



Current correlation functions from a bosonized theory in $3/2 + 1$ dimensions

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Received: 8 October 2019 / Accepted: 2 February 2020 / Published online: 14 February 2020

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Abstract Within the context of a bosonized theory, we evaluate the current-current correlation functions corresponding to a massive Dirac field in $2 + 1$ dimensions, which is constrained to a spatial half-plane. The boundary conditions are imposed on the dual theory, and have the form of perfect-conductor conditions. We also consider, for the sake of comparison, the purely fermionic version of the model and its boundary conditions, in the large-mass limit. We apply the result about the dual theory to the evaluation of induced vacuum currents in the presence of an external field, in a spatial half-plane.

Bosonization is a useful tool which, in $1 + 1$ space-time dimensions, allows for the solution of some non-trivial Quantum Field Theory models (see [1] for a comprehensive review and useful references).

For a massive Dirac field in $2 + 1$ dimensions, the situation we are concerned with here, the path integral bosonization framework may be used to derive the exact bosonization rule for the current. The (dual) bosonic action, is gauge-invariant and, in the massive case, local, what determines the form of the possible terms in a mass expansion.¹ Thus, to the leading order, it is a Chern-Simons term, while the next-to-leading one corresponds, in the Abelian or non Abelian cases, to a (local) Maxwell [2,3] or Yang-Mills term [4,5], respectively. We note that the need for the CS term has been shown explicitly, even in a massless theory, as a consequence of an η function regularization, required to have a consistent gauge invariant theory [6].

In a previous work [7], we have applied the functional bosonization approach to a system consisting of a massive Dirac field constrained to a $2 + 1$ dimensional spacetime manifold \mathcal{U} , with non-trivial conditions on its boundary $\mathcal{M} \equiv \partial\mathcal{U}$. Those conditions, when imposed on the dual (bosonized) ver-

sion of the theory, amounted to the vanishing, at each point of \mathcal{M} , of the normal component of the (bosonized) current. The bosonization rules, formulated in terms of an Abelian gauge field A_μ , were shown to be the same as in the no-boundary case, while the existence of the boundary manifested itself through the fact that the gauge field satisfied *perfect-conductor* conditions on \mathcal{M} . This is one of the benefits of the procedure: the avoidance of the calculation of a fermionic determinant with non-trivial boundary conditions. Indeed, they are converted into conditions for the gauge field, easier to implement.

The exact bosonization of a $1 + 1$ dimensional model with a boundary, i.e., on a half-line, has been implemented in [8]. In this article, following [7], we apply the bosonization approach above to the calculation of current correlation functions, in a concrete geometry: a massive Dirac field confined to a spatial half-plane (so that, following [8], we dub the associated space-time as ' $3/2 + 1$ dimensions'). In this non-supersymmetric model, we do not dwell with a massless theory, where there seems to be, in principle, no natural mass to use in the expansion, and the low energy terms can be non-local. In spite of this, the program could be implemented also in this case (see [9] for a discussion), by using the renormalization mass scale μ as the expansion parameter. Our study of a bosonized Dirac field in $3/2 + 1$ dimensions, which takes into account the leading and sub-leading terms in the mass expansion, encompasses the evaluation of the current-current correlation function, in the context of functional bosonization. Note that this correlation function is tantamount to the vacuum polarization tensor. This vacuum polarization will also be applied to the determination of the induced current in the presence of an external gauge field, presenting the general form of the result, as well as more explicit expressions for some particular cases.

We consider a massive Dirac field in $2 + 1$ dimensions which, in its fermionic incarnation, is described by an

¹ In the massless case, the bosonic action is non-local.

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Euclidean action $\mathcal{S}_f(\bar{\psi}, \psi)$, given by:

$$\mathcal{S}_f(\bar{\psi}, \psi) = \int_{\mathcal{U}} d^3x \bar{\psi}(\not{\partial} + m)\psi, \tag{1}$$

on a spacetime manifold \mathcal{U} which, in terms of the coordinates $x = (x_0, x_1, x_2)$, corresponds to the space-time region $x_2 > 0$. The current is assumed to vanish along the normal direction to the border (see (48) below for a concrete implementation in the fermionic version). In the fermionic version of the model, there are many different ways to achieve the vanishing of the expectation value of the current on the boundary. What we shall see, is that the same dual theory emerges, as soon as one assumes that the boundary conditions on the fermions are such that the model inside the region delimited by the boundary is decoupled from the one outside.

Dirac's γ -matrices are Hermitean and, in our conventions, they satisfy $\gamma_\mu \gamma_\nu = \delta_{\mu\nu} + i \epsilon_{\mu\nu\lambda} \gamma_\lambda$. Letters from the middle of the Greek alphabet are assumed to run over the values 0, 1, 2. The Euclidean metric has been assumed to be the identity matrix $\delta_{\mu\nu}$, and $\epsilon_{\mu\nu\lambda}$ denotes the Levi-Civita symbol, with $\epsilon_{012} = +1$.

The functional bosonization approach, which we briefly review within the framework of a given geometry, begins from the conserved Noether current corresponding to (1), namely, $J_\mu = \bar{\psi} \gamma_\mu \psi$, while the existence of the boundary is reflected in the vanishing of $J_n \equiv \hat{n}_\mu J_\mu|_{\mathcal{M}}$, the normal component of the current on the boundary $\mathcal{M} \equiv (x_\parallel, 0)$, with $x_\parallel = (x_0, x_1)$, and the (outer) unit normal $\hat{n}_\mu = -\delta_{\mu 2}$.

To construct the fermionic generating functional, we need to add two ingredients: first, a term \mathcal{S}_J :

$$\mathcal{S}_J(s, J) = i \int d^3x s_\mu(x) J_\mu(x), \tag{2}$$

which includes a source s_μ , to be able to generate current correlation functions. The integral above² does not need to be restricted to \mathcal{U} if one assumes, as we shall do, that the source s_μ (a field which is not functionally integrated) vanishes outside \mathcal{U} .

A second term, $\mathcal{S}_{\mathcal{M}}$, depending on an auxiliary field $\xi(x_\parallel)$, is added in order to impose the condition on the normal current:

$$\mathcal{S}_{\mathcal{M}}(\xi, J) = -i \int d^2x_\parallel \xi(x_\parallel) J_2(x_\parallel, 0), \tag{3}$$

which can be also written as a term which couples the fermionic current to a vector field $c_\mu(\xi, x)$, which is completely determined by the auxiliary field and the boundary; indeed:

$$\begin{aligned} \mathcal{S}_{\mathcal{M}}(\xi, J) &= i \int d^3x c_\mu(\xi, x) J_\mu(x), \\ c_\mu(\xi, x) &\equiv -\delta_{\mu 2} \xi(x_\parallel) \delta(x_2). \end{aligned} \tag{4}$$

² Integrals are assumed to be unrestricted, unless explicitly stated otherwise. Namely, $\int d^3x \dots$ is assumed to be an integral over $\mathbb{R}^{(3)}$, etc.

Note that the functional integral over ξ yields a (functional) δ of the normal current:

$$\begin{aligned} \delta_{\mathcal{M}}[J_n] &= \int \mathcal{D}\xi e^{i \int d^2x_\parallel \xi(x_\parallel) J_2(x_\parallel, 0)} \\ &= \int \mathcal{D}\xi e^{-\mathcal{S}_{\mathcal{M}}(\xi, J)}. \end{aligned} \tag{5}$$

Note that, assuming the constraint above is due to a boundary condition on the Dirac field which completely determines the problem inside \mathcal{U} , one can extend the fermionic action to the whole of space-time, since the conditions on the current isolate the problem on \mathcal{U} from the one in its complement. In that way, a source which has support on \mathcal{U} will be oblivious to the existence of a fermionic field outside of \mathcal{U} , and the result of the functional integral becomes a product of one depending on the fields inside (and the source) times another one for the fields outside. The latter cancels out when evaluating expectation values.

On the other hand, the important advantage of interpreting $\mathcal{S}_{\mathcal{M}}$ as a coupling between the current and a field c_μ stems from the fact that the fermionic generating functional $\mathcal{Z}(s)$ may be written as follows:

$$\mathcal{Z}(s) = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\xi e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c)}, \tag{6}$$

with

$$\mathcal{S}_f(\bar{\psi}, \psi; s) = \int d^3x \bar{\psi}(\not{\partial} + i \not{s} + m)\psi. \tag{7}$$

Note that the fermionic fields do not have an explicit dependence on the boundary, in the sense that they are not restricted spatially to the region \mathcal{U} .

Following the procedure devised in [7], we now disentangle $s_\mu + c_\mu$ from the fermionic action in (6). Note that this step decoupled the Dirac operator from the boundary, and will allow to evaluate the fermionic determinant in the absence of borders. Of course, the borders will reemerge in the bosonic theory.

To that end, we first perform the change of variables:

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x), \quad \bar{\psi}(x) \rightarrow e^{-i\alpha(x)} \bar{\psi}(x), \tag{8}$$

and integrate over α , to obtain:

$$\mathcal{Z}(s) = \int \mathcal{D}\alpha \mathcal{D}\xi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c+\partial\alpha)}. \tag{9}$$

Then, the integration over α is substituted by one over a vector field $b_\mu \partial_\mu \alpha \rightarrow b_\mu$,

$$\mathcal{Z}(s) = \int \mathcal{D}b \delta[\tilde{f}_\mu(b)] \mathcal{D}\xi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-\mathcal{S}_f(\bar{\psi}, \psi; s+c+b)}, \tag{10}$$

where $\tilde{f}_\mu(b) = \epsilon_{\mu\nu\lambda} \partial_\nu b_\lambda = 0$ (b_μ is a pure gradient).

The $\tilde{f}_\mu(b) = 0$ condition is implemented by means of the representation:

$$\delta[\tilde{f}_\mu(b)] = \int \mathcal{D}A e^{\frac{i}{\sqrt{2\pi}} \int d^3x A_\mu \tilde{f}_\mu(b)}. \tag{11}$$

Thus,

$$\mathcal{Z}(s) = \int \mathcal{D}A \mathcal{D}b \mathcal{D}\xi \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_f(\bar{\psi}, \psi; s+c+b) + \frac{i}{\sqrt{2\pi}} \int d^3x A_\mu \tilde{f}_\mu(b)}. \tag{12}$$

Finally, we make the shift $b \rightarrow b - c - s$, to obtain:

$$\mathcal{Z}(s) = \int \mathcal{D}A \mathcal{D}b \mathcal{D}\xi e^{-W(b) + \frac{i}{\sqrt{2\pi}} \int d^3x A_\mu [\tilde{f}_\mu(b) - \tilde{f}_\mu(c) - \tilde{f}_\mu(s)]}, \tag{13}$$

where $W(b)$ denotes the effective action:

$$e^{-W(b)} = \det(\not{\partial} + i \not{b} + m), \tag{14}$$

which is to be evaluated with trivial boundary conditions, understanding by that that the region is the whole 2 + 1-dimensional spacetime, with the standard conditions for a vacuum to vacuum Euclidean transition amplitude.

This leads to a bosonized representation for the generating functional, which may be rendered as follows:

$$\mathcal{Z}(s) = \int \mathcal{D}A \mathcal{D}\xi e^{-S_B(A) - \frac{i}{\sqrt{2\pi}} \int d^3x c_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda - \frac{i}{\sqrt{2\pi}} \int d^3x s_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda}, \tag{15}$$

where the bosonized action $S_B(A)$ is determined by the expression:

$$e^{-S_B(A)} = \int \mathcal{D}b e^{-W(b) + \frac{i}{\sqrt{2\pi}} \int d^3x A_\mu \tilde{f}_\mu(b)}. \tag{16}$$

This leads to the bosonization rule:

$$J_\mu(x) \rightarrow \frac{1}{\sqrt{2\pi}} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda \equiv \mathcal{J}_\mu(x), \tag{17}$$

with a bosonized action $S_B(A)$ yet to be determined. Since that depends on the knowledge of $W(b)$, an exact expression of which is unknown, we use a possible approximation to it. The usual approach is to use a large-mass expansion, retaining just the leading contribution, a Chern-Simons (CS) term. This term is m -independent. Since we are interested here in dealing with a situation where there is another scale present, namely, the distance to the boundary, and to allow for a possible interplay, we will also include the next-to-leading term, which has the form of a Maxwell action:

$$W(b) = \int d^3x \left[\pm \frac{i}{4\pi} \epsilon_{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda + \frac{1}{48\pi|m|} f_{\mu\nu}(b) f_{\mu\nu}(b) + \mathcal{O}(1/m^3) \right], \tag{18}$$

where the parity-breaking term has a \pm sign, a reflection of the parity anomaly [10].

Inserting this into the expression for the bosonized action $S_B(A)$, (16), and working consistently up to the same order in the mass expansion, leads to:

$$S_B(A) = \int d^3x \left[\mp \frac{i}{2} A_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda + \frac{1}{24|m|} F_{\mu\nu}(A) F_{\mu\nu}(A) \right]. \tag{19}$$

Recalling then (15), the generating functional $\mathcal{Z}(s)$ requires the evaluation of an A_μ integral including the perfect-conductor constraint, what is implemented by the auxiliary field. That integral may be exactly calculated, for example by integrating out A_μ firstly, and then over ξ (a Gaussian).

The integral over A_μ , may be put in the form:

$$\int \mathcal{D}A e^{-S_B(A) - \frac{i}{\sqrt{2\pi}} \int d^3x A_\mu \epsilon_{\mu\nu\rho} (\partial_\nu c_\rho - \partial_\nu s_\rho)}, \tag{20}$$

where $S_B(A)$ is the action (19). It is convenient to write formally this action (using a shorthand notation for the integrals) in the following way

$$S_B(A) = \frac{1}{2} \int_{x,x'} A_\mu(x) \mathcal{K}_{\mu\mu'}(x, x') A_{\mu'}(x'), \tag{21}$$

with the kernel:

$$\mathcal{K}_{\mu\mu'}(x, x') = \pm i \epsilon_{\mu\lambda\mu'} \partial_\lambda^x + \frac{1}{6|m|} \left(-\partial_x^2 \delta_{\mu\mu'} + \partial_\mu^x \partial_{\mu'}^x \right) \delta(x - x'), \tag{22}$$

where we have explicitly indicated which argument of the δ function the derivatives act upon.

Note that the integral is a Gaussian in terms of A_μ , which is coupled to a vector field which has a vanishing divergence. To calculate the integral, it is convenient to decompose the kernel into orthogonal projectors; that can be done by starting from the fact that it can be written in terms of the Fourier space tensors:

$$P_{\mu\nu}(k) = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \quad Q_{\mu\nu}(k) = \epsilon_{\mu\lambda\nu} \frac{k_\lambda}{|k|}. \tag{23}$$

These tensors satisfy relations which in a matrix notation, adopt the form:

$$P^2 = P, \quad Q^2 = -P, \quad PQ = QP = Q. \tag{24}$$

They can then be used to build a complete set of orthogonal projectors for the space of 3×3 Hermitian matrices, which naturally arise in the Fourier representation. Their orthogonality allows one to deal with each invariant subspace separately, decomposing the original problem a set of one-dimensional decoupled problems.

Taking into account the relations above, we see that, defining $P^\pm \equiv \frac{P \pm iQ}{2}$ and $P' \equiv I - P$ (I denotes the identity matrix):

$$\begin{aligned}
 P^+ + P^- + P' &= I, \quad (P^\pm)^2 = P^\pm, \quad P'^2 = P', \\
 P^+ P^- &= P^- P^+ = P^\pm P' = P' P^\pm = 0.
 \end{aligned}
 \tag{25}$$

Then, using the Fourier representation, we have for the kernel:

$$\mathcal{K} = \pm |k| (P^+ - P^-) + \frac{k^2}{6|m|} (P^+ + P^-),
 \tag{26}$$

again in a matrix notation. Gauge fixing can be implemented by adding a term $\frac{\lambda}{2}(\partial \cdot A)^2$ to the bosonized action. This amounts to adding to \mathcal{K} an extra term:

$$\mathcal{K} \rightarrow \mathcal{K}' = \mathcal{K} + \lambda k^2 P'.
 \tag{27}$$

Then, the integral over A_μ yields:

$$\begin{aligned}
 &\int \mathcal{D}A e^{-S_B(A) - \frac{i}{\sqrt{2\pi}} \int d^3x A_\mu \epsilon_{\mu\nu\rho} (\partial_\nu c_\rho - \partial_\nu s_\rho)} \\
 &= \exp \left\{ -\frac{1}{4\pi} \int_{x,x'} \epsilon_{\mu\nu\rho} \partial_\nu (c_\rho(x) \right. \\
 &\quad \left. - s_\rho(x)) [K']_{\mu\mu'}^{-1}(x, x') \epsilon_{\mu'\nu'\rho'} \partial_{\nu'} (c_{\rho'}(x') - s_{\rho'}(x')) \right\},
 \end{aligned}
 \tag{28}$$

where, using the algebraic relations satisfied by the projectors, we see that, in Fourier space,

$$\begin{aligned}
 [K']^{-1} &= (\pm |k| + \frac{k^2}{6|m|})^{-1} P^+ + \left(\mp |k| + \frac{k^2}{6|m|} \right)^{-1} P^- \\
 &\quad + \frac{1}{\lambda k^2} P'.
 \end{aligned}
 \tag{29}$$

It may be seen that P' does not contribute, because $\epsilon_{\mu\nu\rho}(\partial_\nu c_\rho - \partial_\nu s_\rho)$ has zero divergence. Indeed, the result is independent of any gauge fixing, and becomes the exponential of a quadratic action. This quadratic action will evidently contain a term with two c_μ fields, one with two s_μ fields, and a term which mixes them both. The c_μ is dependent on the boundary (recall (4)).

The term quadratic in s_μ is independent of the boundary. There only remains to integrate out ξ , which is again a Gaussian. This produces a term which does depend on the boundary, since c_μ does.

Adding the previously described contributions, the result may be presented as follows:

$$\mathcal{Z}(s) = e^{-T(s)},
 \tag{30}$$

where

$$\begin{aligned}
 T(s) &= \frac{1}{2} \int_{x,x'} s_\mu(x) \Pi_{\mu\mu'}(x, x') s_{\mu'}(x') \\
 \Pi_{\mu\mu'} &= \Pi_{\mu\mu'}^{(1)}(x, x') + \Pi_{\mu\mu'}^{(2)}(x, x'),
 \end{aligned}
 \tag{31}$$

with $\Pi_{\mu\mu'}^{(1)}$ and $\Pi_{\mu\mu'}^{(2)}$ denoting qualitatively different contributions: $\Pi_{\mu\mu'}^{(1)}$ is identical to the contribution one would

obtain for a Dirac field in the absence of boundaries. $\Pi_{\mu\mu'}^{(2)}$, on the other hand, depends on the existence of the boundary. Therefore, it cannot be translation invariant along the x_2 coordinate. We have found it convenient to represent both $\Pi^{(1)}$ and $\Pi^{(2)}$ in terms of their Fourier transforms with respect to the x_\parallel coordinates (for which there is translation invariance).

Note that $\Pi_{\mu\mu'}(x, x')$ is the current-current correlation function, since it is what one gets by taking the functional derivatives with respect to the external sources which couple to the current.

There is a technical detail here: since there is translation invariance along just two of the three spacetime coordinates, and parity is broken, the usual procedure to integrate out Gaussians involving a gauge field had to be generalized. Indeed, following the approach of decomposing the quadratic form in the Gaussian integral into terms involving all the possible tensors compatible with the symmetry, and assuming that indices from the beginning of the Greek alphabet (α, α', \dots) run over the values 0 and 1, the explicit form of those terms (obtained by Gaussian integration) may be shown to be:

$$\Pi_{\mu\mu'}^{(1,2)}(x, x') = \int \frac{d^2 k_\parallel}{(2\pi)^2} e^{ik_\parallel \cdot (x_\parallel - x'_\parallel)} \tilde{\Pi}_{\mu\mu'}^{(1,2)}(k_\parallel; x_2, x'_2)
 \tag{32}$$

with:

$$\begin{aligned}
 \tilde{\Pi}_{\mu\mu'}^{(1)}(k_\parallel; x_2, x'_2) &= \frac{3}{\pi} |m| \\
 &\times \left\{ -\delta_\mu^\alpha \delta_{\mu'}^{\alpha'} \left[\delta_{\alpha\alpha'} \frac{(6m)^2}{2\sqrt{k_\parallel^2 + (6m)^2}} + \frac{k_\alpha k_{\alpha'}}{2\sqrt{k_\parallel^2 + (6m)^2}} \right] \right. \\
 &\quad \left. \pm 3i |m| \epsilon_{\alpha\alpha'} \operatorname{sgn}(x_2 - x'_2) \right\} \\
 &+ \delta_\mu^\alpha \delta_{\mu'}^2 \left[-\frac{i}{2} \operatorname{sgn}(x_2 - x'_2) \delta_{\alpha\alpha'} \pm \frac{3|m|}{\sqrt{k_\parallel^2 + (6m)^2}} \epsilon_{\alpha\alpha'} \right] k_{\alpha'} \\
 &+ \delta_\mu^2 \delta_{\mu'}^{\alpha'} \left[-\frac{i}{2} \operatorname{sgn}(x_2 - x'_2) \delta_{\alpha\alpha'} \mp \frac{3|m|}{\sqrt{k_\parallel^2 + (6m)^2}} \epsilon_{\alpha\alpha'} \right] k_{\alpha'} \\
 &+ \delta_\mu^2 \delta_{\mu'}^2 \frac{k_\parallel^2}{2\sqrt{k_\parallel^2 + (6m)^2}} \left. \right\} \times e^{-\sqrt{k_\parallel^2 + (6m)^2} |x_2 - x'_2|},
 \end{aligned}
 \tag{33}$$

and

$$\begin{aligned}
 \tilde{\Pi}_{\mu\mu'}^{(2)}(k_\parallel; x_2, x'_2) &= -\frac{3}{\pi} |m| \\
 &\times \left\{ \delta_\mu^\alpha \delta_{\mu'}^{\alpha'} \left[-\frac{(6|m|)^2}{2\sqrt{k_\parallel^2 + (6m)^2}} \delta_{\alpha\alpha'} + \frac{k_\alpha k_{\alpha'}}{2\sqrt{k_\parallel^2 + (6m)^2}} \right] \right.
 \end{aligned}$$

$$\begin{aligned}
 & \left. + \frac{(6m)^2}{\sqrt{k_{\parallel}^2 + (6m)^2}} \frac{k_{\alpha} k_{\alpha'}}{k_{\parallel}^2} \pm \frac{3|m|i}{k_{\parallel}^2} (k_{\alpha} \epsilon_{\alpha'\beta'} k_{\beta'} + k_{\alpha'} \epsilon_{\alpha\beta} k_{\beta}) \right] \\
 & + \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \left[-\frac{i}{2} \delta_{\alpha\alpha'} \pm \frac{3|m|}{\sqrt{k_{\parallel}^2 + (6m)^2}} \epsilon_{\alpha\alpha'} \right] k_{\alpha'} \\
 & - \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \left[-\frac{i}{2} \delta_{\alpha\alpha'} \pm \frac{3|m|}{\sqrt{k_{\parallel}^2 + (6m)^2}} \epsilon_{\alpha\alpha'} \right] k_{\alpha'} \\
 & + \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \frac{k_{\parallel}^2}{2\sqrt{k_{\parallel}^2 + (6m)^2}} \left. \right\} \times e^{-\sqrt{k_{\parallel}^2 + (6m)^2} |x_2 + x_2'|}, \tag{34}
 \end{aligned}$$

where we assumed that $x_2 > 0$ and $x_2' > 0$ (which corresponds to the region of interest). The explicit form of each contribution has been obtained by a lengthy but otherwise straightforward procedure, namely, by taking into account the form of the orthogonal projectors arising in the inverted kernel, and decomposing them in order to take into account the reduced symmetry in the system due to the dependence on the boundary. In particular, the integrals over the second component of the momentum have been performed, using residues, in order to express the result in a mixed Fourier representation.

We have explicitly checked that each term, $\tilde{\Pi}_{\mu\mu'}^{(1)}$ and $\tilde{\Pi}_{\mu\mu'}^{(2)}$, satisfies a Ward identity separately. Namely,

$$\begin{cases} ik_{\alpha} \tilde{\Pi}_{\alpha\alpha'}^{(1,2)}(k_{\parallel}; x_2, x_2') + \partial_{x_2} \tilde{\Pi}_{2\alpha'}^{(1,2)}(k_{\parallel}; x_2, x_2') = 0, \\ ik_{\alpha} \tilde{\Pi}_{\alpha 2}^{(1,2)}(k_{\parallel}; x_2, x_2') + \partial_{x_2} \tilde{\Pi}_{22}^{(1,2)}(k_{\parallel}; x_2, x_2') = 0. \end{cases} \tag{35}$$

Note that the full vacuum polarization, the sum of both terms, should satisfy that constraint, since the current is topologically conserved, and the vanishing of the normal current is compatible with conservation; indeed, it follows from current conservation and the divergence theorem. The fact that *each contribution satisfies the identity* can be deduced from the property that one of them satisfies that identity by itself, since it is identical to the one for a conserved current in the absence of boundaries.

In a mass expansion, and keeping just the leading and sub-leading terms, one sees that those two objects adopt the form:

$$\begin{aligned}
 \tilde{\Pi}_{\mu\mu'}^{(1)}(k_{\parallel}; x_2, x_2') &= \frac{1}{2\pi} \left\{ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} [\mp i \epsilon_{\alpha\alpha'} \partial_{x_2} \right. \\
 &+ \frac{1}{6|m|} (k_{\parallel}^2 \delta_{\alpha\alpha'} - k_{\alpha} k_{\alpha'} - \partial_{x_2}^2 \delta_{\alpha\alpha'}) \left. \right\} \\
 &+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \left(\mp \epsilon_{\alpha\alpha'} k_{\alpha'} + \frac{ik_{\alpha}}{6|m|} \partial_{x_2} \right) \\
 &+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \left(\pm \epsilon_{\alpha\alpha'} k_{\alpha''} + \frac{ik_{\alpha'}}{6|m|} \partial_{x_2} \right)
 \end{aligned}$$

$$+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \frac{k_{\parallel}^2}{6|m|} \left. \right\} \delta_m(x_2 - x_2'), \tag{36}$$

and

$$\begin{aligned}
 \tilde{\Pi}_{\mu\mu'}^{(2)}(k_{\parallel}; x_2, x_2') &= -\frac{1}{2\pi} \left\{ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} [\delta_{\alpha\alpha'} \right. \\
 &- 2 \frac{k_{\alpha} k_{\alpha'}}{k_{\parallel}^2} \mp \frac{i}{k_{\parallel}^2} (k_{\alpha} \epsilon_{\alpha'\alpha''} k_{\alpha''} + k_{\alpha'} \epsilon_{\alpha\alpha''} k_{\alpha''}) \partial_{x_2} \\
 &+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} [i \delta_{\alpha'\alpha''} \mp \epsilon_{\alpha'\alpha''}] k_{\alpha''} \\
 &+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} [(-i \delta_{\alpha\alpha''} \pm \epsilon_{\alpha\alpha''}) k_{\alpha''}] \\
 &+ \delta_{\mu}^{\alpha} \delta_{\mu'}^{\alpha'} \frac{k_{\parallel}^2}{6|m|} \left. \right\} \delta_m(x_2 + x_2'), \tag{37}
 \end{aligned}$$

where, in the above two expansions, we have introduced

$$\delta_m(x_2) \equiv 3me^{-6m|x_2|}, \tag{38}$$

which approximates Dirac's δ -function in the large- m limit. We have kept a number of terms which is consistent with the Ward identities (note that, to verify this, one must use the property that $-6m\delta_m(x_2)$ is an approximates of δ' .)

Let us apply the above result to the determination of the induced vacuum currents in the presence of a border and of an external electromagnetic field.

We begin by pointing out that $\Pi_{\mu\mu'}$ satisfies:

$$\Pi_{2\mu'}(x_{\parallel}, 0^+; x_{\parallel}', x_2') = 0, \quad \Pi_{\mu 2}(x_{\parallel}, x_2; x_{\parallel}', 0^+) = 0. \tag{39}$$

This is consistent with the boundary conditions imposed on the normal component of the current. Indeed, the vacuum expectation value of the current in the presence of an external gauge field a_{μ} , is given by:

$$\langle \mathcal{J}_{\mu}(x) \rangle|_a = \frac{\int \mathcal{D}A \delta_{\mathcal{M}}[\mathcal{J}_n] \mathcal{J}_{\mu}(x) e^{-\mathcal{S}_B(A) - i \int d^3x a_{\mu} \mathcal{J}_{\mu}}}{\int \mathcal{D}A \delta_{\mathcal{M}}[\mathcal{J}_n] e^{-\mathcal{S}_B(A)}}, \tag{40}$$

or,

$$\langle \mathcal{J}_{\mu}(x) \rangle|_a = -i \int d^3y \Pi_{\mu\nu}(x, y) a_{\nu}(y). \tag{41}$$

Thus, (39) guarantees that the expectation value of the normal component of the current vanishes on \mathcal{M} . An important point we would like to stress is that, in the presence of borders, the large mass expansion can be problematic, in the sense that the boundary conditions involve a limit, and the current correlation functions contain singular functions. Thus, we argue that in the presence of boundaries it is safer to take the large-mass limit only after calculating observables (for example, an induced current).

Let us apply the general result to the evaluation of the 0-component of the current, i.e., the charge density, in the presence of a point-like static magnetic vortex, located at $(x_1, x_2) = (h_1, h_2)$, which is minimally coupled to the current. Namely, an external field a_μ such that:

$$\partial_1 a_2(x) - \partial_2 a_1(x) = \phi \delta(x_1 - h_1) \delta(x_2 - h_2), \tag{42}$$

where ϕ denotes the magnetic flux piercing the plane at the vortex location.

We chose the gauge: $a_0 = 0, a_1 = 0,$ and $a_2 = \phi \theta(x_1 - h_1) \delta(x_2 - h_2)$ ($\theta \equiv$ Heaviside's step function) to find that $\langle \mathcal{J}_1 \rangle = \langle \mathcal{J}_2 \rangle = 0,$ and

$$\begin{aligned} \langle \mathcal{J}_0(x) \rangle|_a &= -i\phi \int_{-\infty}^{+\infty} dy_0 \int_h^{+\infty} dy_1 \Pi_{02}(x_0, x_1, x_2; y_0, y_1, h_2) \\ &= -\phi \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \frac{e^{ik_1(x_1-h_1)}}{k_1 - i\epsilon} \\ &\times \left[\tilde{\Pi}_{02}^{(1)}(0, k_1; x_2, h_2) + \tilde{\Pi}_{02}^{(2)}(0, k_1; x_2, h_2) \right]. \end{aligned} \tag{43}$$

Using the explicit form of $\tilde{\Pi}^{(1,2)},$ we see that:

$$\begin{aligned} \langle \mathcal{J}_0(x) \rangle|_a &= \mp \phi \frac{(3m)^2}{\pi} \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \frac{e^{ik_1(x_1-h_1)}}{\sqrt{k_1^2 + (6m)^2}} \\ &\times \left(e^{-\sqrt{k_1^2 + (6m)^2}|x_2-h_2|} - e^{-\sqrt{k_1^2 + (6m)^2}|x_2+h_2|} \right). \end{aligned} \tag{44}$$

From this, we see that the interplay between boundary conditions and parity breaking implies that the induced charge density vanishes at the boundary $x_2 = 0,$ since it is the sum of two contributions, one of them being the ‘image’ (in an electrostatic analogy) of the other.

We see that the infinite-mass limit yields the result,

$$\begin{aligned} \langle \mathcal{J}_0(x) \rangle|_a &\Rightarrow \mp \frac{\phi}{2\pi} \delta(x_1 - h_1) \\ &[\delta(x_2 - h_2) - \delta(x_2 + h_2)], \end{aligned} \tag{45}$$

which can be understood as containing the sum of two contributions: one that is the usual charge density induced by a flux, when there is a Chern-Simons term, and the other is due to an (*image*) contribution, in the electrostatic sense, and due to the presence of the conducting plane.

Let us also consider the induced vacuum current in the presence of a electric field of magnitude E in the direction of the x_2 coordinate. Using the gauge field choice $a_0(x_2) = -E x_2,$ it is straightforward to show that the only non-vanishing component of the current is along the x_1 direction: a parity-breaking effect. Since the gauge field is static and translation-invariant along $x_1,$ one sees that:

$$\langle \mathcal{J}_1(x_2) \rangle|_a = -i E \int_0^\infty dx'_2 x'_2 \tilde{\Pi}_{10}(0; x_2, x'_2). \tag{46}$$

Inserting the form of $\tilde{\Pi}_{10}(0; x_2, x'_2),$ we see that:

$$\langle \mathcal{J}_1(x_2) \rangle|_a = \mp \frac{3m^2}{\pi} E \delta_m(x_2), \tag{47}$$

i.e., a Hall current exponentially concentrated on the border.

Discussion

A first issue that we comment here is the form of the current-current correlation function, from the point of view of the fermionic theory. The contribution of a massive fermion may be written in terms of the fermion propagator $\tilde{S}_F(k_\parallel; x_2, x'_2),$ which satisfies bag-like boundary conditions. For the case at hand, that condition adopts the form:

$$(1 + \gamma_2) \tilde{S}_F(k_\parallel; 0^+, x'_2) = 0. \tag{48}$$

Therefore,

$$\begin{aligned} \tilde{\Pi}_{\mu\mu'}(k_\parallel; x_2, x'_2) &= - \int \frac{d^2 p_\parallel}{(2\pi)^2} \text{tr} \\ &\left[\gamma_\mu \tilde{S}_F(p_\parallel + k_\parallel; x_2, x'_2) \gamma_{\mu'} \tilde{S}_F(p_\parallel; x'_2, x_2) \right]. \end{aligned} \tag{49}$$

Following the massive version of the procedure followed in [11] for the massless case, it is rather straightforward to show that the fermion propagator is given by:

$$\begin{aligned} \tilde{S}_F(p_\parallel + k_\parallel; x_2, x'_2) &= \tilde{S}_F^0(p_\parallel + k_\parallel; x_2, x'_2) + \tilde{S}_F^1(p_\parallel + k_\parallel; x_2, x'_2), \end{aligned} \tag{50}$$

where

$$\begin{aligned} \tilde{S}_F^0(p_\parallel; x_2, x'_2) &= \frac{1}{2} [\gamma_2 \text{sign}(x_2 - x'_2) + U(p_\parallel)] \\ &e^{-\omega(p_\parallel)|x_2-x'_2|} \\ \tilde{S}_F^1(p_\parallel; x_2, x'_2) &= \frac{1}{2} \frac{1 - U(p_\parallel)}{1 + U(p_\parallel)} [U(p_\parallel) - \gamma_2] U(p_\parallel) \\ &e^{-\omega(p_\parallel)|x_2+x'_2|}, \end{aligned} \tag{51}$$

where $\omega(p_\parallel) = \sqrt{p_\parallel^2 + m^2},$ and $U(p_\parallel) = (-i \not{p}_\parallel + m)/\omega(p_\parallel).$ Besides the standard, perturbative contribution of a massive fermion, one should include the parity-anomaly term. The form of the anomalous contribution, on the other hand, is again a local Chern-Simons term. Indeed, it may only proceed from the UV-divergent part of the calculation. And that corresponds to a fermion loop involving just the \tilde{S}_F^0 term, in the large-mass limit. Indeed, UV divergences appear in the coincidence ($x_2 \rightarrow x'_2$) limit, and \tilde{S}_F^1 has large-momentum (exponential) suppression for any $x_2, x'_2 > 0.$ At

the border, it may indeed contribute with a localized contribution, which is the form of the terms we have found in the bosonized form of the problem: indeed, $\tilde{\Pi}^{(2)}$ is non vanishing only when $x_2 = x'_2 = 0$ (in the large-mass limit).

We have found an expansion for the current-current correlation function which involves continuous approximations to the δ -function. This exhibits the role of the next to leading term included in the expansion, which here regulates the behaviour of the kernels in the effective action. Besides, note that the effective action for the boundary modes will be modified, by the inclusion of a width (set by $1/m$). In a mass expansion they will of course correspond to higher derivative terms in the Floreanini-Jackiw action, inherited from the extra terms on the induced action. The large mass limit has been considered in [7].

Finally, we have shown that the current-current correlation function may be expanded, for a large mass, in a way that preserves the Ward identity. One of the main lessons to be learnt by the present work, reflected in the concrete realization of the Ward identity in a mass expansion, is that the inclusion of the boundary condition after taking the large mass limit is justified. Indeed, one might have suspected that the presence of a strong spatial variation at the boundary could have put the procedure in jeopardy. We have shown that not to be the case, as long as the effective dual theory is considered, and no fermionic operators are introduced in terms of the bosonic field. Should one be able to do that, they should of course reflect a stronger dependence on the boundary, in particular on the fermionic boundary condition. That information is erased in the present treatment.

In recent years, dualities have been applied to analyze different condensed matter systems, like topological insulators, superconductors, and fractional quantum Hall effect systems [6], [12,13]. In these studies, bosonization in $2 + 1$ dimensions in the presence of a boundary like the one considered here may be relevant to the applications [14–17].

Acknowledgements This research was supported by ANPCyT, CONICET, UNCuyo and UNLP.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: The work presented does not have either experimental or numerical data associated, since it is of a theoretical nature.]

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Funded by SCOAP³.

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