CALDERÓN WEIGHTS AS MUCKENHOUPT WEIGHTS

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ABSTRACT. The Calderón operator S is the sum of the the Hardy averaging operator and its adjoint. The weights w for which S is bounded on $L^p(w)$ are the Calderón weights of the class $\mathcal{C}_p$. We give a new characterization of the weights in $\mathcal{C}_p$ by a single condition which allows us to see that $\mathcal{C}_p$ is the class of Muckenhoupt weights associated to a maximal operator defined through a basis in $(0, \infty)$. The same condition characterizes the weighted weak-type inequalities for $1 < p < \infty$, but that the weights for the strong type and the weak type differ for $p = 1$. We also prove that the weights in $\mathcal{C}_p$ do not behave like the usual A_p weights with respect to some properties and, in particular, we answer an open question on extrapolation for Muckenhoupt bases without the openness property.

1. INTRODUCTION.

Let P and Q be the Hardy averaging operator and its adjoint,

$$Pf(t) = \frac{1}{t}\int_0^t f(x)dx, \quad Qf(t) = \int_t^\infty \frac{f(x)}{x}\,dx, \quad (t > 0).$$

The Calderón operator S is defined as $S = P + Q$. Given $1 \le p < \infty$, it is said that a nonnegative measurable function w defined in $(0, +\infty)$ is a Calderón weight of the class $\mathcal{C}_p$ (see [1]), and we write $w \in \mathcal{C}_p$, if S is bounded on $L^p(w)$, or, equivalently, if P and Q are both bounded

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on $L^p(w)$. For $p>1$ it is known that $w\in\mathcal{C}_p$ if and only if there exists $C>0$ such that for all $t>0$ it holds that

$$M_p:\qquad \left(\int_t^\infty \frac{w(x)}{x^p}\,dx\right)^{1/p}\left(\int_0^t w^{1-p'}(x)\,dx\right)^{1/p'}\le C$$

and

$$M^p:\qquad \left(\int_0^t w(x)\,dx\right)^{1/p}\left(\int_t^\infty \frac{w^{1-p'}(x)}{x^{p'}}\,dx\right)^{1/p'}\le C.$$

The case $p=1$ is easier to describe: $w\in\mathcal{C}_1$ if and only if

$$Sw(x)\le Cw(x) \text{ a.e.} \tag{1.1}$$

If w is a Calderón weight, the best constant in M_p and M^p, that is, the least constant C satisfying M_p and M^p, will be denoted by $[w]_{\mathcal{C}_p}$. Similarly, $[w]_{\mathcal{C}_1}$ will denote the best constant in (1.1). It is known that $[w]_{\mathcal{C}_p}$ is essentially equal to the norm of the Calderón operator on $L^p(w)$, in the sense that the quotient of both quantities is bounded above and below by constants depending only on p. For all these definitions and results, see [1] and [13]. The Calderón operator plays a significant role in the theory of real interpolation. Such theory associated with Calderón weights is developed in [1].

The first aim of this paper is to show that the two conditions M_p and M^p mentioned in the preceding paragraph can be replaced by a single condition.

To this end, we introduce a maximal operator which turns out to be related to the Calderón weights. Given a measurable function f we define the maximal operator

$$Nf(t)=\sup_{b>t}\frac{1}{b}\int_0^b |f(x)|\,dx.$$

Notice that Nf is a decreasing function, and that $Nf\le Sf$ for nonnegative f. Indeed, for $f\ge 0$ and $b>t$ we have

$$\frac{1}{b}\int_0^b f(x)\,dx\le \frac{1}{t}\int_0^t f(x)\,dx+\int_t^b \frac{f(x)}{x}\,dx\le Sf(t).$$

The operator N is the maximal operator on $(0,+\infty)$ associated to the basis of open sets of the form $(0,b)$, for $b>0$. We recall in Section 2 the concept of basis and the corresponding classes of A_p weights. In the case of N we denote as $A_{p,0}$ $(1<p<\infty)$ the class of nonnegative functions defined in $(0,+\infty)$ such that

$$[w]_{p,0}:=\sup_{b>0}\left(\frac{1}{b}\int_0^b w\right)\left(\frac{1}{b}\int_0^b w^{1-p'}\right)^{p-1}<+\infty. \tag{1.2}$$

For $p = 1$, we write $A_{1,0}$ for the class of nonnegative functions such that $Nw(x) \le Cw(x)$ a.e. and $[w]_{1,0}$ denotes the smallest constant for which the inequality holds.

Theorem 1.1. *For $1 \le p < \infty$, N is of weak type (p, p) with respect to the measure $w(t)dt$ if and only if $w \in A_{p,0}$. More precisely,*

$$\sup_{\lambda>0} \lambda w(\{t : Nf(t) > \lambda\})^{1/p} \le [w]_{p,0}^{1/p} \|f\|_{p,w}. \tag{1.3}$$

For $1 < p < \infty$, N is bounded on $L^p(w)$ if and only if $w \in A_{p,0}$. Moreover,

$$\|Nf\|_{p,w} \le \frac{p^{p'}}{p-1} [w]_{p,0}^{p'-1} \|f\|_{p,w}. \tag{1.4}$$

We prove this theorem in Section 2. With such a result in hand we can easily obtain our main theorem whose proof will be provided in Section 3.

Theorem 1.2. *(a) Let $1 < p < \infty$ and let w be a nonnegative measurable function. Then S is bounded on $L^p(w)$ or is of weak-type (p, p) with respect to the measure $w(t)dt$ if and only if $w \in A_{p,0}$. That is, $\mathcal{C}_p$ and $A_{p,0}$ coincide for $1 < p < \infty$. Moreover,*

$$([w]_{p,0})^{1/p} \le ||S||_{L^p(w)} \le 2C(p)([w]_{p,0})^{\max\{1,p'-1\}}, \tag{1.5}$$

where $C(p) = \max\{p' p^{p'-1}, (p')^{p-1} p\}$.

(b) N is bounded on $L^1(w)$ if and only if $w \in \mathcal{C}_1$, and S is of weak-type $(1, 1)$ with respect to the measure $w(t)dt$ if and only if $w \in A_{1,0}$.

So far as we know, (1.2) gives a new characterization of the Calderón weights for $p > 1$ by means of a single condition which is clearly related to Muckenhoupt's A_p condition. It is worth remarking that in the proof of this result (for $p > 1$) we do not need to know the conditions M_p and M^p. On the other hand, note that we can also relate the usual A_p weights to the Calderón weights: $w : \mathbb{R} \to [0, +\infty]$ satisfies A_p if and only if for each $a \in \mathbb{R}$ the weights $w_a(t) = w(t - a)$ restricted to $(0, +\infty)$ are Calderón weights and $\sup_{a\in\mathbb{R}} [w_a]_{\mathcal{C}_p} < +\infty$.

In the case $p = 1$, $\mathcal{C}_1$ is strictly contained in $A_{1,0}$ (for instance, $1 \in A_{1,0} \setminus \mathcal{C}_1$). The theorem says that $\mathcal{C}_1$ characterizes the strong-type inequality and $A_{1,0}$ characterizes the weak-type inequality. A remarkable fact is that, unlike for the usual Hardy-Littlewood maximal operator, there are nontrivial weights for N in the strong $(1, 1)$ case (for instance, $|t|^{-\alpha}$ for $0 < \alpha < 1$).

Once we know that the Calderón weights are of Muckenhoupt type for an appropriate basis, it is natural to wonder whether they share the properties of the usual A_p weights. Strikingly, some of them fail:

- The basis associated to the operator N does not have the openness property with respect to its weights, that is, there exist $w \in \mathcal{C}_p$ such that w is not in $\mathcal{C}_q$ for any $q < p$ (Proposition 4.1).
- Let $\mathcal{C}_{\exp}$ be the class of weights w on $(0, \infty)$ which satisfy for some constant $C > 0$ that

$$\frac{1}{b}\int_0^b w \leq C \exp\left(\frac{1}{b}\int_0^b \log w\right) \tag{1.6}$$

  for all $b > 0$. The inclusion $\cup_p \mathcal{C}_p \subset \mathcal{C}_{\exp}$ is strict (Proposition 4.4).
- There exist weights in $\mathcal{C}_p$ for which the reverse Hölder inequality fails, and there exist weights for which the reverse Hölder inequality holds but do not belong to $\cup_p \mathcal{C}_p$ (Remark 4.5).

Moreover, we also provide a negative answer to an open question related to extrapolation of weighted inequalities for bases without the openness property (Proposition 4.3). All these negative results will be proved in Section 4 along with a result providing sufficient conditions on w to ensure the openness property.

In Section 5 we consider the Riemann-Liouville and Weyl averaging operators from which the operators P and Q are particular cases. They are defined for $\alpha \geq 0$ respectively as

$$I_\alpha f(t) = \frac{\alpha + 1}{t^{\alpha+1}} \int_0^t (t - x)^\alpha f(x)\, dx$$

and

$$J_\alpha f(t) = (\alpha + 1) \int_t^\infty \frac{(x - t)^\alpha}{x^{\alpha+1}} f(x)\, dx,$$

for $t > 0$. Let $S_\alpha := I_\alpha + J_\alpha$. It is clear that $I_0 = P$, $J_0 = Q$ and $S_0 = S$. It is also easy to see that $I_\alpha f \leq Pf$, $J_\alpha f \leq Qf$ and $S_\alpha f \leq Sf$ for nonnegative measurable fuctions f. Therefore, S_α is bounded on $L^p(w)$ for $w \in \mathcal{C}_p$.

The boundedness of I_α on $L^p(w)$ is characterized by two independent conditions and the boundedness of J_α in $L^p(w)$ by two other conditions (see [16] and [12]). Therefore, a priori, the boundedness of S_α is characterized by four conditions. Again, we are able to show that those conditions can be reduced to $w \in \mathcal{C}_p$, hence, the single condition (1.2) is the only one we need to characterize the boundedness of S_α on $L^p(w)$ for $p > 1$ and the condition $\mathcal{C}_1$ for $p = 1$. Moreover, in both cases, the conditions are independent of α.

Theorem 1.3. *Let w be a nonnegative measurable function.*

(1) Let $1 < p < \infty$. The following assertions are equivalent.

(a) $w \in \mathcal{C}_p$ *(or $w \in A_{p,0}$).*

(b) *There exist $\alpha \geq 0$ such that S_α is bounded on $L^p(w)$.*

(c) *For all $\alpha \geq 0$, S_α is bounded on $L^p(w)$.*

(d) *There exist $\alpha \geq 0$ such that S_α is of weak-type (p, p) with respect to $w(t)dt$.*

(e) *For all $\alpha \geq 0$, S_α is of weak-type (p, p) with respect to $w(t)dt$.*

(2) Let $p = 1$. Then (b) *and* (c) *are equivalent to $w \in \mathcal{C}_1$ and* (d) *and* (e) *are equivalent to $w \in A_{1,0}$.*

The fact that $w \in A_{p,0}$ in the form (1.2) is sufficient for (c) to hold when $p > 1$ was observed by H. Heinig in [9], but he did not prove the necessity of the condition. The proof of Theorem 1.3 is in Section 5.

In the last section we introduce higher dimensional versions of the Calderón operator and the maximal function. The extension to such a setting of the results obtained in $(0, +\infty)$ is straightforward.

Let us indicate that the Calderón operator is almost the same as the operator

$$Hf(t) = \int_0^\infty \frac{f(x)}{t+x}dx,$$

arising in the continuous version of Hilbert's inequality (see [8, Chapter IX]). Indeed, it is immediate that for nonnegative f it holds that

$$Hf(t) \leq Sf(t) \leq 2Hf(t).$$

Consequently, all the results we state for S are valid for H.

Let us fix some notation. For a measurable set B, we denote as $|B|$ its Lebesgue measure and as $w(B)$ the integral of the weight w on B. If B is the interval (a, b), we write $w(a, b)$ for its measure. The characteristic function of B is denoted as χ_B.

2. Weights for the maximal operator

Muckenhoupt's theory of weighted inequalities was extended by B. Jawerth and formulated in the framework of bases of open sets in $\mathbb{R}^n$ (see [10]). The concept of basis is fairly general and it was already observed by Jawerth that $\mathbb{R}^n$ can be replaced by a measure space with σ-finite measure. Among the bases, we are interested in those for which the weighted inequalities are characterized by a condition of Muckenhoupt type, called Muckenhoupt bases. We recall here the concept and adapt it to our setting, that is, to the half-line. The reader can find in [3, Chapter 3] the formulation in $\mathbb{R}^n$.

A basis $\mathcal{B}$ in $(0,+\infty)$ is a collection of open sets B contained in $(0,+\infty)$. Given a basis $\mathcal{B}$, the maximal operator associated with $\mathcal{B}$ is defined by

$$M_{\mathcal{B}}f(t)=\sup_{B\in\mathcal{B}:t\in B}\frac{1}{|B|}\int_B |f(x)|dx$$

if $t\in\cup_{B\in\mathcal{B}}B$ and $M_{\mathcal{B}}f(t)=0$ otherwise. Given a basis $\mathcal{B}$ and a weight w, we say that w belongs to the Muckenhoupt class associated to $\mathcal{B}$, $A_{p,\mathcal{B}}$, $1<p<\infty$, if there exists a constant C such that for every $B\in\mathcal{B}$,

$$\left(\int_B w\right)\left(\int_B w^{1-p'}\right)^{p-1}\le C|B|^p.$$

The infimum of all such C is called the $A_{p,\mathcal{B}}$ constant of w.

We say that the basis $\mathcal{B}$ has the $A_{p,\mathcal{B}}$ openness property or is $A_{p,\mathcal{B}}$ open if given any $w\in A_{p,\mathcal{B}}$ for some $p>1$, there exists $q<p$ such that $w\in A_{q,\mathcal{B}}$.

The basis $\mathcal{B}$ is a Muckenhoupt basis if for each p, $1<p<\infty$, and for every $w\in A_{p,\mathcal{B}}$, the maximal operator $M_{\mathcal{B}}$ is bounded on $L^p(w)$, that is,

$$\int_0^\infty M_{\mathcal{B}}f(x)^p w(x)dx\le C\int_0^\infty |f(x)|^p w(x)dx,$$

with a constant C independent of f and depending only on the $A_{p,\mathcal{B}}$ constant of w.

Observe that with these definitions, N can be identified with $M_{\mathcal{B}_0}$ where $\mathcal{B}_0=\{(0,b):b>0\}$. The corresponding class $A_{p,\mathcal{B}_0}$ is denoted in this paper as $A_{p,0}$ (see (1.2)) for simplicity of notation. Theorem 1.1, stated in Section 1 and proved below, says that $\mathcal{B}_0$ is a Muckenhoupt basis. Working with $\mathcal{B}_0$ is particularly simple, because the elements of the family are nested.

Before proceeding with the proof of the theorem we state some properties of the weights.

Proposition 2.1. *Let w be a nonnegative measurable function, not identically zero. Assume that N is of weak-type (p,p) with respect to $w(t)dt$. Then $w(t)>0$ a.e. and $w(0,b)$ is finite for all $b>0$ (in particular, $w\in L^1_{\mathrm{loc}}$).*

Proof. Let E be a bounded measurable set of positive Lebesgue measure. Then

$$N\chi_E(x)=\frac{|E|}{x}\quad\text{for } x\ge\sup E.$$

Using the weak-type (p,p), for $b\ge\sup E$ we have

$$\frac{|E|}{b}w(0,b)^{1/p}\le Cw(E)^{1/p}. \tag{2.1}$$

If $w(E) = 0$, it follows that $w(0,b) = 0$ for all $b > \sup E$, hence $w(0,+\infty) = 0$. On the other hand, since we can choose the right-hand side finite for some E (otherwise, $w(x) = \infty$ a.e.), the inequality also implies that $w(0,b)$ has to be finite for all b. □

In the proof of Theorem 1.1 we shall consider the maximal operator N_g associated to a fixed positive measurable function g. We define N_g as

$$N_g f(t) = \sup_{b>t} \frac{\int_0^b |f(x)|g(x)\,dx}{\int_0^b g(x)\,dx}.$$

The boundedness properties we need are in the following lemma.

Lemma 2.2. *Let g be a nonnegative measurable function such that $0 < g(0,b) < \infty$ for all $b > 0$.*

(i) *N_g is of weak type $(1,1)$ with respect to the measure $g(t)dt$. Actually,*

$$\int_{\{t:N_g(f)(t)>\lambda\}} g \le \frac{1}{\lambda}\int_{\{t:N_g(f)(t)>\lambda\}} |f|g \tag{2.2}$$

for all $\lambda > 0$ and all measurable functions f.

(ii) *N_g is of strong type (p,p), $1 < p < \infty$, with respect to the measure $g(t)dt$. More precisely, it holds that*

$$\int |N_g(f)|^p g \le (p')^p \int |f|^p g.$$

Proof. The proof of the lemma is straightforward. By standard interpolation arguments, it suffices to prove (i) since $||N_g(f)||_\infty \le ||f||_\infty$.

Observe that $N_g f$ is decreasing and continuous. Therefore, if $\{t : N_g(f)(t) > \lambda\}$ is not empty, then it is either a bounded interval $(0,d)$ or all of $(0,+\infty)$. In the first case it holds that

$$\lambda\int_0^d g(x)\,dx = \int_0^d |f(x)|g(x)\,dx, \tag{2.3}$$

whereas in the second case we have

$$\lambda\int_0^\infty g(x)\,dx \le \int_0^\infty |f(x)|g(x)\,dx.$$

Thus we obtain (2.2). Notice that if $g(0,+\infty) = +\infty$ and f is integrable with respect to g, only the first case is possible and the equality holds. □

We proceed to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us prove first the necessity of $A_{p,0}$ for the weak-type inequality. Let $E_k = \{x : w(x) > 1/k\}$ and $w_k = w\chi_{E_k}$. Take $f = w_k^{1-p'}\chi_{(0,b)}$. Then

$$N f(x) \geq \frac{1}{b}\int_0^b w_k^{1-p'} \tag{2.4}$$

for $0 < x < b$. Thus, $(0, b) \subset \{x : N f(x) > \lambda\}$ taking as λ the right-hand side of (2.4). If the weak type inequality holds,

$$\left(\frac{1}{b}\int_0^b w_k^{1-p'}\right)\left(\int_0^b w\right)^{1/p} \leq C\left(\int_0^b w_k^{(1-p')p} w\right)^{1/p},$$

and

$$\left(\frac{1}{b}\int_0^b w_k^{1-p'}\right)^{1/p'}\left(\int_0^b w\right)^{1/p} \leq C.$$

Letting k tend to infinity, $w \in A_{p,0}$ follows.

To show the necessity, arguing as in the proof of the preceding lemma, we have (2.3) with $g \equiv 1$, that is,

$$\lambda d = \int_0^d |f|.$$

Then

$$\lambda\left(\int_0^d w\right)^{1/p} \leq \frac{1}{d}\left(\int_0^d w\right)^{1/p}\left(\int_0^d |f|^p w\right)^{1/p}\left(\int_0^d w^{1-p'}\right)^{1/p'},$$

and (1.3) follows.

Using Lemma 2.2 the proof of the strong type with $A_{p,0}$ weights is a particular case of the results of Jawerth in [10]. The strikingly simple argument introduced by A. Lerner in [11] provides the dependence on the constant as stated in the theorem. Indeed, if $w \in A_{p,0}$, $0 < x < b$, and $\sigma = w^{1-p'}$ we have

$$\begin{aligned}\left(\frac{1}{b}\int_0^b f\right)^{p-1} &\leq [w]_{p,0}\left(\frac{\int_0^b f}{\int_0^b \sigma}\right)^{p-1}\frac{\int_0^b 1}{\int_0^b w}\\ &\leq [w]_{p,0}\frac{\int_0^b |N_\sigma(f\sigma^{-1})|^{p-1}}{\int_0^b w}\\ &\leq [w]_{p,0}N_w(w^{-1}|N_\sigma(f\sigma^{-1})|^{p-1})(x).\end{aligned}$$

Therefore,

$$|N f(x)|^p \leq ([w]_{p,0})^{p'}|N_w(w^{-1}|N_\sigma(f\sigma^{-1})|^{p-1})(x)|^{p'}$$

Consequently, using Lemma 2.2 twice,

$$\int |Nf|^p w \leq ([w]_{p,0})^{p'} p^{p'} \int |N_\sigma(f\sigma^{-1})|^p \sigma$$
$$\leq ([w]_{p,0})^{p'} p^{p'} (p')^p \int |f|^p w.$$

This ends the proof of the theorem. □

Remark 2.3. For the usual Hardy-Littlewood maximal operator, if it is of weak-type (p,p) with respect to a general measure μ, then the measure has to be absolutely continuous with respect to Lebesgue measure, that is, $d\mu(t) = w(t)dt$. For a proof, see [7, Theorem 6.1, Chapter VI]. Then the A_p condition on the density w characterizes all the weighted inequalities.

This is not the case for the operator N. For instance, the measure $d\mu(t) = dt + \delta_1$, where δ_1 is the Dirac mass at 1 serves as a counterexample. Indeed, using the fact that Nf is decreasing and N is bounded on L^p we have

$$\int_0^\infty |Nf(t)|^p dt + |Nf(1)|^p \leq 2 \int_0^\infty |Nf(t)|^p dt$$
$$\leq C \int_0^\infty |f(t)|^p dt \leq C \int_0^\infty |f(t)|^p d\mu(t).$$

We want to make it clear that in this paper we are considering only weighted inequalities with weights that are absolutely continuous with respect to Lebesgue measure.

3. Proof of the main theorem

Proof of Theorem 1.2. (a) Since $Nf(t) \leq Sf(t)$ for nonnegative f, it is enough to show that $w \in A_{p,0}$ is sufficient for the weak-type and the strong-type for S. Clearly, we only need the latter.

Let $w \in A_{p,0}$. Then $\sigma = w^{1-p'} \in A_{p',0}$. Since $Pf \leq Nf$, we deduce that P is bounded on $L^p(w)$ and on $L^{p'}(\sigma)$. By duality, Q is also bounded on $L^p(w)$ and so is S. On the other hand, using (1.4) we also have

$$||P||_{L^p(w)} \leq p' p^{p'-1} ([w]_{p,0})^{p'-1},$$

and

$$||Q||_{L^p(w)} = ||P||_{L^{p'}(\sigma)} \leq (p')^{p-1} p ([\sigma]_{p',0})^{p-1}.$$

Since $[\sigma]_{p',0} = ([w]_{p,0})^{p'-1}$, we have the right-hand side inequality of (1.5). The left-hand side inequality holds for $||N||_{L^p(w)}$, then also for $||S||_{L^p(w)}$.

(b) In the case $p=1$, we only need to show that $w\in\mathcal{C}_1$ is necessary for the strong-type of N and that $w\in A_{1,0}$ is sufficient for the weak-type of S.

Fix $b>0$ and let f be the characteristic function of $(b-\epsilon,b)$ for $\epsilon<b$. Then

$$Nf(t)\geq \epsilon\min\left(\frac{1}{b},\frac{1}{t}\right).$$

If N is bounded on $L^1(w)$, we have

$$\epsilon\left(\frac{1}{b}\int_0^b w+\int_b^\infty \frac{w(x)}{x}dx\right)\leq C\int_{b-\epsilon}^b w.$$

Dividing by ϵ and letting ϵ tend to zero, due to the local integrability of w (see Proposition 2.1) we obtain that $w\in\mathcal{C}_1$ as a consequence of Lebesgue's differentiation theorem.

The sufficiency of $w\in A_{1,0}$ for the weak-type of S is a consequence of the pointwise inequality $Pf(t)\leq Nf(t)$. On the one hand, the inequality implies that if N is of weak-type $(1,1)$, so is P. On the other hand, it also implies that w satisfies $Pw(x)\leq Cw(x)$ and, since P is the adjoint of Q, this implies that Q is bounded on $L^1(w)$. □

The usual extrapolation theorems (see, for instance, [3]) can be applied to the class of Calderón weights, taking either S or N as the positive operator from which we extrapolate. Results about extrapolation theorems for Calderón weights appear in [1]. Note that, unlike for the usual Hardy-Littlewood maximal operator, we obtain nontrivial L^1-weighted inequalities starting from L^p-weighted inequalities. But the boundedness on $L^1(w)$ will hold for $w\in\mathcal{C}_1$, not for $w\in A_{1,0}$.

Concerning factorization of C_p weights, we have two choices. For $1<p<\infty$, $w\in\mathcal{C}_p$ can be written as $w=w_0w_1^{1-p}$, where either $w_0,w_1\in\mathcal{C}_1$ or with $w_0,w_1\in\mathcal{A}_{1,0}$. Since $\mathcal{C}_1\subset\mathcal{A}_{1,0}$, the former choice is also of the latter form, but not conversely. For instance, $w\equiv 1$ is trivially factorized using $A_{1,0}$-weights because $1\in A_{1,0}$, but $1\notin\mathcal{C}_1$.

4. Properties of the Calderón weights

In this section we study some properties of the Calderón weights that are different from those of the usual A_p weights.

Proposition 4.1. *Let $1<p<\infty$. There exists $w\in\mathcal{C}_p$ ($w\in A_{p,0}$) such that $w\notin\mathcal{C}_q$ ($w\notin A_{q,0}$) for any $q<p$.*

Proof. For each natural number i, let $I_i = (2^i + 2^{-i}, 2^i + 1)$, let $\Omega = \cup_{i=1}^{\infty} I_i$ and let Ω^c be its complement. Then we define the weights

$$u(x) = \chi_{\Omega^c}(x) + \sum_{i=1}^{\infty} \frac{1}{(x - 2^i)^2} \chi_{I_i}(x) \quad \text{and} \quad w = u^{1-p}.$$

We shall show that w satisfies (1.2), but not its counterpart for $q < p$.

First we note that if β is negative, $u(x)^\beta \le 1$, so that for any integer $k \ge 1$,

$$2^k < |\Omega^c \cap (0, 2^{k+1})| \le \int_0^{2^{k+1}} u(x)^\beta \, dx < 2^{k+1}. \tag{4.1}$$

For $\alpha \ge 0$ we have

$$\int_{I_i} u(x)^{1+\alpha} \, dx = \frac{2^{i(2\alpha+1)} - 1}{2\alpha + 1}$$

so that

$$c_\alpha 2^{k(2\alpha+1)} \le \sum_{i=1}^{k} \int_{I_i} u(x)^{1+\alpha} \, dx \le C_\alpha 2^{k(2\alpha+1)}.$$

Since $u(x) = 1$ on Ω^c, with different constants c_α and C_α we also have

$$c_\alpha 2^{k(2\alpha+1)} \le \int_0^{2^{k+1}} u(x)^{1+\alpha} \, dx \le C_\alpha 2^{k(2\alpha+1)}. \tag{4.2}$$

Let us check now that w satisfies (1.2). If $b \le 2$ the inequality is obvious with constant $C = 1$ since $w(x) = 1$ in the interval $(0, b)$. Let $b > 2$ and let us choose the natural number k such that $2^k < b \le 2^{k+1}$. Then

$$\left(\int_0^b u^{1-p} \right)^{\frac{1}{p}} \left(\int_0^b u \right)^{\frac{1}{p'}} \le (2^{k+1})^{1/p} (C 2^{k+1})^{1/p'} \le Cb,$$

using (4.1) for the first integrand and (4.2) for the second.

Let us see that w does not satisfy $A_{q,0}$ for $q < p$. Let $k \ge 1$. From (4.2) we have

$$\frac{1}{2^{k+1}} \int_0^{2^{k+1}} u(x)^{(1-p)(1-q')} \, dx \ge c_{p,q} 2^{2k(p-q)/(q-1)}.$$

where $c_{p,q}$ is positive and depends only on p and q. From (4.1) we have

$$\int_0^{2^{k+1}} u(x)^{1-p} \, dx > 2^k.$$

Therefore

$$\left(\frac{1}{2^{k+1}}\int_0^{2^{k+1}} u(x)^{1-p}\,dx\right)^{\frac{1}{q}}\left(\frac{1}{2^{k+1}}\int_0^{2^{k+1}} u(x)^{(1-p)(1-q')}\,dx\right)^{\frac{1}{q'}} \geq \widetilde{c}_{p,q}2^{2k(p-q)/(q)}.$$

Since the right-hand side tends to infinity with k for $q < p$, w does not belong to $\mathcal{C}_q$. □

The Muckenhoupt basis without the openness property shown in [3, Chapter 3] consists of a single set. The basis considered here is composed of infinitely many sets. We find this setting suitable to test the validity of an open question on extrapolation of weighted norm inequalities proposed in [3]. Let us first reproduce the result that motivates the question. We give the statement for operators rather than for pairs of functions.

Proposition 4.2 ([3, Proposition 3.21]). *Let $\mathcal{B}$ be a Muckenhoupt basis which is $A_{p,\mathcal{B}}$ open. The following are equivalent:*

(a) There exists $r_0 > 1$ such that for all $r \in (1, r_0)$ and all $w \in A_{1,\mathcal{B}}$, T is bounded on $L^r(w)$.

(b) For all $p \in (1, +\infty)$ and all $w \in A_{p,\mathcal{B}}$, T is bounded on $L^p(w)$.

The open question proposed in [3] is whether such equivalence holds for bases which are not $A_{p,\mathcal{B}}$ open. Our following result provides an answer in the negative.

Proposition 4.3. *There exists an operator T bounded on $L^r(w)$ for all $w \in A_{1,0}$ and all $r \in (1, +\infty)$ but such that for each $p \in (1, +\infty)$ there exists $w \in A_{p,0}$ for which T is unbounded on $L^p(w)$.*

Proof. Let us define T as

$$Tf(t) = \inf_{0<s<1} 2^{1/s}\; N_{1+s}f(t), \tag{4.3}$$

where $N_q f(t) = N(|f|^q)(t)^{1/q}$.

Let $r > 1$. Choose $s_0 = \min(1, (r-1)/2)$. Then N_{1+s_0} is bounded on $L^r(w)$ for $w \in A_{1,0} \subset A_{r/(1+s_0),0}$, with constant depending only on r and the $A_{1,0}$-constant of w. Since $Tf(t) \leq 2^{1/s_0}\; N_{1+s_0}f(t)$, the same holds for T.

Let $p > 1$. We consider the weight $w = u^{1-p}$ defined in (4.4), for which we know that w is in $A_{p,0}$, but not in $A_{q,0}$ for $q < p$. Our objective is to see that T is not bounded on $L^p(w)$.

Let I_i and Ω be as in the proof of Proposition 4.1. Let us define

$$f(t)=\sum_{i=10}^{\infty}\ \frac{1}{2^{i/p}\ i}\ \frac{1}{(t-2^i)^2}\chi_{I_i}(t).$$

To check that $f\in L^p(w)$, notice that

$$f(t)^p w(t)=\sum_{i=10}^{\infty}\frac{1}{2^i\ i^p}\ \frac{1}{(t-2^i)^2}\chi_{I_i}(t),$$

and hence

$$\int_0^\infty f(t)^p w(t)\,dt\le\sum_{i=10}^{\infty}\frac{1}{i^p}<\infty.$$

Let us seek a suitable lower bound for $Tf(t)$. For $0<s<1$ we have

$$\begin{aligned}\frac{1}{2^{k+1}}\int_0^{2^{k+1}} f(x)^{1+s}dx &\ge \frac{1}{2^{k+1}}\frac{1}{(2^{k/p}\ k)^{1+s}}\ \int_{I_k}\ \frac{dx}{(x-2^k)^{2(1+s)}}\\ &\ge\frac{2^{k(1+2s)-1}}{2^{k+1}(2^{k/p}\ k)^{1+s}(1+2s)}\ge\frac{2^{2ks}}{12(2^{k/p}\ k)^{1+s}}.\end{aligned}$$

Therefore, if $t\in(0,2^{k+1})$,

$$2^{1/s}\ N_{1+s}f(t)\ge 2^{1/s}\left(\frac{1}{2^{k+1}}\int_0^{2^{k+1}} f(x)^{1+s}dx\right)^{\frac{1}{1+s}}\ge\frac{2^{1/s}2^{2ks/(1+s)}}{12(2^{k/p}k)}.$$

Thus

$$Tf(t)\ge\frac{1}{12(2^{k/p}k)}\inf_{0<s<1}2^{1/s}2^{ks}\ge C2^{2\sqrt{k}}(2^{k/p}k)^{-1},$$

for $t\in(0,2^{k+1})$.

Since $w\equiv 1$ in Ω^c and $\left|\Omega^c\cap(0,2^{k+1})\right|>2^k$, we have

$$\int_0^{2^{k+1}}(Tf)^p\,w\ge\int_{\Omega^c\cap(0,2^{k+1})}(Tf)^p\,w\ge C2^{2p\sqrt{k}}k^{-p}.$$

As the last term may be made arbitrarily large with k, we conclude that $Tf\notin L^p(w)$. □

The characterization of the weights in $\mathcal{C}_p$ by (1.2) allows us to prove that they satisfy (1.6), which is a reverse Jensen inequality. This can be done as for the usual A_p weights (see [6, page 405]). Nevertheless, in the case of the Calderón weights we have the following negative result.

Proposition 4.4. *There exists a weight $w\in\mathcal{C}_{\exp}$ such that w does not belong to $\mathcal{C}_p$ for any p.*

Proof. For each natural number i, let $I_i = (2^i, 2^i + 1)$, let $\Omega = \cup_{i=1}^{\infty} I_i$ and let Ω^c be its complement. Then we define the weight

$$w(x) = \chi_{\Omega^c}(x) + \sum_{i=1}^{\infty} (x - 2^i)^i \chi_{I_i}(x). \tag{4.4}$$

We first notice that

$$\frac{1}{2} \le \frac{1}{b} \int_0^b w \le 1 \tag{4.5}$$

for all $b > 0$. Indeed, the second inequality holds because $w(x) \le 1$ for all x, and the first one because $w(x) = 1$ on the set $\Omega^c \cap (0, b)$ whose length is larger than $b/2$.

On the other hand, since $\log w(x) < 0$ on Ω and $\log w(x) = 0$ on Ω^c, for $b \in (2^k, 2^{k+1}]$ we have

$$\int_0^b \log w(x)\, dx \ge \sum_{i=1}^{k} i \int_0^1 \log x\, dx = -\frac{k(k+1)}{2}.$$

Then

$$\frac{1}{b} \int_0^b \log w \ge \min_k -\frac{k(k+1)}{2^{k+1}} = -\frac{3}{4}.$$

This inequality together with (4.5) gives that $w \in \mathcal{C}_{\exp}$.

It remains to check that $w \notin \mathcal{C}_p$ for any $p > 1$. But this is immediate beacuse $w^{1-p'}$ is not integrable on intervals of the form $(0, b)$ for sufficiently large b. □

In spite of the result Proposition 4.4, the class $\mathcal{C}_{\exp}$ plays the same role as the corresponding class $A_{\exp}$ for the usual Muckenhoupt weights in the following characterization: $w \in \mathcal{C}_p$ if and only if w and $w^{1-p'}$ belong to $\mathcal{C}_{\exp}$. The proof is straightforward and we do not include it.

Remark 4.5. (a) A consequence of Proposition 4.1 is that there are weights in the $\mathcal{C}_p$ classes for which the reverse Hölder inequality

$$\left(\frac{1}{b} \int_0^b u^{1+\alpha} \right)^{1/(1+\alpha)} \le \frac{C}{b} \int_0^b u$$

does not hold for any $\alpha > 0$. Indeed, it is obvious that if w satisfies (1.2), the reverse Hölder inequality for $w^{1-p'}$ implies that w is in $A_{q,0}$ for q given by $(q' - 1)(1 + \alpha) = p' - 1$. (We shall make use of this result in Proposition 4.6.) Actually, rather than using an indirect argument, we can propose a precise counterexample using the weight u defined in the proof of Proposition 4.1. Indeed, it is in $\mathcal{C}_p$ for any $p \in (1,\infty)$ and using (4.2) we obtain

$$c_\alpha 2^{2k\alpha/(1+\alpha)} \le \left(\frac{1}{2^{k+1}} \int_0^{2^{k+1}} u(x)^{1+\alpha}\, dx \right)^{1/(1+\alpha)}$$

and

$$\frac{1}{2^{k+1}} \int_0^{2^{k+1}} u(x)\, dx \le C_\alpha,$$

which are incompatible with a reverse Hölder inequality.

(b) On the other hand, the weight w considered in the proof of Proposition 4.4 shows that there exist weights satisfying a reverse Hölder inequality which do not belong to $\cup_p \mathcal{C}_p$. The reverse Hölder inequality for w is obvious because $w^{1+\alpha}$ also satisfies (4.5).

The Coifman-Rochberg characterization of A_1 weights says that $w \in A_1$ if and only if $w(t) = Mg(t)^\theta k(t)$, where Mg is finite almost everywhere, $\theta \in [0,1)$, and k and k^{-1} are bounded (see [2]). The corresponding result for $A_{1,0}$ with N instead of M satisfies the if part of the statement, which can be proved as for A_1, but not necessarily the only if part. Indeed, $w(t) = |1-t|^{-1/2}$ is in $A_{1,0}$, but cannot be equal to $Ng(t)^\theta k(t)$, which is bounded on $(1/2,\infty)$, because $Ng(t)$ is decreasing.

Our next result gives some sufficient conditions on a weight which ensure that it has the oppeness property.

Proposition 4.6. *Let $1 < p < \infty$. Let $\rho \ge 0$ and $w \in \mathcal{C}_p$ such that either $w(t)t^\rho$ is a nondecreasing function or $w(t)t^{-\rho}$ is a nonincreasing function. Then there exists q, $1 < q < p$, such that $w \in \mathcal{C}_q$.*

In the proof of this proposition we shall use the following remark.

Remark 4.7. Taking $E = (b/2, b)$ and $E = (0, b/2)$ in (2.1) we deduce that $w \in \mathcal{C}_p$ satisfies the following doubling conditions:

$$w(0,b) \le (2C)^p w(b/2,b) \quad \text{and} \quad w(0,b) \le (2C)^p w(0,b/2), \tag{4.6}$$

for all $b > 0$. Therefore, for $A = 1 - 1/(2C)^p < 1$ it holds that

$$w(0,b/2) \le A w(0,b).$$

By iteration it follows that

$$\int_0^{b/2^j} w(x)\, dx \le A^j \int_0^b w(x)\, dx \tag{4.7}$$

for all $b > 0$ and all nonnegative integers k.

Furthermore, it can be seen that $w \in \mathcal{C}_p$ if and only if $w \in M_p$ and w satisfies the first inequality in (4.6) (and also if and only if $w \in M^p$ and w satisfies the second inequality in (4.6)).

Proof of Proposition 4.6. We shall assume that $w(t)t^{-\rho}$ is a nonincreasing function. If we have the other hypothesis, we can use a similar argument or, alternatively, we can obtain the conclusion as a consequence of Theorem 2 in [15].

We already mentioned in Remark 4.5 that the openness property is an immediate consequence of a reverse Hölder inequality for $w^{1-p'}$, namely, there exist $\alpha > 0$ and $C > 0$ such that

$$\frac{1}{b}\int_0^b w^{(1+\alpha)(1-p')} \le C\left(\frac{1}{b}\int_0^b w^{1-p'}\right)^{1+\alpha}. \tag{4.8}$$

Let $b > 0$ given and let $\alpha > 0$ to be chosen below. We set $b_j = b/2^j$, and observe that $b_j/2 = b_{j+1} = b_j - b_{j+1}$. Taking into account that $(w(t)t^{-\rho})^{(1+\alpha)(1-p')}$ is a nondecreasing function we have that

$$\begin{aligned}\int_0^b w^{(1+\alpha)(1-p')} &= \sum_{j=0}^{\infty}\int_{b_{j+1}}^{b_j}(w\,(x)\,x^{-\rho})^{(1+\alpha)(1-p')}\; x^{\rho(1+\alpha)(1-p')}dx\\ &\le \sum_{j=0}^{\infty}(w\,(b_j)\,b_j^{-\rho})^{(1+\alpha)(1-p')}\; b_{j+1}^{\rho(1+\alpha)(1-p')+1}.\end{aligned}$$

Using again the same property,

$$\begin{aligned}\int_0^b w^{(1+\alpha)(1-p')} &\le \sum_{j=0}^{\infty}\left(\frac{1}{b_j}\int_{b_j}^{b_{j-1}}(w\,(x)\,x^{-\rho})^{(1-p')}dx\right)^{1+\alpha} b_{j+1}^{\rho(1+\alpha)(1-p')+1}\\ &\le \sum_{j=0}^{\infty}\left(\frac{b_{j-1}^{-\rho(1-p')}}{b_j}\int_{b_j}^{b_{j-1}} w^{1-p'}\right)^{1+\alpha} b_j^{\rho(1+\alpha)(1-p')+1}\\ &= \; C(p,\alpha,\rho)\sum_{j=0}^{\infty}\left(\int_{b_j}^{b_{j-1}} w^{1-p'}\right)^{1+\alpha} b_j^{-\alpha}.\end{aligned}$$

Since $w^{1-p'} \in \mathcal{C}_{p'}$, we have that this weight satisfies the doubling conditions (4.6) and (4.7). Therefore, there exist $C > 0$ and a constant γ such that $0 < \gamma < 1$ and

$$\int_{b_j}^{b_{j-1}} w^{1-p'} \le C\,\gamma^j\;\int_0^b w^{1-p'}.$$

Consequently,

$$\int_0^b w^{(1+\alpha)(1-p')} \leq C \left(\int_0^b w^{1-p'}\right)^{1+\alpha} b^{-\alpha} \sum_{j=0}^{\infty} \left(\gamma^{1+\alpha} 2^{\alpha}\right)^j .$$

Choosing $\alpha > 0$ such that $\gamma^{1+\alpha}2^{\alpha} < 1$, and combining the previous estimates we obtain (4.8) for such α. □

5. Proof of Theorem 1.3

Case $1 < p < \infty$. By using Theorem 1.2, the inequality $S_\alpha f \leq Sf$ (for $f \geq 0$), and the fact that the boundedness on $L^p(w)$ implies the weak-type, it is clear that we only need to prove that (d) implies (a).

Let $b > 0$. Set $E_k = \{x : w(x) > 1/k\}$ and $w_k = w\chi_{E_k}$. Take $f = w_k^{1-p'}\chi_{(0,b)}$. Then for $t \in (2b, 3b)$,

$$S_\alpha f(t) = I_\alpha f(t) \geq \frac{1}{(3b)^{\alpha+1}} \int_0^b (t-x)^\alpha w_k^{1-p'}(x)dx \geq \frac{w_k^{1-p'}(0,b)}{3^{\alpha+1}b}.$$

Since (d) holds we have

$$\frac{w_k^{1-p'}(0,b)}{3^{\alpha+1}b}\, w(2b,3b)^{1/p} \leq C w_k^{1-p'}(0,b)^{1/p}. \tag{5.1}$$

Take now $f = \chi_{(2b,3b)}$. For $t \in (0,b)$ we have

$$S_\alpha f(t) = J_\alpha f(t) \geq \int_{2b}^{3b} \frac{(x-t)^\alpha}{x^{\alpha+1}}\, dx \geq \frac{1}{3^{\alpha+1}}.$$

Using again (d) we get

$$\frac{w(0,b)^{1/p}}{3^{\alpha+1}} \leq C w(2b,3b)^{1/p}. \tag{5.2}$$

Combining (5.1) and (5.2), and letting k tend to infinity, we get the condition $A_{p,0}$.

Case $p = 1$. We need to prove that (b) implies $w \in \mathcal{C}_1$ and that (d) implies $w \in A_{1,0}$. We start with the latter.

Given a weight w and $b > 0$, for $\epsilon > 0$ we consider

$$E = \{x \leq b : w(x) \leq \inf_{t\in(0,b)} w(t) + \epsilon\},$$

which has positive Lebesgue measure. If $f = \chi_E$, for $t \in (2b, 3b)$ we have

$$S_\alpha f(t) \geq \frac{|E|}{3^{\alpha+1}b}.$$

Using the weak-type $(1,1)$ we have

$$\frac{|E|}{3^{\alpha+1}b}\, w(2b,3b) \le Cw(E) \le C(\inf_{t\in(0,b)} w(t)+\epsilon)|E|.$$

Letting ϵ tend to zero we get

$$\frac{w(2b,3b)}{b} \le 3^{\alpha+1}C \inf_{t\in(0,b)} w(t). \tag{5.3}$$

On the other hand, (5.2) is true for $p=1$ if we have the weak-type $(1,1)$. This together with (5.3) shows that $w\in A_{1,0}$.

Finally we prove that (b) implies $w\in\mathcal{C}_1$. More precisely, we shall prove that (b) implies the pointwise inequality $Sw(t)\le CS_\alpha w(t)$ for some constant depending only on α. Since (b) is characterized by $S_\alpha w(t)\le Cw(t)$ a.e., we obtain $w\in\mathcal{C}_1$.

Fix $b>0$ and let $f(x)=b^{-1}\chi_{(b/4,b/2)}(x)$. Then $S_\alpha f(x)\ge c_1/b$ for $x\le b$ and

$$S_\alpha f(x) = I_\alpha f(x) \ge \frac{c_2}{x} \quad \text{for } x>b.$$

Taking $c_\alpha=\min(c_1,c_2)$ and using (b) we obtain

$$c_\alpha Sw(b) \le \frac{C}{b}\int_{b/4}^{b/2} w(x) \le C_\alpha S_\alpha w(b).$$

The theorem is fully proved.

Remark 5.1. We have written directly the proof of both cases, $p=1$ and $p>1$, because of their simplicity. Alternatively, one could use an extrapolation argument and avoid part of the proof. Indeed, let us denote as $W_p(S_\alpha)$ the class of weights for which S_α is bounded on $L^p(w)$. By extrapolation, if an operator T is bounded on $L^{p_0}(w)$ for some $p_0\in[1,\infty)$ and for all weights $w\in W_{p_0}(S_\alpha)$, then T is bounded on $L^p(w)$ for all $1\le p<\infty$ and for all weights $w\in W_p(S_\alpha)$. Using this extrapolation theorem, the result for $p>1$ is a consequence of the case $p=1$.

6. Higher dimensional results

A higher dimensional analogue of the operator N appears in [5] in a different context, together with its fractional version. It is defined for locally integrable functions in $\mathbb{R}^n$ as

$$Nf(x) = \sup_{r\ge|x|} \frac{1}{r^n}\int_{|y|<r} |f(y)|dy.$$

This operator is up to a constant factor the maximal operator associated to the basis of $\mathbb{R}^n$ formed by the Euclidean balls centered at the origin. The corresponding $A_{p,0}$ classes are defined accordingly.

In a similar way, we can define the higher dimensional analogues of the Hardy operator P and its adjoint Q as

$$Pf(x) = \frac{1}{|x|^n} \int_{|y|\leq|x|} f(y)dy \quad \text{and} \quad Qf(x) = \int_{|y|\leq|x|} \frac{f(y)}{|y|^n} dy.$$

Conditions for the boundedness of P and Q on $L^p(w)$ analogous to M_p and M^p are proved in [4]. It is easy to check that all our results concerning N and $S = P + Q$ can be carried out to the higher dimensional setting.

Note that Nf, Pf and Qf are always radial functions, regardless of whether f is radial or not. If, moreover, f is radial, $f(x) = f_0(|x|)$, the n-dimensional operators N, P and Q acting on f and evaluated at x coincide up to a constant factor with the corresponding operators on $(0, +\infty)$ acting on $f_0(t^{1/n})$ and evaluated at $|x|^n$. This is obtained by a simple change of variable and the details are left to the reader.

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