



Drill-string horizontal dynamics with uncertainty on the frictional force

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ARTICLE INFO

Article history:

Received 12 February 2012

Received in revised form

9 August 2012

Accepted 10 August 2012

Handling Editor: W. Lacarbonara

Available online 30 August 2012

ABSTRACT

This paper analyzes the dynamics of a horizontal drill-string. In this dynamics, the frictional forces between the column and the borehole are relevant and uncertain. A stochastic model is proposed for the frictional coefficient: a random field with exponential autocorrelation function. In the numerical analysis, the drill-string is modeled using the bar model (tension/compression), and is discretized by means of the finite element method. An oscillatory force (due to mud motor) is imposed on the system and there is a bit-rock interaction as the column moves forward. We propose a new way to measure the efficiency of the process: an output/input power ratio. The resultant random ratio is analyzed, and it turns out that it presents a bimodal characteristic, an unexpected result.

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1. Introduction

A drill-string is a slender structure used to drill rocks in search of oil. It is composed of thin tubes (drill-pipes) and thicker tubes (drill-collars) with a bit fixed at the end. In the beginning of oil exploration there were mainly vertical wells, but nowadays directional (horizontal) drilling is very common [1,2].

There is a reasonable amount of papers in the literature that analyze the dynamics of a drill-string performing vertical drilling. Each author uses a different approach to model the drill-string dynamics: one-mode approximation [3,4], beam model together with the finite element method [5–7], Cosserat theory [8], discrete system with two degrees of freedom [9].

Concerning horizontal drilling, there are few papers treating the modeling of horizontal drill-string. Actually, to the best of the authors knowledge, there is no previous work on the dynamics of horizontal drilling.

In the present paper, the part of the column composed by the drill-collars (called Bottom Hole Assembly—BHA) is modeled using the bar model (tension/compression), and is discretized by means of the finite element method [10]. The mass of the bit is taken into account and an exponential dissipation model is proposed to model the bit-rock interaction. Also at the bit, an oscillatory force (due to mud motor) is applied and, on the other side of the BHA, a constant force is imposed to push the structure forward.

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The goal of this paper is to analyze how uncertainties on the frictional forces propagate through the system. The frictional forces are difficult to model because there are many uncertainties related to them (friction model, points of contact, friction coefficient, etc.). A stochastic field with exponential correlation function is used to model the frictional coefficient. To approximate the statistics of the response, the Monte Carlo method [11] is applied.

There are few papers treating the stochastic problem of the drill-string dynamics; and they are all related to vertical wells. Among them, we may cite [7,12,13]. Ritto et al. [7] propose a probabilistic model for the bit-rock interaction model, Spanos et al. [12] analyze stochastic lateral forces at the bit, and Kotsonis and Spanos [13] analyze a random weight-on-bit using a simple two degrees of freedom drill-string model.

Another contribution of the present paper (besides the modeling of a horizontal drill-string and the analysis of stochastic frictional forces) is to analyze the performance of the system in terms of output/input power. The relationship between the output power and the input power is used to measure the efficiency of the system. The results are explored for different values of a slenderness parameter (length of the BHA divided by its outer diameter), different excitation frequencies, different levels of uncertainty of the frictional forces, etc.

This paper is organized as follows. The deterministic and stochastic models are presented in Sections 2 and 3, and the numerical results are discussed in Section 4. Finally, the concluding remarks are done in Section 5.

2. Deterministic model

This section describes the features of the model employed in the analysis. It should be noticed that this is a first approach to the horizontal drilling dynamics, therefore the model is the simplest possible continuous model. A sketch of the system analyzed is shown in Fig. 1. A constant force is imposed on the left side of the Bottom Hole Assembly (BHA), there is an oscillatory force imposed at the bit, and there is an interaction force between the bit and the rock on the right side of the structure. Besides these forces, gravity and the normal reaction associated to it (that, in the model, are balanced), and friction (the main force in the process) are acting on the drill-string, as shown in the figure.

Only the axial vibrations of the BHA are considered and the equation of motion is given by

$$\rho A \frac{\partial^2 u(x,t)}{\partial t^2} - EA \frac{\partial^2 u(x,t)}{\partial x^2} = f_{sta}(x,t) + f_{har}(x,t) + f_{bit}(\dot{u}(x,t)) + f_{fric}(\dot{u}(x,t)) + f_{mass}(\ddot{u}(x,t)), \quad (1)$$

where u is the axial displacement, ρ is the density of the column, A is the cross sectional area and E is the elasticity modulus. The space and time variables are $x \in [0, L]$, $t \in [0, T]$, where L is the length of the structure and T is the duration of the time analysis. The right-hand side of the equation presents the forces acting on the system, which are depicted in the sequence. A constant force is imposed at $x=0$

$$f_{sta}(x,t) = F_{sta} \delta(x), \quad (2)$$

where F_{sta} is the force amplitude, and δ is the Dirac-delta function. This force represents the effect of the drill-pipes on the BHA; it is assumed constant since we are computing the displacement of about 0.1 m of penetration.

A harmonic force is imposed on the system because the driving source of a horizontal drilling is the mud motor, which rotates about a given nominal rotational speed (in a steady operation). Therefore, the axial movement should be excited about the same frequency. The harmonic force f_{har} is given by

$$f_{har}(x,t) = F_o \sin(\omega_f t) \delta(x-L), \quad (3)$$

where F_o and ω_f are the amplitude and frequency of the harmonic force; this force is applied at $x=L$. The force (per unit length) related to the friction field f_{fric} is given by

$$f_{fric}(\dot{u}(x,t)) = -\mu(x)(\rho A)g \operatorname{sgn}(\dot{u}(x,t)), \quad (4)$$

where μ is the friction coefficient field, ρA is the mass per unit length of the structure and g is the gravity. Note that the friction depends on the sign of the speed, which makes it discontinuous. The force related to the bit mass is written as

$$f_{mass}(\ddot{u}(x,t)) = -m_{bit} \ddot{u}(x,t) \delta(x-L), \quad (5)$$

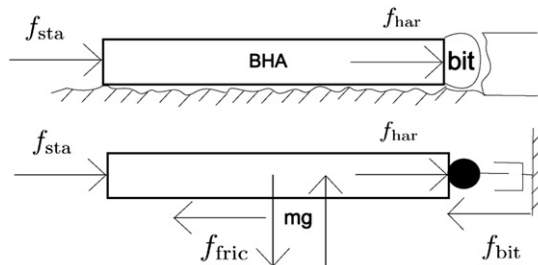


Fig. 1. Sketch of the system analyzed.

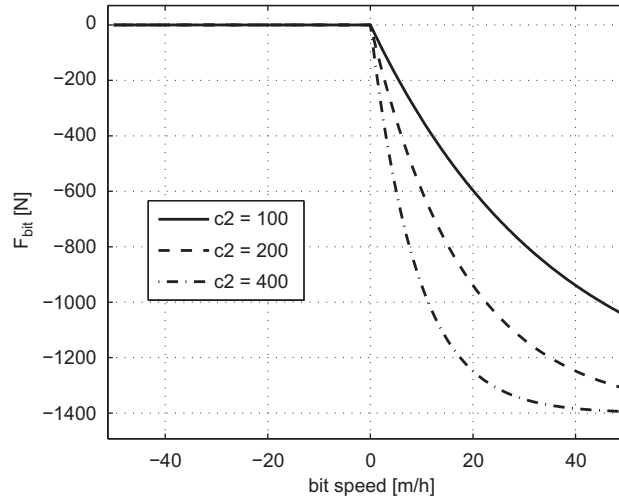


Fig. 2. Bit-rock interaction model for $c_1 = 1.4 \times 10^3$ N and different values of c_2 .

where m_{bit} is the bit mass located at $x=L$. Finally, we propose a model for the bit-rock interaction force f_{bit} . There is no work in the literature (at the best of the authors' knowledge) that considers the modeling of the bit-rock interaction of a horizontal drilling. In this first attempt to construct a bit-rock interaction model we are assuming that there is a limit force for the bit as the speed of the bit increases. We also assume that the bit force approaches this limit asymptotically (with exponential decay); the model was inspired by the work of Wanhein et al. [14], which was developed in another context. The exponential model is given by

$$\begin{cases} f_{\text{bit}}(\dot{u}(x,t)) = [c_1 \exp(-c_2 \dot{u}(x,t)) - c_1] \delta(x-L) & \text{for } \dot{u}(L,t) > 0, \\ f_{\text{bit}}(\dot{u}(x,t)) = 0 & \text{for } \dot{u}(L,t) \leq 0, \end{cases} \quad (6)$$

where c_1 and c_2 are the two constants of the bit-rock interaction; this force is applied at $x=L$. The constant c_1 is chosen such that the bit speed is close to a reasonable value (~ 20 m/h), and c_2 defines the nonlinear shape of the function in the range of analysis. Fig. 2 shows an example of the proposed bit-rock interaction model with $c_1 = 1.3 \times 10^3$ N and different values of c_2 . Initially there is no reaction force at the bit, but as long as the dynamics begins the column moves to the right and f_{bit} starts acting.

Using linear interpolation functions for each element that has one axial degree of freedom per node (see [10] for details), and assembling the global matrices, the deterministic system of equations are written as

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) &= \mathbf{f}_{\text{sta}} + \mathbf{f}_{\text{har}}(t) + \mathbf{f}_{\text{bit}}(\dot{\mathbf{u}}(t)) + \mathbf{f}_{\text{fric}}(\dot{\mathbf{u}}(t)), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \end{aligned} \quad (7)$$

in which \mathbf{u} is the displacement vector, \mathbf{M} is the mass matrix and \mathbf{K} is the stiffness matrix. The proportional damping matrix $\mathbf{C} = \alpha \mathbf{M}$ (α is a positive constant) is added *a posteriori* to the computational model. The initial conditions are given by \mathbf{u}_0 and \mathbf{v}_0 . The force vectors \mathbf{f}_{sta} , \mathbf{f}_{har} , \mathbf{f}_{bit} and \mathbf{f}_{fric} are related to the static, harmonic, bit-rock interaction and frictional forces, respectively. The force related to the bit mass is included in the matrix \mathbf{M} , since it depends linearly on the acceleration.

3. Stochastic model

The frictional force is hard to model, especially in the situation of horizontal drilling, where the points of contact and the characteristics of the surface may vary. Acknowledging this fact, we propose to treat this force as a random force, assuming a random frictional coefficient field. Thus, we define the random field $\{v(x) : x \in [0, L]\}$ as a collection of real-valued random variables from a probability space (Ω, \mathcal{F}, P) , where Ω is the sample space, \mathcal{F} is the σ -algebra and P is the probability measure.

It is assumed that v is a stationary truncated Gaussian random field on $[0, L]$ with exponential autocorrelation function

$$R(x_1, x_2) = \sigma^2 \exp\left(-\frac{|x_2 - x_1|}{b}\right), \quad (8)$$

where b is the correlation length, which measures the decay of the autocorrelation function. At this point we need to justify this hypothesis. Choi et al. [15] have developed a discrete stochastic field to tackle the problem of micro-slip. They derived the probability density function of the discrete friction coefficient field using the Maximum Entropy principle [16] and ended up with a truncated Gaussian field with a given correlation length. However, the random generation of such a

field is not trivial. We propose to use an exponential autocorrelation function and truncated Gaussian field that is very easy to sample from rejection of a Gaussian field; if the random field generated has one or more values lower than zero or greater than 1, this field is rejected. Actually, we need to do another approximation: the truncated Gaussian field is expanded with Karhunen–Loève expansion using standard Gaussian random variables. As long as the probability of having $v(x) \notin [0, 1]$ is very low, this approximation will be good (which is the case of the present analysis).

The stochastic field v is then expanded using the Karhunen–Loève expansion (developed by Karhunen [17] and Loève [18]; for some applications see [19,20]):

$$v(x, \xi) = \underline{\mu}(x) + \sum_{k=1}^N \sqrt{\lambda_k} Z_k(\xi) \phi_k(x), \quad (9)$$

where $\underline{\mu}$ is the mean value of the frictional coefficient, λ_k and ϕ_k are the k -th eigenvalue and k -th eigenvector of the autocorrelation function R , Z_k are independent standard Gaussian random variables and N defines the precision of this representation (as N increases the representation gets better). The random frictional force (per unit length) is written as

$$F_{\text{fric}}(\dot{u}(x, t), \xi) = -v(x, \xi)(\rho A)g \operatorname{sgn}(\dot{u}(x, t)). \quad (10)$$

And the stochastic system is written as

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{U}}(t, \xi) + \mathbf{C}\dot{\mathbf{U}}(t, \xi) + \mathbf{K}\mathbf{U}(t, \xi) &= \mathbf{f}_{\text{sta}} + \mathbf{f}_{\text{har}}(t) + \mathbf{f}_{\text{bit}}(\dot{\mathbf{U}}(t, \xi)) + \mathbf{F}_{\text{fric}}(\dot{\mathbf{U}}(t, \xi), \xi), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \mathbf{v}_0, \end{aligned} \quad (11)$$

where \mathbf{U} is the random response and \mathbf{F}_{fric} is the random vector related to the frictional random field. This is a nonlinear system of equations because the bit–rock interaction is nonlinear and because the frictional force is discontinuous (if the speed at a point changes sign, the friction force of this point also changes sign).

4. Numerical results

The data used in the simulation is given as follows. $E = 210$ GPa, $\rho = 7850$ kg/m³, $g = 9.81$ m/s², $D_i = 0.10$ m (inner diameter), $D_o = 0.15$ m (outer diameter), $L/D_o = 400$, $m_{\text{bit}} = 20$ kg, $\alpha = 0.2$, $c_1 = 1.4 \times 10^3$ N, $c_2 = 400$, $\underline{\mu} = 0.1$, $\sigma = 0.1 \times \underline{\mu}$, $b = 10$ and $\omega_f = 100 \times 2\pi/60$ rad/s, $t \in [0, 10]$ s, $\Delta t = 0.0001$ s, $N = 100$, $f_{\text{sta}} = 5500$ N, $F_o = 550$ N.

The system is discretized with 100 finite elements (after analyzing convergence on the natural frequencies) and the system of differential equations are integrated applying the Newmark integration scheme [10] assuming zero initial condition. It should be noticed that the values of the parameters of the bit–rock interaction model, and of the mean value of v are assumed to be known. These values should be the subject of an identification procedure, such as the one that was done in [21], but they are such that the bit axial speed is coherent with the drilling process.

The above values are for the reference simulation. The response of the system is analyzed for different frequencies of excitation ω_f (100, 150 and 200 RPM), slenderness parameters L/D_o (400, 600, 800), standard deviations σ (0.1 and $0.2 \times \underline{\mu}$), and correlation lengths b (10, 100).

Some deterministic results are presented before discussing the stochastic results. For the reference simulation, the first five natural frequencies of the structures are {42, 86, 120, 174, 220} Hz and the first five damping ratios are about 2 percent. The displacement of the bit is shown in Fig. 3; remark that the column is free to move forward. Fig. 4 shows the bit speed and its Fourier transform. It can be noted that besides the excitation frequency (100 RPM), some harmonics appear

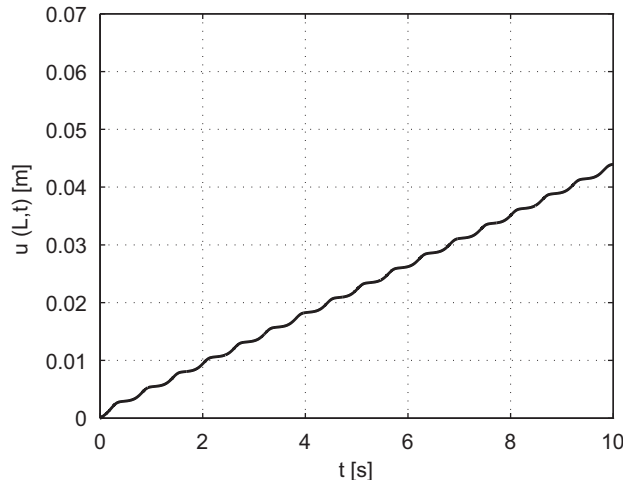


Fig. 3. Displacement of the bit.

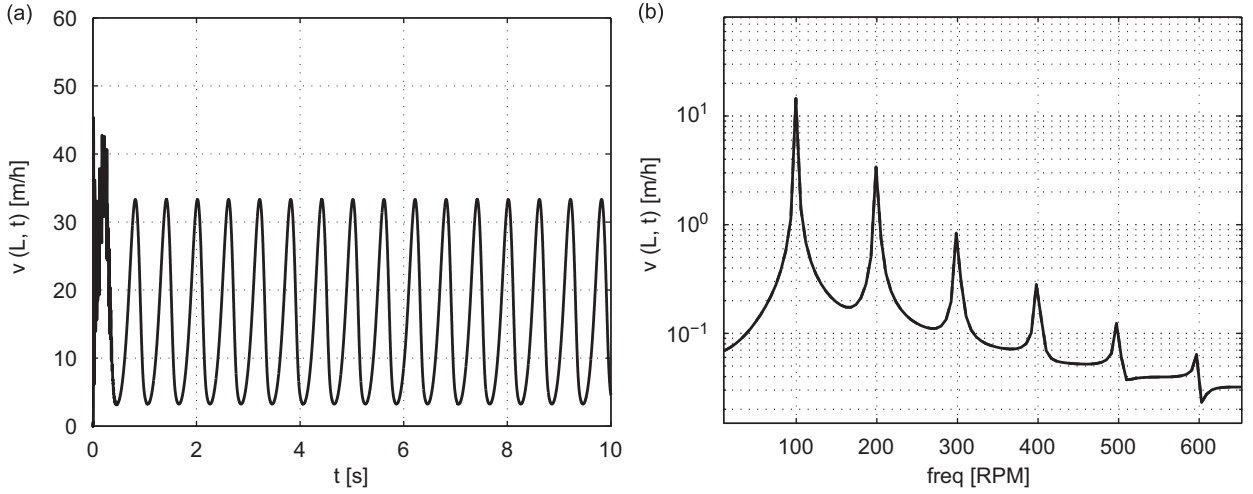


Fig. 4. Bit speed (a) and its Fourier transform (b).

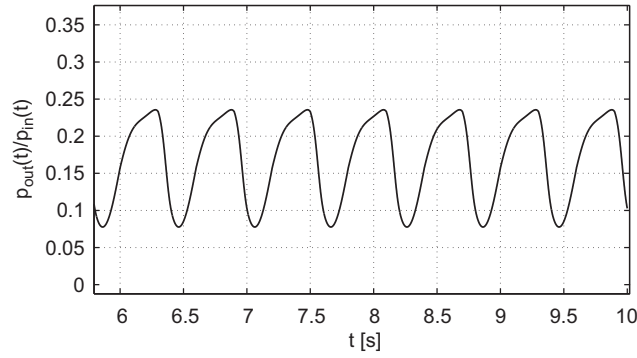


Fig. 5. Ratio between the input and output power.

in the frequency spectrum; this is due to the nonlinearity of the system. Furthermore, 4(a) shows that the system presents some oscillations. There is a phase when the speed is close to zero, and a phase when the speed increases its magnitude. This is close to stick-slip oscillations, but there is no stick (zero speed) phase.

We propose now a new way to analyze the efficiency of the horizontal drilling in the following way. We establish a measure that indicates the ratio between output and input power; if this parameter increases, the efficiency of the process also increases. The input and output power are given by

$$p_{in}(t) = f_{sta} \dot{u}(0, t) + f_{har}(t) \dot{u}(L, t) \quad \text{and} \quad p_{out}(t) = f_{bit}(t) \dot{u}(L, t). \quad (12)$$

And the average power is given by

$$\bar{p}_{in} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} (f_{sta} \dot{u}(0, t) + f_{har}(t) \dot{u}(L, t)) dt,$$

$$\bar{p}_{out} = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} f_{bit}(t) \dot{u}(L, t) dt. \quad (13)$$

Let us define the ratio between the output and input power in the time interval (t_0, t_1) by $y = \bar{p}_{out}/\bar{p}_{in}$. Fig. 5 shows for each instant the ratio $p_{out}(t)/p_{in}(t)$ for the reference simulation. It can be noted that the efficiency of the system (energy used for effective drilling) is lower than 25 percent. Of course, in a real operation, the efficiency may be lower due to other losses in the process.

Now we are going to focus on the stochastic results. Fig. 6 shows a typical convergence curve for the variance of the ratio $Y = \bar{p}_{out}/\bar{p}_{in}$, where $\text{conv}(n_s) = 1/n_s \sum_{i=1}^{n_s} Y_i^2$; in which n_s is the number of Monte Carlo simulations. This is a variable of interest, since it measures the efficiency of the system under analysis. An acceptable convergence is achieved when $n_s = 500$. The probability density function of the ratio Y is plotted for the different cases; see Figs. 7 and 8.

Fig. 7 shows that the performance of the system was similar for different excitation frequencies ω_f and for different correlation lengths b (Fig. 9 shows the autocorrelation function); the approximated probability density functions for these

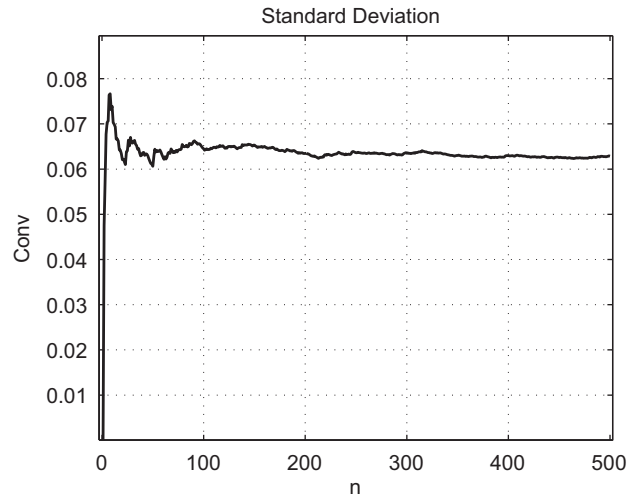


Fig. 6. Variance convergence of the random variable defined by the relation $Y = \bar{P}_{out} / \bar{P}_{in}$.

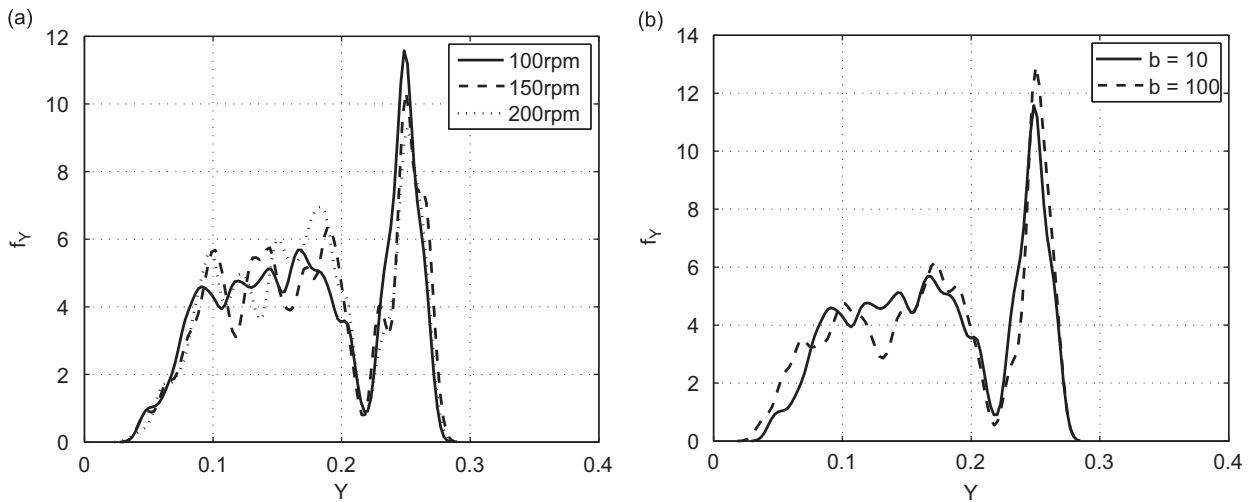


Fig. 7. Probability density functions of Y (a) varying the rotational speed ω_f and (b) varying the correlation length b .

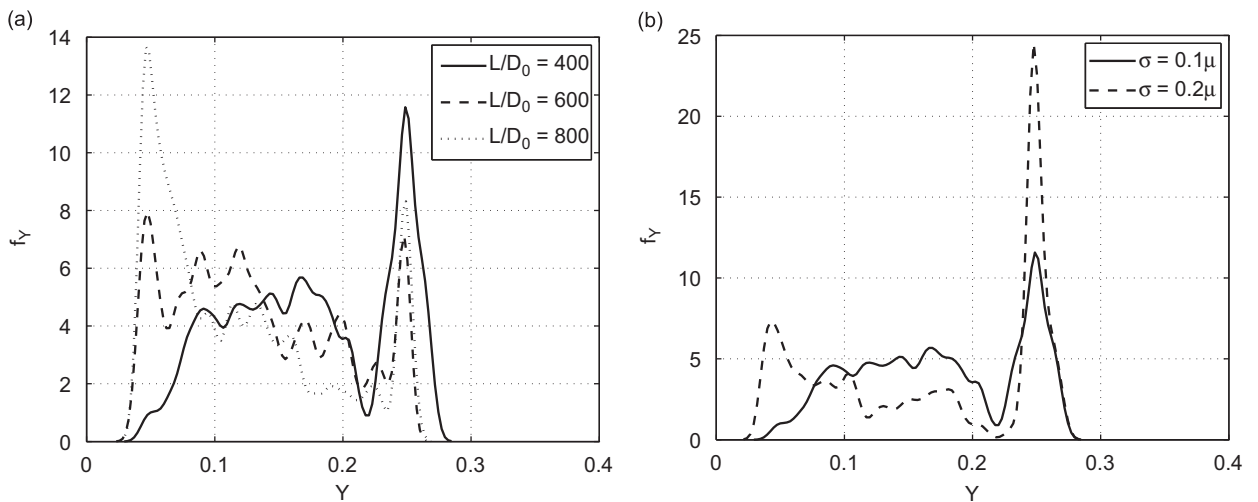


Fig. 8. Probability density functions of Y (a) varying the slenderness parameter L/D_0 and (b) varying the uncertainty level of the random field σ .

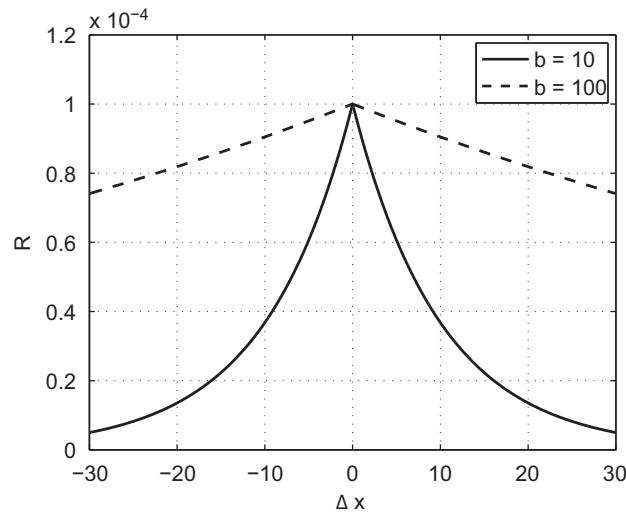


Fig. 9. Autocorrelation function.

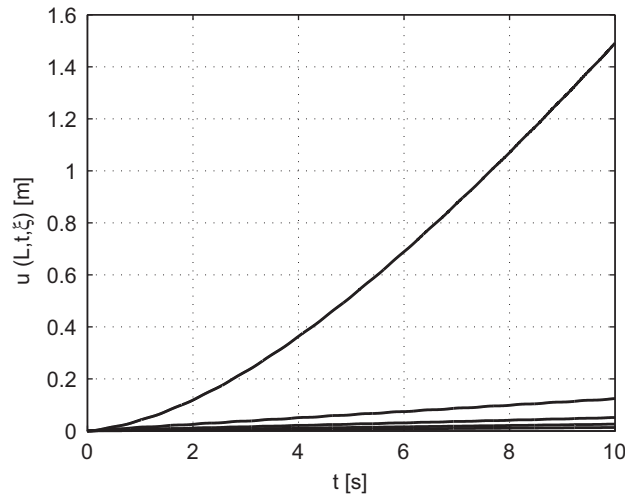


Fig. 10. Five Monte Carlo simulations of the bit displacement.

cases do not vary significantly. Therefore, for the frequencies analyzed, a variation in the load frequency does not change much the magnitude of the displacement of the axial response; it should be remarked that the first natural frequency of the system is far away from the excitation frequencies.

On the other hand, the probability density functions for different slenderness parameters L/D_0 and different levels of uncertainty of the frictional coefficient σ are quite different; see the shape of the probability density functions in Fig. 8. Fig. 8(a) shows that, as the slenderness parameter increases, the efficiency is reduced; note that the values are shifted to the left, indicating less efficiency. This result is expected, since there will be more column to resist the forward movement. A less intuitive result happens when the level of uncertainty (σ) is increased. Although the coefficient of variation of Y (σ_Y/μ_Y) increases from 36 percent to 50 percent, the resulting probability density function is more concentrated around two values (0.04 and 0.25).

About the shape of the probability density function of Y , it should be remarked that this shape is quite unexpected. We observe a bimodal distribution, indicating that there are two sets of behaviors. In this regard it makes no sense to compute the average of the Monte Carlo simulations, because it would simply be meaningless. What makes sense is to understand the behavior of the two sets of behaviors. A small set seems to be more efficient (y around 0.25) than the other (y from 0.05 to 0.20); see Fig. 7, for instance. To understand this point, let us plot some Monte Carlo simulations of the bit displacement; see Fig. 10. For some simulations the bit speed it kept small, and for others (fewer ones) the bit speed increases a lot. Investigating these results, it was noted that there is a limit frictional force for which the axial speed of the drill-string increases significantly. The system will accelerate differently depending on the system characteristics

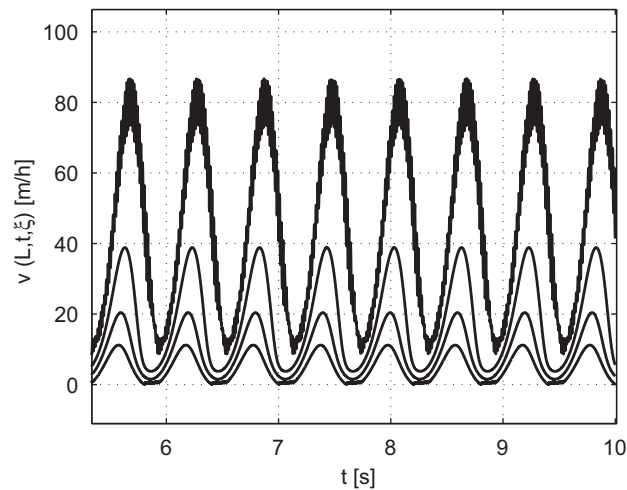


Fig. 11. Four Monte Carlo simulations of the bit speed.

(stiffness, mass, etc.) and on the applied forces (constant force applied on the left, friction force applied along the column and the bit–rock interaction force applied on the right). Fig. 11 shows four Monte Carlo simulations corresponding to one set of simulations (the ones that do not have increasing speed). The stick-slip oscillations is observed for some simulations: the speed reaches the value zero (stick) then it slips. Normally, in petroleum engineering, the term stick-slip is used for the torsional vibration of a drill-string in vertical wells [22]. In that case there is only one contact point at the bit–rock interaction. Nevertheless, in the present case of horizontal drilling, there is stick-slip in the axial direction due to the frictional force, that is acting on all column extension. It is more appropriate to talk about a stick-slip field on $[0, L]$, this of course is akin to a contact problem. To compute the stick-slip we simply regularize the frictional force function, so it is a continuous function of time.

5. Conclusions

In this paper a dynamical model was proposed for the nonlinear dynamics of a horizontal drill-string. It should be noticed that few papers treat this problem. In the present analysis, the drill-string is modeled using a bar model (tension/compression), and is discretized by means of the finite element method. A constant and an oscillatory force is imposed on the system, and there is bit–rock interaction. Friction between the column and the soil is also included in the model, and it is modeled by a random field with an exponential correlation function. The Monte Carlo method is used to approximate the propagation of uncertainty. Other sources of uncertainties, such as bit–rock interaction model, should be the subject of futures works.

The efficiency of the system was measured by the ratio between the output power (obtained from the bit–rock interaction) and the input power. The results were similar for different excitation frequencies and different correlation lengths. But they were different for different slenderness parameters and different uncertainty levels of the random field.

Another interesting result is that the probability density function of the ratio between the output and input power is bimodal. The bimodality indicates that there are two sets of behaviors: one presents considerably higher bit speed than the other. We also observed stick-slip behavior in the axial direction; the usual analysis of stick-slip in drilling dynamics is related to torsional vibrations in vertical wells.

This is an ongoing work, and there are many investigation to perform with this system. For instance, an identification procedure for the parameters, a refinement of the stochastic modeling, a refinement of the bit–rock interaction model, the inclusion of bending and torsional vibrations, and a robust optimization to maximize the performance of the system considering the uncertainties.

Acknowledgements

The authors are grateful to the financial support from the Brazilian agencies: CAPES, CNPq, and FAPERJ; and from the Argentinean support: FRCU-UTN, FRCon-UTN, SGCyT-UNS, MINyT and CONICET.

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