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Explicit solutions for a non-classical heat conduction problem for a semi-infinite strip with a non-uniform heat source

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Abstract

A non-classical initial and boundary value problem for a non-homogeneous one-dimensional heat equation for a semi-infinite material with a zero temperature boundary condition is studied. It is not a standard heat conduction problem because a non-uniform heat source dependent on the heat flux at the boundary is considered. The purpose of this article is to find explicit solutions and analyze how to control their asymptotic temporal behavior through the source term.

Explicit solutions independent of the space or temporal variables, solutions with separated variables and solutions by an integral representation depending on the heat flux at the boundary are given. The controlling effects of the source term are analyzed by comparing the asymptotic temporal behavior of solutions corresponding to the same problem with and without source term. Finally, a relationship between the problem considered here with another non-classical problem for the heat equation is established, and explicit solutions for this second problem are also obtained.

In this article, we give explicit solutions and analyze how to control them through the source term for several non-classical heat equation problems. In addition, our results enable us to compute the asymptotic temporal behavior of the heat flux at the boundary for each explicit solution obtained. As a consequence of our study, several solved non-classical problems for the heat equation that can be used for testing new numerical methods are given.

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Keywords: non-classical heat equation; nonlinear heat conduction problems; explicit solutions; Volterra integral equations

1 Introduction

We consider a one-dimensional isotropic and homogeneous medium with one inaccessible boundary (semi-infinite material) under the effects of a temperature controller device which depends on the heat flux at the accessible boundary (fixed boundary), when the initial distribution of temperature is known and the temperature at the accessible boundary is constant in time. More precisely, we study the following non-classical initial and boundary value problem for a non-homogeneous one-dimensional heat equation (Problem P):

$$u_t(x,t) - u_{xx}(x,t) = -\Phi(x)F\big(u_x(0,t),t\big), \qquad x > 0, t > 0, \tag{1}$$

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$$u(x,0) = h(x), \quad x > 0, \tag{2}$$

$$u(0,t) = 0, \quad t > 0, \tag{3}$$

where $u = u(x,t)$ is the unknown temperature function, defined for $x \geq 0$ and $t \geq 0$, $\Phi = \Phi(x)$, $h = h(x)$ and $F = F(V,t)$ are given functions defined, respectively, for $x > 0$ and $V \in \mathbb{R}$, $t > 0$, and the function h satisfies the following compatibility condition:

$$\lim_{x\to 0^+} h(x) = 0. \tag{4}$$

This problem is motivated by the regulation of the temperature $u = u(x,t)$ through the effects of the non-uniform heat source $-\Phi(x)F(u_x(0,t),t)$, which provides a heater or cooler effect depending on the properties of the function F with respect to the heat flux $u_x(0,t)$ at the boundary $x = 0$ [1, 2]. For example, if

$$\Phi(x) > 0 \quad \text{for } x > 0 \quad \text{and} \quad u_x(0,t)F\big(u_x(0,t),t\big) > 0 \quad \forall t > 0,$$

then the source term is a cooler when $u_x(0,t) > 0$ and a heater when $u_x(0,t) < 0$. Some references in this subject are [3–14].

Our purpose is to find explicit solutions to Problem P and study how to control their asymptotic temporal behavior through the source term $-\Phi(x)F(u_x(0,t),t)$. Exact solutions to initial and boundary value problems for the heat equation allow us to better understand qualitative features of the thermal and diffusive process under them. In particular, this knowledge might give us some insights to develop numerical methods dealing with more complex phenomena related with more complicated equations. Even for cases where a physical interpretation is not obvious, exact solutions are important because of their use for testing accuracy, stability and convergence of numerical methods for solving partial differential equations without any known analytical solution. In addition, how to control the asymptotic behavior of solutions to Problem P through the source term in equation (1) gives us some insights about when it is possible to have stationary solutions. It also gives us a better understanding about how solutions to Problem P are related with the solutions to the same problem but in absence of source term, that is, with solutions to the problem where no cooler or heater term is considered.

Problem P for the slab $0 < x < 1$ was studied in [15]. Recently, free boundary problems (Stefan problems) for the non-classical heat equation have been studied in [16–21], where some explicit solutions are also given, and a first study of non-classical heat conduction problem for an n-dimensional material was given in [22]. There exists a large recent scientific production where exact solutions for heat transfer problems arising from a wide field of applications are given; see, for example, [23–36]. Numerical schemes for Problem P were studied in [37] when a non-homogeneous boundary condition is considered and numerical solutions are given for two particular choices of data function corresponding to problems with known explicit solutions.

The organization of the paper is the following. In Section 2, we give explicit solutions to Problem P. We split this section into three parts. In the first one, we give explicit solutions which are independent of the space variable x or the temporal variable t. In the second part, we find solutions with separated variables when the functions $h = h(x)$ and $\Phi = \Phi(x)$ are proportional to the solution $X = X(x)$ of a linear initial value problem of second order and the function $F = F(V,t)$ is defined from the solution $T = T(t)$ of a non-linear (in general) initial value problem of first order. As a consequence, we give explicit solutions with separated variables corresponding to different definitions of the function F. Finally, in the third part, we find solutions by an integral representation which depends on the heat flux at the boundary $x = 0$ [13] when F is defined by $F(V,t) = \nu V$, with $\nu > 0$. Moreover, we find explicit expressions for the heat flux at the boundary $x = 0$ and for its corresponding solution to Problem P, when $h = h(x)$ is a potential function and $\Phi = \Phi(x)$ is given by $\Phi(x) = \lambda x$, $\Phi(x) = -\mu \sinh(\lambda x)$ or $\Phi(x) = -\mu \sin(\lambda x)$, with $\lambda > 0$ and $\mu > 0$. For this case, where computations are not trivial, we also give the asymptotic temporal behavior of the heat flux at $x = 0$. In Section 3, we deal with the problem of 'controlling' solutions of Problem P through the source term $-\Phi(x)F(u_x(0,t),t)$. We compare the asymptotic temporal behavior of each explicit solution u obtained for Problem P with the asymptotic behavior of the solution u_0 of the same problem but in absence of source term, and we obtain conditions for the parameters involved in the definition of $-\Phi(x)F(V,t)$ under which the asymptotic behavior of u can be controlled with respect to the asymptotic behavior of u_0. Finally, in Section 4, we consider another temperature regulation problem in which the thermostat does not depend on the heat flux at the accessible boundary but on the temperature on it, and a heat flux condition at the accessible boundary is given instead of a temperature boundary condition. More precisely, we consider the following problem (Problem $\widetilde{\mathrm{P}}$):

$$v_t(x,t) - v_{xx}(x,t) = -\widetilde{\Phi}(x)\widetilde{F}\big(v(0,t),t\big), \quad x > 0, t > 0, \tag{5}$$

$$v(x,0) = \tilde{h}(x), \quad x > 0, \tag{6}$$

$$v_x(0,t) = \tilde{g}(t), \quad t > 0. \tag{7}$$

We recall the relationship between Problems P and $\widetilde{\mathrm{P}}$ given in [13], and we find explicit solutions to Problem $\widetilde{\mathrm{P}}$ through what we know about Problem P.

2 Explicit solutions for Problem P

2.1 Explicit solutions independent of space or temporal variables

Theorem 2.1

(1) *Problem P does not admit any non-trivial solution independent of the space variable x.*

(2) *If:*

(a) *F is the zero function and h is defined by*

$$h(x) = \eta x, \quad x \geq 0, \tag{8}$$

with $\eta > 0$,

or

(b) *F is a constant function defined by*

$$F(V,t) = \nu, \quad V \in \mathbb{R}, t > 0, \tag{9}$$

with $\nu \in \mathbb{R} - \{0\}$, and h is a twice differentiable function such that $h(0)$ exists and

$$h''(x) = \nu\Phi(x), \quad x > 0, \tag{10}$$

then the function u defined by

$$u(x,t) = h(x), \quad x \geq 0, t \geq 0, \tag{11}$$

is a solution to Problem P independent of the temporal variable t.

Proof

(1) If Problem P has a solution u independent of the space variable x, then

$$u(x,t) = u(0,t) = 0, \quad x > 0, t > 0 \quad \text{and} \quad u(0,0) = \lim_{x \to 0^+} h(x) = 0. \tag{12}$$

Therefore u is the zero function.

(2) It is easy to check that the function u given in (11) is a solution to Problem P given in this item. □

2.2 Explicit solutions with separated variables

Theorem 2.2 *Let $\lambda, \eta, \delta \in \mathbb{R} - \{0\}$. If Φ, h and F are defined by*

$$\Phi(x) = \lambda X(x), \qquad h(x) = \eta X(x), \quad x > 0 \quad \textit{and} \quad F = F\big(\delta T(t), t\big), \quad t > 0, \tag{13}$$

where X is given by

$$X(x) = \begin{cases} \frac{\delta}{\sqrt{\sigma}} \sinh(\sqrt{\sigma} x) & \textit{if } \sigma > 0, \\ \frac{\delta}{\sqrt{|\sigma|}} \sin(\sqrt{|\sigma|} x) & \textit{if } \sigma < 0, \\ \delta x & \textit{if } \sigma = 0, \end{cases} \quad x \geq 0 \tag{14}$$

and T is the solution of the initial value problem

$$\dot{T}(t) - \sigma T(t) = -\lambda F\big(\delta T(t), t\big), \quad t > 0, \tag{15}$$

$$T(0) = \eta, \tag{16}$$

then the function u given by

$$u(x,t) = X(x) T(t), \quad x \geq 0, t \geq 0 \tag{17}$$

is a solution with separated variables to Problem P.

Proof An easy computation shows that the function u given in (17) is a solution to Problem P. □

Remark 1 We note that the definition of the function $X = X(x)$ given in (14) is not arbitrary. In fact, it is the solution to the following linear initial value problem of second order, which arises naturally when we assume the existence of a solution with separated variables of the form (17) to Problem P:

$$X''(x) - \sigma X(x) = 0, \quad x > 0, \tag{18}$$

$$X(0) = 0, \tag{19}$$

$$X'(0) = \delta. \tag{20}$$

Under the hypothesis of the previous theorem, the problem of finding explicit solutions with separated variables to Problem P reduces to solving the initial value problem (15)-(16).

With the spirit of exhibiting explicit solutions to Problem P, our next result summarizes explicit solutions to the initial value problem (15)-(16) corresponding to three different definitions of the function F.

Proposition 2.1 *If in Theorem* 2.2 *we consider*:

(1) *Function F defined by*

$$F(V,t) = \nu V, \quad V \in \mathbb{R}, t > 0, \tag{21}$$

with $\nu \in \mathbb{R} - \{0\}$, then the function T is given by

$$T(t) = \eta \exp\big((\sigma - \lambda\nu\delta)t\big), \quad t \geq 0. \tag{22}$$

(2) *Function F defined by*

$$F(V,t) = f_1(t) + f_2(t)V, \quad V \in \mathbb{R}, t > 0, \tag{23}$$

with $f_1, f_2 \in L^1_{\text{loc}}(\mathbb{R}^+)$, then the function T is given by

$$T(t) = g_1(t)\exp\big(g_2(t)\big), \quad t \geq 0, \tag{24}$$

where functions g_1 and g_2 are defined by

$$g_1(t) = \eta - \lambda\int_0^t f_1(\tau)\exp\left(\lambda\delta\int_0^\tau f_2(\xi)\,d\xi - \sigma\tau\right)d\tau, \quad t \geq 0, \tag{25}$$

$$g_2(t) = \sigma t - \lambda\delta\int_0^t f_2(\tau)\,d\tau, \quad t \geq 0. \tag{26}$$

(3) *Function F defined by*

$$F(V,t) = V^n f(t), \quad V \in \mathbb{R}, t > 0, \tag{27}$$

with $n < 1$, $0 \leq f, f \in L^1_{\text{loc}}(\mathbb{R}^+)$ and λ, δ and η positive numbers, then the function T is given by

$$T(t) = g(t)\exp(\sigma t), \quad t \geq 0, \tag{28}$$

where the function g is defined by

$$g(t) = \left(\eta^{1-n} + \lambda\delta^n(n-1)\int_0^t f(\tau)\exp\big(\sigma(n-1)\tau\big)\,d\tau\right)^{\frac{1}{1-n}}, \quad t \geq 0. \tag{29}$$

Proof It follows by the application of the integrating factor method to the initial value problem (15)-(16). □

2.3 Explicit solutions obtained from an integral representation

Our next theorem is a restatement of Theorem 1 in [13] for a particular choice of the function F in Problem P.

Theorem 2.3 *Let*:

(1) *h be a continuously differentiable function in $\mathbb{R}^+$ such that $h(0)$ exists and there exist positive numbers ϵ, c_0 and c_1 such that*

$$\left|h(x)\right| \leq c_0 \exp\left(c_1 x^{2-\epsilon}\right), \quad \forall x > 0, \tag{30}$$

(2) *Φ be a locally Hölder continuous function*

and

(3) *F be the function defined by*

$$F = F(V,t) = \nu V, \quad V \in \mathbb{R}, t > 0, \tag{31}$$

with $\nu > 0$.

If there exists a negative monotone decreasing function $f = f(t)$, defined for $t > 0$, such that

$$\int_{t_1}^{t_2} R(t_2 - \tau)\, d\tau \geq f(t_2 - t_1), \quad \forall 0 < t_1 < t_2, \tag{32}$$

where R is defined in function of Φ by (40) (see below), and

$$\lim_{t\to 0^+} f(t) = 0, \tag{33}$$

then the function u defined by

$$u(x,t) = \int_0^{+\infty} G(x,t,\xi,0)h(\xi)\, d\xi - \nu \int_0^t \left(\int_0^{+\infty} G(x,t,\xi,\tau)\Phi(\xi)\, d\xi\right) V(\tau)\, d\tau,$$
$$x \geq 0, t \geq 0 \tag{34}$$

is a solution to Problem P, where G is the Green function:

$$G(x,t,\xi,\tau) = K(x,t,\xi,\tau) - K(-x,t,\xi,\tau), \quad 0 < x, 0 < \xi, 0 < \tau < t, \tag{35}$$

K being the fundamental solution of the one-dimensional heat equation

$$K(x,t,\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left(-(x-\xi)^2/4(t-\tau)\right), \quad 0 < x, 0 < \xi, 0 < \tau < t, \tag{36}$$

and the function V, defined by

$$V(t) = u_x(0,t), \quad t > 0, \tag{37}$$

satisfies the Volterra integral equation

$$V(t) = V_0(t) - \nu \int_0^t R(t-\tau)V(\tau)\,d\tau, \quad t > 0, \tag{38}$$

where

$$V_0(t) = \frac{1}{\sqrt{\pi t}} \int_0^{+\infty} \exp\left(-\xi^2/4t\right) h'(\xi)\,d\xi, \quad t > 0, \tag{39}$$

and

$$R(t) = \frac{1}{2\sqrt{\pi} t^{3/2}} \int_0^{+\infty} \xi \exp\left(-\xi^2/4t\right) \Phi(\xi)\,d\xi, \quad t > 0. \tag{40}$$

Remark 2 The interest of the previous theorem is that it enables us to find an explicit solution $u = u(x,t)$ to Problem P by finding the corresponding heat flux $u_x(0,t)$ at the boundary $x = 0$ as a solution of the integral equation (38).

The remainder of this section will be devoted to the study of Problem P when:

(1) F is given as in (31),

(2) h is defined by

$$h(x) = \eta x^m, \quad x > 0, \tag{41}$$

with $\eta \in \mathbb{R} - \{0\}$ and $m \geq 1$,

and

(3) Φ is given by one of the following expressions:

$$\varphi_1(x) = \lambda x, \qquad \varphi_2(x) = -\mu \sinh(\lambda x) \quad \text{or} \quad \varphi_3(x) = -\mu \sin(\lambda x), \quad x > 0, \tag{42}$$

with $\lambda > 0$ and $\mu > 0$.

It is easy to check that for this choice of functions F, h and Φ, Problem P is under the hypothesis of the previous theorem (see Appendix 1). Therefore, it has the solution $u = u(x,t)$ given in (34).

Proposition 2.2 *If F, h and $\Phi = \varphi_1$ are defined as in* (31), (41) *and* (42), *then the heat flux at the boundary $x = 0$ corresponding to the solution u* (*see* (34)) *to Problem P is given by*

$$u_x(0,t) = \begin{cases} \eta \exp(-\nu\lambda t) & \text{if } m = 1, \\ \frac{c(m-1)}{2} \exp(-\nu\lambda t) \int_0^t \tau^{(m-3)/2} \exp(\nu\lambda\tau)\,d\tau & \text{if } m > 1, \end{cases} \quad t > 0, \tag{43}$$

where

$$c = \frac{2^{m-1} m\eta}{\sqrt{\pi}} \Gamma\left(\frac{m}{2}\right), \tag{44}$$

and Γ is the gamma function, defined by

$$\Gamma(z) = \int_0^{+\infty} \xi^{z-1} \exp(-\xi)\,d\xi, \quad z \in \mathbb{R}. \tag{45}$$

Proof We know from Theorem 2.3 that $u_x(0,t) = V(t)$ satisfies the Volterra integral equation (38), where the function V_0 is given by

$$V_0(t) = ct^{(m-1)/2}, \quad t > 0. \tag{46}$$

Then $V(t)$ is given by (see [38])

$$V(t) = \frac{c(m-1)}{2} \int_0^t \tau^{(m-3)/2} r(\tau)\,d\tau, \quad t > 0, \tag{47}$$

where r satisfies the integral equation

$$r(t) = 1 - \nu\lambda \int_0^t r(\tau)\,d\tau, \quad t > 0, \tag{48}$$

whose solution is given by

$$r(t) = \exp(-\nu\lambda t), \quad t > 0. \tag{49}$$

By replacing (49) in (47), we obtain (43). □

Corollary 2.1 *If in Proposition* 2.2 *we consider m an odd number given by $m = 2p+1$ with $p \in \mathbb{N}$, then we have*

$$u_x(0,t) = p_{1,m}(t) - c_{1,m} \exp(-\nu\lambda t), \quad t > 0, \tag{50}$$

where $c_{1,m}$ is given by

$$c_{1,m} = (-1)^{p-1} \frac{cp!}{(\nu\lambda)^p}, \tag{51}$$

c being the constant given in (44), *and $p_{1,m}(x)$ is the polynomial defined by*

$$p_{1,m}(t) = \begin{cases} c_{1,3} & \text{if } m = 3, \\ -c_{1,5}(\nu\lambda t - 1) & \text{if } m = 5, \\ c_{1,m}\left(\sum_{k=1}^{p-1} \frac{(-\nu\lambda)^k}{k!} t^k + (-1)^{p-1}\right) & \text{if } m \geq 7, \end{cases} \quad t > 0. \tag{52}$$

Proof It follows by solving the integral in the expression of $u_x(0,t)$ given in (43). We do not reproduce these calculations here, but only remark the utility of the identity

$$\int_0^t \tau^n \exp(a\tau)\,d\tau = \frac{n!}{a} \exp(at) \left(\sum_{k=0}^{n-1} \frac{(-1)^k t^{n-k}}{(n-k)!a^k} + \frac{(-1)^n}{a^n} + \frac{(-1)^{n+1}}{a^n} \exp(-at) \right),$$
$$t > 0, n \in \mathbb{N}, n \geq 2, a \in \mathbb{R} \tag{53}$$

when $m \geq 7$. □

The last corollary enables us to obtain the asymptotic behavior of the heat flux $u_x(0,t)$ at the face $x = 0$ when t tends to $+\infty$ for an odd number m. The next result is related to this topic. We do not reproduce here the computations involved in its proof, which follows by taking the limit when t tends to $+\infty$ in the expression of $u_x(0,t)$ given in Corollary 2.1.

Corollary 2.2 *If F, h and $\Phi = \varphi_1$ are defined as in* (31), (41) *and* (42), *where m is an odd number, and u is the solution to Problem P, given in* (34), *then*:

(1) *if $m = 1$, we have*

$$\lim_{t\to+\infty} u_x(0,t) = 0, \tag{54}$$

(2) *if $m = 3$, we have*

$$\lim_{t\to+\infty} u_x(0,t) = \frac{6\eta}{\nu\lambda}, \tag{55}$$

(3) *if $m \geq 5$, we have*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \eta < 0, \\ +\infty & \text{if } \eta > 0. \end{cases} \tag{56}$$

The main idea in the proof of Proposition 2.2 was to find a solution for the integral equation (38) by finding a solution of another integral equation, which was easier to solve. In a more general way, we know that if V satisfies the Volterra integral equation (38), with V_0 an infinitely differentiable function, then $V(t)$ can be written as (see [38])

$$V(t) = V_0(0)r(t) + \int_0^t V_0'(t-\tau)r(\tau)\,d\tau, \quad t > 0, \tag{57}$$

where r satisfies the integral equation

$$r(t) = 1 - \nu\int_0^t R(t-\tau)r(\tau)\,d\tau, \quad t > 0, \tag{58}$$

and R is given in (40). But this last integral equation is not always easy to solve. Nevertheless, in several cases we can find an explicit solution for equation (58) by a formal application of the Laplace transform to their both sides. This is the way which led us to the expressions of $u_x(0,t)$ when $\Phi = \varphi_2$ or $\Phi = \varphi_3$, given in Propositions 2.3 and 2.4.

Proposition 2.3 *Let F, h and $\Phi = \varphi_2$ be defined as in* (31), (41) *and* (42), *and $\sigma = \lambda + \nu\mu$. Then the heat flux at the boundary $x = 0$ corresponding to the solution u* (*see* (34)) *of Problem P is given by*:

(1) *If $\sigma \neq 0$, then*

$$u_x(0,t) = \begin{cases} \frac{\eta}{\sigma}(\lambda + \nu\mu\exp(\lambda\sigma t)) & \text{if } m = 1, \\ \frac{c\lambda}{\sigma}t^{(m-1)/2} + \frac{c(m-1)\nu\mu}{2\sigma}\exp(\lambda\sigma t) & \\ \quad\times\int_0^t \tau^{(m-3)/2}\exp(-\lambda\sigma\tau)\,d\tau & \text{if } m > 1, \end{cases} \quad t > 0. \tag{59}$$

(2) *If $\sigma = 0$, then*

$$u_x(0,t) = \begin{cases} \eta(1-\lambda^2 t) & \text{if } m = 1, \\ ct^{(m-1)/2} - \frac{2c\lambda^2}{m+1}t^{(m+1)/2} & \text{if } m > 1, \end{cases} \quad t > 0. \tag{60}$$

Proof An easy computation shows that the expressions given in (59) and (60) satisfy the integral equation (38). Therefore, they correspond to the heat flux $u_x(0,t)$ at the boundary $x = 0$ for the solution u of Problem P given in (34). □

Corollary 2.3 *If in Proposition* 2.3 *we consider* $\sigma \neq 0$ *and* m *an odd number given by* $m = 2p+1$ *with* $p \in \mathbb{N}$, *then we have*

$$u_x(0,t) = p_{2,m}(t) + c_{2,m}\exp(\lambda\sigma t), \quad t > 0, \tag{61}$$

where $c_{2,m}$ *is given by*

$$c_{2,m} = \frac{c\nu\mu p!}{\sigma(\lambda\sigma)^p}, \tag{62}$$

c *being the constant given in* (44), *and* $p_{2,m}(x)$ *is the polynomial defined by*

$$p_{2,m}(t) = \begin{cases} c_{2,3}(\frac{\lambda^2\sigma}{\nu\mu}t - 1) & \text{if } m = 3, \\ c_{2,5}(\frac{\lambda^3\sigma^2}{2\nu\mu}t^2 - \lambda\sigma t - 1) & \text{if } m = 5, \\ \frac{c\lambda}{\sigma}t^p - c_{2,m}(\sum_{k=1}^{p-1}\frac{(\lambda\sigma)^k}{k!}t^k + 1) & \text{if } m \geq 7, \end{cases} \quad t > 0. \tag{63}$$

Proof It follows by solving the integral in the expression of u given in (59) and the use of identity (53). □

Corollary 2.4 *Let* F, h *and* $\Phi = \varphi_2$ *be defined as in* (31), (41) *and* (42), *with* m *an odd number, and* $\sigma = \lambda + \nu\mu$. *If* u *is the solution of Problem P, given in* (34), *then*:

(1) *If* $\sigma \neq 0$, *then*:

(a) *if* $m = 1$, *we have*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \sigma > 0, \eta < 0, \\ +\infty & \text{if } \sigma > 0, \eta > 0, \\ \frac{\eta\lambda}{\sigma} & \text{if } \sigma < 0, \end{cases} \tag{64}$$

(b) *if* $m \geq 3$, *we have*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \sigma\eta < 0, \\ +\infty & \text{if } \sigma\eta > 0. \end{cases} \tag{65}$$

(2) *If* $\sigma = 0$, *then*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \eta > 0, \\ +\infty & \text{if } \eta < 0. \end{cases} \tag{66}$$

Proposition 2.4 *Let* F, h *and* $\Phi = \varphi_3$ *be defined as in* (31), (41) *and* (42), *and* $\delta = \lambda - \nu\mu$. *Then the heat flux at the boundary* $x = 0$ *corresponding to the solution* u (*see* (34)) *of Problem P is given by*:

(1) *If* $\delta \neq 0$, *then*

$$u_x(0,t) = \begin{cases} \frac{\eta}{\delta}(\lambda - \nu\mu \exp(-\lambda\delta t)) & \text{if } m = 1, \\ \frac{c\lambda}{\delta} t^{(m-1)/2} - \frac{c(m-1)\nu\mu}{2\delta} \exp(-\lambda\delta t) & \\ \quad \times \int_0^t \tau^{(m-3)/2} \exp(\lambda\delta\tau)\, d\tau & \text{if } m > 1, \end{cases} \quad t > 0. \tag{67}$$

(2) *If* $\delta = 0$, *then*

$$u_x(0,t) = \begin{cases} \eta(1 + \lambda^2 t) & \text{if } m = 1, \\ ct^{(m-1)/2} + \frac{2c\lambda^2}{m+1} t^{(m+1)/2} & \text{if } m > 1, \end{cases} \quad t > 0. \tag{68}$$

Proof The proof of (67) and (68) follows by replacing λ^2 by $-\lambda^2$ and σ by δ in the proof of Proposition 2.3. □

Corollary 2.5 *If in Proposition* 2.3 *we consider* $\delta \neq 0$ *and* m *an odd number given by* $m = 2p + 1$ *with* $p \in \mathbb{N}$, *then we have*

$$u_x(0,t) = p_{3,m}(c) + c_{3,m} \exp(-\lambda\delta t), \quad t > 0, \tag{69}$$

where $c_{3,m}$ *is given by*

$$c_{3,m} = (-1)^{p-1} \frac{c\nu\mu p!}{\delta(\lambda\delta)^p}, \tag{70}$$

c being the constant given in (44), *and* $p_{3,m}(x)$ *is the polynomial defined by*

$$p_{3,m}(t) = \begin{cases} c_{3,3}(\frac{\lambda^2\delta}{\nu\mu} t - 1) & \text{if } m = 3, \\ -c_{3,5}(\frac{\lambda^3\delta^2}{2\nu\mu} t^2 - \lambda\sigma t + 1) & \text{if } m = 5, \\ \frac{c\lambda}{\delta} t^p - c_{3,m}(\sum_{k=1}^{p-1} \frac{(-\lambda\delta)^k}{k!} t^k + 1) & \text{if } m \geq 7, \end{cases} \quad t > 0. \tag{71}$$

Proof It follows by solving the corresponding integral in expression (67) and the use of identity (53). □

Corollary 2.6 *Let* F, h *and* $\Phi = \varphi_3$ *be defined as in* (31), (41) *and* (42), *with* m *an odd number, and* $\delta = \lambda - \nu\mu$. *If* u *is the solution of Problem P, given in* (34), *then*:

(1) *If* $\delta \neq 0$, *then*:

(a) *if* $m = 1$, *we have*

$$\lim_{t \to +\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \delta < 0, \eta < 0, \\ +\infty & \text{if } \delta < 0, \eta > 0, \\ \frac{\eta\lambda}{\delta} & \text{if } \delta > 0, \end{cases} \tag{72}$$

(b) *if* $m = 3$ *or* $m = 5$, *we have*

$$\lim_{t \to +\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \eta < 0, \\ +\infty & \text{if } \eta > 0 \end{cases} \tag{73}$$

and

(c) *if $m \geq 7$, we have*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \delta\eta < 0, \\ +\infty & \text{if } \delta\eta > 0. \end{cases} \tag{74}$$

(2) *If $\delta = 0$, then*

$$\lim_{t\to+\infty} u_x(0,t) = \begin{cases} -\infty & \text{if } \eta < 0, \\ +\infty & \text{if } \eta > 0. \end{cases} \tag{75}$$

The next result is related to the behavior of the heat flux $u_x(0,t)$ at the face $x = 0$ when t tends to 0^+, and shows that it is independent of the choice of Φ as any of the functions given in (42).

Corollary 2.7 *If F, h and Φ are given as in* (31), (41) *and any of the expressions in* (42), *respectively, then*

$$\lim_{t\to 0^+} u_x(0,t) = \begin{cases} \eta & \text{if } m = 1, \\ 0 & \text{if } m > 1, \end{cases} \tag{76}$$

where u is the solution of Problem P given in (34).

Proof It follows straightforwardly by computing the limit for the expression of $u_x(0,t)$ given in Propositions 2.2, 2.3 or 2.4, according the definition of Φ. □

We end this section by giving explicit solutions to each Problem P. The proofs of the three following propositions follow from Theorem 2.3 and Corollary 2.1, 2.3 or 2.5, according to the definition of Φ (see Appendix 2).

Proposition 2.5 *If F, h and $\Phi = \varphi_1$ are defined as in* (31), (41) *and* (42), *where m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by*

$$u(x,t) = u_0(x,t) - \nu\Phi(x)\int_0^t V(\tau)\,d\tau, \quad x \geq 0, t \geq 0 \tag{77}$$

is a solution to Problem P, where u_0 is defined by

$$u_0(x,t) = \frac{\eta}{\sqrt{\pi}}\sum_{k=0}^{p}\binom{m}{2k}\Gamma\left(\frac{2k+1}{2}\right)(4t)^k x^{m-2k}, \quad x \geq 0, t \geq 0, \tag{78}$$

and $V(t) = u_x(0,t)$ is given by (50).

Remark 3 If $m = 1$, polynomial $p_{1,m}(x)$ is defined by $p_{1,m}(x) = 0$, $x > 0$.

Proposition 2.6 *If F, h and $\Phi = \varphi_2$ are defined as in* (31), (41) *and* (42), *where $\sigma \neq 0$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by*

$$u(x,t) = u_0(x,t) - \nu\Phi(x)\exp\bigl(\lambda^2 t\bigr)\int_0^t V(\tau)\exp\left(-\lambda^2\tau\right)d\tau, \quad x \geq 0, t \geq 0 \tag{79}$$

is a solution to Problem P, where u_0 and $V(t) = u_x(0,t)$ are given by (78) *and* (61).

Remark 4 If $m = 1$, polynomial $p_{2,m}(x)$ is defined by $p_{2,m}(x) = \frac{\nu\lambda}{\sigma}$, $x > 0$.

Proposition 2.7 *If F, h and $\Phi = \varphi_3$ are defined as in* (31), (41) *and* (42), *where $\delta \neq 0$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$, then the function u defined by*

$$u(x,t) = u_0(x,t) - \nu\Phi(x)\exp(-\lambda^2 t)\int_0^t V(\tau)\exp(\lambda^2\tau)\,d\tau, \quad x \geq 0, t \geq 0 \tag{80}$$

is a solution to Problem P, where u_0 and $V(t) = u_x(0,t)$ are given by (78) *and* (69).

Remark 5 If $m = 1$, polynomial $p_{3,m}(x)$ is defined by $p_{3,m}(x) = \frac{\nu\lambda}{\delta}$, $x > 0$.

3 The controlling problem

This section is devoted to studying the effects introduced by the source term $-\Phi F$ in the asymptotic temporal behavior of the solution u to each Problem P considered in this paper. In particular, this will enable us to control the long term temporal behavior of the temperature u by imposing suitable specifications on the thermostat device through an appropriate choice of the coefficients involved in the definition of $-\Phi F$. We will carry out our study by comparing the asymptotic behavior of u with the asymptotic behavior of the solution u_0 to Problem P in the absence of control (Problem P_0):

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad x > 0, t > 0, \tag{81}$$

$$u(x,0) = h(x), \quad x > 0, \tag{82}$$

$$u(0,t) = 0, \quad t > 0. \tag{83}$$

The study of controlling the solution to Problem P through its source term was done in [3] when Φ is identically equal to 1, $F = F(V)$ is a differentiable function of one real variable which satisfies:

(1) $VF(V) \geq 0$, $\forall V \in \mathbb{R}$,
(2) $F(0) = 0$,
(3) F is convex in $(0, +\infty)$,
(4) $\lim_{V\to+\infty} F'(V) = \kappa > 0$,

and h is a non-negative, continuous and bounded function. They proved that under these hypotheses, both u and u_0 converge to 0 when t tends to $+\infty$ and the control term F has a stabilizing effect because $\lim_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} = 0$, that is, u converges faster to 0 than u_0. None of the cases studied in the previous sections fulfill the hypothesis for Φ, F and h established in [3].

With the aim of supplementing the results given in [3], we will carry out our analysis under conditions which lead us to functions F depending on only one real variable, that is, $F = F(V)$.

Next Theorems 3.1, 3.2 and 3.3 are respectively related with the results obtained in Sections 2.1, 2.2 and 2.3.

Remark 6 For all Problems P studied in this paper, Problem P_0 has the solution u_0 defined by [1]

$$u_0(x,t) = \int_0^{+\infty} G(x,t,\xi,0)h(\xi)\,d\xi, \quad x \geq 0, t \geq 0, \tag{84}$$

where G is the Green function defined in (35).

Theorem 3.1 *Let Φ be identically equal to* 1, *F be a constant function defined by*

$$F(V) = \nu, \quad V \in \mathbb{R}, \tag{85}$$

with $\nu \in \mathbb{R} - \{0\}$, and h be a quadratic function defined by

$$h(x) = \frac{\nu}{2}x^2 + ax, \quad x \geq 0, \tag{86}$$

with $a \in \mathbb{R}$.

For the solution u_0 to Problem P_0 given in (84), *we have*

$$\lim_{t\to+\infty} u_0(x,t) = \infty, \quad \forall x > 0. \tag{87}$$

Furthermore, there exists a solution u to Problem P such that

$$\lim_{t\to+\infty} u(x,t) = h(x), \quad \forall x > 0. \tag{88}$$

Proof By computing the integral in (84) for the function h given in (86), we have that the solution u_0 to Problem P_0 given in (84) is defined by

$$u_0(x,t) = \left(\frac{\nu}{2}x^2 + \nu t\right)\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + \frac{\nu}{\sqrt{\pi}}x\sqrt{t}\exp\left(\frac{-x^2}{4t}\right) + ax, \quad x > 0, t > 0. \tag{89}$$

By taking the limit when t tends to $+\infty$, we have (87).

Since functions Φ, F and h are under the hypothesis of Theorem 2.1, we know that the function u given by

$$u(x,t) = h(x), \quad x \geq 0, t \geq 0, \tag{90}$$

is a solution to Problem P, which satisfies (88). □

Theorem 3.2 *Let Φ, h and F be defined by*

$$\Phi(x) = \lambda X(x), \qquad h(x) = \eta X(x), \quad x > 0 \quad \textit{and} \quad F = F\big(\delta T(t)\big), \quad t > 0, \tag{91}$$

where X is the function given by (14)

$$X(x) = \begin{cases} \frac{\delta}{\sqrt{\sigma}}\sinh(\sqrt{\sigma}x) & \textit{if } \sigma > 0, \\ \frac{\delta}{\sqrt{|\sigma|}}\sin(\sqrt{|\sigma|}x) & \textit{if } \sigma < 0, \quad x > 0 \\ \delta x & \textit{if } \sigma = 0, \end{cases}$$

and T is the solution of the initial value problem (15)-(16)

$$\dot{T}(t) - \sigma T(t) = -\lambda F\big(\delta T(t), t\big), \quad t > 0$$

$$T(0) = \eta,$$

with $\lambda, \eta, \delta \in \mathbb{R} - \{0\}$.

For the solution u_0 *to Problem* P_0 *given in* (84), *we have*

$$\lim_{t\to+\infty} u_0(x,t) = \begin{cases} h(x) & \textit{if } \sigma = 0, \\ \infty & \textit{if } \sigma > 0, \\ 0 & \textit{if } \sigma < 0, \end{cases} \quad \forall x > 0. \tag{92}$$

Furthermore:

(1) *If* F *is defined by*

$$F(V) = \nu V, \quad V \in \mathbb{R}, \tag{93}$$

with $\nu \in \mathbb{R} - \{0\}$, *then there exists a solution* u *to Problem* P *which satisfies*

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} h(x) & \textit{if } \gamma = \sigma, \\ \infty & \textit{if } \gamma < \sigma, \\ 0 & \textit{if } \gamma > \sigma, \end{cases}$$

$$\forall x > 0 \textit{ if } \gamma \geq \sigma \textit{ or } \forall x > 0/h(x) \neq 0 \textit{ if } \gamma < \sigma, \tag{94}$$

where $\gamma = \lambda\nu\delta$.

Therefore,

$$\lim_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} = \begin{cases} \infty & \textit{if } \gamma < 0, \\ 0 & \textit{if } \gamma > 0, \end{cases}$$

$$\forall x > 0 \textit{ if } \sigma \geq 0 \textit{ or } \forall x > 0/h(x) \neq 0 \textit{ if } \sigma < 0. \tag{95}$$

(2) *If* F *is defined by*

$$F(V) = \nu V^n, \quad V \in \mathbb{R}, \tag{96}$$

with $\nu > 0$ *and* $n < 1$, *and we consider* $\lambda > 0$, $\eta > 0$ *and* $\delta > 0$, *then there exists a solution* u *to Problem* P *which satisfies*

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} 0 & \textit{if } \sigma < 0 \vee 0 < n < 1, \\ \theta_1 h(x) & \textit{if } \sigma < 0 \wedge n = 0, \\ \infty & \textit{if } \sigma \geq 0 \vee (\sigma < 0 \wedge n < 0), \end{cases}$$

$$\forall x > 0 \textit{ if } \sigma < 0 \wedge 0 \leq n < 1 \textit{ or } \forall x > 0/h(x) \neq 0 \textit{ if } \sigma \geq 0 \vee (n < 0 \wedge \sigma < 0), \tag{97}$$

where $\theta_1 = \frac{\lambda\mu}{\sigma\eta}$.

Therefore,

$$\lim_{t\to+\infty} \frac{u(x,t)}{u_0(x,t)} = \begin{cases} \infty & \textit{if } \sigma \leq 0, \\ 1 - \frac{\lambda\mu\delta^n}{\sigma\eta^{1-n}} & \textit{if } \sigma > 0, \end{cases}$$

$$\forall x > 0 \textit{ if } \sigma \geq 0 \textit{ or } \forall x > 0/h(x) \neq 0 \textit{ if } \sigma < 0. \tag{98}$$

Proof By computing the integral in (84) for the function h given in (91), we obtain that the solution u_0 to Problem P_0 given in (84) is defined by

$$u_0(x,t) = h(x)\exp\big(|\sigma|t\big), \quad \forall x > 0, t > 0. \tag{99}$$

By taking the limit when t tends to $+\infty$, we have (92).

(1) Since Φ, h and F are under the hypothesis of Corollary 2.1, we know that the function u given by

$$u(x,t) = \eta X(x)\exp\big((\sigma-\gamma)t\big), \quad \forall x > 0, t > 0 \tag{100}$$

is a solution to Problem P, which satisfies (94). Finally, the proof of (95) follows straightforwardly by computing the limit from the explicit expressions of u_0 and u given in (99) and (100).

(2) It follows in the same manner as the proof of the previous item. □

We see from the previous theorem that it is possible to control a solution to Problem P through the parameters involved in the definition of the source term $-\Phi F$. When $F(V) = \nu V$, we can increase ($\gamma < 0 < \sigma$) or decrease ($0 < \gamma < \sigma$) the velocity of convergence to ∞ for u with respect to the velocity of convergence for u_0. We also can stabilize the problem by doing u tending to a constant value ($0 < \sigma \leq \gamma$) when u_0 is going to ∞. When $F(V) = \nu V^n$, we can decrease ($\sigma > 0$ and $1 = \frac{\lambda\mu\delta^n}{\sigma\eta^{1-n}}$) or maintain ($\sigma > 0$ and $1 \neq \frac{\lambda\mu\delta^n}{\sigma\eta^{1-n}}$) the velocity of convergence to ∞ for u with respect to the velocity of convergence for u_0. We also can decrease the velocity of convergence to 0 for u with respect to the velocity of convergence for u_0 ($\sigma < 0$).

Theorem 3.3 *Let Φ be defined by one of the expressions given in* (42)

$$\varphi_1(x) = \lambda x, \qquad \varphi_2(x) = -\mu\sinh(\lambda x) \quad \text{or} \quad \varphi_3(x) = -\mu\sin(\lambda x), \quad x > 0,$$

where $\lambda > 0$ and $\mu > 0$, F is defined by

$$F = F(V) = \nu V, \quad V \in \mathbb{R}, \tag{101}$$

with $\nu > 0$ and h is defined as in (41)

$$h(x) = \eta x^m, \quad x > 0,$$

where $\eta \in \mathbb{R} - \{0\}$ and m is an odd number given by $m = 2p + 1$, with $p \in \mathbb{N}_0$.

For the solution u_0 to Problem P_0 given in (84)*, we have*

$$\lim_{t\to+\infty} u_0(x,t) = \begin{cases} h(x) & \text{if } m = 1, \\ \infty & \text{if } m > 1, \end{cases} \quad \forall x > 0. \tag{102}$$

Furthermore:

(1) *If $\Phi = \varphi_1$, then there exists a solution u to Problem P which satisfies*

$$\lim_{t\to+\infty} u(x,t) = \begin{cases} 0 & \text{if } m = 1, \\ \infty & \text{if } m > 1, \end{cases} \quad \forall x > 0. \tag{103}$$

Therefore,

$$\lim_{t\to+\infty}\frac{u(x,t)}{u_0(x,t)}=\begin{cases}0 & \text{if } m=1,\\ r(x) & \text{if } m>1,\end{cases}\qquad \forall x>0, \tag{104}$$

$r(x)$ being a rational function in the variable x.

(2) *If $\Phi=\varphi_2$, then there exists a solution u to Problem P which satisfies*

$$\lim_{t\to+\infty}u(x,t)=\infty,\qquad \forall x>0. \tag{105}$$

Therefore,

$$\lim_{t\to+\infty}\frac{u(x,t)}{u_0(x,t)}=\infty,\qquad \forall x>0. \tag{106}$$

(3) *If $\Phi=\varphi_3$, then there exists a solution u to Problem P which satisfies*

$$\lim_{t\to+\infty}u(x,t)=\infty,\qquad \forall x>0. \tag{107}$$

Therefore,

$$\lim_{t\to+\infty}\frac{u(x,t)}{u_0(x,t)}=\begin{cases}r(x) & \text{if } \delta>0,\\ \infty & \text{if } \delta\le 0,\end{cases}\qquad \forall x>0, \tag{108}$$

where $r(x)$ is a rational function in the variable x.

Proof It follows in the same manner as the proofs of Theorems 3.1 and 3.2. □

From the previous theorem, we see again that there exist several cases where we can control a solution to Problem P through the source term $-\Phi F$.

4 Explicit solutions for Problem $\widetilde{\mathrm{P}}$

In this section we consider Problem $\widetilde{\mathrm{P}}$ given in (5)-(7) with the aim of finding exact solutions. This problem corresponds to another temperature regulation problem where the temperature controller device depends on the temperature at the fixed boundary of the material instead of the heat flux on it, and a heat flux initial condition is known in place of a temperature condition.

The following theorem states a relationship between Problems P and $\widetilde{\mathrm{P}}$ given in (5)-(7), and it was proved in [13].

Theorem 4.1 *If u is a solution to Problem P, where h and Φ are differentiable functions in $\mathbb{R}^+$, then the function v defined by*

$$v(x,t)=u_x(x,t),\qquad x\ge 0, t\ge 0 \tag{109}$$

is a solution to Problem $\widetilde{\mathrm{P}}$ when $\widetilde{F}$, $\widetilde{\Phi}$, $\tilde{h}$ and $\tilde{g}$ are defined by

$$\begin{aligned}&\widetilde{F}(V,t)=F(V,t),\quad V>0,t>0,\qquad \tilde{g}(t)=\Phi(0)F\big(u_x(0,t),t\big),\quad t>0,\\ &\widetilde{\Phi}(x)=\Phi'(x),\quad x>0,\qquad \tilde{h}(x)=h'(x),\quad x>0.\end{aligned} \tag{110}$$

We see from Theorem 4.1 that we can find exact solutions to Problem $\widetilde{\mathrm{P}}$ from exact solutions to another temperature regulation problem which has the form of Problem P.

We end this section by giving explicit solutions for some particular cases of Problem $\widetilde{\mathrm{P}}$.

Proposition 4.1 *Let $\tilde{g}$ be the zero function and*:

(1)

(a) *$\widetilde{F}$ be the zero function and $\tilde{h}$ be a constant function, or*

(b) *$\widetilde{F}$ be a constant function defined by*

$$\widetilde{F}(V,t) = k, \quad V \in \mathbb{R}, t > 0, \tag{111}$$

with $k \in \mathbb{R} - \{0\}$, $\widetilde{\Phi}$ be a locally integrable function in $\mathbb{R}^+$ and $\tilde{h}$ be a differentiable function such that

$$\tilde{h}(x) = k \int_0^x \widetilde{\Phi}(\xi)\, d\xi, \quad x > 0. \tag{112}$$

Then the function v defined by

$$v(x,t) = \tilde{h}(x), \quad x \geq 0, t \geq 0 \tag{113}$$

is a solution to Problem $\widetilde{\mathrm{P}}$ independent of the temporal variable t.

(2) *$\widetilde{F}$ be given by* (21), (23) *or* (27), *that is*:

$F(V,t) = \nu V$, $V \in \mathbb{R}$, $t > 0$, *with* $\nu \in \mathbb{R} - \{0\}$,

$F(V,t) = f_1(t) + f_2(t)V$, $V \in \mathbb{R}$, $t > 0$, *with* $f_1, f_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, *or*

$F(V,t) = V^n f(t)$, $V \in \mathbb{R}$, $t > 0$, *with* $n < 1$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, $f > 0$ *and* $\lambda, \delta, \eta > 0$,

and $\tilde{h}$ and $\widetilde{\Phi}$ be defined by

$$\tilde{h}(x) = \eta \widetilde{X}(x) \quad \text{and} \quad \widetilde{\Phi}(x) = \lambda \widetilde{X}(x), \quad t > 0, \tag{114}$$

where $\widetilde{X}$ is given by

$$\widetilde{X}(x) = \begin{cases} \delta \cosh(\sqrt{\sigma})x & \text{if } \sigma > 0, \\ \delta \cos(\sqrt{|\sigma|})x & \text{if } \sigma < 0, \\ \delta & \text{if } \sigma = 0, \end{cases} \quad x > 0, \tag{115}$$

$\lambda, \eta, \delta \in \mathbb{R} - \{0\}$.

Then the function v defined by

$$v(x,t) = \widetilde{X}(x)\widetilde{T}(t), \quad x \geq 0, t \geq 0 \tag{116}$$

is a solution with separated variables to Problem $\widetilde{\mathrm{P}}$, where $\widetilde{T}$ is the solution of the initial value problem (15)-(16).

(3) *$\widetilde{F}$ be defined as in* (31)

$$F = F(V,t) = \nu V, \quad V \in \mathbb{R}, t > 0,$$

with $\nu > 0$, $\tilde{h}$ *be defined as*

$$\tilde{h}(x) = \tilde{\eta} x^l, \quad x > 0, \tag{117}$$

with $\tilde{\eta} \in \mathbb{R} - \{0\}$ *and* $l \geq 0$, *and* $\widetilde{\Phi}$ *be given by one of the following expressions*:

$$\widetilde{\varphi}_1(x) = \tilde{\lambda} x, \qquad \varphi_2(x) = -\tilde{\mu} \cosh(\tilde{\lambda} x) \quad \textit{or} \quad \varphi_3(x) = -\tilde{\mu} \cos(\tilde{\lambda} x), \quad x > 0, \tag{118}$$

with $\tilde{\lambda} > 0$ *and* $\tilde{\mu} > 0$. *Then the function* v *defined by*

$$v(x,t) = u_x(x,t), \quad x \geq 0, t \geq 0 \tag{119}$$

is a solution to Problem $\widetilde{\mathrm{P}}$, *where* u *is given by* (77) *if* $\widetilde{\Phi} = \tilde{\varphi}_1$, *by* (79) *if* $\widetilde{\Phi} = \tilde{\varphi}_2$ *or by* (80) *if* $\widetilde{\Phi} = \tilde{\varphi}_3$.

Proof It follows from the previous theorem and the explicit solutions to Problem P obtained in Section 2. □

5 Conclusions

In this paper we consider a non-classical initial and boundary value problem for a non-homogeneous one-dimensional heat equation which represents a temperature regulation problem for a semi-infinite homogeneous isotropic medium where the temperature controller device depends on the heat flux at the fixed boundary, an initial temperature distribution is known and the temperature at the fixed boundary is constant in time. We find explicit solutions for several cases of this problem, which, in particular, enables us to give explicit formulae for the heat flux at the boundary and to compute its asymptotic temporal behavior.

We also analyze how the source term affects the asymptotic temporal behavior of each explicit solution u obtained in this paper by comparing the limits of u and the solution u_0 to the same problem but in absence of source term. As a result, we obtain conditions on the parameters involved in the definition of the source term that enables us to control the solutions u with respect to u_0. In particular, we give conditions on data functions under which stationary solutions exist.

By giving a relationship between the problem considered here with another related non-classical heat equation problem, we obtain explicit solutions for several particular cases of another temperature regulation problem where the thermostat depends on the temperature at the fixed boundary of the material instead on the heat flux on it, and a heat flux initial condition is known in place of a temperature condition.

As a consequence of our study, several solved non-classical problems for the heat equation that can be used for testing new numerical methods are given. In addition, exact solutions given in this article also provide reference values for comparisons in laboratory experiments.

Appendix 1: Problem P is under the hypothesis of Theorem 2.3

Let Problem P with F and h given as in (31) and (41), respectively, and Φ defined by any of the expressions φ_1, φ_2 or φ_3 given in (42).

(1) It is clear that h is a continuously differentiable function such that $h(0)$ exists. We also have

$$\left|h(x)\right| = |\eta|x^m \leq |\eta|(x+1)^m \leq |\eta| \exp(mx), \quad \forall x > 0. \tag{120}$$

Then inequality (30) holds with $\epsilon = 1$, $c_1 = m$ and $c_0 = |\eta|$.

(2) It is easy to check that each of the functions φ given in (42) is uniformly Hölder continuous, with Hölder exponent $\alpha = 1$, on any compact set $K \subset \mathbb{R}$.

(3) Hypothesis 3 holds because of the definition of Problem P.

Furthermore, we have:

(i) If $\Phi = \varphi_1$, then

$$R(t) = \lambda, \quad t > 0. \tag{121}$$

Then inequality (32) holds with a function f defined by

$$f(t) = -\lambda t, \quad t > 0. \tag{122}$$

(ii) If $\Phi = \varphi_2$, then

$$R(t) = -\lambda\mu \exp\left(\lambda^2 t\right), \quad t > 0. \tag{123}$$

Then inequality (32) holds with a function f defined by

$$f(t) = -\frac{\mu}{\lambda}\left(\exp\left(\lambda^2 t\right) - 1\right), \quad t > 0. \tag{124}$$

(iii) If $\Phi = \varphi_3$, then

$$R(t) = -\lambda\mu \exp\left(-\lambda^2 t\right), \quad t > 0. \tag{125}$$

Then inequality (32) holds with a function f defined by

$$f(t) = -\frac{\mu}{\lambda}\left(1 - \exp\left(-\lambda^2 t\right)\right), \quad t > 0. \tag{126}$$

Appendix 2: Proof of Propositions 2.5, 2.6 and 2.7

Let Problem P with F and h given as in (31) and (41), respectively, and Φ given by any of the expressions in (42).

2.1 Computation of $\int_0^{+\infty} G(x,t,\xi,0)h(\xi)\,d\xi$

By the definitions of the functions G and h given in (35) and (41), respectively, we have

$$\begin{aligned}
&\int_0^{+\infty} G(x,t,\xi,0)h(\xi)\,d\xi \\
&\quad = \frac{\eta}{2\sqrt{\pi t}} \int_0^{+\infty} \left(\exp\left(-(x-\xi)^2/4t\right) - \exp\left(-(x+\xi)^2/4t\right)\right)\xi^m\,d\xi.
\end{aligned} \tag{127}$$

We first compute $\int_0^{+\infty} \exp(-(x-\xi)^2/4t)\xi^m\,d\xi$.

By doing the substitution $\zeta = (x-\xi)/2\sqrt{t}$, we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\left(-(x-\xi)^2/4t\right)\xi^m\,d\xi \\
&\quad = 2\sqrt{t}\int_{-\infty}^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)(x-2\sqrt{t}\zeta)^m\,d\zeta \\
&\quad = 2\sqrt{t}\sum_{k=0}^{m}\binom{m}{k}(-2\sqrt{t})^k x^{m-k}\int_{-\infty}^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta.
\end{aligned} \tag{128}$$

Since

$$\begin{aligned}
\int_{-\infty}^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta &= \int_{-\infty}^{0} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta + \int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta \\
&= (-1)^k\int_0^{+\infty} \exp\left(-\sigma^2\right)\sigma^k\,d\sigma + \int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta \\
&= \frac{(-1)^k}{2}\Gamma\left(\frac{k+1}{2}\right) + \int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta,
\end{aligned} \tag{129}$$

then we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\left(-(x-\xi)^2/4t\right)\xi^m\,d\xi \\
&\quad = \sqrt{t}\sum_{k=0}^{m}\binom{m}{k}(2\sqrt{t})^k x^{m-k}\Gamma\left(\frac{k+1}{2}\right) \\
&\qquad + 2\sqrt{t}\sum_{k=0}^{m}\binom{m}{k}(-2\sqrt{t})^k x^{m-k}\int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta.
\end{aligned} \tag{130}$$

By similar calculations, we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\left(-(x+\xi)^2/4t\right)\xi^m\,d\xi \\
&\quad = \sqrt{t}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(2\sqrt{t})^k x^{m-k}\Gamma\left(\frac{k+1}{2}\right) \\
&\qquad + 2\sqrt{t}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}(-2\sqrt{t})^k x^{m-k}\int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta.
\end{aligned} \tag{131}$$

Therefore, we have

$$\begin{aligned}
&\int_0^{+\infty} G(x,t,\xi,0)h(\xi)\,d\xi \\
&\quad = \frac{\eta}{2\sqrt{\pi t}}\Bigg(\sqrt{t}\sum_{k=0}^{m}\left(1-(-1)^{m-k}\right)\binom{m}{k}(2\sqrt{t})^k x^{m-k}\Gamma\left(\frac{k+1}{2}\right) \\
&\qquad + 2\sqrt{t}\sum_{k=0}^{m}\left(1-(-1)^{m+1}\right)\binom{m}{k}(-2\sqrt{t})^k x^{m-k}\int_0^{x/2\sqrt{t}} \exp\left(-\zeta^2\right)\zeta^k\,d\zeta\Bigg)
\end{aligned}$$

$$= \frac{\eta}{2\sqrt{\pi}} \sum_{k=0}^{m} \left(1-(-1)^{k+1}\right) \binom{m}{k} (2\sqrt{t})^k x^{m-k} \Gamma\left(\frac{k+1}{2}\right)$$

$$= \frac{\eta}{\sqrt{\pi}} \sum_{k=0}^{p} \binom{m}{2k} (4t)^k x^{m-2k} \Gamma\left(\frac{2k+1}{2}\right), \quad x>0. \tag{132}$$

2.2 Computation of $\int_0^{+\infty} G(x,t,\xi,\tau)\Phi(\xi)\,d\xi$

(1) By the definitions of the functions G and $\Phi = \varphi_1$ given in (35) and (42), respectively, we have

$$\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_1(\xi)\,d\xi = \frac{\lambda}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} \bigl(\exp\bigl(-(x-\xi)^2/4(t-\tau)\bigr) - \exp\bigl(-(x+\xi)^2/4(t-\tau)\bigr)\bigr)\xi\,d\xi. \tag{133}$$

By replacing t by $(t-\tau)$, η by λ and m by 1 in the precedent calculation, we have

$$\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_1(\xi)\,d\xi = \varphi_1(x), \quad x>0. \tag{134}$$

(2) By the definitions of the functions G and $\Phi = \varphi_2$ given in (35) and (42), respectively, we have

$$\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_2(\xi)\,d\xi = -\frac{\mu}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} \bigl(\exp\bigl(-(x-\xi)^2/4(t-\tau)\bigr) - \exp\bigl(-(x+\xi)^2/4(t-\tau)\bigr)\bigr)\sinh(\lambda\xi)\,d\xi. \tag{135}$$

We first compute $\int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau))\sinh(\lambda\xi)\,d\xi$.

By doing the change of variables $\zeta = (x-\xi)/2\sqrt{t-\tau}$, we have

$$\int_0^{+\infty} \exp\bigl(-(x-\xi)^2/4(t-\tau)\bigr)\exp(\lambda\xi)\,d\xi$$
$$= 2\sqrt{t-\tau}\exp(\lambda x) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\bigl(-\zeta^2 - 2\lambda\sqrt{t-\tau}\,\zeta\bigr)\,d\zeta. \tag{136}$$

By writing

$$\zeta^2 + 2\lambda\sqrt{t-\tau}\,\zeta = (\zeta + \lambda\sqrt{t-\tau})^2 - \lambda^2(t-\tau) \tag{137}$$

and doing the change of variables $\sigma = \zeta + \lambda\sqrt{t-\tau}$, we have

$$2\sqrt{t-\tau}\exp(\lambda x) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\bigl(-\zeta^2 - 2\lambda\sqrt{t-\tau}\,\zeta\bigr)\,d\zeta$$
$$= 2\sqrt{t-\tau}\exp\bigl(\lambda x + \lambda^2(t-\tau)\bigr) \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\bigl(-(\zeta+\lambda\sqrt{t-\tau})^2\bigr)\,d\zeta$$
$$= 2\sqrt{t-\tau}\exp\bigl(\lambda x + \lambda^2(t-\tau)\bigr) \int_{-\infty}^{x/2\sqrt{t-\tau}+\lambda\sqrt{t-\tau}} \exp\bigl(-\sigma^2\bigr)\,d\sigma$$
$$= \sqrt{\pi(t-\tau)}\exp\bigl(\lambda x + \lambda^2(t-\tau)\bigr)\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right), \tag{138}$$

where erf is the error function, defined by

$$\operatorname{erf}(z) = \int_0^z \exp\left(-\xi^2\right) d\xi, \quad z \in \mathbb{R}. \tag{139}$$

Hence, we have

$$\begin{aligned}&\int_0^{+\infty} \exp\left(-(x-\xi)^2/4(t-\tau)\right) \exp(\lambda\xi)\, d\xi \\ &\quad = \sqrt{\pi(t-\tau)} \exp\left(\lambda x + \lambda^2(t-\tau)\right)\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right).\end{aligned} \tag{140}$$

By replacing λ by $-\lambda$ in the previous calculations, we have

$$\begin{aligned}&\int_0^{+\infty} \exp\left(-(x-\xi)^2/4(t-\tau)\right) \exp(-\lambda\xi)\, d\xi \\ &\quad = \sqrt{\pi(t-\tau)} \exp\left(-\lambda x + \lambda^2(t-\tau)\right)\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right).\end{aligned} \tag{141}$$

Therefore, we have

$$\begin{aligned}&\int_0^{+\infty} \exp\left(-(x-\xi)^2/4(t-\tau)\right) \sinh(\lambda\xi)\, d\xi \\ &\quad = \frac{\sqrt{\pi(t-\tau)}}{2} \exp\left(\lambda^2(t-\tau)\right)\left(\exp(\lambda x)\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right)\right. \\ &\qquad \left. - \exp(-\lambda x)\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right)\right).\end{aligned} \tag{142}$$

By similar calculations, we have

$$\begin{aligned}&\int_0^{+\infty} \exp\left(-(x+\xi)^2/4(t-\tau)\right) \sinh(\lambda\xi)\, d\xi \\ &\quad = \frac{\sqrt{\pi(t-\tau)}}{2} \exp\left(\lambda^2(t-\tau)\right)\left(\exp(-\lambda x)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - \lambda\sqrt{t-\tau}\right)\right)\right. \\ &\qquad \left. - \exp(\lambda x)\left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + \lambda\sqrt{t-\tau}\right)\right)\right).\end{aligned} \tag{143}$$

Then we have

$$\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_2(\xi)\, d\xi = \exp\left(\lambda^2(t-\tau)\right)\varphi_2(x), \quad x > 0. \tag{144}$$

(3) By the definitions of the functions G and $\Phi = \varphi_3$ given in (35) and (42), respectively, we have

$$\begin{aligned}\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_3(\xi)\, d\xi = -\frac{\mu}{2\sqrt{\pi(t-\tau)}} \int_0^{+\infty} &\left(\exp\left(-(x-\xi)^2/4(t-\tau)\right)\right. \\ &\left. - \exp\left(-(x+\xi)^2/4(t-\tau)\right)\right) \sin(\lambda\xi)\, d\xi.\end{aligned} \tag{145}$$

We first compute $\int_0^{+\infty} \exp(-(x-\xi)^2/4(t-\tau)) \sin(\lambda\xi)\, d\xi$.
By doing the change of variables $\zeta = (x-\xi)/2\sqrt{t-\tau}$, we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\big(-(x-\xi)^2/4(t-\tau)\big) \sin(\lambda\xi)\, d\xi \\
&\quad = 2\sqrt{t-\tau} \int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\big(-\zeta^2\big) \\
&\qquad \times \big(\sin(\lambda x)\cos(2\lambda\sqrt{t-\tau}\zeta) - \cos(\lambda x)\sin(2\lambda\sqrt{t-\tau}\zeta)\big)\, d\zeta. \qquad (146)
\end{aligned}$$

By using the identities (see [39], p.4)

$$\int \exp\big(-\zeta^2\big)\cos(\alpha\zeta)\, d\zeta = \frac{\sqrt{\pi}}{4}\exp\big(-\alpha^2/4\big)\left(\operatorname{erf}\left(\zeta + \frac{\alpha}{2}i\right) + \operatorname{erf}\left(\zeta - \frac{\alpha}{2}i\right)\right) \qquad (147)$$

and

$$\int \exp\big(-\zeta^2\big)\sin(\alpha\zeta)\, d\zeta = \frac{\sqrt{\pi}i}{4}\exp\big(-\alpha^2/4\big)\left(\operatorname{erf}\left(\zeta + \frac{\alpha}{2}i\right) - \operatorname{erf}\left(\zeta - \frac{\alpha}{2}i\right)\right), \qquad (148)$$

where $\alpha \in \mathbb{R}$ and i denotes the imaginary unit, we have

$$\begin{aligned}
&\int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\big(-\zeta^2\big)\cos(2\lambda\sqrt{t-\tau}\zeta)\, d\zeta \\
&\quad = \frac{\sqrt{\pi}}{4}\exp\big(-\lambda^2(t-\tau)\big)\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right)\right. \\
&\qquad \left. + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) + 2\right) \qquad (149)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{-\infty}^{x/2\sqrt{t-\tau}} \exp\big(-\zeta^2\big)\sin(2\lambda\sqrt{t-\tau}\zeta)\, d\zeta \\
&\quad = \frac{\sqrt{\pi}i}{4}\exp\big(-\lambda^2(t-\tau)\big)\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right)\right. \\
&\qquad \left. - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right)\right). \qquad (150)
\end{aligned}$$

Then we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\big(-(x-\xi)^2/4(t-\tau)\big)\sin(\lambda\xi)\, d\xi \\
&\quad = \frac{\sqrt{\pi(t-\tau)}}{2}\exp\big(-\lambda^2(t-\tau)\big)\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right)\sin(\lambda x)\right. \\
&\qquad + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right)\sin(\lambda x) - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right)i\cos(\lambda x) \\
&\qquad \left. + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right)i\cos(\lambda x) + 2\sin(\lambda x)\right). \qquad (151)
\end{aligned}$$

By similar calculations, we have

$$\begin{aligned}
&\int_0^{+\infty} \exp\left(-(x+\xi)^2/4(t-\tau)\right)\sin(\lambda\xi)\,d\xi \\
&\quad = \frac{\sqrt{\pi(t-\tau)}}{2}\exp\left(-\lambda^2(t-\tau)\right)\left(\operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right) i\cos(\lambda x)\right. \\
&\qquad - \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right) i\cos(\lambda x) + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} + i\lambda\sqrt{t-\tau}\right)\sin(\lambda x) \\
&\qquad \left. + \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} - i\lambda\sqrt{t-\tau}\right)\sin(\lambda x) - 2\sin(\lambda x)\right). \qquad (152)
\end{aligned}$$

Then we have

$$\int_0^{+\infty} G(x,t,\xi,\tau)\varphi_3(\xi)\,d\xi = \exp\left(-\lambda^2(t-\tau)\right)\varphi_3(x), \quad x > 0. \qquad (153)$$

The proofs of Propositions 2.5, 2.6 and 2.7 follow from the expression for u given in (34), the expression for $\int_0^{+\infty} G(x,t,\xi,0)h(\xi)\,d\xi$ obtained in (132) and the expression for $\int_0^{+\infty} G(x,t,\xi,\tau)\Phi(\xi)\,d\xi$ obtained in (134), (144) and (153), respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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