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EXTENSION OF POLYNOMIALS AND JOHN'S THEOREM FOR SYMMETRIC TENSOR PRODUCTS

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ABSTRACT. We show that for every infinite-dimensional normed space E and every $k \geq 3$ there are extendible k-homogeneous polynomials which are not integral. As a consequence, we prove a symmetric version of a result of John.

1. INTRODUCTION

Every continuous linear functional defined over a normed space E can be extended to any superspace $F \supset E$, via the Hahn-Banach Theorem. For continuous k-homogeneous polynomials or k-linear forms (with $k \geq 2$) such an extension is not always possible. This is related to the fact that the projective tensor norm π_s is not injective. Extendible polynomials are defined in [15] as those admitting an extension to any larger space. They can be seen as the dual of the k-fold symmetric tensor product of E with an injective tensor norm η_s [7, 3, 15] given by

$$\bigotimes_{s,\eta_s}^{k} E \overset{1}{\hookrightarrow} \bigotimes_{s,\pi_s}^{k} \ell_\infty(B_{E'}).$$

Since the symmetric tensor norm ε_s is injective, every integral polynomial is extendible. Thus, it is natural to ask about the existence of extendible nonintegral polynomials. In this direction, [4] and [6] proved independently that every 2-homogeneous extendible polynomial over a cotype 2 space is integral. On the other hand, there exist extendible nonintegral k-homogeneous polynomials on ℓ_p, for every $p > 2$ and $k \geq 2$ (see, for example, [5]). These two facts focus the question on the existence of an infinite-dimensional space where all the extendible k-homogeneous polynomials are integral, for some $k \geq 3$. Recently, Pérez-García [16] proved that if $k \geq 4$ there is no such space. We fill in the gap by proving the same result for $k \geq 3$.

In [12], Grothendieck asked about the existence of nonnuclear locally convex spaces E and F such that the injective and projective norm are equivalent on $E \otimes F$ and conjectured a negative answer. Pisier [17] provided a counterexample exhibiting a Banach space P such that $P \otimes_\varepsilon P \simeq P \otimes_\pi P$. In [13, 14], John proved

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that such an example is not possible for tensor products of order $k \geq 3$. For locally convex spaces he showed that $\bigotimes_\varepsilon^k E \simeq \bigotimes_\pi^k E$ if and only if E is nuclear. For normed spaces, the conclusion also holds for tensor products of different spaces.

Theorem 1.1 ([13]). *If the norms ε and π are equivalent in the tensor product $\bigotimes_{i=1}^k E_i$, then all but two of the spaces $E_1, \dots, E_k$ are finite dimensional.*

The fact that extendible and integral k-homogeneous polynomials never coincide for $k \geq 3$ means, in the tensor setting, that the norms η_s and ε_s are never equivalent for k-fold symmetric tensor products. Therefore, the norms π_s and ε_s are not equivalent either, and thus we obtain the symmetric version of John's result for normed spaces.

We refer to [7], [9] and [10] for notation and terminology on tensor products, polynomials and multilinear mappings on normed spaces.

2. The results

In [2], Boas showed estimates for the Bohr radius on $\ell_p^n := (\mathbb{C}^n, \|\cdot\|_p)$. For the upper bounds he proved, for each p, the existence of a symmetric k-linear form on ℓ_p^n as in (2.1) below with "small" norm. For our purpose, we need a symmetric k-linear form having small norm on ℓ_2^n and ℓ_∞^n simultaneously. Therefore, we slightly change the proof of Boas to obtain the following lemma.

Lemma 2.1. *For each $n \geq 2$ there exists a symmetric k-linear form $T_n \in \mathcal{L}(^k\mathbb{C}^n)$ of the form*

$$T_n(x_1, \dots, x_k) = \sum_{j_1, \dots, j_k = 1}^{n} \varepsilon_j \cdot x_1(j_1) \cdots x_k(j_k), \tag{2.1}$$

where $\varepsilon_j = \pm 1$, $j = (j_1, \dots, j_k)$ and $\varepsilon_j = \varepsilon_{\sigma(j)}$ for all permutations σ that verifies

$$\|T_n\|_{\mathcal{L}(^k\ell_2^n)} \leq \sqrt{32k\log(6k)k!}\ n^{1/2}, \tag{2.2}$$

$$\|T_n\|_{\mathcal{L}(^k\ell_\infty^n)} \leq 2\sqrt{32k\log(6k)k!}\ n^{k/2+1/2}. \tag{2.3}$$

Proof. To each choice of $j_1 \leq j_2 \leq \cdots \leq j_k$ we assign a different Rademacher function r_j. We consider for each $t \in [0,1]$ the k-linear form on $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$ given by

$$F_t(x_1, \dots, x_k) = \sum_{j_1 \leq j_2 \leq \cdots \leq j_k} r_j(t) \sum_\sigma x_1(\sigma(j_1)) \cdots x_k(\sigma(j_k)).$$

In the proof of [2, Theorem 4], it is shown that for every positive constant $R_2, \lambda_2, R_\infty, \lambda_\infty$ the following inequalities hold:

$$\mu\left\{t \in [0,1] :\ \|F_t\|_{\mathcal{L}(^k\ell_2^n)} > 2\sqrt{2}R_2\right\} \leq 4(1+4k)^{2nk} e^{-R_2\lambda_2 + \frac{1}{2}\lambda_2^2 k!}, \tag{2.4}$$

$$\mu\left\{t \in [0,1] :\ \|F_t\|_{\mathcal{L}(^k\ell_\infty^n)} > 2\sqrt{2}R_\infty\right\} \leq 4(1+4k)^{2nk} e^{-R_\infty\lambda_\infty + \frac{1}{2}\lambda_\infty^2 k! n^k}, \tag{2.5}$$

where μ denotes the Lebesgue measure on $[0,1]$.

Choosing, as in [2], $R_2 = \sqrt{2k!\log(8(1+4k)^{2nk})}$ and $\lambda_2 = \frac{R_2}{k!}$, one obtains

$$\mu\left\{t \in [0,1] :\ \|F_t\|_{\mathcal{L}(^k\ell_2^n)} > 2\sqrt{2}R_2\right\} \leq \frac{1}{2}.$$

Also, for $R_\infty = \sqrt{2k!n^k \log(8(1+4k)^{2nk})}$ and $\lambda_\infty = \frac{R_\infty}{k!n^k}$, we have

$$\mu\left\{t \in [0,1]:\ \|F_t\|_{\mathcal{L}(^k\ell_\infty^n)} > 2\sqrt{2}R_\infty\right\} \leq \frac{1}{2}.$$

Since the right-hand side of inequality (2.5) is a decreasing function of R_∞ (for a fixed λ_∞), if we take $\widetilde{R}_\infty = 2R_\infty$, we obtain

$$\mu\left\{t \in [0,1]:\ \|F_t\|_{\mathcal{L}(^k\ell_\infty^n)} > 2\sqrt{2}\widetilde{R}_\infty\right\} < \frac{1}{2}.$$

Therefore, $\|F_t\|_{\mathcal{L}(^k\ell_2^n)} \leq 2\sqrt{2}R_2$ and $\|F_t\|_{\mathcal{L}(^k\ell_\infty^n)} \leq 2\sqrt{2}\widetilde{R}_\infty$ simultaneously for t in a positive measure set. For any such t we can define $T_n = F_t$, which verifies inequalities (2.2) and (2.3) since $8(1+4k)^{2nk} < (6k)^{2nk}$, for all $n, k \geq 2$. □

Since ℓ_∞^n is 1-complemented in any larger space, the usual and the extendible norms coincide in $\mathcal{P}(^k\ell_\infty^n)$. This enables us to derive from the previous lemma the following result (where $\mathcal{P}_e$ denotes the space of extendible polynomials and $\mathcal{P}_I$ the space of integral polynomials).

Lemma 2.2. *For each $n \in \mathbb{N}$ there exists a k-homogeneous polynomial $P_n \in \mathcal{P}(^k\ell_2^n)$ such that*

$$\|P_n\|_{\mathcal{P}_e(^k\ell_2^n)} \leq C\ n^{k/2+1/2} \qquad \textit{and} \qquad \|P_n\|_{\mathcal{P}_I(^k\ell_2^n)} \geq D\ n^{k-1/2},$$

where C and D are positive constants independent of n.

Proof. Let P_n be the k-homogeneous polynomial associated to the symmetric k-linear form defined in the previous lemma. We have

$$\begin{aligned} \|P_n\|_{\mathcal{P}_e(^k\ell_2^n)} &\leq \|P_n\|_{\mathcal{P}_e(^k\ell_\infty^n)}\ \|id : \ell_2^n \to \ell_\infty^n\|^k = \|P_n\|_{\mathcal{P}(^k\ell_\infty^n)} \\ &\leq \|T_n\|_{\mathcal{L}(^k\ell_\infty^n)} \leq C\ n^{k/2+1/2}. \end{aligned}$$

For the other inequality, we define the symmetric tensor $s_n \in \bigotimes_s^k \ell_2^n$ by

$$s_n = \sum_j \varepsilon_j\ e_{j_1} \otimes \cdots \otimes e_{j_k},$$

where ε_j are the same signs as in the definition of T_n. Since

$$n^k = P_n(s_n) \leq \|P_n\|_{\mathcal{P}_I(^k\ell_2^n)}\ \|s_n\|_{\bigotimes_{s,\varepsilon_s}^k \ell_2^n}$$

and

$$\|s_n\|_{\bigotimes_{s,\varepsilon_s}^k \ell_2^n} = \|P_n\|_{\mathcal{P}(^k\ell_2^n)} \leq \|T_n\|_{\mathcal{L}(^k\ell_2^n)},$$

from inequality (2.2) we obtain the desired result. □

Now we use Dvoretzky's theorem to extrapolate the situation in ℓ_2^n to an arbitrary infinite-dimensional normed space.

Theorem 2.3. *For any infinite-dimensional normed space E and any $k \geq 3$, there are extendible k-homogeneous polynomials on E which are not integral.*

Proof. Suppose $\mathcal{P}_e(^kE) = \mathcal{P}_I(^kE)$. Then, there must be a constant $M > 0$ such that $\|P\|_{\mathcal{P}_I(^kE)} \leq M\ \|P\|_{\mathcal{P}_e(^kE)}$ for all extendible polynomial on E. If $F \subset E$ is a subspace, any extendible polynomial on F extends to an extendible polynomial on E with the same extendible norm. Therefore, every extendible polynomial Q on F is integral and

$$\|Q\|_{\mathcal{P}_I(^kF)} \leq M\ \|Q\|_{\mathcal{P}_e(^kF)}. \tag{2.6}$$

By Dvoretzky's theorem, for each n there exists an n-dimensional subspace $F_n \subset E$ and an isomorphism $j_n : \ell_2^n \to F_n$ with $\|j_n\| = 1$ and $\|j_n^{-1}\| \leq 2$. Let $P_n \in \mathcal{P}(^k\ell_2^n)$ be as in the previous lemma and define $Q_n = P_n \circ j_n^{-1} \in \mathcal{P}(^kF_n)$. We have

$$\|Q_n\|_{\mathcal{P}_e(^kF_n)} \leq \|P_n\|_{\mathcal{P}_e(^k\ell_2^n)} \|j_n^{-1}\|^k \leq 2^k\ C\ n^{k/2+1/2}.$$

On the other hand,

$$D\ n^{k-1/2} \leq \|P_n\|_{\mathcal{P}_I(^k\ell_2^n)} \leq \|Q_n\|_{\mathcal{P}_I(^kF_n)}.$$

By (2.6), we obtain

$$D\ n^{k-1/2} \leq M\ 2^k\ C\ n^{k/2+1/2} \quad \text{for all } n \in \mathbb{N},$$

which is impossible if $k \geq 3$. □

Clearly, the above result is also valid for k-linear forms. Moreover, with slight modifications we can show the existence of nonintegral extendible k-linear forms on the product of different normed spaces, provided three of them are infinite dimensional. So we have:

Theorem 2.4. *If every extendible k-linear form on $E_1 \times \cdots \times E_k$ is integral, then all the spaces but two are finite dimensional.*

As an application of the previous theorems, first we note that Theorem 1.1 can be deduced from Theorem 2.4. Actually, we have

Corollary 2.5. *If the norms ε and η are equivalent in the tensor product $\bigotimes_{i=1}^k E_i$, then all but two of the spaces $E_1, \ldots, E_k$ are finite dimensional.*

As for the symmetric version of this result, Floret [11] proved it for $E = L_\infty(\mu)$. It is also simple to prove it for stable Banach spaces (i.e., spaces E such that $E \times E \simeq E$). Indeed, for these spaces, $\bigotimes_{s,\varepsilon_s}^k E \simeq \bigotimes_\varepsilon^k E$ and $\bigotimes_{s,\pi_s}^k E \simeq \bigotimes_\pi^k E$. Therefore, for stable spaces, the symmetric version follows from the result for the full tensor product. As a consequence of Theorem 2.3, we obtain this result for arbitrary normed spaces.

Corollary 2.6. *For $k \geq 3$, the norms ε_s and η_s are equivalent in the symmetric tensor product $\bigotimes_s^k E$ if and only if E is finite dimensional. In particular, $\bigotimes_{s,\varepsilon_s}^k E \simeq \bigotimes_{s,\pi_s}^k E$ if and only if E is finite dimensional.*

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