INEQUALITIES FOR ONE-SIDED OPERATORS IN ORLICZ SPACES

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ABSTRACT. In this paper, we get strong type inequalities for one-sided maximal best approximation operators \( \mathcal{M}^\pm \) which are very related to one-sided Hardy-Littlewood maximal functions \( M^\pm \). In order to obtain our results, strong and weak type inequalities for \( M^\pm \) are considered.

1. Introduction

We denote by \( \mathcal{S} \) the set of functions \( \varphi : \mathbb{R} \to \mathbb{R} \) which are nonnegative, even, nondecreasing on \([0,\infty)\), such that \( \varphi(t) > 0 \) for all \( t > 0 \), \( \varphi(0+) = 0 \) and \( \lim_{t \to 0+} \varphi(t) = \infty \).

We say that a nondecreasing function \( \varphi : \mathbb{R}_+^n \to \mathbb{R}_+^n \) satisfies the \( \Delta_2 \) condition, symbolically \( \varphi \in \Delta_2 \), if there exists a constant \( \Lambda_\varphi > 0 \) such that \( \varphi(2a) \leq \Lambda_\varphi \varphi(a) \) for all \( a \geq 0 \).

An even and convex function \( \Phi : \mathbb{R} \to \mathbb{R}_+^n \) such that \( \Phi(a) = 0 \) iff \( a = 0 \) is said to be a Young function. Unless stated otherwise, the Young function \( \Phi \) is the one given by \( \Phi(x) = \int_0^x \varphi(t) \, dt \), where \( \varphi : \mathbb{R}_+^n \to \mathbb{R}_+^n \) is the right-continuous derivative of \( \Phi \).

If \( \varphi \in \mathcal{S} \), we define \( L^\varphi(\mathbb{R}^n) \) as the class of all Lebesgue measurable functions \( f \) defined on \( \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} \varphi(t f) \, dx < \infty \) for some \( t > 0 \) and where \( dx \) denotes the Lebesgue measure on \( \mathbb{R}^n \). If \( \varphi \) is a Young function, then \( L^\varphi(\mathbb{R}^n) \) is an Orlicz space (see [12]).

In the case of \( \Phi \) being a Young function such that \( \Phi \in \Delta_2 \), then \( L^\Phi(\mathbb{R}^n) \) is the space of all measurable functions \( f \) defined on \( \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} \Phi(|f|) \, dx < \infty \).

Also note that if \( \Phi \in C^1 \cap \Delta_2 \) such that \( \Phi(2a) \leq \Lambda_\Phi \Phi(a) \) for all \( a > 0 \), then its derivative function \( \varphi \) satisfies the \( \Delta_2 \) condition and

\[
\frac{1}{2} (\varphi(a) + \varphi(b)) \leq \varphi(a + b) \leq \frac{\Lambda_\Phi^2}{2} (\varphi(a) + \varphi(b)),
\]

for every \( a, b > 0 \).

A nondecreasing function \( \varphi : \mathbb{R}_+^n \to \mathbb{R}_+^n \) satisfies the \( \nabla_2 \) condition, denoted \( \varphi \in \nabla_2 \), if there exists a constant \( \Lambda_\varphi > 2 \) such that \( \varphi(2a) \geq \Lambda_\varphi \varphi(a) \) for all \( a \geq 0 \).

For \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), the classical Hardy-Littlewood maximal function \( M \) defined over cubes \( Q \subset \mathbb{R}^n \) is given by the formula

\[
M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(t)| \, dt.
\]

For \( f \in L^1_{\text{loc}}(\mathbb{R}) \), the one-sided Hardy-Littlewood maximal functions \( M^+ \) and \( M^- \) are introduced in [5] as follows:

\[
M^+ f(x) = \sup_{h > 0} \frac{1}{h} \int_{x}^{x+h} |f(y)| \, dy, \quad \text{with } x \in \mathbb{R},
\]

and

\[
M^- f(x) = \sup_{h > 0} \frac{1}{h} \int_{x-h}^{x} |f(y)| \, dy, \quad \text{with } x \in \mathbb{R}.
\]


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For the sake of simplicity, in the sequel we write $M^\pm$ to refer to $M^+$ or $M^-$. It is well known that $M$ is homogeneous, subadditive, weak type $(1, 1)$ and it also satisfies $\|Mf\|_\infty \leq \|f\|_\infty$. The one-sided maximal functions $M^\pm$ are also homogeneous, subadditive, weak type $(1, 1)$ (see [5]) and strong type $(\infty, \infty)$. In addition, $M$ may be defined from the one-sided maximal functions as follows

$$Mf(x) = \max\{M^+ f(x), M^- f(x)\}.$$  

(2)

In fact,

$$\frac{1}{s+t} \int_{x-s}^{x-t} |f(y)| \, dy \leq \frac{s}{s+t} M^- f(x) + \frac{t}{t+s} M^+ f(x) \leq \max\{M^- f(x), M^+ f(x)\}.$$  

Now, taking supremum over all $s, t > 0$, we have

$$Mf(x) \leq \max\{M^- f(x), M^+ f(x)\}.$$  

On the other hand,

$$Mf(x) = \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(u)| \, du \geq \sup_{s,t>0} \frac{1}{s+t} \int_{x-s}^{x+t} |f(u)| \, du = M^- f(x).$$  

Similarly, we have $Mf(x) \geq M^+ f(x)$. Therefore

$$Mf(x) \geq \max\{M^+ f(x), M^- f(x)\}.$$  

In [1] and [6], weak and strong type inequalities for $M$ in Orlicz spaces were obtained. The one-sided weighted maximal operator on $\mathbb{R}$ in $L^p$ spaces was studied by Sawyer [13], Martín-Reyes, Ortega Salvador and de la Torre [8], and Martín-Reyes [7]. The weighted Orlicz space case was treated in Ortega Salvador [10] assuming the reflexivity of the space, Kokilashvili and Krbec in [6], based on Ortega Salvador [10] and Ortega Salvador and Pick [11], removed the restriction to reflexive spaces and weakened some hypothesis.

In this paper, we follow the idea of Kokilashvili and Krbec in [6] for one-sided maximal functions on $\mathbb{R}$ without dealing with weight functions. Namely, we specify conditions on $\varphi \in \mathcal{F}$ under which the weak type inequalities

$$|\{x \in \mathbb{R} : M^\pm f(x) > \lambda\}| \leq \frac{c_1}{\varphi(\lambda)} \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx,$$  

(3)

and

$$|\{x \in \mathbb{R} : M^\pm f(x) > \lambda\}| \leq c_2 \int_{\mathbb{R}} \varphi \left( \frac{c_2 f(x)}{\lambda} \right) \, dx,$$  

(4)

hold for all $\lambda > 0$ and where $f \in L^1_{\text{loc}}(\mathbb{R})$. We also characterize the strong type inequality

$$\int_{\mathbb{R}} \varphi(M^\pm f(x)) \, dx \leq c \int_{\mathbb{R}} \varphi(cf(x)) \, dx,$$  

(5)

for all $f \in L^1_{\text{loc}}(\mathbb{R})$.

It is worth mentioning that inequalities (3), (4) and (5) are particular cases of results given in [6]; however, as we do not deal with weight functions, we include easier proofs.

Then, we get conditions to assure the validity of strong type inequalities like (5) for one-sided maximal operators $M^\pm$, related to one-sided best $\varphi$-approximation by constants to a function $f \in L^p_{\text{loc}}(\mathbb{R})$.

Last, we get strong type inequalities for lateral maximal operators $M^\pm_p$ related to $p$-averages.
2. **Weak Type Inequalities for \( M^\pm \)**

The next concept is introduced in [6] and we will employ it to set conditions under which (3) and (4) are valid.

**Definition 1.** A function \( \varphi : [0, \infty) \to \mathbb{R} \) is quasiconvex on \([0, \infty)\) if there exist a convex function \( \omega \) and a constant \( c > 0 \) such that
\[
\omega(t) \leq \varphi(t) \leq c \omega(ct),
\]
for all \( t \in [0, \infty) \).

2.1. **Necessary and sufficient condition.** Lemma 1.2.4 in [6] establishes the equivalence between the validity of a weak type inequality like (3) for \( M \) and the quasiconvexity of \( \varphi \). Theorem 2.4.1 in [6] states an analogous equivalence for \( M^\pm \) employing weight functions. The next result is a particular case of this theorem; nevertheless, as we deal without using weights, we include an easier proof.

**Theorem 2.** Let \( \varphi \in \mathcal{S} \). \( \varphi \) is quasiconvex if and only if there exists \( c_1 > 0 \) such that
\[
|\{ x \in \mathbb{R} : M^\pm f(x) > \lambda \}| \leq \frac{c_1}{\varphi(\lambda)} \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx,
\]
for all \( \lambda > 0 \) and for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \).

**Proof.** \( \Rightarrow \) Let \( \varphi \in \mathcal{S} \) be a quasiconvex function. By Lemma 1.2.4 in [6], there exists \( c_1 > 0 \) such that
\[
|\{ x \in \mathbb{R} : Mf(x) > \lambda \}| \leq \frac{c_1}{\varphi(\lambda)} \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx,
\]
for all \( \lambda > 0 \) and for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \). From (2) and the monotonicity of Lebesgue measure, we have
\[
|\{ x \in \mathbb{R} : M^\pm f(x) > \lambda \}| \leq |\{ x \in \mathbb{R} : Mf(x) > \lambda \}|.
\]

Now, by (7) and (8), we get (6).

\( \Leftarrow \) We will prove the statement for \( M^+ \) reasoning as in the proof of Theorem 2.4.1 in [6]. The same argument with a slight modification is also valid for the case of \( M^- \).

Let \( a < b < c \) and assume \( \int_b^c |f(u)| \, du \neq 0 \). If \( x \in (a,b) \) there exists \( h > 0 \) such that \((x, x+h) \supset (b,c)\) and \( x+h = c \), then \( h < c - a \) and
\[
\chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)| \, du < \frac{1}{h} \int_b^c |f(u)| \, du \leq \frac{1}{h} \int_x^{x+h} |f(u)| \, du.
\]

Therefore, if \( x \in (a,b) \) then
\[
\frac{1}{c-a} \int_b^c |f(u)| \, du < M^+ f(x).
\]

On the other hand, if \( x \notin (a,b) \) then
\[
\chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)| \, du \leq M^+ f(x),
\]
as \( M^\pm f(x) \geq 0 \). Eventually, from (9) and (10),
\[
M^+ f(x) \geq \chi_{(a,b)}(x) \frac{1}{c-a} \int_b^c |f(u)| \, du \quad \text{for all } x \in \mathbb{R}.
\]

Let \( \lambda = \frac{1}{c-a} \int_b^c |f(u)| \, du > 0 \). By (6), there exists \( c_1 > 0 \) such that
\[
\left| \left\{ x \in \mathbb{R} : M^+ f(x) > \frac{1}{c-a} \int_b^c |f(u)| \, du \right\} \right| \varphi \left( \frac{1}{c-a} \int_b^c |f(u)| \, du \right) \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx.
\]
From (9) we have
\begin{align*}
(b-a) \leq \left\{ x \in \mathbb{R} : M^+ f(x) > \frac{1}{c-a} \int_b^c |f(u)| \, du \right\},
\end{align*}
then there exists \( c_1 > 0 \) such that
\begin{align*}
(b-a) \varphi \left( \frac{1}{c-a} \int_b^c |f(u)| \, du \right) & \leq c_1 \int_b^c \varphi(c_1 f(x)) \, dx + c_1 \int_{\mathbb{R}-(b,c)} \varphi(c_1 f(x)) \, dx,
\end{align*}
for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \) provided that \( \int_b^c |f(u)| \, du \neq 0 \).

Now, let \( f(x) = f(x) \chi_{(b,c)}(x) \), then there exists \( c_1 > 0 \) such that
\begin{align*}
(b-a) \varphi \left( \frac{1}{c-a} \int_b^c |f(u)| \, du \right) & \leq c_1 \int_b^c \varphi(c_1 f(x)) \, dx, \tag{11}
\end{align*}
with \( f \in L^1_{\text{loc}}(\mathbb{R}) \) such that \( \int_b^c |f(u)| \, du \neq 0 \).

In case of \( \int_b^c |f(u)| \, du = 0 \), (11) holds trivially.

Let \( c > 1 \) and \( a < b < c \) such that \( b-a = c-b \). Let \( t_1, t_2 > 0 \) and \( \theta \in (0, 1) \). We decompose \((b,c)\) into two disjoint sets \( F \) and \( F' \) such that \((b,c) = F \cup F'\), \(|F| = \theta(c-b)\) and \(|F'| = (1-\theta)(c-b)\). Let \( h(x) = t_1 \chi_F(x) + t_2 \chi_{F'}(x) \) for \( x \in (b,c) \), then
\begin{align*}
\frac{1}{c-a} \int_b^c |h(x)| \, dx = \frac{1}{2} [\theta t_1 + (1-\theta) t_2].
\end{align*}
Replacing in the left hand side of (11), there exists \( c_2 > 0 \) such that
\begin{align*}
(b-a) \varphi \left[ \frac{\theta t_1 + (1-\theta) t_2}{2} \right] \leq c_1 \int_b^c \varphi(c_1 h(x)) \, dx = c_1(b-a)[\varphi(c_1 t_1) \theta + (1-\theta) \varphi(c_1 t_2)].
\end{align*}
Let \( 0 < T_1 = \frac{t_1}{2}, 0 < T_2 = \frac{t_2}{2} \), then there exists \( K_2 = 2c_2 > 0 \) independent of \( T_1, T_2 \) and \( h \) such that
\begin{align*}
\varphi[\theta T_1 + (1-\theta) T_2] \leq K_2 [\theta \varphi(K_2 T_1) + (1-\theta) \varphi(K_2 T_2)]. \tag{12}
\end{align*}
Finally, by Lemma 1.1.1 in [6], (12) is equivalent to the fact that \( \varphi \) is quasiconvex. \( \square \)

2.2. **Sufficient conditions.** Next, we set sufficient conditions for (4) to be verified. The next result is a particular case of Theorem 2.4.2 in [6].

**Theorem 3.** Let \( \varphi \in \mathcal{S} \). If \( \varphi \) is quasiconvex, then there exists \( c_2 > 0 \) such that
\begin{align*}
|x| \in \mathbb{R} : M^+ f(x) > \lambda \} \leq c_2 \int_{\mathbb{R}} \varphi \left( \frac{c_2 f(x)}{\lambda} \right) \, dx,
\end{align*}
for all \( \lambda > 0 \) and for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \).

**Proof.** It follows straightforwardly taking \( \rho = \sigma = g = 1 \) in the proof of Theorem 2.4.2 in [6]. \( \square \)

However, the quasiconvexity of \( \varphi \in \mathcal{S} \) is not a necessary condition for the validity of (4). Let \( \varphi(x) = |x|^p \) for \( p \geq 1 \), then \( \varphi \in \mathcal{S} \) and \( \varphi \) is a quasiconvex function on \([0, \infty)\). By Theorem 5 there exists \( c_2 > 0 \) such that
\begin{align*}
|x| \in \mathbb{R} : M^+ f(x) > \lambda \} \leq c_2 \int_{\mathbb{R}} \varphi \left( \frac{c_2 f(x)}{\lambda} \right) \, dx,
\end{align*}
for all \( \lambda > 0 \) and for all \( f \in L^1_{\text{loc}}(\mathbb{R}) \).

Now, let
\begin{align*}
\tilde{\varphi}(x) = \begin{cases} 
|\chi|^p & \text{if } |\chi| \geq 1 \\
|\chi|^{\frac{p}{2}} & \text{if } |\chi| < 1 
\end{cases} 
\quad \text{for } p > 1;
\end{align*}
Finally, we set $y = \lambda$ for all $x \in \mathbb{R}$. Therefore, there exists $c_2 > 0$ such that
\[ |\{x \in \mathbb{R} : M^\pm f(x) > \lambda\}| \leq c_2 \int_{\mathbb{R}} \phi \left( \frac{c_2 f(x)}{\lambda} \right) \, dx, \]
for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$, although $\phi$ is not a quasiconvex function. Hence, the converse of Theorem 5 is not true.

**Remark 4.** Let $\varphi, \bar{\varphi} \in \mathscr{F}$ such that $\varphi(x) \leq \bar{\varphi}(x)$ for all $x \in \mathbb{R}$. If $\varphi$ is quasiconvex on $[0, \infty)$, then there exists $c > 0$ such that
\[ |\{x \in \mathbb{R} : M^\pm (f)(x) > \lambda\}| \leq c \int_{\mathbb{R}} \bar{\varphi} \left( \frac{c f(x)}{\lambda} \right) \, dx, \]
for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$.

Moreover, we determine some characteristics of the class of functions that satisfy (4).

**Theorem 5.** Let $\psi \in \mathscr{F}$. Assume there exist constants $c_1 > 0$ and $x_0 \geq 0$ such that $\psi(x) \geq c_1 x$ for all $x \geq x_0$ and there exists a subinterval $(0, x_v) \subseteq (0, x_0)$ where $\psi$ is either a convex function or a concave one. Then there exists a constant $c > 0$ such that
\[ |\{x \in \mathbb{R} : M^\pm (f)(x) > \lambda\}| \leq c \int_{\mathbb{R}} \psi \left( \frac{c f(x)}{\lambda} \right) \, dx, \]
for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$.

**Proof.** From (2), the monotonicity of Lebesgue measure and Theorem 5.8 in [1].

Therefore, (4) is valid for all $f \in L^1_{\text{loc}}(\mathbb{R})$ and for all $\lambda > 0$, when $\psi \in \mathscr{F}$ belongs to a bigger subset than that of quasiconvex functions.

2.3. Necessary condition. We also find a necessary condition for (4) to be satisfied.

**Theorem 6.** Let $\varphi \in \mathscr{F}$. If there exists $c > 0$ such that
\[ |\{x \in \mathbb{R} : M^\pm f(x) > \lambda\}| \leq c \int_{\mathbb{R}} \varphi \left( \frac{c f(x)}{\lambda} \right) \, dx, \quad (13) \]
for all $\lambda > 0$ and for all $f \in L^1_{\text{loc}}(\mathbb{R})$, then $\frac{\varphi(x)}{c^2} \leq \varphi(y)$ for all $y > c$.

**Proof.** First, we consider the case of $M^\pm$.

Let $0 < t_1 < t_2$, $I = (1 - \frac{t_1}{t_2}, 1)$ and $f(x) = t_2 \chi_I(x)$. For any $x \in (0, 1)$ we have $M^+ f(x) > t_1 > 0$ and then
\[ |\{x \in \mathbb{R} : M^\pm f(x) > t_1\}| \geq 1. \]

Now, with $\lambda = t_1$ and $f(x) = t_2 \chi_I(x)$ in (13), there exists $c > 0$ such that
\[ 1 \leq c \int_{\mathbb{R}} \varphi \left( \frac{ct_2 \chi_I(u)}{t_1} \right) \, du = c \varphi \left( \frac{t_2}{t_1} \right) \frac{t_1}{t_2}, \]
Finally, we set $y = c \frac{t_2}{t_1} > c$, then $y \leq c^2 \varphi(y)$ for all $y > c$.

With the aim of obtaining the result for $M^-$, we set $I = (0, \frac{1}{t_2})$ where $0 < t_1 < t_2$ and we reason as in the case of $M^+$.\[ \square \]
3. Strong type inequality for $M^\pm$

Theorem 1.2.1 in [6] establishes that the validity of a strong type inequality for $M$ is equivalent to the fact that the function involved satisfies the $\nabla_2$ condition. We obtain an analogous result for $M^\pm$.

**Theorem 7.** Let $\varphi \in \mathcal{F}$. The next statements are equivalent:

i) there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}} \varphi(M^+(f(x))) \, dx \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}),$$

(14)

ii) the function $\varphi^\alpha$ is quasiconvex for some $\alpha \in (0, 1)$,

iii) there exists $c_2 > 0$ such that $\int_0^T \varphi^\alpha(t) \, dt \leq c_2 \varphi(c_2) \int_0^T \tau \, d\tau$ for $0 < \sigma < \infty$,

iv) there exists $c_3 > 0$ such that for $t > 0$ $\int_0^t \frac{\varphi(u)}{u} \, du \leq c_3 \varphi(c_3 t)$,

v) there exists $a > 1$ such that

$$\varphi(t) < \frac{1}{2a} \varphi(at), \quad t \geq 0.$$

**Proof.** The proof of Theorem 1.2.1 in [6] follows this scheme: $i) \Rightarrow iii) \Rightarrow v) \Rightarrow ii) \Rightarrow i)$ and $iii) \Leftrightarrow iv)$. In the case of $M^\pm$ it is sufficient to obtain $i) \Rightarrow iii)$ and $ii) \Rightarrow i)$ because the remaining implications are not modified when $M$ is changed by $M^\pm$, as only properties of quasiconvex functions are employed.

i) $\Rightarrow$ iii) Let $f(x) = \chi_{[a,b]}(x)$. After some calculations (see [4] p. 79)), we have

$$Mf(x) = \begin{cases} \frac{b-a}{b-x} & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ \frac{b-a}{x-a} & \text{if } x > b, \end{cases}$$

and

$$M^+(f(x)) = \begin{cases} \frac{b-a}{b-x} & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ 0 & \text{if } x > b, \end{cases}$$

and

$$M^-(f(x)) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } a \leq x \leq b \\ \frac{b-a}{x-a} & \text{if } x > b. \end{cases}$$

Consequently, we can write $M^+(f(x)) = Mf \chi_{[a,b]}(x)$ and $M^-(f(x)) = Mf \chi_{[a,\infty)}(x)$. Then, $\Rightarrow$ iii) of Theorem 1.2.1 in [6].

ii) $\Rightarrow$ i) Due to Theorem 1.2.1 in [6], v) implies that there exists $c_1 > 0$ such that

$$\int_{\mathbb{R}^n} \varphi(Mf(x)) \, dx \leq c_1 \int_{\mathbb{R}} \varphi(c_1 f(x)) \, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}),$$

(15)

thus, by (2) and the monotonicity of $\varphi$ we have

$$\int_{\mathbb{R}} \varphi(M^+(f(x))) \, dx \leq \int_{\mathbb{R}} \varphi(Mf(x)) \, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}).$$

(16)

From (15) and (16), we get the desired inequality (14).

**Remark 8.** Item v) in Theorem 7 is equivalent to say that $\varphi \in \nabla_2$.

We point out that there exists an alternative way to get the strong type inequality (14) applying interpolation techniques.

Theorems 2, 3 and 4 guarantee the existence of classes of functions $\varphi \in \mathcal{F}$ that satisfy weak type inequalities like (3) and (4); in addition, the operators $M^\pm$ are subadditive and strong type $(\infty, \infty)$. Then, by application of Theorem 2.4 in [9] or Theorem 5.2 in [11], we obtain

$$\int_{\mathbb{R}} \Psi(|M^\pm(f(x))|) \, dx \leq K \int_{\mathbb{R}} \Psi(4f(x)) \, dx, \quad \text{for all } f \in L^1_{\text{loc}}(\mathbb{R}),$$

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and for a family of Young functions $\Psi$ such that $\Psi' = \psi$ is related to $\varphi \in \mathcal{F}$ and provided that the function $\varphi$ satisfies additional conditions.

4. **One-sided maximal operators $\mathcal{M}^\pm$**

By $\hat{\mathcal{F}}$ we denote the class of all nondecreasing functions $\varphi$ defined for all real number $t \geq 0$ such that $\varphi(t) > 0$ for all $t > 0$, $\varphi(0+) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

Let $\Phi \in \hat{\mathcal{F}} \cap \Delta_2$ be a convex function and let $B$ be a bounded measurable set of $\mathbb{R}^n$. The next definition is introduced in [2].

**Definition 9.** A real number $c$ is a best $\Phi$-approximation of $f \in L^\Phi(B)$ if and only if

$$
\int_B \Phi(|f(x) - c|) \, dx \leq \int_B \Phi(|f(x) - r|) \, dx, \text{ for all } r \in \mathbb{R}.
$$

With the symbol $\mu_\Phi(f)(B)$ the authors refer to the multivalued operator of all best approximation constants of the function $f \in L^\Phi(B)$. It is well known that $\mu_\Phi(f)(B)$ is a non empty set; and, if $\Phi$ is strictly convex, then $\mu_\Phi(f)(B)$ has an only one element.

In [2] the definition of $\mu_\Phi(f)(B)$ is extended in a continuous way for functions $f \in L^\Phi(B)$ such that $\varphi = \Phi'$ with $\Phi \in C^1$ as follows.

**Definition 10.** Let $\Phi \in \hat{\mathcal{F}} \cap \Delta_2$ be a function in $C^1$ and assume that $\Phi' = \varphi$. If $f \in L^\Phi(B)$, then a constant $c$ is a extended best approximation of $f$ on $B$ if $c$ is a solution of the next inequalities:

a) \[
\int_{\{f > c\}} \Phi(\|f(y) - c\|) \, dy \leq \int_{\{f \leq c\}} \Phi(\|f(y) - c\|) \, dy,
\]

and

b) \[
\int_{\{f < c\}} \Phi(\|f(y) - c\|) \, dy \leq \int_{\{f \geq c\}} \Phi(\|f(y) - c\|) \, dy.
\]

Let $\tilde{\mu}_\Phi(f)(B)$ be the set of all constants $c$.

In the particular case of $B = I^\pm_x(x)$ where $I^\pm_x(x)$ is a bounded one-sided interval of $x \in \mathbb{R}$ with positive Lebesgue measure $\varepsilon$, we write $\tilde{\mu}_\Phi^\pm(f)(x)$ for $\tilde{\mu}_\Phi(f)(I^\pm_x(x))$ which is the one-sided best approximation by constants and we set $\bar{\mu}_\Phi^\pm(f)(x)$ for the set $\tilde{\mu}_\Phi^\pm(f)(I^\pm_x(x))$ which is the extended one-sided best approximation by constants.

We define the one-sided maximal operators $\mathcal{M}_\varepsilon^\pm$, associated to one-sided best approximation by constants, in the following way:

$$
\mathcal{M}_\varepsilon^\pm f(x) = \sup_{\varepsilon > 0} \{f^\pm_{\varepsilon}(x) : f^\pm_{\varepsilon}(x) \in \bar{\mu}_\Phi^\pm(f)(x)\}.
$$

**Remark 11.** If $f^\pm_{\varepsilon}(x) \in \bar{\mu}_\Phi^\pm(f)(x)$, there exists $c^\pm_{\varepsilon} \in \bar{\mu}_\Phi^\pm(|f|)(x)$ such that $|f^\pm_{\varepsilon}(x)| \leq c^\pm_{\varepsilon}$.

In fact, since $|f| \geq f \geq -|f|$ and the extended one-sided best approximation operator is a monotone one (Lemma 12 in [3]), there exist $a^\pm_{\varepsilon} \in \bar{\mu}_\Phi^\pm(-|f|)(x)$ and $b^\pm_{\varepsilon} \in \bar{\mu}_\Phi^\pm(|f|)(x)$ such that $-a^\pm_{\varepsilon} \leq f^\pm_{\varepsilon}(x) \leq b^\pm_{\varepsilon}$.

However, $a^\pm_{\varepsilon} \in \bar{\mu}_\Phi^\pm(|f|)(x)$ and $c^\pm_{\varepsilon} = \max\{a^\pm_{\varepsilon}, b^\pm_{\varepsilon}\} \in \bar{\mu}_\Phi^\pm(|f|)(x)$ because $\bar{\mu}_\Phi^\pm(|f|)(x)$ is a closed set (Lemma 11 in [3]). As $c^\pm_{\varepsilon} \geq a^\pm_{\varepsilon}, b^\pm_{\varepsilon}$, we have $\mathcal{M}_\varepsilon^\pm f(x) \leq \mathcal{M}_\varepsilon^\pm |f| (x)$ and we may assume $f \geq 0$.

Now, we reason as in [2], working on $I^\pm_x$ of $\mathbb{R}$ instead of balls centered at $x \in \mathbb{R}^n$ with radius $\varepsilon$, and we get the following result.
Theorem 12. Let $\Phi \in J \cap \Delta_2$ be a $C^1$ convex function and we assume $\Phi' = \varphi$. Let $f \in L^p_{\text{loc}}(\mathbb{R})$ and we select $f^\pm_\varepsilon(x) \in \bar{\mu}^\pm_\varepsilon(f)(x)$ with $x \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$\frac{1}{C} \varphi(|f^\pm_\varepsilon(x)|) \leq \frac{1}{\varepsilon} \int_{I^\pm_\varepsilon} \varphi(|f(y)|) \, dy \leq C \varphi(|f^\pm_\varepsilon(x)|),$$

and

$$\frac{1}{C} \varphi(|f^\pm_\varepsilon(x) - f(x)|) \leq \frac{1}{\varepsilon} \int_{I^\pm_\varepsilon} \varphi(|f(y) - f(x)|) \, dy,$$

being $\varepsilon$ the Lebesgue measure of the intervals $I^\pm_\varepsilon$ and $C = \frac{3\Lambda^2_\phi}{2}$ where $\Lambda_\phi$ is the constant given by the $\Delta_2$ condition on $\Phi$.

**Proof.** By Remark 11 we can assume $f \geq 0$ and then $f^\pm_\varepsilon(x) \geq 0$. In effect, by a) in Definition 10 if $c < 0$

$$\int_{I^\pm_\varepsilon} \varphi(|f(y) - c|) \, dy = \int_{\{f > c\} \cap I^\pm_\varepsilon} \varphi(|f(y) - x|) \, dy \leq \int_{\{f < c\} \cap I^\pm_\varepsilon} \varphi(|f(y) - x|) \, dy = 0.$$  

(19)

As $\Phi$ is a $C^1$ convex function, then $\varphi(x) > 0$ for $x > 0$; if $c < 0$ then $f(y) - c > -c > 0$, consequently $\varphi(|f(y) - c|) > \varphi(-c) > 0$ and

$$\int_{I^\pm_\varepsilon} \varphi(|f(y) - c|) \, dy > |\varphi(-c)| \varepsilon > 0.$$  

(20)

From (19) and (20) we obtain a contradiction.

Now, applying (1) and $|I^\pm_\varepsilon \cap \{f^\pm_\varepsilon < f\}| \leq \varepsilon$, we have

$$\frac{1}{\varepsilon} \int_{I^\pm_\varepsilon} \varphi(f(y)) \, dy \leq \frac{\Lambda^2_\phi}{2\varepsilon} \int_{I^\pm_\varepsilon \cap \{f^\pm_\varepsilon < f\}} \varphi(f(y) - f^\pm_\varepsilon(x)) \, dy + \frac{\Lambda^2_\phi}{2} \varphi(f^\pm_\varepsilon(x)) + \frac{1}{\varepsilon} \int_{I^\pm_\varepsilon \cap \{f^\pm_\varepsilon \geq f\}} \varphi(f(y)) \, dy.$$  

(21)

Next, by b) of Definition 10 and if we suppose, without loss of generality, that $\Lambda_\Phi \geq \sqrt{2}$, we get

$$\frac{\Lambda^2_\phi}{2\varepsilon} \int_{I^\pm_\varepsilon \cap \{f^\pm_\varepsilon \geq f\}} |\varphi(-f(y) + f^\pm_\varepsilon(x)) + \varphi(f(y))| \, dy + \frac{\Lambda^2_\phi}{2} \varphi(f^\pm_\varepsilon(x)).$$  

(22)

From (1) and as $f^\pm_\varepsilon(x) - f(y) \geq 0$ and $f(y) \geq 0$, then

$$\varphi(f^\pm_\varepsilon(x) - f(y)) + \varphi(f(y)) \leq 2\varphi(f^\pm_\varepsilon(x) - f(y) + f(y)) = 2\varphi(f^\pm_\varepsilon(x)),$$

and since $|I^\pm_\varepsilon \cap \{f^\pm_\varepsilon \geq f\}| \leq \varepsilon$, we obtain

$$\frac{\Lambda^2_\phi}{2\varepsilon} \int_{I^\pm_\varepsilon \cap \{f^\pm_\varepsilon \geq f\}} 2\varphi(f^\pm_\varepsilon(x)) \, dy + \frac{\Lambda^2_\phi}{2} \varphi(f^\pm_\varepsilon(x)) \leq \frac{3\Lambda^2_\phi}{2} \varphi(f^\pm_\varepsilon(x)).$$

(22)

Therefore, there exists $C = \frac{3\Lambda^2_\phi}{2}$ such that

$$\frac{1}{\varepsilon} \int_{I^\pm_\varepsilon} \varphi(f(y)) \, dy \leq C \varphi(f^\pm_\varepsilon(x)).$$  

(23)
On the other hand, applying (1),

$$\varphi(f_\varepsilon^+(x)) = \frac{1}{\varepsilon} \int_{I_\varepsilon^+} \varphi(f_\varepsilon^+(x)) \, dy$$

$$\leq \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ \geq f\}} [\varphi(f_\varepsilon^+(x) - f(y)) + \varphi(f(y))] \, dy + \frac{1}{\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ \leq f\}} \varphi(f_\varepsilon^+(x)) \, dy. \quad (24)$$

Now, we apply a) of Definition 10 and we have

$$\leq \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ \leq f\}} \varphi(f(y) - f_\varepsilon^+(x)) \, dy + \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ > f\}} \varphi(f(y)) \, dy$$

$$+ \frac{1}{\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ < f\}} \varphi(f_\varepsilon^+(x)) \, dy$$

$$\leq \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ \leq f\}} [\varphi(f(y) - f_\varepsilon^+(x)) + \varphi(f_\varepsilon^+(x))] \, dy + \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ > f\}} \varphi(f(y)) \, dy, \quad (25)$$

provided that $1 \leq \Lambda_\Phi^2$. Now, by (1) we get

$$\leq \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ \leq f\}} 2\varphi(f(y)) \, dy + \frac{\Lambda_\Phi^2}{2\varepsilon} \int_{I_\varepsilon^+ \cap \{f_\varepsilon^+ > f\}} \varphi(f(y)) \, dy \leq \frac{\Lambda_\Phi^2}{\varepsilon} \int_{I_\varepsilon^+} \varphi(f(y)) \, dy,$$

because $\frac{\Lambda_\Phi^2}{2} \leq \Lambda_\Phi^2$ and $I_\varepsilon^+ = I_\varepsilon^+ \cap \{f_\varepsilon^+ \leq f\} \cup I_\varepsilon^+ \cap \{f_\varepsilon^+ > f\}$.

Then

$$\frac{1}{C} \varphi(f_\varepsilon^+(x)) \leq \frac{1}{\varepsilon} \int_{I_\varepsilon^+} \varphi(f(y)) \, dy, \quad (26)$$

where $C = \frac{3\Lambda_\Phi^2}{2\varepsilon}$ and (17) follows from (23) and (26).

It remains to prove (18). Note that if $f_\varepsilon^+(x) \in \mu_\varepsilon^+(f)(x)$, then $f_\varepsilon^+(x) - f(x) \in \mu_\varepsilon^+(f - f(x))(x)$. We apply (17) to the function $f - f(x)$ and we obtain

$$\frac{1}{C} \varphi(|f_\varepsilon^+(x) - f(x)|) \leq \frac{1}{\varepsilon} \int_{I_\varepsilon^+} \varphi(|f(y) - f(x)|) \, dy,$$

which is the inequality that we wished to obtain. □

Next, we get an inequality that allows us to compare $M^\pm$ with $\mathcal{M}^\pm$.

**Lemma 1.** Let $\Phi \in \hat{\mathcal{F}} \cap \Delta_2$ be a $C^1$ convex function and let $\Phi' = \varphi$ be such that $A\varphi(t) \leq \varphi(kt)$ for all $t \geq 0$ and some constants $K,A > 1$. Then there exists $C > 0$ such that

$$\frac{1}{K} \varphi^{-1} \left( \frac{1}{C} M^\pm(\varphi(|f|))(x) \right) \leq \mathcal{M}^\pm(|f|)(x) \leq \varphi^{-1}(CM^\pm(\varphi(|f|))(x)), \quad (27)$$

where $M^\pm(f) = \sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_{I_\varepsilon^+} |f(y)| \, dy$.

**Proof.** Let $\varphi^{-1}$ be the generalized inverse of the monotonous function $\varphi$ which is defined by $\varphi^{-1}(s) = \sup\{t : \varphi(t) \leq s\}$, then

$$t \leq \varphi^{-1}(\varphi(t)) \quad \text{for all } t \geq 0, \quad (28)$$

and for every $\tilde{\varepsilon} > 0$

$$\varphi^{-1}(\varphi(t) - \tilde{\varepsilon)) \leq t \quad \text{for all } t \geq 0. \quad (29)$$

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The condition $A\phi(t) \leq \phi(Kt)$ for all $t \geq 0$ and some constants $A, K > 1$, implies that $\phi(0+) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$; therefore, $\phi^{-1}$ is a real valued function and $\phi^{-1} \in \hat{S}$.

From (17) in Theorem 12 we have

$$|f|^+_{\epsilon}(x) \leq \phi^{-1}\left(\frac{C}{\epsilon} \int_{I_{\epsilon}} \phi(|f(y)|) \, dy\right),$$

and since

$$\frac{C}{\epsilon} \int_{I_{\epsilon}} \phi(|f(y)|) \, dy \leq CM^\pm(\phi(|f|))(x),$$

then

$$\mathcal{M}^\pm(|f|) = \sup_{\epsilon > 0} |f|^+_{\epsilon}(x) \leq \phi^{-1}(CM^\pm(\phi(|f|))(x)). \tag{30}$$

Now, by (17) in Theorem 12 and the monotonicity of $\phi$ we have

$$\frac{1}{\epsilon} \int_{I_{\epsilon}} \phi(|f(y)|) \, dy \leq C\phi(|f|^+_{\epsilon}(x)) \leq \phi(M^\pm(\phi(|f|))(x), \quad \text{for all } \epsilon > 0,$$

and therefore

$$M^\pm(\phi(|f|))(x) \leq C\phi(M^\pm(\phi(|f|))(x)). \tag{31}$$

As there exist $K, A > 1$ such that $A\phi(t) \leq \phi(Kt)$ for all $t \geq 0$, then $0 \leq \phi(t) < A\phi(t) \leq \phi(Kt)$ for all $t \geq 0$ and consequently $0 < \phi(Kt) - \phi(t)$ for all $t > 0$. Now, from (29) and taking $0 < \epsilon = \phi(Kt) - \phi(t)$ for all $t > 0$, we get

$$\phi^{-1}(\phi(t)) = \phi^{-1}(\phi(Kt) - \phi(t)) \leq Kt \quad \text{for all } t > 0. \tag{32}$$

From (31), the fact that $\phi^{-1}$ is a nondecreasing function and (32), we get

$$\phi^{-1}\left(\frac{1}{C}M^\pm(\phi(|f|))(x)\right) \leq \phi^{-1}(\mathcal{M}^\pm(\phi(|f|))(x)) \leq K.\mathcal{M}^\pm(\phi(|f|))(x). \tag{33}$$

Therefore, from (30) and (33).

$$\frac{1}{K} \phi^{-1}\left(\frac{1}{C}M^\pm(\phi(|f|))(x)\right) \leq \mathcal{M}^\pm(|f|)(x) \leq \phi^{-1}(CM^\pm(\phi(|f|))(x)). \tag{34}$$

\[\square\]

4.1. Strong type inequalities for $\mathcal{M}^\pm$.

**Theorem 13.** Let $\Phi \in \hat{S} \cap \Delta_2$ be a $C^1$ convex function and let $\Phi' = \phi$ be such that $A\phi(t) \leq \phi(Kt)$ for all $t \geq 0$ and for some constants $K, A > 1$. For a function $\theta \in \hat{S} \cap \Delta_2$, we have that the function $\theta \circ \phi^{-1}$ satisfies the $\nabla_2$ condition if and only if there exists a constant $\bar{C}$ independent of $f$ such that

$$\int_R \theta(M^\pm(|f|))(x) \, dx \leq \bar{C} \int_R \theta(\bar{C}|f(x)|) \, dx,$$

for all $f \in L^\Phi_{\text{loc}}(R)$.

**Proof.** $\Leftrightarrow$ Suppose that $\mathcal{M}^\pm(|f|)$ verifies

$$\int_R \theta(M^\pm(|f|))(x) \, dx \leq \bar{C} \int_R \theta(\bar{C}|f(x)|) \, dx,$$

for all $f \in L^\Phi_{\text{loc}}(R)$.

As $\theta \in \hat{S} \cap \Delta_2$, there exists $K_1 > 0$ such that

$$\int_R \theta(KM^\pm(|f|))(x) \, dx \leq K_1 \int_R \theta(K_1|f(x)|) \, dx, \tag{34}$$

for all $f \in L^\Phi_{\text{loc}}(R)$. 

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From (27), (34) and the fact that $M^\pm$ is homogeneous, we have
\[
\int_{\mathbb{R}} \theta \left( \varphi^{-1} \left( \frac{1}{C} M^\pm (\varphi(\|f\|))(x) \right) \right) \, dx = \int_{\mathbb{R}} \theta \left( \varphi^{-1} \left( M^\pm \left( \frac{1}{C} \varphi(\|f\|) \right)(x) \right) \right) \, dx
\leq \int_{\mathbb{R}} \theta(K(\mathcal{M}^\pm(\|f\|))(x)) \, dx
\leq K_1 \int_{\mathbb{R}} \theta(K(\|f\|))(x) \, dx.
\] (35)

Since $t \leq \varphi^{-1}(\varphi(t))$ for all $t \geq 0$, then $K_1 \|f(x)\| \leq \varphi^{-1}(\varphi(K_1 \|f(x)\|))$; now, by the monotonicity of $\theta$ and the fact that $\varphi \in \mathcal{I} \cap \Delta_2$, there exists $K_2 > 0$ such that
\[
\int_{\mathbb{R}} \theta(K_1 \|f(x)\|) \, dx \leq \int_{\mathbb{R}} \theta(\varphi^{-1}(\varphi(K_1 \|f(x)\|)))(x) \, dx \leq \int_{\mathbb{R}} \theta(\varphi^{-1}K_2(\varphi(\|f(x)\|)))(x) \, dx,
\] (36)

for all $f \in L^0_{\text{loc}}(\mathbb{R})$.

Therefore, from (35) and (36), we have
\[
\int_{\mathbb{R}} \psi(M^\pm(g)(x)) \, dx \leq \tilde{C} \int_{\mathbb{R}} \psi(\tilde{C}g(x)) \, dx,
\] (37)

where $\psi = \theta \circ \varphi^{-1}$, $g = \frac{1}{\tilde{C}} \varphi(\|f\|)$ for any $f \in L^0_{\text{loc}}(\mathbb{R})$ and $\tilde{C} = \max\{K_1, K_2 C\}$.

As the inequality (37) holds for any $f \in L^0_{\text{loc}}(\mathbb{R})$ being $g = \frac{1}{\tilde{C}} \varphi(\|f\|)$, we choose $f = \varphi^{-1}(Cg)$ for any nonnegative function $g \in L^1_{\text{loc}}(\mathbb{R})$ and, using the fact that $\varphi(\varphi^{-1}(t)) = t$ provided that $t \in \text{Im } \varphi \cup \{\inf \text{Im } \varphi, \sup \text{Im } \varphi\}$, we obtain
\[
\int_{\mathbb{R}} \psi(M^\pm(g)(x)) \, dx \leq \tilde{C} \int_{\mathbb{R}} \psi(\tilde{C}g(x)) \, dx,
\]

for all nonnegative functions $g \in L^1_{\text{loc}}(\mathbb{R})$ and where $\tilde{C}$ is independent of $g$. Now, by Theorem 17 we get $\psi = \theta \circ \varphi^{-1} \in \mathcal{V}_2$.

$\Rightarrow$ As $\psi = \theta \circ \varphi^{-1} \in \mathcal{V}_2$, by Theorem 17 there exists $K_1 > 0$ such that
\[
\int_{\mathbb{R}} \psi(M^\pm(g)(x)) \, dx \leq K_1 \int_{\mathbb{R}} \psi(K_1 g(x)) \, dx,
\] (38)

for all nonnegative functions $g \in L^1_{\text{loc}}(\mathbb{R})$. By (27) we have
\[
\mathcal{M}^\pm(\|f\|)(x) \leq \varphi^{-1}(CM^\pm(\varphi(\|f\|))(x)),
\] (39)

and if $K_2 = \max\{C, K_1\}$, both inequalities hold with $K_2$.

Therefore, from (38), the monotonicity of $\theta$, the homogeneity of $M^\pm$ and (39), we have
\[
\int_{\mathbb{R}} \theta(\mathcal{M}^\pm(\|f\|))(x)) \, dx \leq \int_{\mathbb{R}} \psi(K_2 M^\pm(\varphi(\|f\|))(x)) \, dx
\leq \int_{\mathbb{R}} \psi(K_2 M^\pm(K_2 \varphi(\|f\|))(x)) \, dx
\leq K_2 \int_{\mathbb{R}} \psi(K_2^2 \varphi(\|f(x)\|)) \, dx
\leq K_3 \int_{\mathbb{R}} \psi(K_3 \varphi(\|f(x)\|)) \, dx,
\] (40)

with $K_3 = \max\{K_2, C K_2 \}$. Moreover, since $A \varphi(t) \leq \varphi(Kt)$ for all $t \geq 0$ and for some $A, K > 1$, there exists $l$ such that $K_3 \leq A^l$ and, applying the inequality $l$ times, then
\[
K_3 \varphi(x) \leq A^l \varphi(x) \leq A^{l-1} \varphi(Kt) \leq \varphi(K^l t).
\]
Now
\[ K_3 \int_{\mathbb{R}} \psi(K_3 \phi(|f(x)|)) \, dx \leq K_4 \int_{\mathbb{R}} \psi(\phi(K_4 |f(x)|)) \, dx, \]
where \( K_4 = \max\{K_3, K_1^r\} \). By (32) we have \( \phi^{-1}(\phi(t)) \leq Kt \) and since \( \theta \circ \phi^{-1} = \psi \), then \( (\psi \circ \phi)(t) = (\theta \circ \phi^{-1} \circ \phi)(t) \leq \theta(Kt) \); now
\[ K_4 \int_{\mathbb{R}} \psi(\phi(K_4 |f(x)|)) \, dx \leq K_4 \int_{\mathbb{R}} \theta(K \phi(K_4 |f(x)|)) \, dx \leq \tilde{C} \int_{\mathbb{R}} \theta(\tilde{C} |f(x)|) \, dx, \]
(41)
being \( \tilde{C} = \max\{K_4, K K_4\} \).
Consequently, from (40) and (41), we get
\[ \int_{\mathbb{R}} \theta(\mathcal{M}^\pm(|f(x)|)) \, dx \leq \tilde{C} \int_{\mathbb{R}} \theta(\tilde{C} |f(x)|) \, dx. \]
\[ \square \]

Remark 14. If \( \phi \in \mathcal{F} \) such that \( t^p \leq \phi \leq C t^p \) then \( \phi(Kt) > A \phi(t) \) for all \( t \geq 0 \) and for any \( K > 1 \) such that \( A = \frac{K^p}{t^p} > 1 \). In consequence, Theorem 13 allows us to consider \( \phi \in \mathcal{F} \) which is not a strictly increasing function and in this case \( \mu^f_\phi(x) \) may have more than one element.

We also get sufficient conditions to have a strong type inequality for \( \mathcal{M}^\pm \) softening the hypothesis of Theorem 13.

Theorem 15. Let \( \Phi \in \mathcal{F} \cap \Delta_2 \) be a convex function in \( C^1 \) and let \( \Phi' = \phi \) such that \( A \phi(t) \leq \phi(Kt) \) for all \( t \geq 0 \) and for some constants \( K, A > 1 \). Then
\[ \int_{\mathbb{R}} \Phi(\mathcal{M}^\pm(|f(x)|)) \, dx \leq C \int_{\mathbb{R}} \Phi(C |f(x)|) \, dx, \]
for all \( f \in L^p_{\text{loc}}(\mathbb{R}) \) and where the constant \( C \) is independent of \( f \).

Proof. With the aim of applying Theorem 13, we need to show \( \Phi \circ \phi^{-1} \in \nabla_2 \) where \( \Phi \in \mathcal{F} \cap \Delta_2 \) and a proof of this fact is done in [2]. \[ \square \]

4.2. Operators \( M^\pm_p \). If \( \Phi(t) = t^{p+1} \) with \( p > 0 \) in (27), there exists a positive constant \( \tilde{K} \) independent of \( f \) such that
\[ \frac{1}{\tilde{K}} (M^\pm(|f|^p)(x))^{\frac{1}{p}} \leq \mathcal{M}^\pm(|f|)(x) \leq \tilde{K} (M^\pm(|f|^p)(x))^{\frac{1}{p}}. \]
(42)
Let \( M^\pm_p(f)(x) = \left( \sup_{t > 0} \frac{1}{t} \int_{t}^{\infty} |f(t')|^p \, dt' \right)^{\frac{1}{p}} = (M^\pm(|f|^p)(x))^{\frac{1}{p}}. \) The operators \( M^\pm_p \) are related to one-sided \( p \)-averages of a function and they are homogeneous like \( M^\pm \).

A useful and particularly simple characterization of strong type inequalities involving \( M^\pm_p \) may be established for this special case employing Theorem 13.

Corollary 16. Let \( \theta \in \mathcal{F} \) and \( p > 0 \), then there exists \( \tilde{K} > 0 \) such that
\[ \int_{\mathbb{R}} \theta(M^\pm_p(f)(x)) \, dx \leq \tilde{K} \int_{\mathbb{R}} \theta(\tilde{K} |f(x)|) \, dx, \]
(43)
for all \( f \in L^p_{\text{loc}}(\mathbb{R}) \) if and only if \( \theta(t^{1/p}) \in \nabla_2 \).

Proof. It follows from Theorem 13 with \( \Phi(x) = \frac{x^{p+1}}{p+1} \) because \( \Phi \in \mathcal{F} \cap \Delta_2 \) is a \( C^1 \) convex function such that \( A \phi(t) < \phi(Kt) \) for all \( t \geq 0 \) with \( A > 1, K > A^{\frac{1}{p}} \) and where \( \phi = \Phi' \). \[ \square \]
Remark 17. If (43) holds, then
\[ \| M^\pm_p(f) \|_\theta \leq C \| f \|_\theta, \]
where \( \| f \|_\theta \) denotes the Luxemburg norm of \( f \) defined by
\[ \| f \|_\theta = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}} \theta \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}, \]
being \( \theta \) a Young function and \( f \in L^\theta(\mathbb{R}) \).

Proof. The statement follows straightforwardly from Remark 2 in [2]. □

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