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Saddle point problems, Bott–Duffin inverses, abstract splines and oblique projections

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In this note, we present links between several different areas in numerical analysis and operator theory. More precisely, we show that several saddle point problems are solvable in the presence of a property (called compatibility) between a positive operator and a closed subspace of a Hilbert space, and the same happens with certain abstract spline problems. These notions are related to the generalized Bott–Duffin inverse.

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1. Introduction

Given bounded linear operators A and B on a Hilbert space $\mathcal{H}$ with A herein a positive operator, a system

$$\begin{cases} A\sigma + B^*\mu = \eta \\ B\sigma = B\xi \end{cases} \tag{1}$$

describes a type of so-called *saddle-point problems*. These problems appear frequently in numerical analysis, in particular in finite element methods (see [4,7,9,10,17]). They are represented by operator matrices of the form $\left[\begin{smallmatrix} A & B^* \\ B & 0 \end{smallmatrix}\right]$, which are called "*bordered matrices*" (in [8]). System (1) is equivalent to

$$(I-P)A\sigma + B^*\mu = (I-P)\eta \tag{2}$$

$$(PA + I - P)\sigma = P\eta + (I-P)\xi, \tag{3}$$

where P is the orthogonal projection onto $N(B)$, the nullspace of B. For this reduction of system (1) and some basic related theory the reader is referred to [5,24,25]. Y. Chen [14] has solved problems like (1) in the finite dimensional case. He has found conditions on ξ and η in order to (1) admit solutions, found the set of solutions when the problem is solvable and found, in this case, the minimal norm solution of (3). A key role of the study by Chen is played by the operator $AP - I - P$.

Knyazev [25] considers an analogous problem for an infinite dimensional Hilbert space, using the operator $PA + I - P$. Under an assumption that $R(A) + \mathcal{S}^\perp$ is closed or, equivalently $PA+I-P$ has a closed range (see Proposition 2.5), Knyazev derives results which are similar to those by Chen.

In this note, we extend Chen and Knyazev results to systems where $PA + I - P$ may not have a closed range. More precisely, we prove that many of their results hold if $R(PA + I - P) = R(PA) + R(I - P)$. Indeed, this equality means that the operator A and the subspace $R(I - P)$ are compatible in the sense of [20]. This is a geometrical condition between the subspace $R(I-P)$ and $R(AP)$ which has been studied in [18–20], among other papers. Observe that if system (1) has a solution then $\eta \in R(A) + N(B)^\perp$. The compatibility condition mentioned before is optimal, in the sense that Eq. (3) has a solution for every $\xi \in \mathcal{H}$ and $\eta \in R(A) + N(B)^\perp$ if and only if A and $R(I-P) = N(B)^\perp$ are compatible. This is weaker than the assumption that $R(PA + I - P)$ is closed, used by Knyazev [25, Theorem 4.1]: in fact, if $R(PA + I - P)$ is closed then A and $R(I - P)$ are compatible (see [19, Remark 2.7]), but there exist pairs A, P for which the reverse does not hold (see Example 2.6). In the references above, it is proven that compatibility between A and $\mathcal{S}$ involves the existence of bounded oblique (i.e., non-necessarily orthogonal) projections E such that $R(E) = \mathcal{S}$ and $AE = E^*A$. Among these, there exists a unique projection $P_{A,\mathcal{S}}$ such that, for each $\xi \in \mathcal{H}$, $\eta = (I - P_{A,\mathcal{S}})\xi$ is the unique element in $\xi + \mathcal{S}$ with minimal norm which minimizes $\langle A\phi, \phi\rangle$; see [20, Theorem 3.2].

It turns out that operators like $AP + I - P$ and $PA + I - P$ appear in a theory of electric networks developed by R. Bott and R.J. Duffin [11]. For a semidefinite positive $n \times n$ matrix A and a subspace $\mathcal{S}$ of $\mathbb{C}^n$ they define a kind of constrained inverse: A is said to be Bott–Duffin invertible with respect to $\mathcal{S}$ if $AP + I - P$ is invertible where P is the orthogonal projection onto $\mathcal{S}$ and in this case $A^{(-1)}_{(\mathcal{S})} := P(AP + I - P)^{-1}$ is called the Bott–Duffin inverse. The reader is referred to the book by Ben-Israel and Greville [6, Chapter 2] for many results and references on the subject, and to the paper by Chen [14], where he has extended Bott–Duffin theory by means of the Moore–Penrose generalized inverse of $AP + I - P$ obtaining the generalized Bott–Duffin inverse $A^{(\dagger)}_{(\mathcal{S})} = P(AP+I-P)^{\dagger}$. In the infinite dimensional case, these generalized inverses are unbounded unless $R(AP + I - P)$ is closed. This is the reason why Knyazev has restricted his study to this case. One of the main contributions of this paper is, on one side, the extension of Chen and Knyazev results to the case where A and $R(I - P)$ are compatible and, on the other side, to use projections like the $P_{A,\mathcal{S}}$ mentioned above which can be expressed by means of the (possible unbounded) Moore–Penrose generalized inverse of $AP + I - P$ and $PA + I - P$. This unifies the approaches by Chen and Knyazev.

Another contribution is a comparison between $P_{A,\mathcal{S}}$ and $A^{(\dagger)}_{(\mathcal{S})}$. In [20] it is proven that the abstract splines theory of Atteia [3] can be studied by means of the compatibility methods. At the beginning of Section 4 we state this assertion in precise terms as Theorem 4.1. We prove that if $T \in L(\mathcal{H}, \mathcal{K})$, $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, $\xi, \eta \in \mathcal{H}$ and $f(\psi) = \|T\psi\|^2 - 2Re(\langle \eta, \psi\rangle)$, $\psi \in \mathcal{H}$ then

$$sp_g(T, \mathcal{S}, \xi, \eta) = \Big\{\sigma \in \xi + \mathcal{S} : f(\sigma) = \min_{\psi \in \xi + \mathcal{S}} f(\psi)\Big\}$$

is exactly the set of solutions of Eq. (3) where $A = T^*T$. Moreover, the minimal norm solution is $P(AP + I - P)^{\dagger}\eta + (I - P_{A,\mathcal{S}})\xi$, where $P = P_{\mathcal{S}}$. For $\eta = 0$, we re-obtain the results of [20].

The content of the rest of the paper is the following. Section 2 contains a quite complete study of the operators $PA+I-P$ and $AP+I-P$ and their relationship with the compatibility. Some new characterizations of compatible pairs are obtained. In Section 3, we relate, in the case of compatible pairs, the distinguished projection $P_{A,\mathcal{S}}$ with the generalized Bott–Duffin inverse $A^{(\dagger)}_{(\mathcal{S})}$. New explicit formulas for $P_{A,\mathcal{S}}$ are obtained. In particular, the formulas $P_{A,\mathcal{S}} = I-(PA+I-P)^{\dagger}(I-P) = P(I+(AP+I-P)^{\dagger}A(I-P))$ seem to be among the simplest expressions of $P_{A,\mathcal{S}}$ in terms of A and $P = P_{\mathcal{S}}$. Section 4 is devoted to generalized abstract splines.

2. The operator $PA + I - P$

Let $\mathcal{H}$ and $\mathcal{K}$ denote complex Hilbert spaces and $L(\mathcal{H}, \mathcal{K})$ be a space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. The algebra $L(\mathcal{H}, \mathcal{H})$ is abbreviated by $L(\mathcal{H})$. For every $T \in L(\mathcal{H}, \mathcal{K})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. By $L(\mathcal{H})^+$ we denote the cone of positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ := \{A \in L(\mathcal{H}) : A = A^* \text{ and } \langle A\xi, \xi\rangle \geqslant 0\ \forall \xi \in \mathcal{H}\}$. Furthermore, given $B \in L(\mathcal{H},\mathcal{K})$ we denote by $B^\dagger$ the Moore–Penrose generalized inverse of B (see [6]). Recall that $B^\dagger \in L(\mathcal{K},\mathcal{H})$ if and only if $R(B)$ is closed; otherwise, $B^\dagger$ is defined on $R(B)+R(B)^\perp$, and it is not bounded; in any case $R(B^\dagger) = \overline{R(B^*)}$ and $N(B^\dagger) = N(B^*)$. In addition, if $B \in L(\mathcal{H},\mathcal{K})$ with $R(B) \subseteq R(C)$ then $C^\dagger B \in L(\mathcal{H})$ even if C has a non-closed range. We include the proof of this fact here for completeness.

Lemma 2.1. *Let $B, C \in L(\mathcal{H},\mathcal{K})$. If $R(B) \subseteq R(C)$ then $C^\dagger B \in L(\mathcal{H})$.*

Proof. Let $(\xi_n, C^\dagger B\xi_n) \underset{n\to\infty}{\longrightarrow} (\xi,\eta)$. Hence, as $\xi_n \underset{n\to\infty}{\longrightarrow} \xi$ we have that $B\xi_n \underset{n\to\infty}{\longrightarrow} B\xi$. On the other side, as $C^\dagger B\xi_n \underset{n\to\infty}{\longrightarrow} \eta$ then $\eta \in R(C^\dagger)$ and $B\xi_n = CC^\dagger B\xi_n \underset{n\to\infty}{\longrightarrow} C\eta$ where we use that $R(B) \subseteq R(C)$. Thus, $B\xi = C\eta$ and so $C^\dagger B\xi = C^\dagger C\eta = \eta$. Therefore, by the Closed Graph Theorem, we have that $C^\dagger B \in L(\mathcal{H})$. $\square$

Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ the aim of this section is to study the operator $PA + I - P$, where $P = P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$.

Lemma 2.2. *The next equalities hold: $N(PA + I - P) = N(AP + I - P) = N(A) \cap \mathcal{S}$.*

Proof. Let us show that $N(PA + I - P) = N(A) \cap \mathcal{S}$. Indeed, if $(PA + I - P)\psi = 0$, $PA\psi = 0$ and $(I - P)\psi = 0$, i.e., $\psi = P\psi \in \mathcal{S}$. Then, $0 = PA\psi = PAP\psi$ and so $0 = A^{1/2}P\psi = A^{1/2}\psi$. Hence, $\psi \in N(A) \cap \mathcal{S}$. The other inclusion is trivial.

On the other hand, let us see that $N(AP+I-P) = N(A)\cap\mathcal{S}$. Let $\psi \in N(AP+I-P)$. Thus, $AP\psi = -(I - P)\psi$ and so $PAP\psi = 0$, i.e., $AP\psi = 0$ and so $(I - P)\psi = 0$. Therefore, $\psi = P\psi \in R(P)$ and $A\psi = AP\psi = 0$, i.e., $\psi \in N(A)$. For the other inclusion, let $\psi \in N(A)\cap\mathcal{S}$. Hence, $(AP+I-P)\psi = AP\psi = A\psi = 0$ and the assertion follows. $\square$

Lemma 2.3. *There exists an invertible $G \in L(\mathcal{H})$ such that $G(PA+I-P) = PAP+I-P$.*

Proof. Let $G = P - PA(I - P) + I - P$, then $G^{-1} = P + PA(I - P) + I - P$, i.e., $G \in Gl(\mathcal{H})$. Furthermore, $G(PA + I - P) = PAP + I - P$. $\square$

Proposition 2.4. *The following conditions are equivalent:*

1. *$PA + I - P$ is invertible.*
2. *$PAP + I - P$ is invertible.*
3. *$R(PA + I - P) = \mathcal{H}$.*
4. *$AP + I - P$ is invertible.*
5. *$R(AP + I - P) = \mathcal{H}$.*
6. *$R(PA) = \mathcal{S}$.*
7. *$R(PAP) = \mathcal{S}$.*
8. *$PA|_{\mathcal{S}}$ is invertible.*

9. $R(A) + \mathcal{S}^\perp = \mathcal{H}$.
10. $A + I - P$ *is invertible.*

Proof. $1 \Leftrightarrow 2$. It follows by Lemma 2.3.

$1 \Leftrightarrow 3$. Assume that $R(PA + I - P) = \mathcal{H}$. Hence, by Lemma 2.3, $R(PAP + I - P) = R(G(PA + I - P)) = \mathcal{H}$ and, since $PAP + I - P \in L(\mathcal{H})^+$, $PAP + I - P$ is invertible and so $PA + I - P = G^{-1}(PAP + I - P)$ is invertible. The converse is trivial.

$1 \Leftrightarrow 4$. It is trivial.

$4 \Leftrightarrow 5$. The implication $4 \Rightarrow 5$ is obvious. Let $R(AP + I - P) = \mathcal{H}$. Hence, $N(PA + I - P) = \{0\}$ and, by Lemma 2.2, $N(AP + I - P) = \{0\}$. Therefore, $AP + I - P$ is invertible.

$1 \Rightarrow 6$. If $PA + I - P$ is invertible then $\mathcal{H} = R(PA + I - P) \subseteq R(PA) \dot{+} \mathcal{S}^\perp$ and so $R(PA) = \mathcal{S}$.

$6 \Rightarrow 7$. Since $N(AP) = N(A^{1/2}P)$ then, as $R(PA) = \mathcal{S}$ is closed, we get that $R(PA) = R(PA^{1/2})$ and so $\mathcal{S} = R(PA) = R(PAP)$.

$7 \Rightarrow 8$. If $R(PAP) = \mathcal{S}$ then $PA|_\mathcal{S} : \mathcal{S} \to \mathcal{S}$ is a surjective positive operator, i.e., $PA|_\mathcal{S}$ is invertible.

$8 \Rightarrow 9$. If $PA|_\mathcal{S}$ is invertible then $R(PAP) = \mathcal{S}$. Hence, given $\xi \in \mathcal{H}$, $P\xi = PAP\zeta$ for some $\zeta \in \mathcal{H}$ and so $\xi - AP\zeta \in \mathcal{S}^\perp$, i.e., $\xi \in R(A) + \mathcal{S}^\perp$.

$9 \Rightarrow 10$. If $R(A) + \mathcal{S}^\perp = \mathcal{H}$ then, by [2, Theorem 3.3], $A + I - P$ is invertible.

$10 \Rightarrow 2$. If $A + I - P$ is invertible then $R(A) + \mathcal{S}^\perp = \mathcal{H}$. Therefore $\mathcal{S} = R(P) = P(\mathcal{H}) = R(PA)$ and $R(PAP) = \mathcal{S}$ because of $6 \Rightarrow 7$. Thus, $\mathcal{H} = R(PAP) + \mathcal{S}^\perp = R(PAP) + R(I - P)$. Hence, by [2, Theorem 3.3], $R(PAP + I - P) = \mathcal{H}$ and $PAP + I - P$ is invertible. $\square$

In a similar manner we obtain the following result. Conditions 4, 6 and 7 appear in [25, Theorem 4.1].

Proposition 2.5. *The following conditions are equivalent:*

1. $PA + I - P$ *has closed range.*
2. $PAP + I - P$ *has closed range.*
3. $AP + I - P$ *has closed range.*
4. $R(PAP)$ *is closed.*
5. $PA|_\mathcal{S}$ *has closed range.*
6. $R(PA)$ *is closed.*
7. $R(A) + \mathcal{S}^\perp$ *is closed.*
8. $A + I - P$ *has closed range.*

Proof. $1 \Leftrightarrow 2$. It follows by Lemma 2.3.

$1 \Leftrightarrow 3$. It is trivial.

$2 \Rightarrow 4$. If $PAP + I - P$ has closed range then, by [2, Theorem 3.3], $R(PAP) \dot{+} R(I - P) = R(PAP + I - P)$ is closed and, by [22, Theorem 2.2], $R(PAP)$ is closed.

$4 \Rightarrow 5$. It is clear.

$5 \Rightarrow 6$. If $PA|_{\mathcal{S}}$ has closed range then $R(PAP)$ is closed. We claim that $R(PAP) = R(PA)$. Indeed, considering the Hilbert space decomposition $\mathcal{H} = S + S^\perp$ let $A = \left[\begin{smallmatrix} a & b \\ b^* & c \end{smallmatrix}\right]$ where $R(b) \subseteq R(a^{1/2})$ because $A \in L(\mathcal{H})^+$ (see [28]). Then, as $R(PAP)$ is closed we get that $R(a)$ is closed and so $R(b) \subseteq R(a)$. Therefore, $R(PA) = R(a) + R(b) = R(a) = R(PAP)$ and so $R(PA)$ is closed.

$6 \Rightarrow 7$. Notice that $R(A) + \mathcal{S}^\perp = R(PA) + \mathcal{S}^\perp$. Now, since $R(PA)$ is closed and $R(PA) \subseteq \mathcal{S}$ then $R(PA) + \mathcal{S}^\perp$ is closed and so $R(A) + \mathcal{S}^\perp$ is closed.

$7 \Rightarrow 8$. It follows from [2, Theorem 3.3].

$8 \Rightarrow 2$. Assume that $A + I - P$ has closed range. Then, by [2, Theorem 3.3], we get that $R(PA) \dot{+} \mathcal{S}^\perp = R(A) + \mathcal{S}^\perp = R(A + I - P)$ is closed. Therefore, by [22, Theorem 2.3], $R(PA)$ is closed. Hence, $R(PAP)$ is closed and, by [21, Theorem 13], $R(PAP) + R(I - P)$ is closed. Applying [2, Theorem 3.3] again, we obtain that $R(PAP + I - P)$ is closed and the result follows. $\square$

It is interesting to notice that the behavior of $AP + I - P$ with respect to the range additivity is completely different to that of $PA + I - P$. In fact, for every $A \in L(\mathcal{H})^+$ and orthogonal projection P it holds that $R(AP + I - P) = R(AP) + R(I - P) = A\mathcal{S} + \mathcal{S}^\perp$. However, the next example shows that the range additivity does not hold in general for $PA + I - P$.

Example 2.6. Let $C \in L(\mathcal{H})^+$ have a dense non-closed range and define $A = \left(\begin{smallmatrix} C & C^{1/2} \\ C^{1/2} & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} C^{1/2} & 0 \\ I & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} C^{1/2} & 0 \\ I & 0 \end{smallmatrix}\right)^* \in L(\mathcal{H} \oplus \mathcal{H})^+$. Consider the closed subspace $\mathcal{S} = \mathcal{H} \oplus \{0\}$ of $\mathcal{H} \oplus \mathcal{H}$ and $P = P_{\mathcal{S}}$. Then, $R(PA) + S^\perp = R(C^{1/2}) \oplus \mathcal{H}$. We claim that $R(PA + I - P) \neq R(C^{1/2}) \oplus \mathcal{H}$. In fact, since $R(C^{1/2})$ properly contains $R(C)$, there exists $\xi \in \mathcal{H}$ such that $C^{1/2}\xi \notin R(C)$. Let us see that $\eta = (C^{1/2}\xi, C^{1/2}\xi) \notin R(PA + I - P)$. Indeed, suppose that there exists $\nu = (\psi, \zeta) \in \mathcal{H} \oplus \mathcal{H}$ such that $\eta = (PA + I - P)\nu$. Then, $C^{1/2}\xi = C\psi + C^{1/2}\zeta$ and $C^{1/2}\xi = \zeta$ and so $C^{1/2}\xi = C\psi + C\xi \in R(C)$ which is a contradiction. Thus, $R(PA + I - P) \neq R(PA) + \mathcal{S}^\perp$.

Definition 2.7. Given $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, we say that the pair $(A, \mathcal{S})$ is **compatible** if $\mathcal{P}(A, \mathcal{S}) := \{Q \in L(\mathcal{H}) : Q^2 = Q,\ R(Q) = \mathcal{S} \text{ and } AQ = Q^*A\}$ is not empty.

The compatibility of a pair $(A, \mathcal{S})$ means that there exists a (bounded linear) projection with image $\mathcal{S}$ which is Hermitian with respect to the semi-inner product $\langle \cdot\,, \cdot \rangle_A$ defined by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$. If the pair $(A, \mathcal{S})$ is compatible then the unique element in $\mathcal{P}(A, \mathcal{S})$ with nullspace $(A\mathcal{S})^\perp \ominus \mathcal{N}$, where $\mathcal{N} = N(A) \cap \mathcal{S}$, is denoted by $P_{A,\mathcal{S}}$. Furthermore, $\mathcal{P}(A, \mathcal{S}) = P_{A,S} + L(S, N(A) \cap \mathcal{S})$. The reader will find in [18] some results on compatibility which are useful in the sequel.

We shall prove that $R(PA + I - P) = R(PA) + R(I - P)$ if and only if the pair $(A, R(P))$ is compatible. Before that we present the following result about ranges of operators due to A. Maestripieri which will be useful in the sequel. We include the proof for completeness.

Proposition 2.8. *Let $T_1, T_2 \in L(\mathcal{H})$ such that $R(T_1) \cap R(T_2) = \{0\}$. Then, $R(T_1 + T_2) = R(T_1) + R(T_2)$ if and only if $N(T_1) + N(T_2) = \mathcal{H}$.*

Proof. Assume that $R(T_1 + T_2) = R(T_1) + R(T_2)$ and let $\xi \in \mathcal{H}$. Write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in N(T_1), \xi_2 \in N(T_1)^\perp$. Then, $(T_1 + T_2)\xi = T_1\xi_2 + T_2\xi_1 + T_2\xi_2$. Now, since $R(T_1+T_2) = R(T_1)+R(T_2)$ there exists $\eta \in \mathcal{H}$ such that $(T_1+T_2)\xi = (T_1+T_2)\eta+T_2\xi_2$ or, equivalently, $T_1(\xi - \eta) = T_2(\eta + \xi_2 - \xi) \in R(T_1) \cap R(T_2) = \{0\}$. Hence, $\xi - \eta \in N(T_1)$ and $\zeta = \eta + \xi_2 - \xi \in N(T_2)$. Thus $\xi_2 = \xi - \eta + \zeta \in N(T_1) + N(T_2)$ and so $\xi = \xi_1 + \xi_2 \in N(T_1) + N(T_2)$, i.e., $N(T_1) + N(T_2) = \mathcal{H}$.

Conversely, suppose that $N(T_1) + N(T_2) = \mathcal{H}$ and let $\zeta = T_1\xi + T_2\eta \in R(T_1) + R(T_2)$. Therefore, there exist $\xi_1, \eta_1 \in N(T_1)$ and $\xi_2, \eta_2 \in N(T_2)$ such that $\xi = \xi_1 + \xi_2$ and $\eta = \eta_1 + \eta_2$. Hence, $\zeta = T_1\xi + T_2\eta = (T_1 + T_2)(\xi_2 + \eta_1) \in R(T_1 + T_2)$ and so $R(T_1 + T_2) = R(T_1) + R(T_2)$. $\square$

We present now six conditions on A and $P = P_{\mathcal{S}}$ which are equivalent to the compatibility of the pair $(A, \mathcal{S})$. Conditions 2 and 4 appeared in [18, Proposition 3.3], condition 5 in [1, Proposition 5.1] and the others are new.

Proposition 2.9. *The following conditions are equivalent:*

1. *$(A, \mathcal{S})$ is compatible.*
2. *$\mathcal{S} + (A\mathcal{S})^\perp = \mathcal{H}$.*
3. *$R(PA + I - P) = R(PA) + \mathcal{S}^\perp$.*
4. *$R(PA) = R(PAP)$.*
5. *$\mathcal{H} = \overline{R(PA)} \dot{+} N(PA)$.*
6. *$R(A) \subseteq A\mathcal{S} + \mathcal{S}^\perp$.*
7. *$R(A) \subseteq R(AP + I - P)$.*

Proof. $1 \Leftrightarrow 2$. See [18, Proposition 3.3].

$2 \Leftrightarrow 3$. Note that $R(PA) \cap R(I - P) = \{0\}$. Then, by Proposition 2.8, we get $R(PA + I - P) = R(PA) + \mathcal{S}^\perp$ if and only if $\mathcal{H} = N(PA) + \mathcal{S}$ or, equivalently, $\mathcal{H} = R(AP)^\perp + \mathcal{S} = (A\mathcal{S})^\perp + \mathcal{S}$.

$1 \Leftrightarrow 4$. See [18, Proposition 3.3].

$1 \Leftrightarrow 5$. [1, Proposition 5.1].

$2 \Leftrightarrow 6$. Notice that $(A\mathcal{S})^\perp = A^{-1}(\mathcal{S}^\perp)$.

Assume that $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$ and let $\eta = A\xi \in R(A)$. Thus, $\xi = \sigma + \tau$ with $\sigma \in \mathcal{S}$ and $\tau \in A^{-1}(\mathcal{S}^\perp)$ and so $\eta = A\xi = A\sigma + A\tau \in A(\mathcal{S}) + \mathcal{S}^\perp$, i.e., $R(A) \subseteq A\mathcal{S} + \mathcal{S}^\perp$. Conversely, assume that $R(A) \subseteq A\mathcal{S} + \mathcal{S}^\perp$ and let $\xi \in \mathcal{H}$. Then, there exist $\sigma \in \mathcal{S}$ and

$\psi \in \mathcal{S}^\perp$ such that $A\xi = A\sigma + \psi$. Hence, $A(\xi - \sigma) = \psi$, i.e., $\xi - \sigma \in A^{-1}(\mathcal{S}^\perp)$ and so $\xi \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$. Therefore, $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{S} + (A\mathcal{S})^\perp$.

$6 \Leftrightarrow 7$. The equivalence follows from the fact that $A\mathcal{S} + \mathcal{S}^\perp = R(AP + I - P)$. □

Remark 2.10. It is clear that any condition of Proposition 2.4 implies everyone of Proposition 2.5. Moreover, by the proof of Proposition 2.5, it holds that any condition of Proposition 2.5 implies everyone of Proposition 2.9.

If $R(A)$ is closed then all items of Proposition 2.5 and Proposition 2.9 are equivalent (see [18, Section 6]). In particular, if $R(A)$ is closed the $(A, \mathcal{S})$ is compatible if and only if $(P_{R(A)}, \mathcal{S})$ is compatible, which means that the Friedrichs angle between $\mathcal{S}$ and $N(A)$ is non-zero. We refer the reader to [21] and [26] for many results on angle between subspaces in Hilbert spaces.

3. Compatible pairs and generalized Bott–Duffin inverses

If $(A, \mathcal{S})$ is compatible then the projection $P_{A,\mathcal{S}}$ plays a relevant role similar to that of the orthogonal projection $P_\mathcal{S}$ among all the projections with range $\mathcal{S}$. The next explicit formulas of $P_{A,\mathcal{S}}$ will be useful in the sequel. The first one is particularly simple since it only depends on P and A. Compare with previous formulas [20, Propositions 4.1 and 4.2].

In the sequel, given two closed subspaces $\mathcal{S}$ and $\mathcal{W}$ in $\mathcal{H}$, such that $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{W}$ we shall denote by $Q_{\mathcal{S}//\mathcal{W}}$ the projection onto $\mathcal{S}$ with nullspace $\mathcal{W}$. If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{W}$ with $\mathcal{S}$ and $\mathcal{W}$ closed subspaces then the angle α between $\mathcal{S}$ and $\mathcal{W}$ is automatically non-zero and the projection $Q_{\mathcal{S}//\mathcal{W}}$ has norm $1/\sin(\alpha)$; if $\alpha = 0$ and $\mathcal{S} \cap \mathcal{W} = \{0\}$ then $\mathcal{S} \dot{+} \mathcal{W}$ is not closed in $\mathcal{H}$ and $Q_{\mathcal{S}//\mathcal{W}}$, defined on $\mathcal{S} \dot{+} \mathcal{W}$ is unbounded (see [13]).

Proposition 3.1. *Let $(A, \mathcal{S})$ be compatible. Then,*

$$P_{A,\mathcal{S}} = I - (PA + I - P)^\dagger (I - P). \tag{4}$$

$$= Q_{\overline{R(PA)}//N(PA)} + P_{N(A)\cap\mathcal{S}} \tag{5}$$

Proof. Assume that $(A, \mathcal{S})$ is compatible and let us prove equality (4). Before that observe that $PP_{A,\mathcal{S}} = P_{A,\mathcal{S}}$ and $PAP_{A,\mathcal{S}} = PA$. In fact, for two projections E, F it holds $EF = F$ if $R(E) = R(F)$ and $EF = E$ if $N(E) = N(F)$. In our case, $PAP_{A,\mathcal{S}} = PP^*_{A,\mathcal{S}}A = PA$ because $N(P^*_{A,\mathcal{S}}) = \mathcal{S}^\perp$.

Now, taking this into account, we have that $(PA + I - P)(I - P_{A,\mathcal{S}}) = PA + I - P - PAP_{A,\mathcal{S}} - (I - P)P_{A,\mathcal{S}} = PA + I - P - PA = I - P$. Now, since $R(I - P_{A,\mathcal{S}}) = N(P_{A,\mathcal{S}}) \subseteq (N(A) \cap \mathcal{S})^\perp = N(PA + I - P)^\perp$ we obtain that $I - P_{A,\mathcal{S}} = (PA + I - P)^\dagger (I - P)$ and equality (4) is proved.

On the other hand, by [1, Proposition 5.3], we get that $P_{A,\mathcal{S}} = Q_{\overline{R(PA)}//N(PA)} + P_\mathcal{M}$ with $\mathcal{S} = \overline{R(PA)} \oplus \mathcal{M}$ and $\mathcal{M} \subseteq N(A)$. Let us show that $\mathcal{M} = N(A) \cap \mathcal{S}$. Clearly, $\mathcal{M} \subseteq N(A) \cap \mathcal{S}$. For the other inclusion, take $y \in N(A) \cap \mathcal{S}$ then $y = P_{A,\mathcal{S}}y = Q_{\overline{R(PA)}//N(PA)}y + P_\mathcal{M}y = P_\mathcal{M}y$, i.e., $y \in \mathcal{M}$. Therefore, $\mathcal{M} = N(A) \cap \mathcal{S}$ and (5) is proved. □

Definition 3.2. The operator $A^{(\dagger)}_{(\mathcal{S})} := P(AP+I-P)^\dagger$ is called the generalized Bott–Duffin inverse of A respect to $\mathcal{S} = R(P)$.

Note that $A^{(\dagger)}_{(\mathcal{S})} \in L(\mathcal{H})$ if and only if $AP + I - P$ has closed range. However, we shall see that if $(A, \mathcal{S})$ is compatible then $A^{(\dagger)}_{(\mathcal{S})}A \in L(\mathcal{H})$ even if $A^{(\dagger)}_{(\mathcal{S})} \notin L(\mathcal{H})$.

The next lemma is known. For the sake of completeness, we include its proof.

Lemma 3.3. *Let $Q_1, Q_2 \in L(\mathcal{H})$ be two projections. If $R(Q_1) \subseteq R(Q_2)$ and $N(Q_1) \subseteq N(Q_2)$ then $Q_1 = Q_2$.*

Proof. If $R(Q_1) \subseteq R(Q_2)$ then $Q_2Q_1 = Q_1$. On the other side, if $N(Q_1) \subseteq N(Q_2)$ then $Q_2Q_1 = Q_2$. Therefore, $Q_1 = Q_2$. $\square$

Proposition 3.4. *Let $(A, \mathcal{S})$ be a compatible pair. Then,*

$$Q_{\overline{R(PA)}//N(PA)} = A^{(\dagger)}_{(\mathcal{S})}A \tag{6}$$

Proof. By Proposition 2.9, the projection $Q_{\overline{R(PA)}//N(PA)}$ is well-defined. Moreover, by Proposition 2.9, we have that $R(A) \subseteq R(AP + I - P)$. Hence, by Lemma 2.1, $(AP + I - P)^\dagger A \in L(\mathcal{H})$ and then $A^{(\dagger)}_{(\mathcal{S})}A \in L(\mathcal{H})$. Therefore, by Lemma 3.3, we remain to show that:

1. $A^{(\dagger)}_{(\mathcal{S})}A$ is a projection.
2. $R(PA) \subseteq R(A^{(\dagger)}_{(\mathcal{S})}A)$.
3. $N(PA) \subseteq N(A^{(\dagger)}_{(\mathcal{S})}A)$.

First, let us observe that $A^{(\dagger)}_{(\mathcal{S})}(I - P) = 0$. Indeed, $A^{(\dagger)}_{(\mathcal{S})}(I - P) = P(AP + I - P)^\dagger(I - P) = P(AP + I - P)^\dagger(AP + I - P)(I - P) = PP_{N(AP+I-P)^\perp}(I - P) = PP_{(N(A)\cap\mathcal{S})^\perp}(I - P) = P(I - P) = 0$. In the sequel we shall use that $A^{(\dagger)}_{(\mathcal{S})}(I - P) = 0$ without any mention.

1. $(A^{(\dagger)}_{(\mathcal{S})}A)^2 = P(AP+I-P)^\dagger AP(AP+I-P)^\dagger A = P(AP+I-P)^\dagger(AP+I-P)(AP+I - P)^\dagger A = P(AP + I - P)^\dagger A = A^{(\dagger)}_{(\mathcal{S})}A$.
2. Let $\eta = PA\xi \in R(PA)$. Hence, $A^{(\dagger)}_{(\mathcal{S})}A\eta = P(AP + I - P)^\dagger A\eta = P(AP + I - P)^\dagger APA\xi = P(AP + I - P)^\dagger(AP + I - P)A\xi = PP_{(N(A)\cap\mathcal{S})^\perp}A\xi = PA\xi = \eta$, i.e., $\eta \in R(A^{(\dagger)}_{(\mathcal{S})}A)$.
3. Let $\xi \in N(PA)$. Then, $PA\xi = 0$, and so $A\xi = (I - P)A\xi = (AP + I - P)(I - P)A\xi$. Therefore, $A^{(\dagger)}_{(\mathcal{S})}A\xi = P(AP+I-P)^\dagger A\xi = P(AP+I-P)^\dagger(AP+I-P)(I-P)A\xi = PP_{(N(A)\cap\mathcal{S})^\perp}(I - P)A\xi = P(I - P)A\xi = 0$, i.e., $\xi \in N(A^{(\dagger)}_{(\mathcal{S})}A)$ and the result is proved. $\square$

Proposition 3.5. *Let $(A, \mathcal{S})$ be compatible. Hence,*

$$I - P_{A,\mathcal{S}} = \big(I - A^{(\dagger)}_{(\mathcal{S})}A\big)(I - P). \tag{7}$$

Proof. By Propositions 3.1 and 3.4, we have that $I - P_{A,\mathcal{S}} = I - A^{(\dagger)}_{(\mathcal{S})}A - P_{N(A)\cap\mathcal{S}}$. Hence, as $N(I-P) = N(I-P_{A,\mathcal{S}})$, we have that $I-P_{A,\mathcal{S}} = (I-P_{A,\mathcal{S}})(I-P) = (I-A^{(\dagger)}_{(\mathcal{S})}A)(I-P)$ as desired. □

4. Generalized abstract splines

Let $T \in L(\mathcal{H}, \mathcal{W})$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and $\xi, \eta \in \mathcal{H}$. In this section we shall consider the functional: $f(\psi) = \|T\psi\|^2 - 2Re(\langle \eta, \psi \rangle)$, $\psi \in \mathcal{H}$. More precisely, we are interested in the next set, called *generalized abstract spline*:

$$sp_g(T, \mathcal{S}, \xi, \eta) = \Big\{\sigma \in \xi + \mathcal{S} : f(\sigma) = \min_{\psi \in \xi + \mathcal{S}} f(\psi)\Big\}. \tag{8}$$

If $\eta = 0$ then $sp_g(T, \mathcal{S}, \xi, \eta)$ is called an abstract spline or a $(T, \mathcal{S})$-spline interpolant to ξ. This notion which unifies the treatment of many spline-like functions was introduced by M. Atteia [3]. See the surveys of Champion, Lenard and Mills [15,16] and Deutsch [21] and the papers by Shekhtman [27], de Boor [12] and Izumino [23]. The relationship between $sp_g(T, \mathcal{S}, \xi, 0)$ and the compatibility of the pair $(T^*T, \mathcal{S})$ was studied in [20].

Theorem 4.1. *(See [20, Theorem 3.2].) Let $T \in L(\mathcal{H}, \mathcal{W})$, $A = T^*T$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace.*

1. *If $\xi \in \mathcal{H}$, $sp_g(T, \mathcal{S}, \xi, 0)$ is not empty if and only if $\xi \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$.*
2. *The following conditions are equivalent:*
   (a) *$sp_g(T, \mathcal{S}, \xi, 0)$ is not empty for every $\xi \in \mathcal{H}$.*
   (b) *$\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$.*
   (c) *The pair (A, S) is compatible.*
   Moreover, if $(A, \mathcal{S})$ is compatible then
3. *$sp_g(T, \mathcal{S}, \xi, 0) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}$.*
4. *$(I - P_{A,\mathcal{S}})\xi$ is the unique vector in $sp_g(T, \mathcal{S}, \xi, 0)$ with minimal norm.*

Our goal is to study the relationship between $sp_g(T, \mathcal{S}, \xi, \eta)$ and the compatibility of the pair $(T^*T, \mathcal{S})$ for $\eta \neq 0$. The proof of the next result follows the lines of [14, Theorem 5].

Theorem 4.2. *Let $T \in L(\mathcal{H}, \mathcal{W})$, $A = T^*T$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$, $P = P_\mathcal{S}$ and $\xi, \eta \in \mathcal{H}$. The following conditions are equivalent:*

1. *$\sigma \in sp_g(T, \mathcal{S}, \xi, \eta)$.*
2. *$(PA + I - P)\sigma = P\eta + (I - P)\xi$.*

Proof. 1 $\Rightarrow$ 2. Let $\sigma \in sp_g(T, \mathcal{S}, \xi, \eta)$ and $\phi = \sigma + P\psi$ with $\psi \in \mathcal{H}$. Hence, from $f(\phi) \geqslant f(\sigma)$ we have that

$$\langle AP\psi, P\psi\rangle + \langle A\sigma - \eta, P\psi\rangle + \langle P\psi, A\sigma - \eta\rangle \geqslant 0, \tag{9}$$

for all $\psi \in \mathcal{H}$. Put $\zeta = P(A\sigma - \eta)$ and take $\psi = -\lambda\zeta$ in (9) with $\lambda > 0$; we get: $\lambda\langle AP\zeta, P\zeta\rangle - 2\|\zeta\|^2 \geqslant 0$ for all $\lambda > 0$. Now, if $\zeta \neq 0$ then we can find λ small enough so that $\lambda\langle AP\zeta, P\zeta\rangle - 2\|\zeta\|^2 < 0$ which is a contradiction. Therefore, $\zeta = 0$, i.e., $PA\sigma = P\eta$. On the other side, since $\sigma \in \xi + \mathcal{S}$ we have that $(I - P)\sigma = (I - P)\xi$. Combining this two last equalities we obtain that $(PA + I - P)\sigma = P\eta + (I - P)\xi$ and the result follows.

2 $\Rightarrow$ 1. Let $\sigma \in \mathcal{H}$ such that $(PA + I - P)\sigma = P\eta + (I - P)\xi$. Hence,

$$PA\sigma = P\eta \quad \text{and} \quad (I - P)\sigma = (I - P)\xi.$$

From $(I - P)\sigma = (I - P)\xi$ we have that $\sigma \in \xi + \mathcal{S}$. Consider $\psi \in \xi + \mathcal{S}$. Hence, $\psi = \sigma + \zeta$ with $\zeta \in \mathcal{S}$ and $f(\psi) = f(\sigma) + \langle A\zeta, \zeta\rangle - 2Re(\langle A\sigma, \zeta\rangle) + 2Re(\langle\eta, \zeta\rangle)$. Now, as $PA\sigma = P\eta$ then $A\sigma = \eta + \phi$ with $\phi \in \mathcal{S}^\perp$ and so $\langle A\sigma, \zeta\rangle = \langle\eta, \zeta\rangle$. Hence, $f(\psi) = f(\sigma) + \langle A\zeta, \zeta\rangle \geqslant f(\sigma)$ and $\sigma \in sp_g(T, \mathcal{S}, \xi, \eta)$. $\square$

Motivated by Theorem 4.2, our goal in what follows is to study the equation:

$$(PA + I - P)\sigma = P\eta + (I - P)\xi. \tag{10}$$

Proposition 4.3. *Let $A \in L(\mathcal{H})^+$, $\xi, \eta \in \mathcal{H}$. The following conditions are equivalent:*

(a) there exists $\sigma \in \mathcal{H}$ such that $(PA + I - P)\sigma = P\eta + (I - P)\xi$,
(b) there exists $\psi \in \mathcal{H}$ such that $(AP + I - P)\psi = \eta - A\xi$.

Proof. Given $\sigma \in \mathcal{H}$ such that (a) holds, then $\psi = \sigma - \xi + \eta - A\sigma$ solves (b). Conversely, given $\psi \in \mathcal{H}$ such that (b) holds, then $\sigma = \xi + P\psi$ solves (a). $\square$

Proposition 4.4. *(See [25, Lemma 4.2].) Suppose that for some $\xi, \eta \in \mathcal{H}$ there exists a solution σ of (10). Then, all possible solutions are $\sigma + N(A) \cap \mathcal{S}$. In particular, there exists a unique solution of (10) provided that $N(A) \cap \mathcal{S} = \{0\}$.*

Proof. It follows from Lemma 2.2. $\square$

Now, applying the results of the previous section we obtain:

Theorem 4.5.

1. *The next conditions are equivalent:*
   (a) *For all $\xi, \eta \in \mathcal{H}$ Eq. (10) has a unique solution which depends continuously on the data.*

(b) *For all* $\xi, \eta \in \mathcal{H}$ *Eq.* (10) *has a unique solution.*
(c) *For all* $\xi, \eta \in \mathcal{H}$ *Eq.* (10) *has a solution.*
(d) $R(PAP) = \mathcal{S}$.

2. *The next conditions are equivalent:*
   (a) $\{(\xi, \eta) : Eq.\ (10)\ is\ solvable\} = \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$ *and the unique solution in* $(N(A) \cap \mathcal{S})^\perp$ *depends continuously on the data.*
   (b) $R(PAP)$ *is closed.*
3. *The next conditions are equivalent:*
   (a) $\{(\xi, \eta) : Eq.\ (10)\ is\ solvable\} = \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$.
   (b) $R(PAP) = R(PA)$.
4. *Any condition of 1 implies everyone of 2; any condition of 2 implies everyone of 3.*

Proof.

1. $a \Rightarrow b \Rightarrow c$. Trivial.
   $c \Rightarrow d$. If for all $\xi, \eta \in \mathcal{H}$ Eq. (10) has a solution then $R(PA + I - P) = \mathcal{H}$ or, equivalently, by Proposition 2.4, $R(PAP) = \mathcal{S}$.
   $d \Rightarrow a$. By Proposition 2.4, if $R(PAP) = \mathcal{S}$ then $PA + I - P$ is invertible and the result follows.
2. $a \Rightarrow b$. If $\{(\xi, \eta) :$ Eq. (10) is solvable$\} = \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$ then $R(PA) + \mathcal{S}^\perp = \{P\eta + (I - P)\xi : (\xi, \eta) \in \mathcal{H} \times (R(A) + \mathcal{S}^\perp)\} = R(PA + I - P)$. Now, since the unique solution in $(N(A) \cap \mathcal{S})^\perp$ is given by $(PA + I - P)^\dagger(P\eta + (I - P)\xi)$ and it depends continuously on the data then $(PA + I - P)^\dagger|_{R(PA+I-P)}$ is bounded or, equivalently, $R(PA + I - P)$ is closed. Therefore, by Proposition 2.5, $R(PAP)$ is closed.
   $b \Rightarrow a$. Suppose that $R(PAP)$ is closed then, by Remark 2.10, $R(PA + I - P) = R(PA) + \mathcal{S}^\perp$. Take (ξ, η) such that (10) is solvable, i.e., there exists $\sigma \in \mathcal{H}$ such that $(PA + I - P)\sigma = P\eta + (I - P)\xi$. Therefore, $PA\sigma = P\eta$, i.e., $\eta \in R(A) + \mathcal{S}^\perp$. For the other inclusion, let $(\xi, \eta) \in \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$. Then, $P\eta + (I - P)\xi \in R(PA) + \mathcal{S}^\perp = R(PA + I - P)$, i.e., Eq. (10) is solvable. Now, the unique solution in $(N(A) \cap \mathcal{S})^\perp$ is obtained by $(PA + I - P)^\dagger(P\eta + (I - P)\xi)$ which depends continuously on the data since $(PA + I - P)^\dagger$ is a bounded operator because of Proposition 2.5.
3. $a \Rightarrow b$. Assume that item *(a)* holds. We claim that $R(PA + I - P) = R(PA) + \mathcal{S}^\perp$. Indeed, let $\zeta = PA\psi + \phi$ with $\phi \in \mathcal{S}^\perp$. Then, $(\phi, A\psi) \in \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$ and so there exists $\sigma \in \mathcal{H}$ such that $(PA + I - P)\sigma = \zeta$, i.e., $\zeta \in R(PA + I - P)$. Therefore, $R(PA + I - P) = R(PA) + \mathcal{S}^\perp$ and the conclusion follows by Proposition 2.9.
   $b \Rightarrow a$. If $R(PAP) = R(PA)$ then, by Proposition 2.9, $R(PA + I - P) = R(PA) + \mathcal{S}^\perp$ and the proof follows as in the previous item.
4. It is clear. □

Remark 4.6. Knyazev [25, Lemma 4.3] asserts that for every $\xi \in \mathcal{H}$ and every $\eta \in R(A) + \mathcal{S}^\perp$ Eq. (10) has a solution. The assertion is false; Knyazev's proof fails because identity $R(PA + I - P) = PR(A) + R(I - P)$ holds only if A and $R(P)$ are compatible. We have shown examples for which the identity does not hold (see Example 2.6). The right assertion should read: for every $\xi \in \mathcal{H}$ and every $\eta \in R(A) + \mathcal{S}^\perp$ Eq. (10) has a solution if and only if $R(PA) = R(PAP)$ or, equivalently $(A, R(P))$ is compatible. However, this little mistake does not affect the validity of the main assertions of his paper, because in general one only considers pairs A, P for which $R(PA + I - P)$ is closed; as we have seen, in such cases the identity $R(PA + I - P) = PR(A) + R(I - P)$ holds. Warning: Knyazev denotes by P the orthogonal projection onto $\mathcal{S}^\perp$ so that in order to translate his results to our notation one must interchange P and $I - P$.

By the previous corollary and Proposition 2.9 we obtain:

Corollary 4.7. *Let $T \in L(\mathcal{H})$ and $A = T^*T$. The set $sp_g(T, \mathcal{S}, \xi, \eta)$ is not empty for all $\xi \in \mathcal{H}$ and $\eta \in R(A) + \mathcal{S}^\perp$ if and only if $(A, \mathcal{S})$ is compatible.*

Under the assumption of compatibility, we characterize the solutions of (10):

Proposition 4.8. *Let $T \in L(\mathcal{H})$ and $A = T^*T$ such that $(A, \mathcal{S})$ be compatible and $(\xi, \eta) \in \mathcal{H} \times (R(A) + \mathcal{S}^\perp)$. Therefore, if $\eta = A\psi + \sigma^\perp$ with $\sigma^\perp \in \mathcal{S}^\perp$ then*

$$sp_g(T, \mathcal{S}, \xi, \eta) = \{P_{A,\mathcal{S}}\psi + (I - P_{A,\mathcal{S}})\xi + \zeta :\ \zeta \in N(A) \cap \mathcal{S}\} \tag{11}$$

$$= \{Q\psi + (I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\}, \tag{12}$$

$$= \{A^{(\dagger)}_{(\mathcal{S})}\eta + (I - P_{A,\mathcal{S}})\xi + \zeta :\ \zeta \in N(A) \cap \mathcal{S}\} \tag{13}$$

Moreover, $A^{(\dagger)}_{(\mathcal{S})}\eta + (I - P_{A,\mathcal{S}})\xi$ is the unique element in $sp_g(T, \mathcal{S}, \xi, \eta)$ with minimal norm.

Proof. First notice that, by Theorem 4.2 and Corollary 4.7, it holds that the set $sp_g(T, \mathcal{S}, \xi, \eta) = \{\sigma \in \mathcal{H} : \text{equality (10) holds}\}$ is not empty.

Now, let us show that $P_{A,\mathcal{S}}(\psi - \xi) + \xi$ is a solution of (10). Indeed, since $PAP_{A,\mathcal{S}} = PA$ we obtain that $(PA + I - P)(P_{A,\mathcal{S}}(\psi - \xi) + \xi) = PAP_{A,\mathcal{S}}(\psi - \xi) + PA\xi + (I - P)\xi = PA\psi + (I - P)\xi = P\eta + (I - P)\xi$. Therefore, $P_{A,\mathcal{S}}\psi + (I - P_{A,\mathcal{S}})\xi = P_{A,\mathcal{S}}(\psi - \xi) + \xi$ is a solution of (10) and (11) holds because of Proposition 4.4.

Equality (12) is consequence of $\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}, N(A) \cap \mathcal{S})$.

For equality (13), note that by (11), $\tilde{\sigma} := P_{A,\mathcal{S}}\psi + (I - P_{A,\mathcal{S}})\xi - P_{N(A)\cap\mathcal{S}}\psi \in sp_g(T, \mathcal{S}, \xi, \eta)$. Now, by (5) and (6), we have that $\tilde{\sigma} = P(AP + I - P)^\dagger A\psi + (I - P_{A,\mathcal{S}})\xi = P(AP + I - P)^\dagger(A\psi + \sigma^\perp) + (I - P_{A,\mathcal{S}})\xi$ where the last equality holds because $P(AP + I - P)^\dagger(I - P) = 0$. Thus, $\tilde{\sigma} = P(AP + I - P)^\dagger\eta + (I - P_{A,\mathcal{S}})\xi \in sp_g(T, \mathcal{S}, \xi, \eta)$ and (13) follows by Proposition 4.4.

Finally, since $sp_g(T, \mathcal{S}, \xi, \eta)$ is the set of solutions of Eq. (10), then the element in $sp_g(T, \mathcal{S}, \xi, \eta)$ with minimal norm is the unique element in $sp_g(T, \mathcal{S}, \xi, \eta) \cap N(PA + I - P)^\perp = sp_g(T, \mathcal{S}, \xi, \eta) \cap (N(A) \cap \mathcal{S})^\perp$. Therefore, let us show that $\tilde{\sigma} = P_{A,\mathcal{S}}\psi + (I - P_{A,\mathcal{S}})\xi - P_{N(A)\cap\mathcal{S}}\psi \in (N(A) \cap \mathcal{S})^\perp$. Now, since $P_{N(A)\cap\mathcal{S}}(P_{A,\mathcal{S}}\psi - P_{N(A)\cap\mathcal{S}}\psi) = 0$ we get that $P_{A,\mathcal{S}}\psi - P_{N(A)\cap\mathcal{S}}\psi \in (N(A)\cap\mathcal{S})^\perp$. On the other hand, $(I - P_{A,\mathcal{S}})\xi \in N(P_{A,\mathcal{S}}) \subseteq (N(A)\cap\mathcal{S})^\perp$. Therefore, $\tilde{\sigma} \in (N(A)\cap\mathcal{S})^\perp$ as desired. $\square$

Remark 4.9. In [14, Theorem 5] it is also described the elements of the set $sp_g(T, \mathcal{S}, \xi, \eta)$ and the element of minimal norm but for the finite dimensional case. Note that our expression of the minimal norm solution coincides with the one obtained by Chen because of Proposition 3.5. It should be noticed that Chen works with pairs $(A, \mathcal{S})$ such that A is $\mathcal{S}$-positive, in the sense that $A = A^*$, $\langle A\phi, \phi\rangle \geqslant 0$ for all $\phi \in \mathcal{S}$ and if $\langle A\phi, \phi\rangle = 0$ with $\phi \in \mathcal{S}$ then $A\phi = 0$. Most results on compatible pairs can be extended to this type of pairs.

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References

[1] M.L. Arias, G. Corach, M.C. Gonzalez, Products of projections and positive operators, Linear Algebra Appl. 439 (2013) 1730–1741.
[2] M.L. Arias, G. Corach, M.C. Gonzalez, Additivity properties of operator ranges, Linear Algebra Appl. 439 (2013) 3581–3590.
[3] M. Atteia, Generalization de la définition et des propriétés des “splinefonction”, C. R. Acad. Sci. Paris 260 (1965) 3550–3553.
[4] C. Bacuta, A unified approach for Uzawa algorithms, SIAM J. Numer. Anal. 44 (2006) 2633–2649.
[5] N.S. Bakhvalov, A.V. Knyazev, A new iterative algorithm for solving problems of the fictitious flow method for elliptic equations, Soviet Math. Dokl. 41 (3) (1990) 481–485.
[6] A. Ben-Israel, T.N.E. Greville, Generalized Inverses: Theory and Applications, Wiley, New York, 1974.
[7] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, Acta Numer. 44 (2005) 1–137.
[8] J.W. Blattner, Bordered matrices, J. Soc. Indust. Appl. Math. 19 (1962) 528–536.
[9] F. Brezzi, On the existence, uniqueness and approximation of saddle point problems arising from Lagrange multipliers, RAIRO Math. Model. Numer. Anal. 8 (1974) 129–151.
[10] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, Springer Ser. Comput. Math., vol. 44, Springer, Berlin, 2013.
[11] R. Bott, R.J. Duffin, On the algebra of networks, Trans. Amer. Math. Soc. 74 (1953) 99–109.
[12] C. de Boor, Convergence of abstract splines, J. Approx. Theory 31 (1981) 80–89.
[13] D. Buckholtz, Hilbert space idempotents and involutions, Proc. Amer. Math. Soc. 128 (2000) 1415–1418.
[14] Y. Chen, The generalized Bott–Duffin inverse and its application, Linear Algebra Appl. 134 (1990) 71–91.
[15] R. Champion, C.T. Lenard, T.M. Mills, An introduction to abstract splines, Math. Sci. 21 (1996) 8–26.
[16] R. Champion, C.T. Lenard, T.M. Mills, A variational approach to splines, ANZIAM J. 42 (2000) 119–135.
[17] P. Ciarlet Jr., J. Huang, J. Zou, Some observations on generalized saddle-point problems, SIAM J. Matrix Anal. Appl. 25 (2003) 224–236.
[18] G. Corach, A. Maestripieri, D. Stojanoff, Oblique projections and Schur complements, Acta Sci. Math. (Szeged) 67 (2001) 439–459.
[19] G. Corach, A. Maestripieri, D. Stojanoff, Generalized Schur complements and oblique projections, Linear Algebra Appl. 341 (2002) 259–272.

[20] G. Corach, A. Maestripieri, D. Stojanoff, Oblique projections and abstract splines, J. Approx. Theory 117 (2002) 189–206.
[21] F. Deutsch, The angles between subspaces of a Hilbert space, in: S.P. Singh (Ed.), Approximation Theory, Wavelets and Applications, Kluwer, Netherlands, 1995, pp. 107–130.
[22] P.A. Fillmore, J.P. Williams, On operator ranges, Adv. Math. 7 (1971) 254–281.
[23] S. Izumino, Convergence of generalized inverses and spline projectors, J. Approx. Theory 38 (1983) 269–278.
[24] A.V. Knyazev, Iterative solution of PDE with strongly varying coefficients: algebraic version, in: Iterative Methods in Linear Algebra, Brussels, 1991, North-Holland, Amsterdam, 1992, pp. 85–89.
[25] A.V. Knyazev, Observations on degenerate saddle point problems, Comput. Methods Appl. Mech. Engrg. 196 (2007) 3742–3749.
[26] A.V. Knyazev, A. Jujunashvili, M.E. Argentati, Angles between infinite dimensional subspaces with applications to the Rayleigh–Ritz and alternating projectors methods, J. Funct. Anal. 259 (2010) 1323–1345.
[27] B. Shekhtman, Unconditional convergence of abstract splines, J. Approx. Theory 30 (1980) 237–246.
[28] J.L. Šmul'jan, A Hellinger operator integral, Mat. Sb. (N. S.) 49 (91) (1959) 381–430 (in Russian); English transl. Amer. Math. Soc. Transl. 22 (1962) 289–337.