# Trajectory Tracking Control of a PVTOL Aircraft Based on Linear Algebra Theory 

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#### Abstract

In this work, a trajectory tracking control design is proposed for the planar vertical takeoff and landing (PVTOL) aircraft using Linear Algebra Theory. The resulting control law has an easy implementation since not complex equation must be solved. The tracking is achieved providing convergence of the tracking errors to zero and simulations results show the good performance of the proposed controller.


Keywords - PVTOL aircraft, Markov Propriety, Linear algebra based controller, internal states, tracking error.

## I. Introduction

The Planar Vertical Take Off and Landing (PVTOL) aircraft represents a challenging nonlinear systems control problem due to the fact that this system is under-actuated, nonlinear, and is non-minimum-phase when controlling some specific outputs. The dynamic model of the PVTOL has three degrees of freedom and two control inputs. Although this particular system is a simplified aircraft with a minimal number of states and inputs, it retains the main features that must be considered when designing control laws for a real aircraft.

Numerous design methods for the flight control of the PVTOL aircraft model exist in the literature. In [1] a state feedback controller was designed by employing the use of a finite-time convergent control law; then, a finite-time observer such that the unknown states can be recovered in finite time and thus reducing the output feedback stabilization problem to the state feedback one. In [4] Olfati-Saber gave a globally stabilizing smooth static state feedback law in explicit form for the VTOL aircraft with arbitrary $\varepsilon \neq 0$ and in the same way as [5], it presents backstepping designs with the aid of special saturation functions. In 2011, [6] proposed an output feedback sliding mode controller of a PVTOL where a sliding mode observer is used to estimate the velocity. [7] proposed a feedback design based on a new bounded backstepping method with applicability to cases where the velocity measurements may no be available. In 2007, Huawen Ye [2] transformed a PVTOL subsystem into 4-D chains of integrators with nonlinear perturbations and then the bounded control for the chain of integrators used in [3] was moderately
modified and applied to the perturbed case. [8] has used an optimal control approach to design a robust hovering control. Saif Al-Hiddabi [9] has studied the execution of a maneuver for which the aircraft is intended to follow a circular path in a vertical plane using the approach development in [10] which is based on a decomposition of the aircraft vertical dynamics and the aircraft horizontal and roll dynamics.

We have shown in the state of the art that numerous design methods for the flight control of PVTOL aircraft model exist in the literature and all of them improve the performance of the system control in a particular interest point of view, however the resulting control law has a relative implementation complexity. In this paper, we propose a tracking control of a PVTOL aircraft using a technique based on linear algebra theory to achieve an easy implementation control law. The convergence to zero of tracking errors related to all system states variables are taken into account and demonstrated.

The paper is organized as follows. In section II some necessary mathematical preliminaries are briefly mentioned. In section III the PVTOL aircraft model is described. In section IV the proposed linear algebra based controller is presented. In section V the convergence of tracking errors to zero is demonstrated. Section VI shows the performance of the proposed controller through simulations results and finally in section VII brief conclusions are commented.

## II. PRELIMINARIES

Here some mathematical preliminaries are briefly mentioned to have a better understanding of the proposed controller.

## A. Markov Property

A process with a Markov property means that, given the present state, future states are independent of the past states.

In the case that one process takes discrete values and is indexed by a discrete time, this can be reformulated as follows:

$$
R\left(X_{n}=x_{n} \mid X_{(n-1)}=x_{(n-1)} \ldots X_{0}=x_{0}\right)=R\left(X_{n}=x_{n} \mid X_{(n-1)}=x_{(n-1)}\right)
$$

In other words, the description of the present state fully captures all the information that could influence the future evolution of the process [19].

## B. Theory of simultaneous linear equations

Let us consider the problem of solving the system of an equations $A x=b$, where $A$ is an $m \times n$ matrix and $b$ an $m \times 1$ vector. The system will have solution if and only if it is possible to express the vector $b$ as a linear combination of the columns of $A$, that is if $b$ belongs to the column space of $A$. The column space of $A$ is a subspace constructed with all combinations of the columns of $A$. For each $m \times n$ matrix there will be a subspace of $\mathfrak{R}^{m}$ [17].

## III. THE PVTOL AIRCRAFT MODEL

We are interested in a trajectory tracking control of a PVTOL aircraft. This system has the natural restriction of movement in a vertical-lateral plane as shown in fig. 1. The aircraft states are the position of the aircraft center of mass $x$, $y$, the roll angle $\theta$ of the aircraft, and the corresponding velocities $\dot{x}, \dot{y}, \dot{\theta}$. The control inputs $u_{1}$ and $u_{2}$ are respectively, the thrust (directed out the bottom of the aircraft) and the rolling moment about the aircraft center of mass.


Fig. 1 Front view of PVTOL aircraft.
The PVTOL aircraft dynamics from fig. 1 is modelled by the following equations:

$$
\begin{align*}
& -m \ddot{x}=-U_{1} \sin (\theta)+\varepsilon_{0} U_{2} \cos (\theta)  \tag{1}\\
& -m \ddot{y}=-U_{1} \cos (\theta)+\varepsilon_{0} U_{2} \sin (\theta)-m g  \tag{2}\\
& J \ddot{\theta}=U_{2} \tag{3}
\end{align*}
$$

Dividing (2) and (3) by $m g$ and (4) by $J$ we obtain:

$$
\begin{align*}
& \frac{d^{2}}{d t^{2}}\left[\begin{array}{c}
-x / g \\
-y / g
\end{array}\right]=\left[\begin{array}{cc}
-\sin (\theta) & \cos (\theta) \\
\cos (\theta) & \sin (\theta)
\end{array}\right]\left[\begin{array}{c}
U_{1} / m g \\
\frac{\varepsilon_{0} J U_{2}}{m g J}
\end{array}\right]+\left[\begin{array}{c}
0 \\
-1
\end{array}\right]  \tag{4}\\
& \ddot{\theta}=U_{2} / J \tag{5}
\end{align*}
$$

Where $J$ is the mass moment of inertia about the axis through the aircraft center of mass and along the fuselage and
$m g$ represents the gravitational force exerted on the aircraft center of mass. If we define:

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
-x / g \\
-y / g
\end{array}\right] ;\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]=\left[\begin{array}{c}
U_{1} / m g \\
U_{2} / J
\end{array}\right] \text { and } \varepsilon=\frac{\varepsilon_{0} J}{m g}
$$

Then, the rescaled dynamics becomes:

$$
\begin{align*}
& {\left[\begin{array}{c}
\ddot{x} \\
\ddot{y}
\end{array}\right]=\left[\begin{array}{ll}
-\sin (\theta) & \cos (\theta) \\
\cos (\theta) & \sin (\theta)
\end{array}\right]\left[\begin{array}{c}
u_{1} \\
\varepsilon u_{2}
\end{array}\right]+\left[\begin{array}{c}
0 \\
-1
\end{array}\right]}  \tag{6}\\
& \ddot{\theta}=u_{2} \tag{7}
\end{align*}
$$

For our PVTOL system, the outputs are the position of the aircraft center of mass $x$ and $y$. The rolling angle $\theta$, his derivate $\dot{\theta}$ and $\dot{x}, \dot{y}$, are internal states.

The parameter $\varepsilon$ is a small coefficient which characterizes the coupling between the rolling moment and the lateral acceleration of the aircraft which is usually negligible and not always well-known. Therefore, it is possible to suppose that $\varepsilon=0,[11],[12],[13]$, then:

$$
\left\{\begin{array}{l}
\ddot{x}=-\sin (\theta) u_{l}  \tag{8}\\
\ddot{y}=\cos (\theta) u_{l}-1 \\
\ddot{\theta}=u_{2}
\end{array}\right.
$$

On the other hand, several authors [14], [15], [16] have shown that by an appropriate change of coordinates it is possible to obtain a representation of the system without the coupling $\varepsilon$ term.

An $n_{t h}$ order differential equation can be converted into an n -dimensional system of first order equations. There are various reasons for doing this, one being that a first order system is much easier to solve numerically (using computer software) and most differential equations you encounter in "real life" (physics, engineering etc) don't have nice exact solutions. Taking this into account, the representation of the system in spaces states become a useful tool. For the PVTOL aircraft system the representation is shown bellow.

$$
\left\{\begin{array}{l}
x_{1}=x \rightarrow \dot{x}_{1}=x_{2}  \tag{9}\\
x_{2}=\dot{x} \rightarrow \dot{x}_{2}=-\sin \left(x_{5}\right) u_{1} \\
x_{3}=y \rightarrow \dot{x}_{3}=x_{4} \\
x_{4}=\dot{y} \rightarrow \dot{x}_{4}=\cos \left(x_{5}\right) u_{1}-1 \\
x_{5}=\theta \rightarrow \dot{x}_{5}=x_{6} \\
x_{6}=\dot{\theta} \rightarrow \dot{x}_{6}=u_{2}
\end{array}\right.
$$

Note that the $x_{1}$ state represents the $x$ lateral displacement of the aircraft and $x_{3}$ represents the vertical $y$ displacement and these are the variables to be controlled.

## IV. LINEAR ALGEBRA BASED CONTROLLER

Let us consider the general following differential equation:

$$
\begin{equation*}
\dot{y}=f(y, u, t) \quad ; y(0)=y_{0} \tag{10}
\end{equation*}
$$

Where $y$ represents the output of the system to be controlled, $u$ the control action, and $t$ the time. The values of $y(t)$ at discrete time $t=n T_{0}$, where $T_{0}$ is the sampling period, and $n \in\{0,1,2,3, \ldots\}$ will be denoted as $y_{n}$. Thus, when computing $y_{(n+1)}$ by knowing $y_{(n)}$, (10) should be integrated over the time interval $n T_{0} \leq t \leq(n+1) T_{0}$ as follows:

$$
\begin{equation*}
y_{(n+l)}=y_{(n)}+\int_{n T_{0}}^{(n+l) T_{0}} f(y, u, t) d t \tag{11}
\end{equation*}
$$

There are several numerical integration methods to calculate $y_{(n+1)}[18]$. For instance, the Euler and trapezoidal method approaches can be used [(12) and (13) respectively].

$$
\begin{align*}
& y_{(n+1)} \cong y_{n}+T_{0} f\left(y_{n}, u_{n}, t_{n}\right)  \tag{12}\\
& y_{(n+1)} \cong y_{n}+\frac{T_{0}}{2}\left\{f\left(y_{n}, u_{n}, t_{n}\right)+f\left(y_{n+1}, u_{n+1}, t_{n+1}\right)\right\} \tag{13}
\end{align*}
$$

Where $y_{n+1}$ on the right-hand side of (13) is not known and, therefore, can be estimated by (11). The use of numerical methods in the simulation of the system is based mainly on the possibility to determine the state of the system at instant $n$ +1 from the state, the control action, and other variables at instant $n$ (Markov property) [19]. So, $y_{n+1}$ can be substituted by the desired trajectory and then the control action to make the output system evolve from the current value $y_{n}$ to the desired one can be calculated. To accomplish this, it is necessary to solve a system of linear equations for each sampling period, as we show later. This work proposes applying this approximation to the model of a PVTOL aircraft and thus obtaining the control action that enables the aircraft to follow a preestablished trajectory during its navigation.

Through the Euler's approximation of the PVTOL aircraft system (9), the following set of equations is obtained:

$$
\left\{\begin{array}{l}
x_{1(n+l)}=x_{1(n)}+T_{0} x_{2(n)}  \tag{14}\\
x_{2(n+l)}=x_{2(n)}-T_{0} \sin \left(x_{5(n)}\right) u_{1(n)} \\
x_{3(n+l)}=x_{3(n)}+T_{0} x_{4(n)} \\
x_{4(n+1)}=x_{4(n)}+T_{0}\left\{\cos \left(x_{5(n)}\right) u_{l(n)}-1\right\} \\
x_{5(n+l)}=x_{5(n)}+T_{0} x_{6(n)} \\
x_{6(n+1)}=x_{6(n)}+T_{0} u_{2(n)}
\end{array}\right.
$$

This can be expressed in vectorial form as:

$$
\left[\begin{array}{l}
x_{l(n+1)}-x_{l(n)}  \tag{15}\\
x_{2(n+1)}-x_{2(n)} \\
x_{3(n+1)}-x_{3(n)} \\
x_{4(n+1)}-x_{4(n)} \\
x_{5(n+1)}-x_{5(n)} \\
x_{6(n+1)}-x_{6(n)}
\end{array}\right]=T_{0}\left\{\left[\begin{array}{c}
x_{2(n)} \\
0 \\
x_{4(n)} \\
-1 \\
x_{6(n)} \\
0
\end{array}\right]+\left[\begin{array}{cc}
0 & 0 \\
-\sin \left(x_{5(n)}\right) & 0 \\
0 & 0 \\
\cos \left(x_{5(n)}\right) & 0 \\
0 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{l(n)} \\
u_{2(n)}
\end{array}\right]\right\}
$$

If the desired trajectory $\left[x d_{1(n+1)}, x d_{3(n+1)}\right]^{T}$ is known, then $\left[x_{1(n+1)}, x_{3(n+1)}\right]^{T} \quad$ in (15) can be substituted by $\left[x d_{1(n+1)}, x d_{3(n+1)}\right]^{T}$ and thus it will be possible to calculate the control actions $u_{1}, u_{2}$ necessary to make the aircraft go from the current state $\left[x_{1(n)}, x_{3(n)}\right]^{T}$ to the desired one $\left[x d_{1(n+1)}, x d_{3(n+1)}\right]^{T}$. So we can write:
$\left[\begin{array}{l}x d_{l(n+1)}-x_{l(n)} \\ x_{2(n+1)}-x_{2(n)} \\ x d_{3(n+1)}-x_{3(n)} \\ x_{4(n+1)}-x_{f(n)} \\ x_{5(n+1)}-x_{S(n)} \\ x_{6(n+1)}-x_{\sigma(n)}\end{array}\right]=T_{0}\left\{\left[\begin{array}{c}x_{2(n)} \\ 0 \\ x_{f(n)} \\ -1 \\ x_{\sigma(n)} \\ 0\end{array}\right]+\left[\begin{array}{cc}0 & 0 \\ -\sin \left(x_{S(n)}\right) & 0 \\ 0 & 0 \\ \cos \left(x_{5(n)}\right) & 0 \\ 0 & 0 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}u_{1(n)} \\ u_{2(n)}\end{array}\right]\right\}$

Then, from (16) we can operate to get the following system of linear equations:

$$
\underbrace{\left[\begin{array}{cc}
0 & 0  \tag{17}\\
-\sin \left(x_{5(n)}\right) & 0 \\
0 & 0 \\
\cos \left(x_{5(n)}\right) & 0 \\
0 & 0 \\
0 & 1
\end{array}\right]}_{A} \underbrace{\left[\begin{array}{l}
u_{1(n)} \\
u_{2(n)}
\end{array}\right]}_{u}=\underbrace{\frac{x}{T_{0}}-x_{2(n)}}_{b} \begin{array}{l}
\frac{x d_{1(n+1)}-x_{1(n)}}{T_{0}} \\
\frac{x d_{3(n+1)}-x_{2(n)}}{T_{0}}-x_{4(n)} \\
\frac{x_{4(n+1)}-x_{4(n)}}{T_{0}}-1 \\
\frac{x_{5(n+1)}-x_{5(n)}}{T_{0}}-x_{\sigma(n)} \\
\frac{x_{6(n+1)}-x_{6(n)}}{T_{0}}
\end{array}]
$$

Equation (17) represents a system of six linear equations, with two unknown variables $\left(u_{1}, u_{2}\right)$ and allows in each sampling instant, calculating the control actions in order that the aircraft achieves the desired trajectory. Note that if we denote the vector which is on the right side of (17) as $b$ and the vector that contains the control actions as $u$, the system can be written in the classic system equation form $A u=b$.

Now, the condition so that this system can have exact solution is that vector $b$ belong to column space of $A$. In order for the aircraft to follow a desired trajectory, the angular position must be calculated from de previous equations system and we denote it as $x_{5} r(n)$. The equation system expressed in (17) can be partitioned as show (18) and (19):

$$
\left\{\begin{array}{l}
x_{2(n)}=\frac{x d_{1(n+1)}-x_{1(n)}}{T_{0}}  \tag{18}\\
x_{4(n)}=\frac{x d_{3(n+1)}-x_{3(n)}}{T_{0}} \\
x_{6(n)}=\frac{x_{5(n+1)}-x_{S(n)}}{T_{0}}
\end{array}\right.
$$

$$
\left[\begin{array}{cc}
-\sin \left(x_{S(n)}\right) & 0  \tag{19}\\
\cos \left(x_{S(n)}\right) & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{(n n)} \\
u_{2(n)}
\end{array}\right]=\left[\begin{array}{c}
\frac{x_{2(n+1)}-x_{2(n)}}{T_{0}} \\
\frac{x_{4(n+1)}-x_{4(n)}}{T_{0}}+1 \\
\frac{x_{\sigma(n+1)}-x_{\sigma(n)}}{T_{0}}
\end{array}\right]
$$

From (19) it is possible to obtain:

$$
\begin{equation*}
\frac{-\sin \left(x_{5(n)}\right)}{\cos \left(x_{S(n)}\right)}=-\operatorname{tg}\left(x_{S(n)}\right)=\frac{x_{2(n+1)}-x_{2(n)}}{x_{A(n+1)}-x_{4(n)}+T_{0}} \tag{20}
\end{equation*}
$$

Equation (20) represents the necessary orientation (angle $\theta$ ) to make the linear system equations in (17) have exact solution and the aircraft tend to the reference trajectory.

In order for the tracking errors tend to zero in a smoothly way, (18), (19) and (20) are rewritten as:

$$
\begin{align*}
& \left\{\begin{array}{l}
x r_{2(n+1)}=\frac{x d_{l(n+1)}-K_{l}\left[x d_{l(n)}-x_{I(n)}\right]-x_{l(n)}}{T_{0}} \\
x r_{4(n+1)}=\frac{x d_{3(n+1)}-K_{3}\left[x d_{3(n)}-x_{3(n)}\right]-x_{3(n)}}{T_{0}} \\
x r_{6(n+1)}=\frac{x r_{5(n+1)}-K_{5}\left[x r_{5(n)}-x_{5(n)}\right]-x_{5(n)}}{T_{0}} \\
\operatorname{tg}\left(x r_{5(n)}\right)=-\frac{x r_{2(n+1)}-K_{2}\left[x r_{(n)}-x_{2(n)}\right]-x_{2(n)}}{x r_{4(n+1)}-K_{4}\left[x r_{4(n)}-x_{4(n)}\right]-x_{4(n)}+T_{0}}
\end{array} .\right. \tag{21}
\end{align*}
$$

$$
\underbrace{\left[\begin{array}{cc}
-\sin \left(x r_{5(n)}\right) & 0  \tag{23}\\
\cos \left(x r_{5(n)}\right) & 0 \\
0 & 1
\end{array}\right]}_{A_{l}} u=\underbrace{\left[\begin{array}{c}
\frac{x r_{2(n+1)}-K_{2}\left[x r_{2(n)}-x_{2(n)}\right]-x_{2(n)}}{T_{0}} \\
\frac{x r_{4(n+1)}-K_{4}\left[x r_{4(n)}-x_{4(n)}\right]-x_{4(n)}}{T_{0}}+1 \\
\frac{x r_{6(n+1)}-K_{6}\left[x r_{\sigma(n)}-x_{6(n)}\right]-x_{6(n)}}{T_{0}}
\end{array}\right]}_{b_{1}}
$$

Note that $x r_{i(n+1)}, i=2,4,6$ represent the speed references that the aircraft should have in the instant $(n+1)$ to achieve
the desired trajectory and $x r_{5}$ represents the necessary angular position.

The constants $0<K_{1}, K_{2}, K_{3}, K_{4}, K_{5}, K_{6}<1$ are design parameters of the proposed controller and they adjust in some way the convergence rate of actual states to the desired states. Using (21) and (22) it is possible to calculate $b_{1}$ and then solve the system $A_{1} u=b_{1}$ to get the necessary control action vector $u$ as follows:

$$
\begin{equation*}
A_{l}^{T} A u=A_{l}^{T} b_{l} \tag{24}
\end{equation*}
$$

Operating (24) we get:

$$
\left[\begin{array}{ll}
1 & 0  \tag{25}\\
0 & 1
\end{array}\right]\left[\begin{array}{l}
u_{l(n)} \\
u_{2(n)}
\end{array}\right]=\left[\begin{array}{c}
-\sin \left(x r_{5(n)}\right) b_{l(l, l)}+\cos \left(x r_{5(n)}\right) b_{l(2, l)} \\
b_{l(3, l)}
\end{array}\right]
$$

Where $b_{1(i, 1)}, i=1,2,3$ represents the columns of vector $b_{1}$. Finally, rewriting (25) the necessary control effort in order that the aircraft follow a desired trajectory, can be expressed as:

$$
\left\{\begin{array}{l}
u_{l(n)}=-\sin \left(x r_{5(n)}\right) b_{l(l, l)}+\cos \left(x r_{5(n)}\right) b_{l(2, l)}  \tag{26}\\
u_{2(n)}=\frac{x r_{\sigma(n+l)}-K_{6}\left[x r_{\sigma(n)}-x_{\sigma(n)}\right]-x_{\sigma(n)}}{T_{0}}
\end{array}\right.
$$

Were:

$$
\begin{align*}
& b_{1(l, l)}=\frac{x r_{2(n+l)}-K_{2}\left[x r_{2(n)}-x_{2(n)}\right]-x_{2(n)}}{T_{0}} \\
& b_{l(2, l)}=\frac{x r_{4(n+l)}-K_{4}\left[x r_{4(n)}-x_{4(n)}\right]-x_{4(n)}}{T_{0}}+1 \tag{27}
\end{align*}
$$

Note that the resulting control law has not complicated terms to solve because most of them are simple operations and they can be easily implemented in a simple microcontroller.

## V. CONVERGE TO ZERO OF TRAKING ERRORS

In this section, the convergence to zero of control errors is demonstrated. As it was shown in (14), the state $\mathrm{X}_{6(\mathrm{n+1)}}$ can be written as:

$$
x_{\sigma(n+l)}=x_{\sigma(n)}+T_{0} u_{2(n)}
$$

And replacing the proposed control law expressed in (26):

$$
\begin{align*}
& x_{\sigma(n+1)}=x_{\sigma(n)}+T_{0} \frac{x r_{\sigma(n+1)}-K_{6}\left[x r_{\sigma(n)}-x_{\sigma(n)}\right]-x_{\sigma(n)}}{T_{0}} \\
& x r_{\sigma(n+1)}-x_{\sigma(n+1)}=K_{\sigma}\left(x r_{\sigma(n)}-x_{\sigma(n)}\right) \tag{28}
\end{align*}
$$

Note that the right and left side of (28) are the error of $\mathrm{x}_{6}$ state (the real state respect to the calculated reference value) at the instant $(n+1)$ and $(n)$ respectively, so we can write:

$$
\begin{equation*}
e_{\sigma(n+1)}=K_{6} e_{\sigma(n)} \tag{29}
\end{equation*}
$$

The equation (29) represents the time evolution of error and as we know $0<K_{6}<1$ then $e_{\sigma(n)} \rightarrow 0$ when $n \rightarrow \infty$. A similar analysis is made for the state $\mathrm{x}_{5(\mathrm{n})}$.

$$
x_{5(n+l)}=x_{5(n)}+T_{0} x_{6(n)}
$$

Replacing $x_{\sigma(n)}$ by the respective reference value added to its errors, the previous equation is rewritten.

$$
\begin{equation*}
x_{5(n+1)}=x_{5(n)}+T_{0}\left(x r_{\sigma(n)}+e_{\sigma(n)}\right) \tag{30}
\end{equation*}
$$

Using (21) and (30)

$$
\begin{equation*}
x_{5(n+1)}=x_{5(n)}+T_{0}\left\{\left[\frac{x r_{5(n+1)}-K_{5}\left(x r_{5(n)}-x_{5(n)}\right)-x_{5(n)}}{T_{0}}\right]+e_{\sigma(n)}\right\} \tag{31}
\end{equation*}
$$

Operating

$$
\begin{equation*}
e_{5(n+1)}=k_{5} e_{5(n)}+T_{0} e_{6(n)} \tag{32}
\end{equation*}
$$

As we know $0<K_{5}<1$ and $e_{\sigma(n)} \rightarrow 0$ then $e_{5(n)} \rightarrow 0$ when $n \rightarrow \infty$.

For the states $x_{2(n)}$ and $x_{4(n)}$ the demonstration is a little different. From (14)

$$
\begin{equation*}
x_{2(n+l)}=x_{2(n)}-T_{0} \sin \left(x_{5(n)}\right) u_{1(n)} \tag{33}
\end{equation*}
$$

Using Taylor series the sin function can be expressed as follows:

$$
\begin{equation*}
\sin \left(x_{5(n)}\right)=\sin \left(x r_{5(n)}\right)+\cos \left(x_{5 \xi(n)}\right) e_{5(n)} \tag{34}
\end{equation*}
$$

Were:

$$
\begin{align*}
& e_{5(n)}=\left(x_{5(n)}-x r_{5(n)}\right)  \tag{35}\\
& x_{5 \xi(n)}=x r_{5(n)}+\xi\left(x_{5(n)}-x r_{5(n)}\right), \text { and } 0<\xi<1
\end{align*}
$$

Replacing (34) in (33) we get:

$$
\begin{equation*}
x_{2(n+1)}=x_{2(n)}-\left\{\sin \left(x r_{5(n)}\right)+\cos \left(x_{5 \xi(n)}\right) e_{5(n)}\right\} u_{l(n)} T_{0} \tag{36}
\end{equation*}
$$

Operating with (26) and (22) it is possible to get:

$$
\begin{equation*}
u_{l(n)} T_{o}=-\frac{b_{l(l, l)}}{\sin \left(x r_{(n)}\right)} \tag{37}
\end{equation*}
$$

Replacing (37) in (36) and operating

$$
\begin{equation*}
e_{2(n+l)}=K_{2} e_{2(n)}+\cos \left(x_{5 \xi(n)}\right) u_{l(n)} T_{0} e_{5(n)} \tag{38}
\end{equation*}
$$

Where:

$$
\begin{aligned}
& e_{2(n+1)}=x r_{2(n+1)}-x_{2(n+1)} \\
& e_{2(n)}=x r_{2(n)}-x_{2(n)}
\end{aligned}
$$

Equation (38) represents a time evolution of $x_{2}$ state error and, doing a similar analysis, the analog expression can be found to $x_{4}$ state:

$$
\begin{equation*}
e_{4(n+l)}=K_{4} e_{4(n)}+\sin \left(x_{5 \psi(n)}\right) u_{l(n)} T_{0} e_{5(n)} \tag{39}
\end{equation*}
$$

Where:

$$
\begin{aligned}
& x_{5 \psi(n)}=x r_{5(n)}+\psi\left(x_{5(n)}-x r_{5(n)}\right), \text { and } 0<\psi<1 \\
& e_{4(n+1)}=x r_{4(n+1)}-x_{4(n+1)} \\
& e_{4(n)}=x r_{4(n)}-x_{4(n)}
\end{aligned}
$$

Putting the equations (38) and (39) in matrix form:

$$
\left[\begin{array}{l}
e_{2(n+l)}  \tag{40}\\
e_{4(n+1)}
\end{array}\right]=\left[\begin{array}{cc}
K_{2} & 0 \\
0 & K_{4}
\end{array}\right]\left[\begin{array}{c}
e_{2(n)} \\
e_{4(n)}
\end{array}\right]+\underbrace{\left[\begin{array}{cc}
\cos \left(x_{5 \xi(n)}\right) u_{1(n)} T_{0} \\
-\sin \left(x_{5 \psi(n)}\right) u_{1(n)} & T_{0}
\end{array}\right]}_{M} e_{5(n)}
$$

The coefficients of matrix $M$ in (40) are non-linear and they are bounded (see appendix on [21]).The coefficients $0<K_{2}<1,0<K_{4}<1$ and as we showed before $e_{5(n)} \rightarrow 0$, then according to (40) $e_{2(n)} \rightarrow 0$ and $e_{4(n)} \rightarrow 0$ when $n \rightarrow \infty$.

The expression of state $x_{I}$ is defined by (14):

$$
x_{l(n+l)}=x_{l(n)}+T_{0} x_{2(n)}
$$

And replacing $x_{2(n)}$ in a similar way to (30) it is possible to get:

$$
\begin{equation*}
x_{l(n+1)}=x_{l(n)}+T_{0}\left(x r_{2(n)}+e_{2(n)}\right) \tag{41}
\end{equation*}
$$

$$
\begin{equation*}
x_{l(n+l)}=x_{l(n)}+T_{0} \frac{x d_{l(n+l)}-K_{l}\left[x d_{l(n)}-x_{l(n)}\right]-x_{l(n)}}{T_{0}}+T_{0} e_{2(n)} \tag{42}
\end{equation*}
$$

And then,

$$
\begin{equation*}
e_{1(n+l)}=k_{1} e_{1(n)}+T_{0} e_{2(n)} \tag{43}
\end{equation*}
$$

In the same way, the time evolution error of $x_{3}$ state can be found:

$$
\begin{equation*}
e_{3(n+1)}=k_{3} e_{3(n)}+T_{0} e_{4(n)} \tag{44}
\end{equation*}
$$

The coefficients $0<K_{1}<1,0<K_{3}<1$ and as we showed before $e_{2(n)} \rightarrow 0, e_{4(n)} \rightarrow 0$, then according to (43) and (44) $e_{1(n)} \rightarrow 0$ and $e_{3(n)} \rightarrow 0$ when $n \rightarrow \infty$.

## SIMULATIONS RESULTS

In this section, we present some simulation results using MATLAB and SIMULINK in order to observe the performance of the proposed control law. In the simulation we have considered two different examples with the same constants $K_{1}=0.96, K_{2}=0.7, K_{3}=0.96, K_{4}=0.8, K_{5}=$ $0.7, K_{6}=0.7$. For the first example we took a senoidal reference profile for the aircraft with a null initial condition of all states. The reference and real position of the PVTOL aircraft on the $x-y$ plane is shown in Fig. 1 and the time evolution of lateral and vertical displacement is plotted in Fig. 2. As we mentioned before, the internal dynamics (the states which are not directly reflected on the output system) should stay bounded along the resulting tracking. Fig. 3 shows the real internal states of the system (which are linear and angular velocities of the aircraft $x_{2}, x_{4}, x_{6}$ and the angular position $x_{5}$ ) and the generated references that should be followed $x_{2} r$, $x_{4} r, x_{5} r, x_{6} r$. Finally the necessary controls efforts to achieve the desired tracking are shown in Fig.4. Note that in each case the solid blue line in the figures is the real value of states and the doted red line the reference one.


Fig. 1 Reference and real position of the PVTOL aircraft on the $x-y$ plane

In the second example we are considering the initial condition $x_{1}(0)=6, x_{2}(0)=-2, x_{3}(0)=4, x_{4}(0)=1$, $x_{5}(0)=0.1, x_{6}(0)=0.9$ and the reference is just a point in
the origin, so the aircraft should go from $x=6, y=4$ to $x=0, y=0$. The trajectory described by the aircraft is practically a line between the initial and desired position which is shown in fig. 5 . The proposed control law has reached the shortest way to reach the desired point (line) and consequently a good way to save energy. The time evolution of lateral and vertical displacement is plotted in Fig. 6 which shows the finite time convergence of the state to the desired values. Fig. 7 shows the real internal states of the system and the generated references that should be followed and Fig. 8 the resulting controls laws.



Fig. 2 Time evolution of lateral and vertical displacement of the aircraft.


Fig. 3 Linear and angular velocities of the aircraft and angular position


Fig. 4 Control actions $u_{l}$ (thrust) and $u_{l}$ (rolling moment).


Fig. 5 Trajectory described by the aircraft with no null initial conditions.



Fig. 6 Time evolution of lateral and vertical displacement of the aircraft with no null initial conditions.


Fig. 7 Linear and angular velocities of the aircraft and angular position with no null initial conditions.


Fig. 8 Control actions $u_{l}$ (thrust) and $u_{l}$ (rolling moment) with no null initial conditions.

## VI. CONCLUSIONS

In this work, a new approach to control The Planar Vertical Take Off and Landing (PVTOL) aircraft by using linear algebra theory and numerical methods has been presented. The design of the proposed control law by using Linear Algebra tools is intuitive, and the final expression for the control signals, which will be directly implemented on the aircraft, is presented. The proposed control law leads the trajectory-tracking errors to zero as it was demonstrated. The presented simulations results show that the trajectory error between the desired and the real trajectory of the aircraft is very small and has a good tracking performance showing the feasibility of the developed algorithms. The required precision of the proposed numerical method for the system approximation is smaller than the one needed to simulate the behavior of the system. This is because, when the states for the feedback are available, in each sampling time, any difference from accumulative errors is corrected (e.g., rounding errors). Thus, the approach is used to find the best
way to go from one state to the next one according to the availability of the system model.

An appealing characteristic of this controller is its simple implementation in any programming language, besides the proposed methodology for the controller design can be applied to other types of systems.

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