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Läuchli's Completeness Theorem from a Topos-Theoretic Perspective

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Abstract We prove a variant of Läuchli's completeness theorem for intuitionistic predicate calculus. The formulation of the result relies on the observation (due to Lawvere) that Läuchli's theorem is related to the logic of the canonical indexing of the atomic topos of $\mathbb{Z}$-sets. We show that the process that transforms Kripke-counter-models into Läuchli-counter-models is (essentially) the inverse image of a geometric morphism. Completeness follows because this geometric morphism is an open surjection.

Keywords Läuchli's completeness theorem · Topos theory

Mathematics Subject Classifications (2000) 03G30 · 18B25 · 03B60

1 Introduction

In Section 3 of [10], commutative triangles

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

of subobjects are presented as the content of a "logic in a narrow sense" which is contrasted with the fact that "in topology, geometry, combinatorics, etc. there often arises the need to compare $E_i \to B$ which are not necessarily monomorphisms". The contrast suggests the need for a "logic in a broader sense" that can deal with cases where the attributes of B form non-posetal categories.

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Section 4 of [10] relates these narrow and broad aspects of logic via the following result: the inclusion of the bicategory of posets into the bicategory of categories has a left adjoint which is a bifunctor (hence preserves adjointness) and which preserves products. This left adjoint is meant to embody the central idea of "there exists a proof" and is referred to as the Curry–Läuchli adjoint, in reference to Curry's observation (concerning the analogy between modus ponens and laws of functionality) and Läuchli's completeness result [8].

The results in this paper can be seen as an application but, in a sense, they are not new. We will use some of the things that the broad sense of logic has taught us in order to better understand one of its sources of inspiration.

In the author's commentary of [11] "Läuchli's 1967 success in finding a completeness theorem for Heyting predicate calculus lurking in the category of ordinary permutations" is recalled as inspiration for the introduction of hyperdoctrines. In the introduction to [9] it is mentioned that hyperdoctrines appear to be related to Läuchli's result (see [8]) but that "the precise relation is yet to be worked out".

The first attempts to establish the precise relation were in lectures by Lawvere "at the AMS Los Angeles meeting in August 1967 as well as another AMS meeting in New York (which became the 'hyperdoctrines' paper) and at the 1968 Batelle meeting in Seattle" (Kock and Lawvere, private communication). Further work in this direction is [6] which unfortunately was never published. There, Kock uses predoctrines to formulate and prove a variant of Läuchli's result in the form of a predoctrinal representation theorem. (Kock's work was done in 1970 at Dalhousie while he was participating in the seminar which gave rise to the theory of elementary toposes and, according to Lawvere, it gave the participants of the seminar "much-needed encouragement" that the ideas being developed were "potentially fruitful" [Kock and Lawvere, private communication].)

The relation between Läuchli's theorem and hyperdoctrines is also explored in [14] where fibrations are used to give a uniform treatment of completeness theorems for intuitionistic predicate calculus. In this framework, Läuchli's result also appears as a corollary of a representation theorem. (See also [3].)

Both in [14] and [6] the structure used to study Läuchli's work is the hyperdoctrine determined by the canonical indexing of the topos $\mathbf{Set}^{\mathbb{Z}}$ of $\mathbb{Z}$-sets. That is, the hyperdoctrine determined by the pseudo-functor which assigns $\mathbf{Set}^{\mathbb{Z}}/X$ to each $\mathbb{Z}$-set X.

It is our impression that the approaches just cited disregard a lot of structure that is present in the categories involved in the original context. The purpose of this paper is also to study Läuchli's completeness theorem and its relation to the canonical indexing of $\mathbf{Set}^{\mathbb{Z}}$; but in contrast to the approaches in [14] and [6] we will rely more on topos-theoretic machinery. We believe that the topos theoretic tools available nowadays allow a more 'geometric' proof which we hope the readers will find a good complement to those in [6, 8, 14].

Very roughly, Läuchli's result and its proof can be described as follows. He defines a new kind of model which he calls *proof assignments*. The completeness theorem then states that for any closed formula A that is not provable in Heyting's predicate calculus there is a "counter-proof-assignment" for A. The proof relies on Kripke's work to produce a Kripke model Φ where A does not hold and then, out of Φ, Läuchli constructs a proof assignment where A 'does not hold'.

We will show that this process of building proof assignments out of Kripke models can be seen as the inverse image of a geometric morphism which is an open surjection. General results about exact completions will allow us to relate this geometric morphism with $\mathbf{Set}^{\mathbb{Z}}$ and to deduce completeness.

Concerning the organization of the paper. In Section 2 we recall Läuchli's work in more detail. Most of this section is taken from [8]. It is included here to ease the comparison with our approach.

Section 3 recalls the necessary notions to formulate our version of Läuchli's result. Section 4 proves the main result and Section 5 compares the work reported here with that in [6] and [14].

2 Läuchli's Theorem

Consider formulas containing n-place predicate letters, a distinguished propositional constant $\mathbf{f}$ ('false'), individual constants, variables u, v, w, ... and connectives $\wedge$, $\vee$, $\Rightarrow$, $\exists$, $\forall$. The set of closed formulas with individual constants from Γ is denoted by $F(\Gamma)$. Let Γ and Π be countably infinite sets and $c_0 \in \Gamma$ a designated element. To each formula A associate a set $S(A)$, the set of 'possible proofs of A':

$S(A) = \Pi$ if A is atomic,
$S(A \wedge B) = S(A) \times S(B)$
$S(A \vee B) = S(A) + S(B)$
$S(A \Rightarrow B) = S(B)^{S(A)}$
$S(\forall v A) = S(A)^{\Gamma}$
$S(\exists v A) = \Gamma \times S(A)$

Läuchli notes that since $S(A^v_c) = S(A)$ for all individual constants c, $S(\forall v A)$ can be interpreted as the set of all choice functions which assign to each $c \in \Gamma$ an element of $S(A^v_c)$. Now define a *proof assignment* as any function p which assigns to each closed formula $A \in F(\Gamma)$ a set $p[A]$ such that:

$p[\mathbf{f}] \subseteq p[A] \subseteq \Pi$ if A is atomic,
$p[A \wedge B] = p[A] \times p[B]$
$p[A \vee B] = p[A] + p[B]$
$p[A \Rightarrow B] = \{x \in S(A \Rightarrow B) \mid xy \in p[B] \quad \text{for all} \quad y \in p[A]\}$
$p[\forall v A] = \{x \in S(\forall v A) \mid xc \in p[A^v_c] \quad \text{for all} \quad c \in \Gamma\}$
$p[\exists v A] = \{(c, x) \mid c \in \Gamma \quad \text{and} \quad x \in p[A^v_c]\}$

Läuchli notes that $p[A] \subseteq S(A)$ for all p and A, that the elements of $p[A \Rightarrow B]$ are functions with domain $S(A)$ and that, in particular, the identity function on $S(A)$ belongs to $p[A \Rightarrow A]$. Notice also that contrary to a natural expectation, $p[\mathbf{f}]$ need not be empty. This is essential and we will discuss it further below.

Now define $\mathcal{D}$ to be the least class containing the sets $\{0, 1\}$, Γ, Π, such that whenever $D_1, D_2 \in \mathcal{D}$, then $D_1 \times D_2$, $D_1 \cup D_2$ and $D_1^{D_2}$ are also in $\mathcal{D}$. Then let $\mathcal{F} = \bigcup \mathcal{D}$ and call the elements of $\mathcal{F}$ *functionals*. A functional is called *simple* if it can be described by a closed term definable from 0, 1, c_0, variables, application, pair-formation and λ^D-abstraction relative to a domain $D \in \mathcal{D}$.

Let σ be a permutation on $\Gamma \cup \Pi$ which leaves invariant the sets Γ and Π and the designated element c_0. Läuchli extends σ to a permutation on $\mathcal{F}$ as follows: $\sigma 0 = 0$, $\sigma 1 = 1$; if g is a function then $(\sigma g)x = \sigma(g(\sigma^{-1}x))$. A functional $\Theta \in \mathcal{F}$ is called *invariant*, if $\sigma\Theta = \Theta$; that is if Θ is a *fixed-point* of σ. Läuchli notes that simple functionals are invariant but that there are uncountable many invariant functionals that are not simple. If we denote derivability in the intuitionistic predicate calculus by $\vdash$ then the main result in [8] can be stated as follows.

Theorem 2.1 (Läuchli) *Let A be a closed formula containing no individual constants other than* c_0.

1. *If* $\vdash A$ *then there is simple functional* Θ *such that* $\Theta \in p[A]$ *for all proof assignments p.*
2. *If* $\nvdash A$ *then there is a p such that* $p[A]$ *contains no invariant functional.*

So, classically, $\vdash A$ if and only if for every p there is an invariant $\Theta \in p[A]$.

After stating Theorem 2.1, Läuchli shows that the theorem is not true if one restricts to proof assignments p with $p[\mathbf{f}] = \emptyset$. As an example, he shows that for $B = \forall v(Rv \vee \neg Rv)$, $p[\neg\neg B]$ contains an invariant functional for every p with $p[\mathbf{f}] = \emptyset$.

Concerning the proof of Theorem 2.1, Läuchli first states that "the first item of his theorem is a routine variation on the proof of Theorem 62, [5] p. 504". In order to prove the second item, Läuchli relies on Kripke's completeness theorem in the following way.

Let Σ be the set of all finite sequences of natural numbers (including the empty sequence Λ) together with an ideal element U. Let R be the binary relation on Σ such that sRs' iff either $s' = U$ or s is an initial segment of s'. Let Ψ be a function with domain Σ and countable sets as values, such that whenever sRs' and $s \neq s'$, then $\Psi s \subseteq \Psi s'$ and $(\Psi s') - (\Psi s)$ is infinite. It is also assumed that $\Psi\Lambda$ is infinite. Then define $F(\Psi s)$ to be the set of formulas with individual constants from Ψs.

A *model* is defined as a binary function $\Phi : F(\Psi U) \times \Sigma \to \{T, F\}$ satisfying the following conditions:

if $\Phi(A, s) = T$ then $A \in F(\Psi s)$;
if $\Phi(A, s) = T$ and sRs' then $\Phi(A, s') = T$;
(!!) if $\Phi(\mathbf{f}, s) = T$ then $\Phi(A, s) = T$ for all $A \in F(\Psi s)$;
$\Phi(A \wedge B, s) = T$ iff $A \wedge B \in F(\Psi s)$, $\Phi(A, s) = T$ and $\Phi(B, s) = T$;
$\Phi(A \vee B, s) = T$ iff $A \vee B \in F(\Psi s)$ and either $\Phi(A, s) = T$ or $\Phi(B, s) = T$;
$\Phi(A \Rightarrow B, s) = T$ iff $A \Rightarrow B \in F(\Psi s)$ and for all s' with sRs', if $\Phi(A, s') = T$ then $\Phi(B, s') = T$;
$\Phi((\forall v)A, s) = T$ iff $\Phi(A^v_c, s') = T$ for all s' with sRs' and all $c \in \Psi s'$;
$\Phi((\exists v)A, s) = T$ iff $\Phi(A^v_c, s) = T$ for some $c \in \Psi s$.

After defining the notion of model in this way Läuchli states the following.

Lemma 2.2 *Let* $A \in F(\Psi\Lambda)$ *with not* $\vdash A$. *Then* (*and only then*) *there is a model* Φ *such that* $\Phi(A, \Lambda) = F$.

He then comments that the proof of Lemma 2.2 "is clear from Kripke's work [7]. The element U is no bother: Any Φ which is defined on $\Sigma - \{U\}$ can be extended to Σ by setting $\Phi(A, U) = T$ for all $A \in F(\Psi U)$".

The extent to which the proof is clear from Kripke's work may be a matter of taste and/or background. In any case, the reader should be warned that Kripke works with formulas that may have negation $\neg$ but which do not have **f** and, more importantly, the condition that Kripke uses to define $\Phi(\neg A, s)$ does not coincide with Läuchli's $\Phi(A \Rightarrow \mathbf{f}, s)$. This warning is, of course, related to the highlighted condition on $\Phi(\mathbf{f}, s)$ and to the fact that $p[\mathbf{f}]$ need not be empty for a proof assignment p. (See also the last two paragraphs in p. 223 of [3].)

In order to relate models and proof assignments Läuchli introduces the set of prime numbers $\mathbf{P}$, the poset $\mathbb{D}$ of natural numbers ordered by divisibility and fixes a bijection $q : \Sigma \to \mathbf{P}$. Then he defines a function $\varphi : \Sigma \to \mathbb{D}$ by $\varphi\Lambda = 1$, $\varphi(s * n) = (\varphi s)q(s * n)$ and $\varphi U = 0$ where $s * n$ denotes the sequence obtained by adding the number n to the end of the sequence s. Moreover, in the opposite direction, Läuchli defines a function $\mathbf{s} : \mathbb{D} \to \Sigma$ by $\mathbf{s}n = glb\{s \mid n \text{ divides } \varphi s\}$. After that he states five simple consequences of the definitions which essentially say the following. (Here we are considering Σ as a set partially ordered by the relation R. See the definition of Σ above.)

Lemma 2.3 *The functor* $\varphi : \Sigma \to \mathbb{D}$ *is full* (*as well as faithful*) *and is right adjoint to* $\mathbf{s} : \mathbb{D} \to \Sigma$.

(Lawvere commented that the fact that the connection between Σ and $\mathbb{D}$ was given by an adjoint functor "was further confirmation that categorical logic, while analogous to its posetal reflection, actually serves moreover as an objective logic that produces not only statements but needed things for the statements to be about. Similarly the fixed point operation crucial to Läuchli is really the right adjoint part of a geometric morphism")

After the facts that we have resumed as Lemma 2.3 comes the more intricate part of [8]: it is explained how to transform a model Φ into a proof assignment p and then it is proved by induction on the complexity of a formula A that (roughly) if Φ is a counter-model for A then $p[A]$ has no fixed points (see Lemma 2 in [8]).

3 Categorical Formulation

Having recalled Läuchli's result in some detail let us go back to its categorical formulation. We will assume that the reader is familiar with the interpretation of first order logic in hyperdoctrines (see [11] and [17]).

We will only work with two types of hyperdoctrines (always over toposes). The first type is the standard hyperdoctrine which assigns to each object X in the underlying topos $\mathcal{E}$, the poset of subobjects of X (see [4, 13]). We will denote this hyperdoctrine by $\mathbf{s}\mathcal{E}$. The second type is the one determined by the assignment $X \mapsto \mathcal{E}/X$. This hyperdoctrine will be denoted by $\mathbf{p}\mathcal{E}$. When interpreting formulas in $\mathbf{p}\mathcal{E}$, relation symbols of type X are interpreted as maps with codomain X, disjunctions, conjunctions and implications are interpreted using coproducts, products and exponentials (in the slices $\mathcal{E}/X$) respectively and quantifiers are interpreted using the adjoints to change of base $\pi_0^* : \mathcal{E}/Y \to \mathcal{E}/(Y \times Z)$ along projections.

Remark 3.1 The fundamental observation (due to Lawvere) for the categorical understanding of Läuchli's work is that the interpretation of logical symbols in $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ is the same as that in the definition of proof assignment. This is clear for the propositional connectives. The issue for quantifiers is less clear but only because Läuchli avoids open formulas (and introduces the set Γ to get away with it). Related to this, let us mention as a curiosity, that in the key part of his proof, when the time comes to build a proof assignment with certain properties, Läuchli takes $\Gamma = \Pi$.

For a fixed signature **Sg** and a hyperdoctrine $\mathcal{H}$, **Sg**-structures in $\mathcal{H}$ are defined as usual (see e.g. Section 5.1 in [17] or Section D1.2 in [4]). If M is a **Sg**-structure and S is a sort in **Sg**, MS will denote the interpretation of S and we extend this notation to contexts as usual. If $\overrightarrow{x}.\phi$ is a formula-in-context the interpretation $[\![\overrightarrow{x}.\phi]\!]$ (also denoted by $[\![\phi]\!]$ if no confusion arises) will be defined as usual except that, for essentially the same reasons that the $p[\mathbf{f}]$ should not be required to be empty, we need to slightly adjust the way to interpret **f**. (In the following definition, $! : M(S_1, \ldots, S_n) \to 1$ is the unique map to the terminal object and $!^*$ is the usual notation for $\mathcal{H}! : \mathcal{H}1 \to \mathcal{H}M(S_1, \ldots, S_n)$ where $\mathcal{H}$ is the underlying hyperdoctrine.)

Definition 3.2 A *rich* **Sg**-structure is a **Sg**-structure M together with an object $M\mathbf{f}$ in $\mathcal{H}1$ such that for every relation symbol $R \to S_1, \ldots, S_n$ in **Sg**, there is a map $!^*(M\mathbf{f}) \to MR$ in $M(S_1, \ldots, S_n)$.

Given a rich **Sg**-structure M, the interpretation $[\![\overrightarrow{x}.\phi]\!]$ of a formula-in-context is defined inductively as usual except that $[\![\overrightarrow{x}.\mathbf{f}]\!]$ is interpreted as $!^*(M\mathbf{f})$. We say that that $\overrightarrow{x}.\phi$ *holds* in a **Sg**-structure M if $[\![\overrightarrow{x}.\phi]\!]$ is terminal in the category $\mathcal{H}M(S_1, \ldots, S_n)$, where $S_1, \ldots, S_n$ is the sequence of sorts associated with the context $\overrightarrow{x}$. A standard inductive argument shows the following.

Lemma 3.3 *If $\overrightarrow{x}.\phi$ is a formula-in-context with $x_i : S_i$ then there is a morphism $[\![\mathbf{f}]\!] \to [\![\phi]\!]$ in $M\overrightarrow{S_i}$.*

So a rich **Sg**-structure allows to interpret formulas ϕ in such a way that $\mathbf{f} \Rightarrow \phi$ always holds. As the logical connectives and quantifiers are interpreted as usual, it should be clear that first order intuitionistic logic is sound for its interpretation in terms of rich **Sg**-structures. The reason for considering rich **Sg**-structures is that they allow new counter-models to appear.

Example 3.4 Consider the topos $\mathbf{Set}^{\mathbb{Z}}$ of $\mathbb{Z}$-sets. We denote the essentially unique representable also by $\mathbb{Z}$. For each positive integer n, the integers modulo n (denoted here by $\mathbb{Z}_n$ and with underlying set $\{0, \ldots n-1\}$) have a canonical structure of $\mathbb{Z}$-set. We denote the resulting object in the topos $\mathbf{Set}^{\mathbb{Z}}$ also by $\mathbb{Z}_n$. Consider the hyperdoctrine $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ and the signature given by a single propositional constant A. It is not difficult to check that, for any structure M with $M\mathbf{f} = \emptyset$, $[\![\neg A \vee \neg\neg A]\!]$ has a fixed-point. On the other hand, consider the rich structure M with $M\mathbf{f} = \mathbb{Z}_6$ and $MA = \mathbb{Z}_2$. In this case, $[\![\neg A \vee \neg\neg A]\!]$ has a fixed point if and only if there is a map $1 \to {\mathbb{Z}_6}^{\mathbb{Z}_2}$ or a map $1 \to {\mathbb{Z}_6}^{({\mathbb{Z}_6}^{\mathbb{Z}_2})}$. Clearly, there is no map $\mathbb{Z}_2 \to \mathbb{Z}_6$ so we are left to check if there is a map ${\mathbb{Z}_6}^{\mathbb{Z}_2} \to \mathbb{Z}_6$. But ${\mathbb{Z}_6}^{\mathbb{Z}_2}$ has a cycle of length 3 so a map as above can not exist and hence $[\![\neg A \vee \neg\neg A]\!]$ does not have a fixed point. (The use of

$\neg A \vee \neg\neg A$ as an example for the need to consider proof assignments with $p[\mathbf{f}] \neq \emptyset$ is attributed to Läuchli in the Remark after Theorem 3.8 in [3].)

Finally, we will consider sequents-in-context $\phi \vdash_{\vec{x}} \psi$ (see, for example, Definition D1.1.5 in [4]) and we will say that one such is *satisfied* or that it *holds* in a rich structure M if the formula in context $\vec{x}.(\phi \Rightarrow \psi)$ holds in M or, equivalently, if there is a morphism $[\![\phi]\!] \to [\![\psi]\!]$ in the fiber determined by the context.

The version of Läuchli's result that emerges from the work of Kock, Lawvere and Harnik–Makkai can be stated as follows.

Theorem 3.5 *Let* $\mathbf{Sg}$ *be a countable signature and let* $\phi \vdash_{\vec{x}} \psi$ *be a sequent in that signature. If* $\phi \vdash_{\vec{x}} \psi$ *is not provable then there is a rich* $\mathbf{Sg}$*-structure* M *in* $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ *such that* $\phi \vdash_{\vec{x}} \psi$ *does not hold in* M.

In particular, let α be a closed formula in the empty context. If $\nvdash \alpha$ then the result above says that there is a rich structure where the sequent does not hold. Now, the interpretation of α is a morphism $a : A \to 1$ in $\mathbf{Set}^{\mathbb{Z}}$ and the fact that the sequent does not hold means that a does not have a section (that is, A has no fixed-points). In other words, if we let $\Delta \dashv \Gamma : \mathbf{Set}^{\mathbb{Z}} \to \mathbf{Set}$ be the unique geometric morphism to the category of sets then we have the following.

Corollary 3.6 *Let* α *be a closed formula in the empty context. Then* $\vdash \alpha$ *if and only if for every rich structure in* $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$, *the object* $[\![\alpha]\!]$ *has a fixed point* (*i.e.* $\Gamma[\![\alpha]\!] \neq \emptyset$; *recall comment after Lemma* 2.3).

Our proof of Theorem 3.5 will also rely on Kripke's work [7]. But instead of seeing models as Läuchli does, we are simply going to think of Kripke models as objects in the presheaf topos $\mathbf{Set}^{\Sigma}$ (see the first paragraph of p. 318 in [13]). So we reformulate Lemma 2.2 as follows.

Lemma 3.7 *Let* $\mathbf{Sg}$ *be a countable signature and let* $\phi \vdash_{\vec{x}} \psi$ *be a sequent in that signature. If* $\phi \vdash_{\vec{x}} \psi$ *is not provable then there is a rich* $\mathbf{Sg}$*-structure* M *in* $\mathbf{s}(\mathbf{Set}^{\Sigma})$ *such that* $\phi \vdash_{\vec{x}} \psi$ *does not hold in* M.

It seems fair to say that our work does not clarify the reasons why rich structures (or other models with the interpretation of $\mathbf{f}$ different from empty) are necessary. Läuchli's comment cited after Lemma 2.2 suggests that such structures are already present, at least implicitly, in Kripke's work. Or perhaps, in completeness theorems with respect to a restricted class of Kripke models (see Proposition 5.6 in [14]).

4 Our Proof

Let us now outline our proof of Theorem 3.5. The main idea is that there exists:

1. A topos $\mathcal{G}$ together with
2. A geometric inclusion $\mathbf{Set}^{\mathbb{Z}} \to \mathcal{G}$ which strongly relates the hyperdoctrines $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ and $\mathbf{s}\mathcal{G}$ and
3. An open surjection $\mathcal{G} \to \mathbf{Set}^{\Sigma}$.

Recall that open geometric morphisms can be described as those whose inverse images preserve first order logic. On the other hand, a geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is a surjection if and only if for any pair of subobjects U and V of X in $\mathcal{F}$, $f^*U \leq f^*V$ in $\mathcal{E}$ implies $U \leq V$ in $\mathcal{F}$. So, from a logical point of view, an open surjection $f : \mathcal{E} \to \mathcal{F}$ can be thought of as saying that $\mathcal{E}$ is 'as complete as $\mathcal{F}$' (at least, in the sense that if there is a model M in $\mathcal{F}$ where the sequent $\phi \vdash \psi$ is not satisfied then there is a model f^*M in $\mathcal{E}$ where the same sequent is not satisfied either). As $\mathbf{Set}^\Sigma$ is complete by Lemma 3.7, the existence of an open surjection $\mathcal{G} \to \mathbf{Set}^\Sigma$ implies that $\mathcal{G}$ is complete. The strong relation between $\mathbf{s}\mathcal{G}$ and $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ mentioned above allows to transfer completeness of the first hyperdoctrine to that of the second. So the result will follow.

The construction of $\mathcal{G}$ and the precise nature of the relation between $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ and $\mathbf{s}\mathcal{G}$ is better explained as an application of results about coproduct and exact completions. We discuss these issues in Section 4.1. In Section 4.2 we explain how to build the open surjection $\mathcal{G} \to \mathbf{Set}^\Sigma$. Finally, in Section 4.3 we review the whole proof.

4.1 The Exact and Coproduct Completions

In this section we recall the facts about exact and coproduct completions that we need. First, consider coproduct completions. For any category $\mathcal{C}$ there exists a category $Fam\mathcal{C}$ and a full and faithful functor $\mathcal{C} \to Fam\mathcal{C}$ which is universal, in the right 2-categorical sense, among functors $\mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ has small coproducts (see Section 4 in [2]).

In particular, consider the atomic topos $\mathbf{Set}^{\mathbb{Z}}$ of $\mathbb{Z}$-sets. Let $\mathbb{C}$ be the full subcategory of $\mathbf{Set}^{\mathbb{Z}}$ determined by the representable $\mathbb{Z}$ together with the $\mathbb{Z}_n$ for all $n \geq 1$ (recall Example 3.4). The category $\mathbb{C}$ is small and equivalent to the full subcategory of indecomposable objects of $\mathbf{Set}^{\mathbb{Z}}$. Since $\mathbf{Set}^{\mathbb{Z}}$ has small coproducts, the universal property of the coproduct completion induces a functor $Fam\mathbb{C} \to \mathbf{Set}^{\mathbb{Z}}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & Fam\mathbb{C} \\ & \searrow & \downarrow \\ & & \mathbf{Set}^{\mathbb{Z}} \end{array}$$

Using Lemma 4.1 in [2], and the fact that every $\mathbb{Z}$-set is a coproduct of connected ones, we can conclude the following.

Lemma 4.1 *The functor $Fam\mathbb{C} \to \mathbf{Set}^{\mathbb{Z}}$ is an equivalence.*

Consider now the forgetful 2-functor from the category of exact categories to that of categories with finite limits. This functor has a left bi-adjoint with a very simple explicit description discussed in Section 2 of [2]. For any category with finite limits $\mathcal{C}$, the result of applying this construction is usually denoted by $\mathcal{C}_{ex/lex}$. This notation was devised to distinguish the construction from a related one denoted by $\mathcal{D} \mapsto \mathcal{D}_{ex/reg}$ where $\mathcal{D}$ is a regular category (see Section 6 of [2]). We will not need ex/reg-completions, so we denote $\mathcal{C}_{ex/lex}$ simply by $\mathcal{C}_{ex}$.

Proposition 4.2 *Let $\mathcal{C}$ be a category with finite limits.*

1. *There is a full embedding $\mathbf{y} : \mathcal{C} \to \mathcal{C}_{ex}$ which preserves finite limits and whose image is essentially the subcategory of projectives of $\mathcal{C}_{ex}$. Moreover, $\mathcal{C}$ is exact if and only if $\mathbf{y}$ has a left adjoint which preserves finite limits.*
2. *Every object of $\mathcal{C}_{ex}$ is covered by a projective.*
3. *If $\mathcal{C}$ is a locally cartesian closed pretopos then so is $\mathcal{C}_{ex}$.*
4. *For every X in $\mathcal{C}$, the poset of subobjects of $\mathbf{y}X$ is (iso to) poset reflection of $\mathcal{C}/X$.*
5. *If $\mathcal{A}$ is small and Fam$\mathcal{A}$ has finite limits then $(Fam\mathcal{A})_{ex}$ is equivalent to the presheaf topos $\mathbf{Set}^{\mathcal{A}^{op}}$.*

Proof All these facts are proved in [2]. Indeed, the first three items are proved in Lemmas 2.1 and 2.2, the fourth is proved in the discussion just before Section 3 and the fifth is the Corollary after Lemma 4.1. □

The third item of Proposition 4.2 implies that $\mathbf{s}(\mathcal{E}_{ex})$ is a hyperdoctrine when $\mathcal{E}$ is a topos. The fourth item allows us to relate this hyperdoctrine with $\mathbf{p}\mathcal{E}$. Indeed, let $a : A \to X$ and $b : B \to X$ in $\mathcal{E}$. These induce subobjects $\overline{a}$ and $\overline{b}$ in the poset of subobjects of X as an object in $\mathcal{E}_{ex}$. Moreover, $\overline{a} \leq \overline{b}$ if and only if there is a map $f : A \to B$ such that $bf = a$ in $\mathcal{E}$. So, if we have a counter-model in $\mathbf{p}\mathcal{E}$ then we have one in $\mathbf{s}(\mathcal{E}_{ex})$. The simple lemma below shows (using the first four items of Proposition 4.2) that we can go the other way round.

Lemma 4.3 *Let* **Sg** *be a signature and let M be a rich* **Sg***-structure in $\mathbf{s}(\mathcal{E}_{ex})$ for a topos $\mathcal{E}$. Then there exists a rich* **Sg***-structure N in $\mathbf{p}(\mathcal{E})$ such that for every sequent σ, σ is satisfied in M if and only if it is satisfied in N.*

Proof For each sort S in **Sg**, choose a projective cover $e_S : NS \to MS$ of MS. Using that projectives are closed under finite products it is clear that for every context X the original choices of projective covers induce a cover $e : NX \to MX$. Projectivity of the covers allows one to lift interpretation of function symbols and by pulling back along the chosen projective covers we can obtain interpretation of relation symbols. Denote the resulting **Sg**-structure by N. It should be clear that it is also a rich **Sg**-structure.

Now let ϕ be a formula in context X and let $[\![\phi]\!] \rightarrowtail MX$ be the interpretation in M. Then, the result $e^*[\![\phi]\!] \rightarrowtail NX$ of pulling back $[\![\phi]\!]$ along e is the interpretation of ϕ in N (because pulling back preserves Heyting structure and quantification). Moreover, if $\phi \vdash \psi$ holds in NX then it must hold in MX because $[\![\phi]\!]_N \to [\![\phi]\!]_M$ is a regular epi.

All this in $\mathbf{s}(\mathcal{E}_{ex})$. But for each context X, the poset of subobjects of NX (in $\mathcal{E}_{ex}$) is iso to the poset reflection of $\mathcal{E}/(NX)$ so it must be the case that $\phi \vdash \psi$ holds in $\mathbf{p}\mathcal{E}$. □

We still have not used the last item of Proposition 4.2. But notice that together with Lemma 4.1, it implies that $(\mathbf{Set}^{\mathbb{Z}})_{ex}$ is equivalent to $\mathbf{Set}^{\mathbb{C}^{op}}$. The discussion preceding Lemma 4.3 shows that we can reduce completeness of $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ to completeness of $\mathbf{s}(\mathbf{Set}^{\mathbb{C}^{op}})$.

Remark 4.4 It is perhaps interesting to mention that the relation between $\mathbf{Set}^{\mathbb{Z}}$ and $\mathbf{Set}^{\mathbb{C}^{op}}$ is an instance of a more general fact: every atomic topos has a presheaf topos as its exact completion. Indeed, since every atomic topos over **Set** is the coproduct completion of its full (and essentially small) subcategory of atoms, the exact completion of such a topos is equivalent, by the last item of Proposition 4.2, to the topos of presheaves on the category of atoms. See also [15] for recent results concerning the characterization of toposes whose exact completions are toposes.

4.2 The Open Surjection

We must now prove the following.

Proposition 4.5 *There exists an open surjection* $\mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Set}^{\Sigma}$.

In contrast with the situation in the late sixties or early seventies, we have nowadays established results about such geometric morphisms. Let us state a sufficient condition quickly applicable to the cases we need.

Lemma 4.6 *Let* $f : \mathcal{C} \to \mathcal{D}$ *be a functor between small categories. If*

1. f *is surjective on objects and*
2. *For any morphism* $b : fU \to V$ *in* $\mathcal{D}$ *there exists an* $a : U \to U'$ *in* $\mathcal{C}$ *such that* $fU' = V$ *and* $fa = b$

then the geometric morphism $f^* \dashv f_* : \mathbf{Set}^{\mathcal{C}} \to \mathbf{Set}^{\mathcal{D}}$ *is an open surjection.*

Proof Example A4.2.7(b) in [4] implies that the geometric morphism is a surjection and Lemma C3.1.2 in [4] implies that it is open. □

It is easy to check that Lemma 4.6 is applicable to the left adjoint $\mathbf{s} : \mathbb{D} \to \Sigma$ discussed in Lemma 4.6 so we obtain an open surjection $\mathbf{s}^* \dashv \mathbf{s}_* : \mathbf{Set}^{\mathbb{D}} \to \mathbf{Set}^{\Sigma}$. To complete the proof of Proposition 4.5 we prove that there is an open surjection $\mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Set}^{\mathbb{D}}$. Denote by $\mathbf{r} : \mathbb{C}^{op} \to \mathbb{D}$ the obvious functor presenting $\mathbb{D}$ as the poset reflection of $\mathbb{C}^{op}$. We can apply Lemma 4.6 again, this time to $\mathbf{r}$, and obtain an open surjection $\mathbf{r}^* \dashv \mathbf{r}_* : \mathbf{Set}^{\mathbb{C}^{op}} \to \mathbf{Set}^{\mathbb{D}}$.

At this point, Proposition 4.5 is proved and hence so is Theorem 3.5. It seems useful, though, to give another look at the proof without concentrating on the technicalities.

4.3 Recapitulation

First, imitating Läuchli, start assuming Kripke's completeness theorem in the form of Lemma 3.7 which states that $\mathbf{Set}^{\Sigma}$ has 'enough counter-models' to make it complete. Then apply Lemma 4.6 to conclude that the functors on the left below

$$\mathbb{C}^{op} \xrightarrow{\ \mathbf{r}\ } \mathbb{D} \xrightarrow{\ \mathbf{s}\ } \Sigma \qquad\qquad \mathbf{Set}^{\mathbb{C}^{op}} \longrightarrow \mathbf{Set}^{\mathbb{D}} \longrightarrow \mathbf{Set}^{\Sigma}$$

induce open surjections as on the right above.

In this way we obtain that both $\mathbf{Set}^{\mathbb{D}}$ and $\mathbf{Set}^{\mathbb{C}^{op}}$ also have enough counter-models. (The fact for $\mathbf{Set}^{\mathbb{D}}$ is stated explicitely both in [6] and [14] but the case of $\mathbf{Set}^{\mathbb{C}^{op}}$ does not seem to have attracted much attention.)

Finally, apply the results on exact completions (recalled in Section 4.1) to observe that $\mathbf{Set}^{\mathbb{C}^{op}} \cong (\mathbf{Set}^{\mathbb{Z}})_{ex}$ and then 'push down' the counter-models in $\mathbf{s}(\mathbf{Set}^{\mathbb{C}^{op}})$ to counter-models in $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$.

Remark 4.7 We have already mentioned that it may be arguable how "clear from Kripke's work" is Lemma 2.2. So some readers may find Lemma 3.7 not a very good place to start. For those readers, it may be useful to point out that Proposition 5.6 in [14] implies that $\mathbf{Set}^{\mathbb{D}}$ is complete. So we could have started from there.

5 Comparison

In this section we compare the work reported here with that of Läuchli [8], Kock [6] and Harnik–Makkai [3, 14].

Theorem 2.1 is a completeness result with respect to the semantical structures that Läuchli calls 'proof assignments'. In perspective, and after the work of Lawvere, Kock and Harnik–Makkai, proof assignments appear as a clever artifact to capture the logic of the hyperdoctrine $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$ at a time when hyperdoctrines had not been discovered. Although Läuchli's proof is highly technical and concrete I believe that the construction of proof assignments out of (Kripke) models is essentially the same as the one produced by the open surjections discussed above. This is not really evident but it looks more plausible after reading Kock's work.

In [6], Kock states the main results in terms of representations of pre-doctrines but the key technical step is to construct, out of a presheaf P in $\mathbf{Set}^{\mathbb{D}}$, a morphism in $\mathbf{Set}^{\mathbb{Z}}$ over the $\mathbb{Z}$-set $P' = \sum_{\lambda\in\mathbb{D}} P\lambda \times \mathbb{Z}_\lambda$, that is, an object in the fiber over P' of the hyperdoctrine $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$. Then, Kock proves (through a long induction on the complexity of formulas) Theorem 3.1 in [6], which is analogous to Lemma 2 in [8], and attributes to Läuchli the observation that $\mathbf{Set}^{\mathbb{D}}$ has enough counter-models. Kock's version of Läuchli's main result is then a corollary of this and of the assignment $P \mapsto P'$. It must be said, though, that it is not clear how Kock deals with the issues around $\mathbf{f}$.

Notice that for a presheaf P in $\mathbf{Set}^{\mathbb{D}}$, $(\mathbf{r}^* P)\mathbb{Z}_\lambda = P\lambda$ so that P can be covered by $\sum_{\lambda\in\mathbb{D}} P\lambda \times \overline{\lambda}$ where $\overline{\lambda}$ is the representable associated to λ. In turn, each $\overline{\lambda}$ has $\mathbb{Z}_\lambda$ as projective cover, so Kock's assignment $P \mapsto P'$ can be seen as an application of $\mathbf{r}^*$ followed by a natural choice of projective cover (Recall Lemma 4.3.)

In [14], Makkai uses hyperdoctrines to give an algebraic framework for the proof theory of intuitionistic predicate calculus. In this setting, completeness results appear as representation theorems, much as in the work of Kock. Makkai aims at a great level of generality and we do not intend to give here an account of his work. We only wish to point out certain analogies with the proofs we have already discussed.

From the very beginning, Makkai considers h^--*fibrations* which are hyperdoctrines that are not required to have initial objects in the fibers. This is, of course, related to the fact that $p[\mathbf{f}]$ must not be required to be empty. In Section 5 of [14] Makkai deals with Kripke's completeness result. In particular, Proposition 5.6 proves a result which can be seen as a version of the completeness of $\mathbf{Set}^{\mathbb{D}}$. (The proof uses a very interesting property of $\mathbb{D}$ which is proved in [3].)

In Section 6 of [14] a general result is proved which implies roughly that the freely generated h^--fibration can be weakly represented in $\mathbf{p}(\mathbf{Set}^{\mathbb{Z}})$. The proof involves the version of Kripke's completeness mentioned in the previous paragraph and an explicit calculation of the poset reflection of $\mathbf{Set}^{\mathbb{Z}}$. The last calculation naturally involves $\mathbb{Z}$-sets of the form $\mathbb{Z}_n$. (See the paragraph before Proposition 6.4 in [14] for a brief plan of the proof of the main result.)

Altogether, all the proofs discussed involve more or less the same fundamental ingredients. First, a version of Kripke's completeness in the form of enough counter-models in $\mathbf{Set}^{\mathbb{D}}$. (As we have already mentioned, Kock attributes this observation to Läuchli.) Second, an explicit consideration of the indecomposable objects in $\mathbf{Set}^{\mathbb{Z}}$. In this perspective, our factorization of the proof through $(\mathbf{Set}^{\mathbb{Z}})_{ex} \cong \mathbf{Set}^{\mathbb{C}^{op}}$ seems like a natural intermediate step. Moreover, this intermediate step is the one that allows the Kripke-to-Läuchli construction to be the inverse image of a geometric morphism. In turn, this presentation is the one that allows to use topos theory (through openness and surjectivity) to take care of the 'conservativity' of the construction (compare with Theorem 3.1 in [6] and Lemma 6.6 in [14]).

Further work related to Läuchli's theorem appears, for example, in [1, 12, 16]. A discussion of this work falls outside the scope of the present paper but it is worth mentioning that, on top of the results which form the core of [1], this paper contains very interesting historical information on how Läuchli's theorem influenced important work in the area of semantics of programming languages.

We hope that the organization of the fundamental ingredients of Läuchli's theorem in terms of toposes, exact completions and open surjections provides an interesting perspective.

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References

1. Blute, R.F., Scott, P.J.: Linear Läuchli semantics. Ann. Pure Appl. Logic **77**(2), 101–142 (1996)
2. Carboni, A.: Some free constructions in realizability and proof theory. J. Pure Appl. Algebra **103**(2), 117–148 (1995)
3. Harnik, V., Makkai, M.: Lambek's categorical proof theory and Läuchli's realizability. J. Symbolic Logic **57**(1), 200–230 (March 1992)
4. Johnstone, P.T.: Sketches of an Elephant: A Topos Theory Compendium, vol. 43–44 of Oxford Logic Guides. The Clarendon Press Oxford University Press, New York (2002)
5. Kleene, S.C.: Introduction to Metamathematics. North Holland (1952)
6. Kock, A.: On a theorem of Läuchli concerning proof bundles. Unpublished (August 1970)
7. Kripke, S.A.: Semantical analysis of intuitionistic logic I. In: Formal Systems and Recursive Functions, pp. 92–130. Amsterdam, North Holland (1965)
8. Läuchli, H.: An abstract notion of realizability for which intuitionistic predicate calculus is complete. In: Kino, A., Myhill, J., Vesley, R.E. (eds.) Intuitionism and Proof Theory (Proc. Conf., Buffalo, N. Y., 1968), pp. 227–234. North Holland (1970)
9. Lawvere, F.W.: Equality in hyperdoctrines and the comprehension schema as an adjoint functor. In: Heller, A. (ed.) Proceedings of the New York Symposium on Applications of Categorical Logic, pp. 1–14. Providence (1970)

10. Lawvere, F.W.: Adjoints in and among bicategories. In: Logic and Algebra, Lecture Notes in Pure and Applied Algebra, vol. 180, pp. 181–189. Proceedings of the 1994 Siena Conference in Memory of Roberto Magari. Marcel Dekker, Inc. (1996)
11. Lawvere, F.W.: Adjointness in foundations. Reprints in Theory and Applications of Categories, pp. 1–16 (2006), Originally published: Dialectica 23 (1969)
12. Lipton, J., O'Donnell, M.J.: Some intuitions behind realizability semantics for constructive logic: Tableaux and Läuchli countermodels. Ann. Pure Appl. Logic **81**(1–3), 187–239 (1996)
13. Mac Lane, S., Moerdijk, I.: Sheaves in Geometry and Logic: a First Introduction to Topos Theory. Universitext. Springer Verlag (1992)
14. Makkai, M.: The fibrational formulation of intuitionistic predicate logic 1: completeness according to Gödel, Kripke and Läuchli. Parts 1 and 2. Notre Dame J. Formal Logic **34**(3 and 4 resp.), 334–377, 471–498 (1993)
15. Menni, M.: Cocomplete toposes whose exact completions are toposes. J. Pure Appl. Algebra **210**, 511–520 (2007)
16. Paris, A.: Läuchli realizability and gauge theories. Abstract in the 2000 annual meeting of the Association for Symbolic Logic, June 2000. Bull. Symbolic Logic **6**(3)
17. Pitts, A.M.: Categorical logic. In: Abramsky, S., Gabbay, D.M., Maibaum, T.S.E. (eds.) Handbook of Logic in Computer Science, vol. 5. Algebraic and Logical Structures, chapter 2. Oxford University Press (2000)