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Connection coefficients of interval wavelets satisfying boundary conditions*

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Abstract. The computation of connection coefficients is an important issue in the wavelet numerical solution of partial differential equations. We study this problem for the orthonormal interval wavelets bases, satisfying homogeneous boundary conditions, introduced by Monasse and Perrier. We first obtain explicit expressions to compute the connection coefficients involving (derivatives of) scaling functions at the same level. Then we describe how to compute connection coefficients when we have (derivatives of) scaling functions and/or wavelets at different levels, using local refinement relations.

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1. Introduction

The applicability of wavelets in several different areas in pure and applied mathematics depend upon their ability to represent efficiently a wide class of functions and operators (for a treatment of the theory of wavelets see, e. g., [17, 23]). In particular, this fact is useful in the development of fast and adaptive algorithms for the numerical solution of partial differential equations (see, e. g., [12, 31]). These wavelets-based schemes can require the computation of integrals involving products of (derivatives of)

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scaling functions and/or wavelets. These integrals are usually known as *connection coefficients*.

The computation of connection coefficients was considered in several papers (see, e. g., [3, 5, 8, 14, 18, 19]). In particular, for wavelets that have no closed analytic representation and that are only determined by their refinement coefficients the quadrature rules to compute these integrals are in general expensive or not applicable. So, in this case one approach is to reduce the calculus of the connection coefficients to an algebraic eigenvector-eigenvalue problem.

Wavelets approaches for the numerical solution of partial differential equations in non trivial geometries is a challenger problem. These cases are studied using the unit interval $[0,1]$ as the starting point in domain decomposition approaches (see, e. g., [3, 6, 7, 11, 15]). The periodization of compactly supported wavelets on the real line is the simplest way to consider the multiresolution analysis of a function on an interval (see, e. g., [17, 23]). Nevertheless, if we need to consider boundary conditions, a more efficient approach is to use compactly supported wavelets on the real line, retaining all those scaling functions which support is contained in the interval, and adding linear combinations of those scaling functions which cross the endpoints of the interval (for some examples of this type of construction see, e. g., [1, 4, 9, 10, 20]).

In this article we compute exactly (up to round-off errors) the connection coefficients for the interval orthonormal scaling functions and wavelets, satisfying homogeneous boundary conditions, introduced by Monasse and Perrier in [20]. These wavelets result from an adaptation of the ideas of Auscher [2] to impose boundary conditions on the construction of orthonormal interval wavelets proposed by Cohen, Daubechies and Vial [10]. Monasse-Perrier wavelets have the same regularity and vanishing moments that Daubechies wavelets [16]. They have no closed analytic representation and they are only determined by their refinement coefficients.

The rest of the paper is organized as follows. In section 2, we review multiresolution analysis and wavelet bases of both $L^2(\mathbf{R})$ and $L^2([0,1])$. In particular, we consider the definition and properties of Monasse-Perrier wavelets [20]. In section 3. we compute the connection coefficients involving (derivatives of) Monasse-Perrier scaling functions at the same level. Then, we describe how to compute connection coefficients with (derivatives of) scaling functions and/or wavelets at different levels, using local refinement relations.

2. Orthonormal wavelet bases for $L^2([0,1])$

In this section we review some aspects about multiresolution analysis (MRA) and wavelets bases. We begin considering in subsection 2.1. MRA and wavelet bases of $L^2(\mathbf{R})$ [17]. In particular, we briefly discuss compactly supported wavelets constructed by Daubechies in [16]. In subsections 2.2. and 2.3., we consider the construction proposed by Monasse and Perrier in [20] for the spaces $L^2([0,\infty[)$ and $L^2([0,1])$, respectively.

2.1. MRA and wavelet bases of $L^2(\mathbf{R})$

A MRA of $L^2(\mathbf{R})$ is a sequence of closed subspaces $(V_j)_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R})$ such that

(i) $\{0\} = \bigcap_{j\in\mathbf{Z}} V_j \subset ... \subset V_{-1} \subset V_0 \subset V_1 \subset ... \subset \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$.

(ii) $f(x) \in V_0 \Leftrightarrow f(2^j x) \in V_j$.

(iii) $\exists g \in V_0$, such that $\{g(.-k); k \in \mathbf{Z}\}$ is a Riesz basis of V_0.

From g, it is possible to obtain a function ϕ, called the *scaling function*, such that $\{\phi(.-k); k\in\mathbf{Z}\}$ is an orthonormal basis of V_0. Since $\phi \in V_0 \subset V_1$, from (ii) there exists a sequence $\{h_k\}_{k\in\mathbf{Z}}$ such that

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k). \tag{1}$$

Equation (1) is known as the *dilation equation*, the *two-scale difference equation*, or the *refinement equation*. We shall refer to it by the later name. The collection of functions $\{2^{j/2}\phi(2^j.-k); k\in\mathbf{Z}\}$ is an orthonormal basis of V_j.

Associated with V_j is the space W_j defined as the orthogonal complement of V_j in V_{j+1}, i. e., W_j is the space that satisfies $V_{j+1} = V_j \oplus W_j$. We have $L^2(\mathbf{R}) = \bigoplus_j W_j$. A *wavelet* is a function ψ such that $\{\psi(.-k); k \in$

$\mathbf{Z}\}$ is an orthonormal basis of W_0. Therefore, the collection of functions $\{2^{j/2}\psi(2^j. - k); j, k \in \mathbf{Z}\}$ is an orthonormal basis of $L^2(\mathbf{R})$. The wavelet ψ satisfies an equation similar to the refinement equation for the scaling function ϕ,

$$\psi(x) = \sqrt{2}\sum_k g_k \phi(2x-k), \quad g_k = (-1)^k h_{-k+1}.$$

Now we present Daubechies's compactly supported wavelets (for more details we refer to [16]). In Daubechies's construction,$h_k = g_k = 0$ for $k < -N+1$ and for $k > N$, therefore the scaling and wavelet functions satisfy the equations

$$\phi(x) = \sqrt{2}\sum_{m=-N+1}^{N} h_m \phi(2x-m), \quad \psi(x) = \sqrt{2}\sum_{m=-N+1}^{N} g_m \phi(2x-m). \quad (2)$$

Both ϕ and ψ have support in $[-N+1, N]$, $\int_{-\infty}^{\infty}\phi(x)dx = 1$, ψ has N vanishing moments, i. e.,

$$\int_{-\infty}^{\infty}\psi(x)x^l dx = 0, l = 0, 1, ..., N-1, \quad (3)$$

and ϕ has N^{th}-order approximation, i. e.,

$$\frac{x^l}{l!} = \sum_{k=-\infty}^{\infty} P_l(k)\phi(x-k), l = 0, 1, ..., N-1, \quad (4)$$

with

$$P_l(k) := \int_{-\infty}^{\infty}\frac{x^l}{l!}\phi(x-k)dx = \sum_{s=0}^{l}\frac{C_{l-s}k^s}{s!}, \quad (5)$$

where $C_l := \int_{-\infty}^{\infty}\phi(x)\frac{x^l}{l!}dx$ are recursively given by

$$C_0 = 1 \; ; \; C_l = \frac{1}{2^l - 1}\sum_{n=1}^{l}\left(\frac{1}{\sqrt{2}}\sum_{m\in\mathbf{Z}} h_m \frac{m^n}{n!}\right) C_{l-n} \; (l > 0). \quad (6)$$

For sufficient large N, $\phi, \psi \in C^{\mu N}$ with $\mu \simeq 0.2$.

2.2. Scaling and wavelet functions of $L^2([0,\infty[)$

The edge scaling functions $\widetilde{\phi}_l$, $l = 0, ..., N-1$, defined by

$$\widetilde{\phi}_l(x) := \sum_{k=-N+1}^{N-1-\alpha} P_l(k)\phi(x-k)\chi_{[0,\infty[}(x), \tag{7}$$

for $\alpha \in \{0,1\}$, are linearly independent and are orthogonal to the functions $\phi(.-k)$ for $k \geq N-\alpha$.

Let

$$V_j^{[0,\infty[} := \overline{span\left\{\left(\widetilde{\phi}_l(2^j.)\right)_{l=0,...,N-1}, (\phi(2^j.-k))_{k\geq N-\alpha}\right\}}.$$

For $\Lambda \subset \{0, ..., N-1\}$ let $BC(\Lambda)$ be the space of functions $f \in L^2([0,\infty[)$ such that $f^\lambda(0) = 0$, $\lambda \in \Lambda$, and

$$V_j^{[0,\infty[}(\Lambda) := V_j^{[0,\infty[} \cap BC(\Lambda). \tag{8}$$

Then $\left\{\left(\widetilde{\phi}_l(2^j.)\right)_{l\notin\Lambda}, \left(\phi(2^j.-k)\right)_{k\geq N-\alpha}\right\}$ is a basis for $V_j^{[0,\infty[}(\Lambda)$.

Remark 2.1. If $\alpha = 0$ then the edge scaling functions are derivable at 0 since they are polynomial near 0. If $\alpha = 1$ the regularity of the edge scaling functions depends on N. So, in this case we must suppose that N is large enough to assure that they are derivable at 0.

Let b be the matrix with entries

$$b(i+1, j-N+\alpha+1) = \sqrt{2}\sum_{m=\lceil\frac{j-N}{2}\rceil}^{N-\alpha-1} P_i(m)h_{j-2m},$$

for $i = 0, ..., N-1$ and $j = N-\alpha, ..., 3N-2-2\alpha$, where $\lceil x\rceil$ ($\lfloor x\rfloor$) is the nearest integer greater (less) than x, and let D be the diagonal matrix with entries $D(i,j) = \delta_{i-j}2^{1-i}$, for $i, j = 1, ..., N$. The Gram matrix $G^{\widetilde{\phi}}$ of the edge scaling functions $\widetilde{\phi}_l$, $l = 0, ..., N-1$, is given by

$$2G^{\widetilde{\phi}} = DG^{\widetilde{\phi}}D + bb^t.$$

Remark 2.2. $G^{\widetilde{\phi}}$ can be obtained by dividing term by term the matrix bb^t by the matrix $2M_1 - M$ where M_1 and M are of order $N \times N$ with $M_1(i,j) = 1$ and $M(i,j) = 2^{2-i-j}$ for $i, j = 1, ..., N$.

Let $G^{\widetilde{\phi}}_{\Lambda}$ and D_{Λ} be the matrices obtained from $G^{\widetilde{\phi}}$ and D, respectively, by keeping only rows and columns of index not in $1+\Lambda$, and let b_{Λ} be the matrix obtained from b by keeping only rows of index not in $1+\Lambda$ Then $G^{\widetilde{\phi}}_{\Lambda}$ is the Gram matrix of $\widetilde{\phi}_l$, $l \notin \Lambda$. If $G^{\widetilde{\phi}}_{\Lambda} = R^{\widetilde{\phi}}_{\Lambda}(R^{\widetilde{\phi}}_{\Lambda})^*$ is a factorization of $G^{\widetilde{\phi}}_{\Lambda}$, then the orthonormal family $\widetilde{\widetilde{\phi}}_l$, $l \notin \Lambda$, defined by

$$\begin{pmatrix} \widetilde{\widetilde{\phi}}_0 \\ \vdots \\ \widetilde{\widetilde{\phi}}_{N-1} \end{pmatrix}_{l\notin\Lambda} = (R^{\widetilde{\phi}}_{\Lambda})^{-1} \begin{pmatrix} \widetilde{\phi}_0 \\ \vdots \\ \widetilde{\phi}_{N-1} \end{pmatrix}_{l\notin\Lambda}, \tag{9}$$

satisfies the refinement equation:

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{pmatrix} \widetilde{\widetilde{\phi}}_0(\frac{x}{2}) \\ \vdots \\ \widetilde{\widetilde{\phi}}_{N-1}(\frac{x}{2}) \end{pmatrix}_{l\notin\Lambda} &= H_0 \begin{pmatrix} \widetilde{\widetilde{\phi}}_0(x) \\ \vdots \\ \widetilde{\widetilde{\phi}}_{N-1}(x) \end{pmatrix}_{l\notin\Lambda} + \\ &\quad + h_0 \begin{pmatrix} \phi(x-N+\alpha) \\ \vdots \\ \phi(x-3N+2+2\alpha) \end{pmatrix}, \end{aligned} \tag{10}$$

where $H_0 = \frac{1}{\sqrt{2}}(R^{\widetilde{\phi}}_{\Lambda})^{-1} D_{\Lambda} R^{\widetilde{\phi}}_{\Lambda}$ and $h_0 = \frac{1}{\sqrt{2}}(R^{\widetilde{\phi}}_{\Lambda})^{-1} b_{\Lambda}$.

2.2.1. The wavelets

The edge wavelets $\widetilde{\psi}_l$, $l = 0, ..., N-1$, are defined by

$$\widetilde{\psi}_l(x) = \sqrt{2}\left(I - P_{V_j^{[0,\infty[}(\Lambda)}\right)\left(\widetilde{\phi}_l(2x) - 2^l \widetilde{\phi}_l(x)\right)\chi_{[0,\infty[}(x),$$

or, equivalently,

$$\begin{aligned} &\begin{pmatrix} \widetilde{\psi}_0(x) \\ \vdots \\ \widetilde{\psi}_{N-1}(x) \end{pmatrix} \\ &= \quad -\sqrt{2}\left(I - P_{V_j^{[0,\infty[}(\Lambda)}\right) D^{-1} b \begin{pmatrix} \phi(2x-N+\alpha) \\ \vdots \\ \phi(2x-3N+2+2\alpha) \end{pmatrix}, \end{aligned} \tag{11}$$

where I and $P_{V_j^{[0,\infty[}(\Lambda)}$ are the identity and the orthogonal projection onto $V_j^{[0,\infty[}(\Lambda)$, respectively. Let $W_j^{[0,\infty[}(\Lambda)$ be the closure of the subspace of

$L^2\left([0,\infty[\right)$ orthogonal to $V_j^{[0,\infty[}(\Lambda)$ in $V_{j+1}^{[0,\infty[}(\Lambda)$. If $\Gamma \subset \{0,...,N-1\}$ with $\#\Gamma = N-\alpha$, then $\left\{ \left(\widetilde{\psi}_l(2^j.)\right)_{l\in\Gamma}, \left(\psi(2^j.-k)\right)_{k\geq N-\alpha} \right\}$ is a basis of $W_j^{[0,\infty[}(\Lambda)$.

The Gram matrix $G^{\widetilde{\psi}}$ of the edge scaling functions is given by

$$G^{\widetilde{\psi}} = g_1 g_1^t + g_2 g_2^t,$$

where $g_1 = D^{-1} b h_0^t H_0$ and $g_2 = D^{-1} b (h_0^t h_0 - I)$. Let $G_\Gamma^{\widetilde{\psi}}$ be the matrix obtained from $G^{\widetilde{\psi}}$ by keeping only rows and columns of index in $1+\Gamma$. Then $G_\Gamma^{\widetilde{\psi}}$ is the Gram matrix of $\widetilde{\psi}_l$, $l \in \Gamma$. If $G_\Gamma^{\widetilde{\psi}} = R_\Gamma^{\widetilde{\psi}} (R_\Gamma^{\widetilde{\psi}})^*$ is a factorization of $G_\Gamma^{\widetilde{\psi}}$, then the orthonormal family $\widetilde{\widetilde{\psi}}_l$, $l \in \Gamma$, defined by

$$\begin{pmatrix} \widetilde{\widetilde{\psi}}_0 \\ \vdots \\ \widetilde{\widetilde{\psi}}_{N-1} \end{pmatrix}_{l\in\Gamma} = (R_\Gamma^{\widetilde{\psi}})^{-1} \begin{pmatrix} \widetilde{\psi}_0 \\ \vdots \\ \widetilde{\psi}_{N-1} \end{pmatrix}_{l\in\Gamma}, \tag{12}$$

satisfies the refinement equation

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{pmatrix} \widetilde{\widetilde{\psi}}_0(\frac{x}{2}) \\ \vdots \\ \widetilde{\widetilde{\psi}}_{N-1}(\frac{x}{2}) \end{pmatrix}_{l\in\Gamma} &= G_0 \begin{pmatrix} \widetilde{\widetilde{\phi}}_0(x) \\ \vdots \\ \widetilde{\widetilde{\phi}}_{N-1}(x) \end{pmatrix}_{l\notin\Lambda} + \\ &\quad + g_0 \begin{pmatrix} \phi(x-N+\alpha) \\ \vdots \\ \phi(x-3N+2+2\alpha) \end{pmatrix}, \end{aligned} \tag{13}$$

where $G_0 = (R_\Gamma^{\widetilde{\psi}})^{-1} g_1$ and $g_0 = (R_\Gamma^{\widetilde{\psi}})^{-1} g_2$.

2.3. MRA of $L^2\left([0,1]\right)$

Let $\mathcal{T} : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be given by $\mathcal{T}\{(\S) = \{(\infty - \S)$. Then the function $\mathcal{T}\phi$ has support $[-N+1, N]$ and satisfies the refinement equation

$$\mathcal{T}\phi(x) = \sqrt{2} \sum_{k=-N+1}^{N} \check{h}_k \mathcal{T}\phi(2x-k), \tag{14}$$

with coefficient mask $\check{h}_k := h_{1-k}$. Using the procedure described previously, with scaling function $\mathcal{T}\phi$, the mask $\check{h}_k$, $\alpha_1 \in \{0,1\}$, $\Lambda_1 \subseteq \{0,...,N-1\}$ and

$\Gamma_1 \subset \{0, ..., N-1\}$ with $\#\Gamma_1 = N - \alpha_1$ we arrive at a MRA of $L^2\left([0,\infty[\right)$ satisfying boundary conditions with edge scaling functions $\widetilde{\widetilde{\phi}}^{\sharp}_l$, $l \notin \Lambda_1$, interior scaling functions $\mathcal{T}\phi(. - k)$, $k \geq N - \alpha_1$, edge wavelets functions $\widetilde{\widetilde{\psi}}^{\sharp}_l$, $l \in \Gamma_1$ and interior wavelets functions $\mathcal{T}\psi(. - k)$, $k \geq N - \alpha_1$. Then the family of functions

$$\mathcal{T}[\widetilde{\widetilde{\phi}}^{\sharp}_l(2^j.)] = \widetilde{\widetilde{\phi}}^{\sharp}_l(2^j(1-.)), \mathcal{T}[\mathcal{T}\phi(2^j. - k)] = \phi(2^j. - (2^j - 1 - k)),$$

constitutes an orthonormal basis for a MRA of $L^2\left(]-\infty, 1]\right)$. Moreover,

$$\mathcal{T}[\widetilde{\widetilde{\psi}}^{\sharp}_l(2^j.)] = \widetilde{\widetilde{\psi}}^{\sharp}_l(2^j(1-.)), \mathcal{T}[\mathcal{T}\psi(2^j. - k)] = \psi(2^j. - (2^j - 1 - k)),$$

are the associated orthonormal wavelets.

Let $P^{\sharp}_l(k)$, $b^{\sharp}$, $G^{\widetilde{\phi}^{\sharp}}$, $G^{\widetilde{\psi}^{\sharp}}$, $H^{\sharp}_1$, $h^{\sharp}_1$, $G^{\sharp}_1$ and $g^{\sharp}_1$ obtained from the mask $\check{h}_k$ as $P_l(k)$, b, $G^{\widetilde{\phi}}$, $G^{\widetilde{\psi}}$, H_0, h_0, G_0 and g_0 were obtained form the mask h_k. Let H_1, h_1, G_1 and g_1 be the matrices $H^{\sharp}_1$, $h^{\sharp}_1$, $G^{\sharp}_1$ and $g^{\sharp}_1$, respectively, with the rows and columns in reversed order. Then

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \widetilde{\widetilde{\phi}}^{\sharp}_{N-1}(2^j(1-x)) \\ \vdots \\ \widetilde{\widetilde{\phi}}^{\sharp}_0(2^j(1-x)) \end{pmatrix}_{l \notin \Lambda_1} = H_1 \begin{pmatrix} \widetilde{\widetilde{\phi}}^{\sharp}_{N-1}(2^{j+1}(1-x)) \\ \vdots \\ \widetilde{\widetilde{\phi}}^{\sharp}_0(2^{j+1}(1-x)) \end{pmatrix}_{l \notin \Lambda_1} + \\ + h_1 \begin{pmatrix} \phi(2^{j+1}x - 2^{j+1} + 3N - 1 - 2\alpha_1) \\ \vdots \\ \phi(2^{j+1}x - 2^{j+1} + N + 1 - \alpha_1) \end{pmatrix}, \tag{15}$$

and

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \widetilde{\widetilde{\psi}}^{\sharp}_{N-1}(2^j(1-x)) \\ \vdots \\ \widetilde{\widetilde{\psi}}^{\sharp}_0(2^j(1-x)) \end{pmatrix}_{l \in \Gamma_1} = G_1 \begin{pmatrix} \widetilde{\widetilde{\phi}}^{\sharp}_{N-1}(2^{j+1}(1-x)) \\ \vdots \\ \widetilde{\widetilde{\phi}}^{\sharp}_0(2^{j+1}(1-x)) \end{pmatrix}_{l \notin \Lambda_1} + \\ + g_1 \begin{pmatrix} \phi(2^{j+1}x - 2^{j+1} + 3N - 1 - 2\alpha_1) \\ \vdots \\ \phi(2^{j+1}x - 2^{j+1} + N + 1 - \alpha_1) \end{pmatrix}. \tag{16}$$

If $j_{min} = \lceil \log_2 4N \rceil$ we have a MRA of $L^2([0,1])$, $\left(V_j^{[0,1]}(\Lambda_1,\Lambda_2)\right)_{j\geq j_{min}}$, where $V_j^{[0,1]}(\Lambda_1,\Lambda_2)$ has dimension $2^j - \#\Lambda_0 - \#\Lambda_1 + \alpha_0 + \alpha_1$ with orthonormal basis

$$\left(2^{j/2}\widetilde{\widetilde{\phi}}_l(2^jx)\right)_{l\notin\Lambda_0}, \left(2^{j/2}\phi(2^jx-k)\right)_{k=N-\alpha_0}^{2^j-1-N+\alpha_1}, \left(2^{j/2}\widetilde{\widetilde{\phi}}_l^{\sharp}(2^j(1-x))\right)_{l\notin\Lambda_1},$$

and $W_j^{[0,1]}(\Lambda_1,\Lambda_2)$ has dimension 2^j with orthonormal basis

$$\left(2^{j/2}\widetilde{\widetilde{\psi}}_l(2^jx)\right)_{l\in\Gamma_0}, \left(2^{j/2}\psi(2^jx-k)\right)_{k=N-\alpha_0}^{2^j-1-N+\alpha_1}, \left(2^{j/2}\widetilde{\widetilde{\psi}}_l^{\sharp}(2^j(1-x))\right)_{l\in\Gamma_1}.$$

It is convenient to choose $\alpha_0 = \delta_{\#\Lambda_0-1}, \alpha_1 = \delta_{\#\Lambda_1-1}$.

Remark 2.3.1. We can consider different decompositions of the Gram matrices $G^{\widetilde{\phi}}$ and $G^{\widetilde{\psi}}$ to obtain orthonormal scaling and wavelets functions. Let G denote the Gram matrix associated with edge scaling or wavelets functions. For example, we can use the following options for R such that $G = RR^t$:

- If $G = QPQ^t$ where Q is an orthogonal matrix and P is a diagonal matrix with positive diagonal entries, we consider $R = QP^{1/2}$. This choice of R corresponds to the Schweinler-Wigner orthonomalization procedure [21].
- $R = G^{1/2} = QP^{1/2}Q^t$. This R was considered in [20].
- If $G = U^tU$ where U is an upper triangular matrix with positive diagonal entries (i. e., the Cholesky decomposition of G), we consider $R = U^t$ which corresponds to the Gram-Smith orthonomalization procedure [22].

2.3.1. Wavelet transforms

Consider the column vector Φ_j with entries

- $\left(2^{j/2}\widetilde{\widetilde{\phi}}_0(2^j.), ..., 2^{j/2}\widetilde{\widetilde{\phi}}_{N-1}(2^j.)\right)_{l\notin\Lambda_0}$,
- $\left(2^{j/2}\phi(2^j. - N + \alpha_0), ..., 2^{j/2}\phi(2^j. - 2^j + 1 + N - \alpha_1)\right)$,

- $\left(2^{j/2}\widetilde{\widetilde{\phi}}^{\sharp}_{N-1}(2^j(1-.)),...,2^{j/2}\widetilde{\widetilde{\phi}}^{\sharp}_{0}(2^j(1-.))\right)_{l\notin\Lambda_1}$,

and the column vector Ψ_j with entries

- $\left(2^{j/2}\widetilde{\widetilde{\psi}}_0(2^j.),...,2^{j/2}\widetilde{\widetilde{\psi}}_{N-1}(2^j.)\right)_{l\in\Gamma_0}$,
- $\left(2^{j/2}\psi(2^j.-N+\alpha_0),...,2^{j/2}\psi(2^j.-2^j+1+N-\alpha_1)\right)$,
- $\left(2^{j/2}\widetilde{\widetilde{\psi}}^{\sharp}_{N-1}(2^j(1-.)),...,2^{j/2}\widetilde{\widetilde{\psi}}^{\sharp}_{0}(2^j(1-.))\right)_{l\in\Gamma_1}$.

If $\mathcal{H}^j$ and $\mathcal{G}^j$ are the matrices of order $(2^{j-1}-2N+\alpha_0+\alpha_1)\times(2^j-2N+\alpha_0+\alpha_1)$ with entries $\mathcal{H}^j(k,l)=h_{-N+1+\alpha_0+l-2k}$ and $\mathcal{G}^j(k,l)=g_{-N+1+\alpha_0+l-2k}$, respectively, we set

$$H_j=\begin{pmatrix} H_0 & h_0 & 0 \\ 0 & \mathcal{H}^| & 0 \\ 0 & h_1 & H_1 \end{pmatrix},\ G_j=\begin{pmatrix} G_0 & g_0 & 0 \\ 0 & \mathcal{G}^| & 0 \\ 0 & g_1 & G_1 \end{pmatrix},$$

where h_0 and g_0 are completed with columns of zeros at the right, whereas h_1 and g_1 are completed with columns of zeros at the left, to fit the size of $\mathcal{H}^j$ and $\mathcal{G}^j$.

We have

$$\Phi^{j-1}=H_j\Phi^j,\ \Psi^{j-1}=G_j\Phi^j, \tag{17}$$

and

$$\Phi^j=\begin{pmatrix} H_j^t & G_j^t \end{pmatrix}\begin{pmatrix} \Phi^{j-1} \\ \Psi^{j-1} \end{pmatrix}. \tag{18}$$

Let $f\in L^2([0,1])$, and consider the column vector s^j with entries

- $\left(\left\langle f,2^{j/2}\widetilde{\widetilde{\phi}}_0(2^j.)\right\rangle,...,\left\langle f,2^{j/2}\widetilde{\widetilde{\phi}}_{N-1}(2^j.)\right\rangle\right)_{l\notin\Lambda_0}$,
- $\left(\left\langle f,2^{j/2}\phi(2^j.-N+\alpha_0)\right\rangle,...,\left\langle f,2^{j/2}\phi(2^j.-2^j+1+N-\alpha_1)\right\rangle\right)$,
- $\left(\left\langle f,2^{j/2}\widetilde{\widetilde{\phi}}^{\sharp}_{N-1}(2^j(1-.))\right\rangle,...,\left\langle f,2^{j/2}\widetilde{\widetilde{\phi}}^{\sharp}_{0}(2^j(1-.))\right\rangle\right)_{l\notin\Lambda_1}$,

and the column vector d^j with entries

- $\left(\left\langle f, 2^{j/2}\widetilde{\widetilde{\psi}}_0(2^j.)\right\rangle, ..., \left\langle f, 2^{j/2}\widetilde{\widetilde{\psi}}_{N-1}(2^j.)\right\rangle\right)_{l\in\Gamma_0}$,
- $\left(\left\langle f, 2^{j/2}\psi(2^j. - N + \alpha_0)\right\rangle, ..., \left\langle f, 2^{j/2}\psi(2^j. - 2^j + 1 + N - \alpha_1)\right\rangle\right)$,
- $\left(\left\langle f, 2^{j/2}\widetilde{\widetilde{\psi}}^{\sharp}_{N-1}(2^j(1-.))\right\rangle, ..., \left\langle f, 2^{j/2}\widetilde{\widetilde{\psi}}^{\sharp}_0(2^j(1-.))\right\rangle\right)_{l\in\Gamma_1}$.

From (17) and (18), we obtain the following wavelet transforms

$$s^{j-1} = H_j s^j,\ d^{j-1} = G_j s^j, \tag{19}$$

and

$$s^j = \begin{pmatrix} H_j^t & G_j^t \end{pmatrix} \begin{pmatrix} s^{j-1} \\ d^{j-1} \end{pmatrix}. \tag{20}$$

2.3.2. Moments of the edge scaling functions

For edge scaling functions at 0 the moments are given by

$$\int_0^\infty x^p \widetilde{\widetilde{\phi}}_l(x)dx = p! X_p(l),\ l \notin \Lambda_0, \tag{21}$$

where $X_p(l)$ are obtained from

$$\left(2^{p+1/2}I - H_0\right) X_p = h_0 \begin{pmatrix} \frac{(N-\alpha_0)^p}{p!} & \cdots & \frac{(N-\alpha_0)^0}{0!} \\ \vdots & & \vdots \\ \frac{(3N-2-2\alpha_0)^p}{p!} & \cdots & \frac{(3N-2-2\alpha_0)^0}{0!} \end{pmatrix} \begin{pmatrix} C_0 \\ \vdots \\ C_p \end{pmatrix},$$

and for edge scaling functions at 1,

$$\int_0^\infty \widetilde{\widetilde{\phi}}^{\sharp}_l(x)dx = p! X_p^{\sharp}(l) = ,\ l \notin \Lambda_1, \tag{22}$$

where $X_p^{\sharp}(l)$ are obtained from

$$\begin{aligned} (2^{p+1/2}I - H_1^{\sharp})X_p^{\sharp} \quad &= \quad h_1^{\sharp} \begin{pmatrix} \frac{(-1)^0(1+N-\alpha_1)^p}{p!} & \cdots & \frac{(-1)^p(1+N-\alpha_1)^0}{0!} \\ \vdots & & \vdots \\ \frac{(-1)^0(3N-1-2\alpha_1)^p}{p!} & \cdots & \frac{(-1)^p(3N-1-2\alpha_1)^0}{0!} \end{pmatrix} \times \\ &\times \begin{pmatrix} C_0 \\ \vdots \\ C_p \end{pmatrix}. \end{aligned} \tag{23}$$

3. Connection coefficients

In this section we prove that the calculus of the connection coefficients involving products of (derivatives of) Monasse-Perrier scaling functions and/or wavelets can be reduced to the calculus of the connection coefficients, $r_{k,l}^{(m,n)}$, $l \in \mathbf{Z}$, of the Daubechies's scaling functions given by

$$r_{k,l}^{(m,n)} = \int_{-\infty}^{\infty} \frac{d^m \phi}{dx^m}(x-k)\frac{d^n \phi}{dx^n}(x-l)dx,\ k,l \in \mathbf{Z}. \tag{24}$$

These integrals can be computed simultaneously, for all k, l such that $r_{k,l}^{(m,n)} \neq 0$, by solving an eigenvector-eigenvalue problem (see [14]).

3.1. Scaling functions at the same level

For Monasse-Perrier scaling functions we consider the no null connection coefficients as the entries of the following matrices:

$${}_{m,n}\widetilde{R}_0(k+1,l+1) = \int_0^{\infty} \frac{d^m}{dx^m}\tilde{\tilde{\phi}}_k(x)\frac{d^n}{dx^n}\tilde{\tilde{\phi}}_l(x)dx, \tag{25}$$

with $k,l = 0,...,N-1$, $k,l \notin \Lambda_0$.

$${}_{m,n}S_0(k+1,l-N+\alpha_0+1) = \int_0^{\infty} \frac{d^m}{dx^m}\tilde{\tilde{\phi}}_k(x)\frac{d^n}{dx^n}\phi(x-l)dx, \tag{26}$$

with $k = 0,...,N-1$, $k \notin \Lambda_0$, $l = N-\alpha_0,...,2^j-N-1+\alpha_1$.

$$\begin{aligned} &{}_{m,n}R_j(k-N+\alpha_0+1, k-N+\alpha_0+1) \\ = &\int_0^{\infty} \frac{d^m}{dx^m}\phi(x-k)\frac{d^n}{dx^n}\phi(x-l)dx, \end{aligned} \tag{27}$$

with $k,l = N-\alpha_0,...,2^j-N-1+\alpha_1$.

$${}_{m,n}S_1(N-k, 2^j-N+\alpha_1-l) = \int_{-\infty}^{1} \frac{d^m}{dx^m}\tilde{\tilde{\phi}}^{\sharp}_k(1-x)\frac{d^n}{dx^n}\phi(x-l)dx, \tag{28}$$

with $k = N-1,....,0$, $k \notin \Lambda_1$, $l = 2^j-N-1+\alpha_1,...,N-\alpha_0$.

$${}_{m,n}\widetilde{R}_1(N-k,N-l) = \int_{-\infty}^{1} \frac{d^m}{dx^m}\tilde{\tilde{\phi}}^{\sharp}_k(1-x)\frac{d^n}{dx^n}\tilde{\tilde{\phi}}^{\sharp}_l(1-x)dx, \tag{29}$$

with $k,l = N-1,....,0$, $k,l \notin \Lambda_1$.

We show now that we can express all the above connection coefficients in terms of the $r_{k,l}^{(m,n)}$. The particular case $m = 0$ with $n = 1, 2$, was considered in [20]. For the general case we have:

- By (7), ${}_{m,n}S_0 := (G^{\widetilde{\phi}}_{\Lambda_0})^{-1/2}\,{}_nS^{\widetilde{\phi}}_0$ where

$$ {}_{m,n}S^{\widetilde{\phi}}_0(k+1, l-N+\alpha_0+1) = \sum_{i=-N+1}^{N-1-\alpha_0} P_k(i) r^{(m,n)}_{i,l}, $$

  $k = 0, ..., N-1$, $k \notin \Lambda_0$, $l = N-\alpha_0, ..., 3N-3-\alpha_0$.

- Similarly, if ${}_{m,n}S^\sharp_1 := (G^{\widetilde{\phi^\sharp}}_{\Lambda_1})^{-1/2}\,{}_{m,n}S^{\widetilde{\phi^\sharp}}_1$ where

$$ {}_{m,n}S^{\widetilde{\phi^\sharp}}_1(k+1, l-N+\alpha_1+1) = \sum_{i=-N+1}^{N-1-\alpha_1} P^\sharp_k(i) r^{(m,n)}_{-i,-l}, $$

  $k = 0, ..., N-1$, $k \notin \Lambda_1$, $l = N-\alpha_1, ..., 3N-3-\alpha_1$, then ${}_{m,n}S_1$ is the matrix ${}_{m,n}S^\sharp_1$ with the rows and columns in reversed order.

- From (10), ${}_{m,n}\widetilde{R}_0 := (G^{\widetilde{\phi}}_{\Lambda_0})^{-1/2}\,{}_{m,n}R^{\widetilde{\phi}}_0(G^{\widetilde{\phi}}_{\Lambda_0})^{-1/2}$, where ${}_{m,n}R^{\widetilde{\phi}}_0$ can be compute by a term by term division of the matrix

$$ D_{\Lambda_0}\,{}_{m,n}S^{\widetilde{\phi}}_0 b^t_{\Lambda_0} + b_{\Lambda_0}({}_{n,m}S^{\widetilde{\phi}}_0)D_{\Lambda_0} + b_{\Lambda_0}\,{}_{m,n}R b^t_{\Lambda_0}, \tag{30} $$

  with ${}_{m,n}R(k-N+\alpha_0+1, l-N+\alpha_0+1) = r^{(n)}_{k,l}$, $k, l = N-\alpha_0, ..., 3N-2-2\alpha_0$, by the matrix $\left(\frac{1}{2^{m+n-1}}M_1 - M\right)_{\Lambda_0}$.

- ${}_{m,n}\widetilde{R}_1$ is the matrix ${}_{m,n}\widetilde{R}^\sharp_1$ with the rows and columns in reversed order with ${}_{m,n}\widetilde{R}^\sharp_1 := (-1)^n(G^{\widetilde{\phi^\sharp}}_{\Lambda_1})^{-1/2}\,{}_{m,n}R^{\widetilde{\phi^\sharp}}_1(G^{\widetilde{\phi^\sharp}}_{\Lambda_1})^{-1/2}$, where ${}_{m,n}R^{\widetilde{\phi^\sharp}}_1$ can be compute by a term by term division of the matrix

$$ D_{\Lambda_1}\,{}_{m,n}S^{\widetilde{\phi^\sharp}}_1(b^\sharp_{\Lambda_1})^t + b^\sharp_{\Lambda_1}({}_{n,m}S^{\widetilde{\phi^\sharp}}_1)D_{\Lambda_1} + b^\sharp_{\Lambda_1}\,{}_{m,n}R(b^\sharp_{\Lambda_1})^t, \tag{31} $$

  with ${}_{m,n}R(k-N+\alpha_1+1, l-N+\alpha_1+1) = r^{(n)}_{k,l}$, $k, l = N-\alpha_1, ..., 3N-2-2\alpha_1$, by the matrix $\left(\frac{1}{2^{m+n-1}}M_1 - M\right)_{\Lambda_1}$.

- ${}_{m,n}R_j(k-N+\alpha_0+1, k-N+\alpha_0+1) = r^{(m,n)}_{k,l}$, $k, l = N-\alpha_0, ..., 2^j-N-1+\alpha_1$,

Remark 3.1. If $\alpha_0 = 0$, we add a column of 0 at the right to ${}_{m,n}S^{\widetilde{\phi}}_0$ to obtain ${}_{m,n}R^{\widetilde{\phi}}_0$ from the matrix (30). A similar consideration is valid for computing ${}_{m,n}R^{\widetilde{\phi^\sharp}}_1$ from the matrix (31).

Remark 3.2. The entries ${}_{m,n}R_0^{\widetilde{\phi}}(k,l)$ with $k-1, l-1 \in \Lambda_0$ and $k+l = m+n+1$, can not be computed using the term by term division described above. If $k \geq m+1$, by (24) and (27) for these entries we have,

$$\begin{aligned} {}_{m,n}R_0^{\widetilde{\phi}}(k,l) &= \int_0^\infty \frac{d^m\widetilde{\phi}_{k-1}}{dx^m}(x)\frac{d^n\widetilde{\phi}_{l-1}}{dx^n}(x)dx \\ &= \int_0^\infty \frac{x^{k-m-1}}{(k-m)!}\frac{d^n\widetilde{\phi}_{l-1}}{dx^n}(x) \\ &\quad - \sum_{m=N-\alpha_0}^{3N-3-\alpha_0} P_{k-1}(m)\,{}_{n,m}S_0^{\widetilde{\phi}}(l, m-N+\alpha_0+1). \end{aligned} \tag{32}$$

Integrating by parts $k-m-1$ times and using $\frac{d^n\widetilde{\phi}_l}{dx^n}(0) = \delta_{n-l}$, $0 \leq l \leq N-1$, we obtain

$${}_{m,n}R_0^{\widetilde{\phi}}(k,l) = (-1)^{k-m} - \sum_{m=N-\alpha_0}^{3N-3-\alpha_0} P_{k-1}(m)\,{}_{n,m}S_0^{\widetilde{\phi}}(l, m-N+\alpha_0+1).$$

Similarly, if $k < m+1$ then

$${}_{m,n}R_0^{\widetilde{\phi}}(k,l) = - \sum_{m=N-\alpha_0}^{3N-3-\alpha_0} P_{k-1}(m)\,{}_{n,m}S_0^{\widetilde{\phi}}(l, m-N+\alpha_0+1).$$

Analogously, if $k-1, l-1 \in \Lambda_1$ and $k+l = m+n+1$, then

$${}_{m,n}R_1^{\widetilde{\phi}}(k,l) = (-1)^{k-m} - \sum_{m=N-\alpha_1}^{3N-3-\alpha_1} P^\sharp_{k-1}(m)\,{}_{n,m}S_1^{\widetilde{\phi^\sharp}}(l, m-N+\alpha_1+1),$$

for $k \geq m+1$, and

$${}_{m,n}R_1^{\widetilde{\phi}}(k,l) = - \sum_{m=N-\alpha_1}^{3N-3-\alpha_1} P^\sharp_{k-1}(m)\,{}_{n,m}S_1^{\widetilde{\phi^\sharp}}(l, m-N+\alpha_1+1),$$

for $k < m+1$.

Remark 3.3. For the case $m = 0$ we note that

$${}_{0,n}S_i^{\widetilde{\phi}}(l, m-N+\alpha_0+1) = (-1)^n\,{}_{n,0}S_i^{\widetilde{\phi}}(l, m-N+\alpha_0+1),\ i = 0, 1,$$

and $r_l^{(n)} := r_{l,0}^{(0,n)}$, $l \in \mathbf{Z}$, can be computed solving the following system of linear algebraic equations (see [5] for more details):

$$r_l^{(n)} = 2^n \left[r_{2l}^{(n)} + \frac{1}{2}\sum_{k=1}^{N} a_{2k-1}\left(r_{2l-2k+1}^{(n)} + r_{2l+2k-1}^{(n)} \right) \right], \tag{33}$$

where $a_k := 2\sum_{m=-N+1}^{N-k} h_m h_{m+k}$, $k = 1, ..., 2N-1$, are the autocorrelation coefficients of $\{h_m\}_{m=-N+1}^{N}$, and

$$\sum_l l^n r_l^{(n)} = (-1)^n n!. \tag{34}$$

We also have, $r_{i,l}^{(0,n)} = r_{i-l}^{(n)}$ and $r_l^{(n)} = (-1)^n r_{-l}^{(n)}$.

3.2. Scaling functions and wavelets at different levels

We analyze now how to express the connection coefficients involving products (of derivatives) of scaling functions and/or wavelets at different scales j, j', in terms of connection coefficients with products (of derivatives) of scaling functions at the same scale, using local refinement relations (see, e. g., [3]). In all cases the amount of work is proportional to $|j - j'|$. We only consider connection coefficients with the edge scaling functions and wavelets at 0. At 1 we can proceed in a similar manner.

We define $ind_1(l)$ as the number in $\{1, ..., M - \#\Lambda_0\}$ giving the position of l in the set $\{0, ..., M-1\} \setminus \Lambda_0$ considered with its elements in an increasing order. For $l \in \Gamma_0$, the number $ind_2(l) \in \{1, ..., \#\Gamma_0\}$ is defined in a similar manner. In order to obtain the results of this subsection we rewrite (10) and (13) as

$$\begin{aligned}\tilde{\tilde{\phi}}_l(2^j x) &= \sqrt{2}\sum_{\substack{i=0\\ i\notin\Lambda_0}}^{M-1} H_0(ind_1(l), ind_1(i))\tilde{\tilde{\phi}}_i(2^{j+1}x) + \\ &\quad+\sqrt{2}\sum_{i=N-\alpha_0}^{3N-2-2\alpha_0} h_0(ind_1(l), i-N+\alpha_0+1)\phi_i(2^{j+1}x),\end{aligned} \tag{35}$$

$$\begin{aligned}\tilde{\tilde{\phi}}_l(2^j x) &= \sqrt{2}\sum_{\substack{i=0\\ i\notin\Lambda_0}}^{M-1} G_0(ind_2(l), ind_1(i))\tilde{\tilde{\phi}}_i(2^{j+1}x) + \\ &\quad+\sqrt{2}\sum_{i=N-\alpha_0}^{3N-2-2\alpha_0} g_0(ind_2(l), i-N+\alpha_0+1)\phi_i(2^{j+1}x).\end{aligned} \tag{36}$$

Case 1. If $j > j'$, $k, l = N - \alpha_0, ..., 2^j - N - 1 + \alpha_1$, since $\text{supp}(\phi^{(m)}(2^j x -$

$k)) = [\frac{k-N+1}{2^j}, \frac{k+N}{2^j}]$, by (2)

$$\begin{aligned}&\int_{-\infty}^{\infty} \phi^{(m)}(2^j x-k)\phi^{(n)}(2^{j'}x-l)dx \\ = \;& 2^{n+\frac{1}{2}} \sum_{i\in\Delta_{k,l}^{j,j',1}} \int_{-\infty}^{\infty} \phi^{(m)}(2^j x-k)\phi^{(n)}(2^{j'+1}x-i)dx.\end{aligned}$$

where $\nabla_{k,l}^{j,j',r}$ is the set of integers $2l+p$ with $p\in\{-N+1,...,N\}$ and $\frac{k-N+1}{2^{j-j'-r}} < p-N < \frac{k+N}{2^{j-j'-r}}-1$. We have $\#\nabla_{k,l}^{j,j',r} \leq C$. Repeating this procedure $j-j'$ times, we finally express the connection coefficient $\int_{-\infty}^{\infty} \phi^{(m)}(2^j x-k)\phi^{(n)}(2^{j'}x-l)dx$ in terms of a sum of $\sum_{r=1}^{j-j'} \#\Delta_{k,l}^{j,j',r} \leq (j-j')C_k$ terms that involve the connection coefficients $r_{k,l}^{(m,n)}$.

Case 2. If $j>j'$, $k=0,...,N-1$, $k\notin\Lambda_0$, $l=N-\alpha_0,...,2^j-N-1+\alpha_1$, by (2) and taking into account that $\text{supp}(\overset{\approx}{\phi}{}_k^{(m)}(2^j x)) = [0, \frac{2N-1-\alpha_0}{2^j}]$,

$$\begin{aligned}&\int_0^{\infty} \overset{\approx}{\phi}{}_k^{(m)}(2^j x)\phi^{(n)}(2^{j'}x-l)dx \\ = \;& 2^{n+\frac{1}{2}} \sum_{i\in\Delta_l^{j,j',1}} h_i \int_0^{\infty} \overset{\approx}{\phi}{}_k^{(m)}(2^j x)\phi^{(n)}(2^{j'+1}x-i)dx,\end{aligned}$$

where $\Delta_l^{j,j',r}$ is the set of integers $2l+p$ with $p\in\{-N+1,...,N\}$, $p<\frac{2N-1-\alpha_0}{2^{j-j'-r}}+N-1$. Clearly, $\#\Delta_l^{j,j',r}\leq C$. Repeating this procedure $j-j'$ times, we finally express $\int_0^{\infty} \overset{\approx}{\phi}{}_k^{(m)}(2^j x)\phi^{(n)}(2^{j'}x-l)dx$ in terms of a sum of $\sum_{r=1}^{j-j'} \#\Delta_l^{j,j',r} \leq (j-j')C$ terms involving connection coefficients $\int_0^{\infty} \overset{\approx}{\phi}{}_k^{(m)}(x)\phi^{(n)}(x-l)dx$.

Case 3. Similarly, using (35), if $j<j'$, $k=0,...,N-1$, $k\notin\Lambda_0$, $l=N-\alpha_0,...,2^j-N-1+\alpha_1$, then $\int_0^{\infty} \overset{\approx}{\phi}{}_k^{(m)}(2^j x)\phi^{(n)}(2^{j'}x-l)dx$ can be expressed in terms of the connection coefficients $\int_0^{\infty} \overset{\approx}{\phi}{}_i^{(m)}(x)\phi^{(n)}(x-l)dx$, $\int_0^{\infty} \phi^{(m)}(2^{j+k}x-i)\phi^{(n)}(2^{j'}x-l)dx$, $k=1,...,j'-j$. Finally these last connection coefficients are treated as in the Case 1.

Similarly, using (35) and (36), the rest of the connection coefficients can be expressed in terms of that considered in the above cases and/or in the previous subsection.

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