation not expressible in finite terms when p or q are not dyadic numbers.

Equation (30) becomes vacuous for particular values of the weights p, q. In the cases in which $p = 0$, $q = 1$ or $p = 1/2 = q$, (30) reduces to the identity $M(x, y) = M(x, y)$; while the equation of symmetric means $M(y, x) = M(x, y)$ is obtained when $p = 1$, $q = 0$. Restrictions of a different kind arise when the *w.p.* is monotonic, as shown by the following:

Lemma 9 *Assume that the weighting M^w is strictly monotonic, then, equation (30) has no solution in $\mathcal{M}_S(I)$ when p, $q \in (0, 1/2)$ or p, $q \in (1/2, 1)$.*

Proof. Assume that equation (30) with weights p, $q \in [0, 1]$, $p < q$, is solved by a mean M. Setting $x < y$, both the strict internality of M and the monotonicity of the *w.p.* yield

$$M^p(x, y) < M(M^p(x, y), M^q(x, y)) < M^q(x, y),$$

or, in view M solves equation (30),

$$M^p(x, y) < M(x, y) = M^{1/2}(x, y) < M^q(x, y).$$

Hence, the inequalities

$$p < \frac{1}{2} < q.$$

follows from the strict monotonicity of the *w.p.*. ■

If the solution M were not strict, a similar reasoning proves that $p \leq 1/2 \leq q$, being strict one of the inequality signs at least.

Now, pay attention to the arguments of the mean M in the left member of equation (30). For a continuous mean M and $0 < p < 1/2 < q < 1$, a transformation $T_M : I^2 \to I^2$ is defined as follows:

$$T_M(x, y) = (M^p(x, y), M^q(x, y)), \ (x, y) \in I^2. \tag{31}$$

In view of M^p and M^q are means, T_M turns out to be a *mean-type map* (cf. [11]); furthermore, since M^p and M^q are strict means provided that the *w.p.* M^w is strictly monotonic, it can be proved that the iterations T_M^n (pointwise) converge, when $n \uparrow \infty$, to another mean-type map of the form $(K(x, y), K(x, y))$, where K is a continuous (strict) mean satisfying

$$K(M^p(x, y), M^q(x, y)) = K(x, y); \tag{32}$$

i.e., K is invariant under the transformation T_M (see [11] and the references cited therein). Using these facts and terminology, it can be affirmed that Aumann generalized equation is satisfied by a (strict) mean M if and only if M is T_M-invariant. A formal statement of this result is written for future reference.

Proposition 10 *Let M^w be a strictly monotonic and continuous* w.p.*.The Aumann generalized equation (30) is solved by a continuous mean M if and only if*

$$\lim_{n\uparrow\infty} T_M^n(x, y) = (M(x, y), M(x, y)), \ x, y \in I,$$

where T_M is the mean-type map defined by (31). Furthermore, the solution M is a strict mean.

The invariance properties of a *w.p.* are generally inherited by the solutions to the Aumann generalized equation (30). For instance, by the scale invariance of the dyadic *w.p.*, if a continuous mean M is a solution to the Aumann equation (6), then, the conjugated mean $f^{-1} \circ M \circ (f \times f)$ is a solution as well for every continuous and strictly monotonic function $f : I \to I$ (recall that $f \times f$ denote the cartesian product $(f \times f)(x, y) = (f(x), f(y)),\ x, y \in I)$. Of course, the same is true for the Aumann generalized equation for the dyadic *w.p.*. In this case, if the equality $p + q = 1$ is satisfied by the weights p, q, then, a simple computation shows that the entire family of quasiarithmetic means is contained in the general solution to the equation.

4 Simple weighting procedures

Suppose that M^w is a continuous (weight-continuous and sequentially mean-continuous) and strictly monotonic *w.p.* defined on a class of continuous means $\mathcal{M}(I) \subseteq \mathcal{GM}^0(I)$. The weight representation Λ of the *w.p.* M^w will be indistinctly used throughout this section. For a pair of weights (p, q) satisfying $0 < p < 1/2 < q < 1$, the *w.p.* Λ is said to be (p, q)-*simple* when the quotient $(\lambda - \Lambda(\lambda; p)) \,/\, (\Lambda(\lambda; q) - \Lambda(\lambda; p))$ depends on (x, y) only through the functional weight λ or; expressed in symbols, when there exists a function $F_{p,q}$ such that

$$\frac{\lambda - \Lambda(\lambda; p)}{\Lambda(\lambda; q) - \Lambda(\lambda; p)} = F_{p,q}(\lambda),\ \lambda \in \mathcal{F}(I) \setminus \{0, 1\}. \tag{33}$$

When a *w.p.* is (p, q)-simple for every pair (p, q), $0 < p < 1/2 < q < 1$, then it is said to be a *simple w.p.*.

It is emphasized that the function $F_{p,q}$ is not generally defined at the extremal weights $\lambda = 0$ or $\lambda = 1$, where the quotient of the left hand side of (33) may not exist. However, some examples are shown below of weighting procedures in which $F_{p,q}$ is defined and continuous on $[0, 1]$. Moreover, taking into account that $0 < F_{p,q} < 1,\ \lambda \in \mathcal{F}(I) \setminus \{0, 1\}$, by the strict monotonicity of Λ; it turns out to be that $F_{p,q}$ is a self-map of the interval $(0, 1)$ which, in view of the sequential mean continuity of M^w, is continuous. In regards to the dependence of $F_{p,q}$ on the parameters p, q, note that $(p, q) \mapsto F_{p,q}(t)$ is continuous on $(0, 1/2) \times (1/2, 1)$ and that $p \mapsto F_{p,q}(t)$ and $q \mapsto F_{p,q}(t)$ are (strictly) decreasing functions.

If λ is the functional weight of the mean M, an easy computation based on (19) yields

$$\frac{\lambda - \Lambda(\lambda; p)}{\Lambda(\lambda; q) - \Lambda(\lambda; p)} = \frac{M - M^p}{M^q - M^p},\ x \neq y. \tag{34}$$

Proposition 11 i) *If the* w.p. $\mathcal{W}$ *is* (p, q)*-simple on a class of means which includes the weighted arithmetic means* $L_t(x, y) = (1-t)x + ty,\ 0 < t < 1,$ *then,*

$$F_{p,q}(t) = \frac{t - \Lambda(t; p)}{\Lambda(t; q) - \Lambda(t; p)},\ t \in (0, 1),$$

where $\Lambda(t;w)$ stands for the weight representation of $\mathcal{W}(L_t;w)$.

ii) *Every functional* w.p. *is simple.*

Proof. i) is immediate from the definition of (p,q)-simple *w.p.*. If Λ denotes the weight representation of a functional *w. p.* associated to the function Φ and $0<p<1/2<q<1$, then,

$$\frac{\lambda-\Lambda(\lambda;p)}{\Lambda(\lambda;q)-\Lambda(\lambda;p)}=\frac{\lambda-\Phi(\lambda;p)}{\Phi(\lambda;q)-(\lambda;p)}$$

is a function of $\lambda\in\mathcal{F}(I)\setminus\{0,1\}$ and therefore, Λ is (p,q)-simple. Since (p,q) was arbitrary, this proves **ii)**. ∎

As an application of this result, let us compute the map $F_{p,q}$ for the functional weighting procedures of the Examples 4 and 5. In the first place, for the sectionally linear *w.p.*,

$$\begin{aligned}\frac{t-\Phi(t;p)}{\Phi(t;q)-\Phi(t;p)} &= \frac{t-2pt}{t-(1-2q)(1-t)-2pt}\\ &= \frac{(1-2p)t}{2\,(1-q-p)\,t+(2q-1)};\end{aligned}$$

so that, the corresponding function $F_{p,q}$ is the homographic map given by

$$F_{p,q}(t)=\frac{(1-2p)t}{2\,(1-q-p)\,t+(2q-1)}. \tag{35}$$

In the case of the homographic *w.p.*, $F_{p,q}$ is the following affine function:

$$\begin{aligned}F_{p,q}(t) &= \frac{t-\frac{pt}{(1-p)(1-t)+pt}}{\frac{qt}{(1-q)(1-t)+qt}-\frac{pt}{(1-p)(1-t)+pt}}\\ &= \frac{(1-2p)\,(2q-1)}{q-p}t+\frac{1-2p}{q-p}\,(1-q)\,. \end{aligned}\tag{36}$$

Observe that in both cases the function $F_{p,q}$ is also defined at the extremal weights $t=0,\,1$.

Whichever be the pair (p,q), the dyadic *w.p.* is not a (p,q)-simple *w.p.*. To prove this assertion let us consider the Aumann equation (6) with $I=\mathbb{R}^+$ as domain of its variables. For the harmonic mean $H(x,y)=2xy/(x+y)$, whose functional weight λ_H is given by

$$\lambda_H(x,y)=\frac{x}{x+y},$$

the equality

$$\Lambda(\lambda_H;w)=\frac{wx}{wx+(1-w)y}$$

is derived from (28) and (15). Thus, in computing the quotient

$$\frac{\lambda_H - \Lambda(\lambda_H;p)}{\Lambda(\lambda_H;q) - \Lambda(\lambda_H;p)},$$

the affine function (36) takes part again; i.e.,

$$\frac{\lambda_H - \Lambda(\lambda_H;p)}{\Lambda(\lambda_H;q) - \Lambda(\lambda_H;p)} = F_{p,q}(\lambda_H),$$

where $F_{p,q}$ is given by (36). An analogous computation, this time for the geometric mean $G(x,y) = xy$, gives

$$\lambda_G(x,y) = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}}, \ \Lambda(\lambda_H;w) = \frac{x^{1-w}y^w - x}{y-x};$$

and therefore,

$$\begin{aligned}\frac{\lambda_H - \Lambda(\lambda_H;p)}{\Lambda(\lambda_H;q) - \Lambda(\lambda_H;p)} &= \frac{(y/x)^{\frac{1}{2}-p} - 1}{(y/x)^{q-p} - 1} \\ &= F^*_{p,q}(\lambda_G),\end{aligned}$$

where

$$F^*_{p,q}(t) = \frac{(1-t)^{1-2p} - t^{1-2p}}{(1-t)^{2(q-p)} - t^{2(q-p)}}.$$

Since $F^*_{p,q}(0) = 1 = F^*_{p,q}(1)$, if $F^*_{p,q}(t)$ was an affine function, then, it should be $F^*_{p,q}(t) \equiv 1$; hence, $F^*_{p,q} \neq F_{p,q}$ and the dyadic *w.p.* is not (p,q)-simple for any pair (p,q), as affirmed. The same computations prove the assertion for a domain of the variables of the form $I = |a,b| \subseteq \mathbb{R}^+$, while the case $I = \mathbb{R}$ is managed by a previous introduction of a logarithmic change of coordinates.

The study of simple weighting procedures in connection with the Aumann generalized equation is justified by the following:

Proposition 12 *Let Λ be a (p,q)-simple* w.p.. *A mean M with functional weight λ solves the Aumann generalized equation (30) if and only if the equality*

$$(\lambda \circ T_M)(x,y) = F_{p,q}(\lambda(x,y)), \tag{37}$$

holds for every x, $y \in I$, $x \neq y$.

In the equality (37), T_M is the mean-type map defined by (31) and $F_{p,q}$ is the function given by (33).

Proof. From (16), (19) and (31), it is derived

$$\begin{aligned}M(M^p(x,y), M^q(x,y)) &= (1-(\lambda \circ T_M)(x,y))M^p(x,y) + (\lambda \circ T_M)(x,y)M^q(x,y) \\ &= M^p(x,y) + (M^q(x,y) - M^p(x,y))(\lambda \circ T_M)(x,y);\end{aligned}$$

thus, the equation (30) is satisfied by M if and only if

$$\frac{M(x,y)-M^p(x,y)}{M^q(x,y)-M^p(x,y)}=(\lambda\circ T_M)(x,y),\ x\neq y;$$

whence, after equality (15), equality (37) is obtained. ■

A first consequence of Proposition 12 is stated in the following:

Corollary 13 *Let Λ be a (p,q)-simple* w.p. *defined on a class of means containing the weighted arithmetic means $\{L_\alpha : 0<\alpha<1\}$. Then, L_α is a solution of the Aumann generalized equation (30) if and only if α is fixed point of the map $F_{p,q}$.*

Proof. Since the functional weight of the weighted arithmetic mean L_α is $\lambda(x,y)\equiv\alpha$, the equality (37) of Proposition 12 take the form

$$\alpha=F_{p,q}(\alpha)$$

and it is satisfied if and only if α is a fixed point of $F_{p,q}$. ■

Observe that the corollary is also true for the extremal weights $\alpha=0, 1$ provided that $F_{p,q}$ is defined on $[0,1]$.

The equality (37) can be iterated; namely,

$$(\lambda\circ T_M^2)(x,y)=(\lambda\circ T_M)(T_M(x,y))=F_{p,q}(\lambda(T_M(x,y)))=F_{p,q}^2(\lambda(x,y)),$$

and, after an inductive reasoning, for every $n\in\mathbb{N}$,

$$(\lambda\circ T_M^n)(x,y)=F_{p,q}^n(\lambda(x,y)),\ x\neq y. \tag{38}$$

It is clear from (38) that the solutions of the Aumann generalized equation closely depend on the asymptotic behavior of the map $F_{p,q}$.

In order to pass to the limit in (37) and (38), in the remaining of this section let us frequently consider means M which are regular enough so as to their corresponding functional weights λ_M are continuously extendable to the whole I^2. Under this assumption, the images $\lambda(I^2)$ and $\lambda(\Delta(I^2))$ both are intervals; clearly $\lambda(\Delta(I^2))\subseteq\lambda(I^2)\subseteq[0,1]$.

Theorem 14 *Let M be a mean with functional weight λ and suppose that M solves the Aumann generalized equation (30) for a (p,q)-simple* w.p. *M^w. Then,*

i) *the restriction $F_{p,q}|_{\lambda(I^2)}=f_{p,q}$ is a self map of $\lambda(I^2)$; i.e., $f_{p,q}:\lambda(I^2)\to\lambda(I^2)$;*

moreover, assuming that λ is (extendable to a) continuous on I^2,

ii) *for every $\alpha\in\lambda(I^2)$, $\alpha\in\mathrm{Fix}(f_{p,q})$ or $f_{p,q}^n(\alpha)$ converges, when $n\uparrow\infty$, to a fixed point of $f_{p,q}$; and*

iii) $\mathrm{Fix}(f_{p,q})=\lambda(\Delta(I^2))$*.*

Loosely speaking, the theorem says that the asymptotic behavior of the map $F_{p,q}$ is extremely simple on the image $\lambda(I^2)$ of a regular solution of the Aumann generalized equation.

Proof. If the Aumann generalized equation is solved by a mean M with functional weight λ and $\alpha \in \lambda(I^2)$, then, there exists $(x_0, y_0) \in I^2$ such that $\alpha = \lambda(x_0, y_0)$, and Proposition 12 shows that $F_{p,q}(\alpha) = F_{p,q}(\lambda(x_0, y_0)) = \lambda(T_M(x_0, y_0)) \in \lambda(I^2)$. This proves the assertion **i)**. Now, assuming that λ is continuous on I^2 and that $\alpha \notin \mathrm{Fix}(f_{p,q})$, it is deduced from (38) that

$$f_{p,q}^n(\alpha) = \lambda(T_M^n(x_0, y_0)),\ n \in \mathbb{N}.$$

The sequence of the second of this equality is convergent by Proposition 10 and the continuity of λ; thus, the sequence $\{f_{p,q}^n(\alpha)\}$ is convergent and clearly converges to a fixed point of $f_{p,q}$, as affirmed by **ii)**. To prove **iii)**, let us take limits in the equalities (37) and (38) as follows: first, for a given $x \in I$, make y approach x in (37) to obtain

$$\lambda(x, x) = \lambda(T_M(x, x)) = \lim_{y \to x} (\lambda \circ T_M)(x, y) = \lim_{y \to x} f_{p,q}(\lambda(x, y)) = f_{p,q}(\lambda(x, x)),$$

whence, in view of the arbitrariness of x, it is deduced that $\lambda(\Delta(I^2)) \subseteq \mathrm{Fix}(f_{p,q})$. The opposite inclusion follows from observing that if $\alpha = \lambda(x, y) \in \mathrm{Fix}(f_{p,q})$, then, making $n \uparrow \infty$ in (38) yields

$$\lambda(M(x, y), M(x, y)) = \lim_{n \uparrow \infty} (\lambda \circ T_M^n)(x, y) = \lim_{n \uparrow \infty} f_{p,q}^n(\lambda(x, y)) = \alpha;$$

i.e., $\alpha \in \lambda(\Delta(I^2))$. ∎

The assertion **iii)** of the previous theorem implies that $\mathrm{Fix}(f_{p,q}) = \lambda(\Delta(I^2))$ is a closed interval contained in $\lambda(I^2)$. Thus, writing $\lambda(I^2) = |\alpha, \beta|$, $\lambda(\Delta(I^2)) = [\alpha_0, \beta_0]$, $L = |\alpha, \alpha_0)$ and $R = (\beta_0, \beta|$, the interval $\lambda(I^2)$ is expressed as the disjoint union

$$\lambda(I^2) = L \cup \lambda(\Delta(I^2)) \cup R,$$

and the following corollary of Theorem 14 can be stated.

Corollary 15 *Under the hypotheses of Theorem 14, the inequalities*

$$f_{p,q}(t) \begin{cases} > t, & t \in L \\ = t, & t \in \lambda(\Delta(I^2)) \\ < t, & t \in R \end{cases} \tag{39}$$

are satisfied by the restriction $F_{p,q}|_{\lambda(I^2)} = f_{p,q}$.

Proof. If $L \neq \varnothing$, then, $f_{p,q}(t) \lesseqgtr t,\ t \in L$, by Theorem 14, **iii)**. Assuming that $f_{p,q}(t) < t,\ t \in L$, the sequence $\{f_{p,q}^n(t)\}$ would decrease to a fixed point t_0 of $f_{p,q}$ such that $t_0 < \alpha_0$, an absurdity. Thus, $f_{p,q}(t) > t,\ t \in L$, as affirmed. The proof of the inequality for $t \in R$ is similar provided that $R \neq \varnothing$. ∎

As shown at the end of the next section, the inequalities in (39) are closely related to the stability properties of the fixed points α_0, β_0.

Now, suppose that $F_{p,q}$ has a unique fixed point $\alpha \in (0,1)$; i.e., $\mathrm{Fix}(F_{p,q}) = \{\alpha\}$. If M is a mean solving the Aumann generalized equation (30) and its functional weight λ is continuous on I^2, then, Theorem 14 **ii)-iii)** yields $\lambda(x,x) = \alpha$ and

$$\lim_{n\uparrow\infty} F^n_{p,q}(\lambda(x,y)) = \alpha,\ \ (x,y) \in I^2. \tag{40}$$

The equality (40) expresses the fact that, under the assumptions of Theorem 14, α must be an *attractive* fixed point of $f_{p,q}$. Thus, it was proved the following:

Corollary 16 *Assuming that there exists a solution M to the Aumann generalized equation (30) whose functional weight λ is continuous on I^2, if $F_{p,q}$ has a unique fixed point $\alpha \in (0,1)$, then, α must be an attractive fixed point of the map $f_{p,q}$.*

To end this section, let us remark that the regularity condition on the mean M can be relaxed in Theorem 14 and its corollaries. The following observation serves as a clue: given a pair of weights $p, q,\ p < q$, the inequality

$$M^p(x,y) < M^q(x,y)$$

holds among the coordinate functions M^p, M^q of the map T_M when $x < y$ by the strict monotonicity of the *w.p.*. Clearly, a similar inequality holds among the first and second coordinate functions of the iterated map $T^n_M(x,y),\ n \in \mathbb{N}$; in other words, when $x < y$, the point $T^n_M(x,y) \in \Delta^+(I^2) = \{(x,y) \in I^2 : x < y\},\ n \in \mathbb{N}$. In this way, the passage to the limit in the equalities (37) and (38) take, respectively, the forms

$$\lim_{y\downarrow x} \lambda(x,y) = \lim_{y\downarrow x}(\lambda \circ T_M)(x,y) = \lim_{y\downarrow x} f_{p,q}(\lambda(x,y)) = f_{p,q}(\lim_{y\downarrow x} \lambda(x,y)) \tag{41}$$

and

$$\lim_{z\downarrow M(x,y)} \lambda(M(x,y),z) = \lim_{n\uparrow\infty}(\lambda \circ T^n_M)(x,y) = \lim_{n\uparrow\infty} f^n_{p,q}(\lambda(x,y)) \tag{42}$$

provided that there exist the limits involved in their terms. Since $T^n_M(x,y) \in \Delta^-(I^2) = \{(x,y) \in I^2 : x > y\},\ n \in \mathbb{N}$, equalities like (41) and (42) with $\lim_{y\uparrow x} \lambda(x,y)$ replacing $\lim_{y\downarrow x} \lambda(x,y)$ are obtained when $x > y$. In this way, the restriction $F_{p,q}|_{\lambda(\Delta^+(I^2))} = f^+_{p,q}$ turns out to be a self map of $\lambda(\Delta^+(I^2))$; i.e., $f^+_{p,q} : \lambda(\Delta^+(I^2)) \to \lambda(\Delta^+(I^2))$. Moreover, assuming that $x \mapsto \lim_{y\downarrow x} \lambda(x,y) = \lambda^+(x)$ exists and is continuous for every $x \in I$, it can be proved that $\mathrm{Fix}(f^+_{p,q}) = \lambda^+(I)$

Summarizing, these observations enable us to extend both Theorem 14 and its corollaries to the cases in which the limits $\lim_{y\uparrow x} \lambda(x,y)$ and $\lim_{y\downarrow x} \lambda(x,y)$ may differ one each other like, for instance, in the means of the form

$$M(x,y) = (1-\alpha)\min\{x,y\} + \alpha \max\{x,y\},\ \ (\alpha \in [0,1] \setminus \{1/2\}).$$

5 The Aumann generalized equation for simple weighting procedures

In this final section, the theory developed in Section 4 is applied to solve some instances of the Aumann generalized equation for simple weighting procedures. Solutions belonging to the classes $\mathcal{SM}^1(I)$ or $\mathcal{SM}_S^1(I)$ are sough for the equation with the purpose of simplifying the statements, which will be eventually exemplified by the weighting procedures introduced at the end of Section 2. Let us begin with the following:

Theorem 17 *Let M^w be a (p,q)-simple* w.p. *defined on $\mathcal{GM}^0(I)$ such that the map $F_{p,q}$ reduces to the identity; i.e., $F_{p,q}(t) = t,\ t \in (0,1)$. Then, the arithmetic mean $A(x,y) = (x+y)/2$ is the general solution in the class $\mathcal{SM}^1(I)$ to the Aumann generalized equation (30) for the* w.p. *M^w.*

Proof. That the arithmetic mean $A = L_{1/2}$ solves the equation (30) is a consequence of Corollary 13 and the hypothesis on $F_{p,q}$. Conversely, suppose that a symmetric and continuous mean M with functional weight λ is a solution to (30). By Proposition 1, **iv)**, the symmetry of M is equivalent to the equality

$$\lambda(x,y) + \lambda(y,x) = 1,\ \ x \neq y. \tag{43}$$

Moreover, Proposition 12 and the hypothesis on $F_{p,q}$ yield

$$(\lambda \circ T_M)(x,y) = F_{p,q}(\lambda(x,y)) = \lambda(x,y),\ \ x \neq y;$$

whence, for every $n \in \mathbb{N}$,

$$(\lambda \circ T_M^n)(x,y) = \lambda(x,y),\ \ x \neq y. \tag{44}$$

Since M is a $\mathcal{C}^1$ mean, its functional weight λ can be continuously extended to I^2 by Proposition 1, **ii),** and therefore, a passage to the limit $n \uparrow \infty$ in (44) yields, after Proposition 10,

$$\lambda(M(x,y), M(x,y)) = \lambda(x,y),\ \ x \neq y.$$

From this equality and the symmetry of M, it is deduced that the functional weight λ is also a symmetric function; i.e.,

$$\lambda(x,y) = \lambda(y,x). \tag{45}$$

Finally, equalities (43) and (45) yield

$$\lambda(x,y) = \frac{1}{2};$$

i.e., $M = A$. ■

A result similar to Theorem 17 can be established in the non symmetric case if the symmetry condition is replaced by $\lambda(x,x) \equiv \alpha \in [0,1]$. Observe that the weighted arithmetic mean L_a becomes the general solution in this case.

A slight modification of Theorem 17 applies to the sectionally linear *w.p.*. In fact, it is easily derived from (35) that $F_{p,q}(0) = 0,\ F_{p,q}(1) = 1$ and

$$F_{p,q}(t) \begin{cases} > t, & \text{if } p+q<1 \\ = t, & \text{if } p+q=1 \\ < t, & \text{if } p+q>1 \end{cases}, \ t \in (0,1); \tag{46}$$

so that the following result can be stated.

Proposition 18 *The Aumann generalized equation (30) with weights p, q $(0 < p < 1/2 < q < 1)$ for the sectionally linear* w.p. *has the arithmetic mean A as its general solution in the class $\mathcal{SM}^1_S(I)$ provided that $p+q=1$. There are no solutions to the equation belonging to the class $\mathcal{SM}^1_S(I)$ when $p+q \neq 1$.*

Proof. As established in Example 4, the sectionally linear *w.p.* is strictly monotonic on the class $\mathcal{GM}^0_S(I)$. Now, if $p+q=1$, the proof follows from (46) and, mutatis mutandis, from the proof of Theorem 17. When $p+q<1$, $\lim_{n\uparrow\infty} F^n_{p,q}(t) = 1$, while $\lim_{n\uparrow\infty} F^n_{p,q}(t) = 0$ in the case in which $p+q>1$; therefore, if a mean $M \in \mathcal{SM}^1_S(I)$ with functional weight λ solves equation (30), then

$$\frac{1}{2} = \lambda(M(x,y), M(x,y)) = \lim_{n\uparrow\infty} F^n_{p,q}(\lambda(x,y)) = \begin{cases} 0 \\ 1 \end{cases},$$

an absurdity. Thence, there are no solutions belonging to the class $\mathcal{SM}^1_S(I)$ to equation (30) in these cases. ■

Other solutions to equation (30) do appear for the sectionally linear *w.p.* when they are sough in more ample classes of means. For instance, $M(x,y) = \max\{x,y\} \in \mathcal{SM}^0(I)$ solves equation (30) whichever be $0<p<1/2<q<1$.

Now, pay attention to the function $F_{p,q}$ corresponding to the homographic *w.p.* as given by (36). In view of $0<p<1/2<q<1$, this affine function satisfies

$$0 < F_{p,q}(0) = \frac{1-2p}{q-p}(1-q) < \ q\frac{1-2p}{q-p} = F_{p,q}(1) < 1;$$

and therefore, $F_{p,q}$ has a unique attractive fixed point in $[0,1]$ given by

$$\alpha = \frac{(1-2p)(1-q)}{q-p-(1-2p)(2q-1)}. \tag{47}$$

Observe that $\alpha = 1/2$ if and only if $p+q=1$. All these facts are employed in the proof of the following result.

Proposition 19 *The Aumann generalized equation (30) with weights p, q $(0 < p < 1/2 < q < 1)$ for the homographic* w.p. *admits solution belonging to $\mathcal{SM}^1_S(I)$ if and only if $p+q=1$. In any other case, the equality $\lambda(x,x) = \alpha,\ x \in I$, with α given by (47) is satisfied by every $\mathcal{C}^1$ solution to the equation.*

Proof. As seen in Example 5, the homographic *w.p.* is strictly monotonic on the class $\mathcal{GM}_S^0(I)$. If $p+q=1$, then $\alpha = 1/2$ is the unique fixed point of $F_{p,q}$ so that, by Corollary 13, the arithmetic mean A solves the equation (30). Conversely, if the equation (30) admits a solution in $\mathcal{SM}_S^1(I)$, then, by Theorem 14, **ii)**,

$$\frac{1}{2} = \lambda(M(x,y), M(x,y)) = \lim_{n\uparrow\infty} F_{p,q}^n(\lambda(x,y)) = \alpha,$$

hence $p+q=1$.

When $p+q \neq 1$, by Theorem 14, **iii)**, the equality

$$\lambda(x,x) = \alpha, \ x \in I,$$

holds for the functional weight λ of a $\mathcal{C}^1$ solution to the equation. ■

In the example furnished by the homographic *w.p.*, the equation (30) with weights p, q satisfying $p+q=1$ is solved by other means different from the arithmetic one; v.g., the harmonic mean $H(x,y) = 2xy/(x+y)$. Thus, when $F_{p,q}$ has a unique fixed point, the general solution in the class $\mathcal{SM}^1(I)$ to the Aumann generalized equation does not typically reduces to the arithmetic mean. Nevertheless, the uniqueness of solution which is characteristic of the case $F_{p,q}(t) \equiv t$ can be recovered by imposing additional restrictions on the dynamic of the map $F_{p,q}$. As a matter of fact, suppose that $F_{p,q}$ has a unique fixed point $\alpha \in (0,1)$ which is *repulsive*. Thus, if M were a $\mathcal{C}^1$ solution to the Aumann generalized equation with $\lambda \neq \alpha$, then, given a small enough $\delta > 0$, there would be a certain $(x_0, y_0) \in I^2$ such that $0 < |\lambda(x_0,y_0) - \alpha| < \delta$. It follows that the sequence $\{f_{p,q}^n(\lambda(x_0,y_0))\}$ would not converge to α when $n \uparrow \infty$, which contradicts Corollary 16 of the previous section. This observation proves the following:

Theorem 20 *Let M^w be a (p,q)-simple* w.p. *defined on $\mathcal{GM}^0(I)$ such that the map $F_{p,q}$ has in $(0,1)$ a unique fixed point α; i.e.,* $\mathrm{Fix}(F_{p,q}) = \{\alpha\}$*. If α is a repulsive fixed point, then, the weighted arithmetic mean $L_a(x,y) = (1-\alpha)x + \alpha y$ is the general solution in the class $\mathcal{GM}^1(I)$ to the Aumann generalized equation (30) for the* w.p. M^w.

Proof. See the previous discussion. ■

The last result of this section is concerned with a version of the previous theorem for the remaining case in which $\mathrm{Fix}(F_{p,q})$ is an interval; concretely, $\mathrm{Fix}(F_{p,q}) = [\alpha_0, \beta_0] \subsetneq [0,1]$. If $\alpha_0 > 0$ and there exists $\delta > 0$ such that $\{F_{p,q}^n(t)\}$ is not convergent for any $t \in (\alpha_0 - \delta, \alpha_0)$, then, it is said that α_0 is *repulsive from the left*. The notion of repulsion (from the right) for the fixed point β_0 at the right end of the interval is analogously defined provided that $\beta_0 < 1$. For brevity, it will be said that α_0 or β_0 are repulsive fixed points. The situation should be compared with that one described by Corollary 15 of Section 4.

Theorem 21 *Let M^w be a (p,q)-simple* w.p. *defined on $\mathcal{GM}^0(I)$ such that the map $F_{p,q}$ satisfies* $\mathrm{Fix}(F_{p,q}) = [\alpha_0, \beta_0] \subsetneq [0,1]$*. Assuming that the end points α_0*

and β_0 are both repulsive, then, the inequalities

$$\min\{L_{\alpha_0}(x,y), L_{\beta_0}(x,y)\} \leq M(x,y) \leq \max\{L_{\alpha_0}(x,y), L_{\beta_0}(x,y)\},\ x,y \in I, \tag{48}$$

are satisfied by every solution $M \in \mathcal{GM}^1(I)$ to the Aumann generalized equation (30) for the w.p. *M^w.*

Proof. By Proposition 2, the inequalities (48) are equivalent to the inequalities

$$\alpha_0 \leq \lambda(x,y) \leq \beta_0,\ x,y \in I,$$

for the functional weight λ of a mean M. Thus, in view of Theorem 14, **iii)**, the theorem affirms that the equality $\lambda(I^2) = \lambda(\Delta(I^2))$ holds for every solution $M \in \mathcal{GM}^1(I)$. This last equality follows from an argument similar to that used to prove Theorem 20. ∎

Acknowledgements: Prof. Osvaldo Méndez from the University of Texas at El Paso kindly provide me English translations of the two articles by Aumann.

References

[1] G. Aumann, *Aufbau von Mittelwerten mehrerer Argumente I*, Math. Ann. **109**, (1933), 235-253.

[2] G. Aumann, *Aufbau von Mittelwerten mehrerer Argumente II (Analytische Mittelwerte)*, Math. Ann. **111**, (1935), 713-730.

[3] L. R. Berrone, *A dynamical characterization of quasilinear means*, Aequationes Math. **84**, Issue **1** (2012), 51-70.

[4] L. R. Berrone, A. L. Lombardi, *A note on equivalence of means*, Publ. Math. Debrecen **58**, Fasc. **1-2**, (2001), 49-56.

[5] L. R. Berrone, G. E. Sbérgamo, *La familia de bases de una media continua y la representación de las medias cuasiaritméticas*, Rev. de la Soc. Venezolana de Matemática, Vol. **XIX**, No. **1**, (2012), 3-18.

[6] L. R. Berrone, G. E. Sbérgamo, *Weighting general means by iteration*, (to appear).

[7] P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic Publishers, Dordrecht, 2010.

[8] P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordrecht, 1988.

[9] A. Horwitz, *Invariant means*, J. Math. Anal. Appl. **270**, (2002), 499-518.

[10] J. Lawson, Y. Lim, *A general framework for extending means to higher orders*, Colloq. Math. **113**, No. **2**, 191-221, (2008).

[11] J. Matkowski, *Iterations of mean-type mappings and invariant means*, Ann. Math. Siles **13**, 211-226, (1999).