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DISTRIBUTIVE NEARLATTICES WITH A NECESSITY MODAL OPERATOR

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ABSTRACT. The aim of this paper is to study the class of distributive nearlattices with a necessity modal operator. We develop a full duality to the category of distributive nearlattices whose morphisms are applications that preserving the infimum when exists and, as special case, we obtain a representation and duality for distributive nearlattices with a necessity modal operator. We study certain particular subclasses and give some applications.

1. Introduction and preliminaries

The Tarski algebras are the algebraic counterpart of the $\{\rightarrow\}$-fragment of the classical propositional logic. In [1], Abbott establishes a correspondence between Tarski algebras and join-semilattices with greatest element which every principal filter is a Boolean lattice. A natural generalization of the Tarski algebras are the nearlattices: join-semilattices with greatest element which every principal filter is a bounded lattice. A special class are the distributive nearlattices. This algebras have been studied by different authors. We refer the reader to the following bibliography: [18], [21], [12], [13], [14], [20] and [5]. A full duality between distributive nearlattices with greatest element and certain topological spaces was developed in [9], extending the results given in [8] for Tarski algebras and the duality of Stone for bounded distributive lattices [22].

In [6], the author studies the algebraic semantics of the $\{\rightarrow, \Box\}$-fragment of the classical modal logic introducing the class of Tarski algebras with a modal operator as a generalization of the Boolean algebras with a modal operator. Since the distributive nearlattices are a generalization of the Tarski algebras, it is natural to extend some of these results. The main purpose of this paper is to study the class of distributive nearlattices with a necessity modal operator $\Box$. As in a nearlattice the infimum of any two elements automatically exists when they share a common lower bound, we define the notion of modal operator as an operator $\Box$ that preserves existing finite infimum. Recall that a modal operator in a Boolean algebra $\mathbf{B}$ can be considered as a meet-homomorphism defined in $\mathbf{B}$. In [19], Halmos works with join-homomorphisms which generalize the modal operator $\Diamond$, while meet-homomorphisms generalize the modal operator $\Box$. In the same way we can define a modal operator $\Box$ in a nearlattice $\mathbf{A}$ as a particular class of function. Taking into account this observation we introduce the notion of $\wedge$-semi-homomorphisms between distributive nearlattices.

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Consequently, a distributive nearlattice with a necessity modal operator is a distributive nearlattice endowed with a $\wedge$-semi-homomorphism.

In the remaining part of this section we review some definitions and the topological duality developed in [9]. In Section 2 we define the category of distributive nearlattices with $\wedge$-semi-homomorphisms and we prove that there exists a correspondence between $\wedge$-semi-homomorphisms of distributive nearlattices and certain binary relations. In Section 3 we introduce the distributive nearlattices with a necessity modal operator $\Box$, or $\Box$-distributive nearlattices. We develop a topological representation and duality for $\Box$-distributive nearlattices using the simplified representation given in the Section 2. In Section 4 we study certain classes of $\Box$-distributive nearlattices, in particular, the variety of **S5**-nearlattices which is a generalization of the class of monadic algebras. Finally, in Section 5, we apply the duality to characterize the lattices of the congruences, subalgebras and the free $\Box$-distributive lattice extension of a $\Box$-distributive nearlattice.

Let $\mathbf{A} = \langle A, \vee, 1\rangle$ be a join-semilattice with greatest element, or simply semilattice. Recall that the binary relation $\leq$ defined by $x \leq y$ if and only if $x \vee y = y$ is a partial order. A filter of $\mathbf{A}$ is a non-empty subset $F \subseteq A$ such that $1 \in F$, if $x \leq y$ and $x \in F$ then $y \in F$, and if $x, y \in F$ then $x \wedge y \in F$, whenever $x \wedge y$ exists. The intersection of any collection of filters is a filter. For any non-empty subset $X \subseteq A$, the filter generated by X, in symbols $F(X)$, is the smallest filter containing X. A filter F is said to be finitely generated if $F = F(X)$ for some finite non-empty subset X. If $X = \{a\}$ then $F(\{a\}) = [a) = \{x \in A \colon a \leq x\}$, called the principal filter of a. Let us denote by $\mathrm{Fi}(\mathbf{A})$ and $\mathrm{Fi}_f(\mathbf{A})$ the set of all filters and filters finitely generated of $\mathbf{A}$, respectively. A subset $I \subseteq A$ is called an ideal if for every $x, y \in A$, if $x \leq y$ and $y \in I$ then $x \in I$, and for all $x, y \in I$ then $x \vee y \in I$. The smallest ideal containing X is called ideal generated by X and will be denoted by $I(X)$. A non-empty proper ideal P is prime if for every $x, y \in A$, if $x \wedge y \in P$, whenever $x \wedge y$ exists, then $x \in P$ or $y \in P$. The set of all ideals and prime ideals of $\mathbf{A}$ will be denoted by $\mathrm{Id}(\mathbf{A})$ and $X(\mathbf{A})$, respectively.

Definition 1. Let $\mathbf{A}$ be a semilattice. We say that $\mathbf{A}$ is a *nearlattice* if each principal filter is a bounded lattice with respect to the induced order.

The meet is not everywhere defined and the class of nearlattices consists of partial algebras only. However, nearlattices can be regarded as total algebras through a ternary operation. Hickman in [21] and Chajda and Kolařík in [14] proved that the nearlattices form a variety. In [3] the authors found a smaller equational base.

Lemma 1.1 ([3]). *Let $\mathbf{A}$ be a nearlattice. Let $m\colon A^3 \to A$ be the ternary operation given by $m(x, y, z) = (x \vee z) \wedge_z (y \vee z)$. The following identities are satisfied:*

(1) $m(x, y, x) = x$,

(2) $m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))$,

(3) $m(x, x, 1) = 1$.

Conversely, let $\mathbf{A} = \langle A, m, 1\rangle$ be an algebra of type $(3, 0)$ satisfying the identities (1)–(3). If we define $x \vee y = m(x, x, y)$ then $\mathbf{A}$ is a semilattice. Moreover, for each $a \in A$, $[a)$ is a bounded lattice where for $x, y \in [a)$ their infimum is $x \wedge_a y = m(x, y, a)$. Hence, $\mathbf{A}$ is a nearlattice.

We are interested in a particular class of nearlattices.

Definition 2. Let $\mathbf{A}$ be a nearlattice. We say that $\mathbf{A}$ is *distributive* if each principal filter is a bounded distributive lattice.

***Example* 1.** A Tarski algebra can be defined as a join-semilattice with greatest element such that each principal filter is a Boolean lattice with respect to the induced order. Indeed, if $\langle A, \to, 1\rangle$ is a Tarski algebra then the join of two elements x and y is given by $x \vee y = (x \to y) \to y$ and for each $a \in A$, $[a) = \{x \in A \colon a \leq x\}$ is a Boolean lattice where for $x, y \in [a)$ the meet is given by $x \wedge_a y = (x \to (y \to a)) \to a$ and $x \to a$ is the complement of x in $[a)$. Thus, every Tarski algebra is in particular a distributive nearlattice. For more details see [1] and [2].

We denote by $\mathcal{DN}$ the variety of distributive nearlattices. From the results given in [18], if $\mathbf{A} \in \mathcal{DN}$ then we have the following characterization of the filter generated by a non-empty subset X of A:

$$F(X) = \{a \in A \colon \exists x_1, \ldots, x_n \in [X), \exists x_1 \wedge \cdots \wedge x_n \ (x_1 \wedge \cdots \wedge x_n = a)\}.$$

Theorem 1.1 ([20]). *Let $\mathbf{A} \in \mathcal{DN}$. Let $I \in \mathrm{Id}(\mathbf{A})$ and let $F \in \mathrm{Fi}(\mathbf{A})$ such that $I \cap F = \emptyset$. Then there exists $P \in X(\mathbf{A})$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

The Stone's representation theorem for distributive lattices states that every distributive lattice is isomorphic to a ring of sets. We recall that a ring of sets $\mathcal{A}$ is a family of subsets of a set X that is closed under the operations of unions and intersections. These results are the basis for topological dualities developed by Stone in [22]. For distributive nearlattices we have a similar representation theorem proved by Halaš [20].

Theorem 1.2 ([12]). *Let $\mathbf{A} \in \mathcal{DN}$. Then the map $\varphi_{\mathbf{A}} \colon A \to \mathcal{P}_d(X(\mathbf{A}))$ defined by $\varphi_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) : a \notin P\}$ is an embedding of $\mathbf{A}$ into $\mathcal{P}_d(X(\mathbf{A}))$. Thus, $\mathbf{A}$ is isomorphic to the subalgebra $\varphi_{\mathbf{A}}[\mathbf{A}] = \{\varphi_{\mathbf{A}}(a) \colon a \in A\}$ of $\mathcal{P}_d(X(\mathbf{A}))$.*

Recall the Stone style duality for distributive nearlattices developed in [9]. First we give some topological concepts. A topological space with a base $\mathcal{K}$ will be denoted by $\langle X, \mathcal{K}\rangle$. We consider the family

$$D_{\mathcal{K}}(X) = \{U \colon U^c \in \mathcal{K}\}.$$

A subset $Y \subseteq X$ is basic saturated if $Y = \bigcap\{U_i \in \mathcal{K} \colon Y \subseteq U_i\}$. The basic saturation of Y is the smallest basic saturated containing Y and will be denoted by $\mathrm{Sb}(Y)$. Analogously, the closure of Y is the smallest closed containing Y and will be denoted by $\mathrm{Cl}(Y)$. If $Y = \{y\}$ we write $\mathrm{Sb}(\{y\}) = \mathrm{Sb}(y)$ and $\mathrm{Cl}(\{y\}) = \mathrm{Cl}(y)$. On X is defined a binary relation $\leq$ as $x \leq y$ if and only if $y \in \mathrm{Sb}(x)$. It is easy to see that the relation $\leq$ is a partial order if and only if $\langle X, \mathcal{K}\rangle$ is T_0. The dual order of $\leq$ is denoted by $\preceq$, i.e., $x \preceq y$ if and only if $x \in \mathrm{Sb}(y)$. Note that $x \preceq y$ if and only if $y \in \mathrm{Cl}(x)$. The order $\leq$ and the dual order $\preceq$ will be indexed by the set where it is used. It follows that if $U \in D_{\mathcal{K}}(X)$ then U is increasing with respect to the dual order. We say that $Y \subseteq X$ is irreducible if for every $U, V \in D_{\mathcal{K}}(X)$ such that $U \cap V \in D_{\mathcal{K}}(X)$ and $Y \cap (U \cap V) = \emptyset$ implies $Y \cap U = \emptyset$ or $Y \cap V = \emptyset$ and we say that Y is dually compact if for every family $\mathcal{F} = \{U_i \colon i \in I\} \subseteq \mathcal{K}$ such that $\bigcap\{U_i \colon i \in I\} \subseteq Y$ implies that there exists a finite family $\{U_1, \ldots, U_n\}$ of $\mathcal{F}$ such that $U_1 \cap \cdots \cap U_n \subseteq Y$.

Definition 3. Let $\langle X, \mathcal{K}\rangle$ be a topological space. Then $\langle X, \mathcal{K}\rangle$ is an *N-space* if:

(1) $\mathcal{K}$ is a basis of open, compact and dually compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X.
(2) For every $U, V, W \in \mathcal{K}$, $(U \cap W) \cup (V \cap W) \in \mathcal{K}$.
(3) For every irreducible basic saturated subset Y of X there exists a unique $x \in X$ such that $\mathrm{Sb}(x) = Y$.

Theorem 1.3 ([9]). *Let $\langle X, \mathcal{K}\rangle$ be a topological space where $\mathcal{K}$ is a basis of open and compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X. Suppose that $(U \cap W) \cup (V \cap W) \in \mathcal{K}$ for all $U, V, W \in \mathcal{K}$. The following conditions are equivalent:*

(1) *$\langle X, \mathcal{K}\rangle$ is T_0, and if $A = \{U_i \colon i \in I\}$ and $B = \{V_j \colon j \in J\}$ are non-empty families of $D_{\mathcal{K}}(X)$ such that $\bigcap\{U_i \colon i \in I\} \subseteq \bigcup\{V_j \colon j \in J\}$ then there exist $U_1, \ldots, U_n \in [A)$ and $V_1, \ldots, V_k \in B$ such that $U_1 \cap \cdots \cap U_n \in D_{\mathcal{K}}(X)$ and $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_k$.*

(2) *$\langle X, \mathcal{K}\rangle$ is T_0, every $U \in \mathcal{K}$ is dually compact and the assignment $H_X\colon X \to X(D_{\mathcal{K}}(X))$ defined by*

$$H_X(x) = \{U \in D_{\mathcal{K}}(X)\colon x \notin U\},$$

for each $x \in X$, is onto.

(3) *Every $U \in \mathcal{K}$ is dually compact and for every irreducible basic saturated subset Y of X there exists a unique $x \in X$ such that $Y = \mathrm{Sb}(x)$.*

If $\langle X, \mathcal{K}\rangle$ is an N-space then $\langle D_{\mathcal{K}}(X), \cup, X\rangle$ is a distributive nearlattice. So, the map H_X defined in the Theorem 1.3 is an homeomorphism such that $x \leq y$ if and only if $H_X(x) \subseteq H_X(y)$. Conversely, if $\mathbf{A} \in \mathcal{DN}$ then the family $\mathcal{K}_{\mathbf{A}} = \{\varphi_{\mathbf{A}}(a)^c\colon a \in A\} \subseteq \mathcal{P}_d(X(\mathbf{A}))$ is a basis for a topology $\mathcal{T}_{\mathcal{K}_{\mathbf{A}}}$ on $X(\mathbf{A})$. The topological space $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle$ is an N-space called the dual space of $\mathbf{A}$. For more details see [9].

2. $\wedge$-semi-homomorphisms

In this section we define the notion of $\wedge$-semi-homomorphism between distributive nearlattices which we use in the following section to present a representation and duality for distributive nearlattices with a necessity modal operator.

Definition 4. Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$. Let $h\colon A \to B$ be a map. We say that h is a *$\wedge$-semi-homomorphism* if it verifies the following conditions:

(1) $h(1) = 1$,

(2) $h(a \wedge b) = h(a) \wedge h(b)$ for all $a, b \in A$ such that $a \wedge b$ exists.

Let us denote by $\mathcal{SDN}_{\wedge}[\mathbf{A}, \mathbf{B}]$ the set of all $\wedge$-semi-homomorphisms from $\mathbf{A}$ into $\mathbf{B}$.

Remark 1. Note that if $h \in \mathcal{SDN}_{\wedge}[\mathbf{A}, \mathbf{B}]$ then h preserves the natural order. On the other hand, if $a, b \in A$ such that $a \wedge b$ exists then $h(a) \wedge h(b)$ exists. Indeed, since $a \wedge b \leq a, b$ then $h(a), h(b) \in [h(a \wedge b))$ and as $\mathbf{B}$ is a nearlattice, $h(a) \wedge h(b)$ exists.

We denote by $\mathcal{SDN}_{\wedge}$ the category whose objects are distributive nearlattices and morphisms $\wedge$-semi-homomorphisms between them.

Lemma 2.1. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$. Let $h\colon A \to B$ be a map. The following conditions are equivalents:*

(1) $h \in \mathcal{SDN}_{\wedge}[\mathbf{A}, \mathbf{B}]$.

(2) $h^{-1}(P)^c \in \mathrm{Fi}(\mathbf{A})$ *for all* $P \in X(\mathbf{B})$.

Proof. (1) $\Rightarrow$ (2) It is clear that $1 \in h^{-1}(P)^c$. Let $a, b \in A$ such that $a \leq b$ and $a \in h^{-1}(P)^c$. Then $h(a) \notin P$. As h preserves the natural order, $h(a) \leq h(b)$. Since P is an ideal, $h(b) \notin P$. So, $b \in h^{-1}(P)^c$ and $h^{-1}(P)^c$ is increasing. Let $a, b \in h^{-1}(P)^c$ such that $a \wedge b$ exists. Then $h(a), h(b) \notin P$. If $h(a \wedge b) = h(a) \wedge h(b) \in P$ then for primality of P, $h(a) \in P$ or $h(b) \in P$ which is impossible. Thus,

$$h(a) \wedge h(b) \notin P \qquad \text{and} \qquad a \wedge b \in h^{-1}(P)^c.$$

Therefore, $h^{-1}(P)^c \in \mathrm{Fi}(\mathbf{A})$.

(2) $\Rightarrow$ (1) We assume that $h^{-1}(P)^c \in \mathrm{Fi}(\mathbf{A})$ for all $P \in X(\mathbf{B})$. If $h(1) < 1$ then by Theorem 1.1 there exists $P \in X(\mathbf{B})$ such that $h(1) \in P$ and $1 \notin P$. So, $1 \notin h^{-1}(P)^c$ which is a contradiction because $h^{-1}(P)^c$ is a filter. Then $h(1) = 1$. Let $a, b \in A$ such that $a \wedge b$ exists. It is easy to see that h preserves the order. Thus, $h(a \wedge b) \leq h(a) \wedge h(b)$. We prove that $h(a) \wedge h(b) \leq h(a \wedge b)$.

If $h(a) \wedge h(b) \nleq h(a \wedge b)$ then $(h(a \wedge b)] \cap [h(a) \wedge h(b)) = \emptyset$ and by Theorem 1.1 there exists $P \in X(\mathbf{B})$ such that $h(a \wedge b) \in P$ and $h(a) \wedge h(b) \notin P$. So, $a \wedge b \notin h^{-1}(P)^c$. On the other hand, $h(a) \wedge h(b) \notin P$ and $a, b \in h^{-1}(P)^c$. Since $a \wedge b$ exists and $h^{-1}(P)^c$ is a filter, $a \wedge b \in h^{-1}(P)^c$ which is a contradiction. Thus,

$$h(a \wedge b) = h(a) \wedge h(b) \qquad \text{and} \qquad h \in \mathcal{SDN}_\wedge[\mathbf{A}, \mathbf{B}]. \qquad \square$$

Let X_1 and X_2 be two sets and let $R \subseteq X_1 \times X_2$ be a binary relation. We consider

$$R[C] = \{y \in X_2 \colon \exists x \in C\,((x, y) \in R)\},$$

where C is a subset of X_1. We write $R(x)$ instead of $R[\{x\}]$. Let $h_R \colon \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ be given by $h_R(U) = \{x \in X_1 \colon R(x) \subseteq U\}$. It is easy to check that $h_R \in \mathcal{SDN}_\wedge[\mathcal{P}(X_2), \mathcal{P}(X_1)]$.

Definition 5. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation. We say that R is an *$N_\wedge$-relation* if it verifies the following conditions:

(1) $h_R(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$,

(2) $R(x)$ is a closed subset of X_2 for all $x \in X_1$.

Denote by $\mathcal{NR}_\wedge[X_1, X_2]$ the set of all $N_\wedge$-relations between $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$.

Remark 2. If $R \in \mathcal{NR}_\wedge[X_1, X_2]$ and $x \preceq_1 y$ then $R(y) \subseteq R(x)$ for all $x, y \in X_1$.

The following result gives some equivalences of item (2) of the Definition 5.

Proposition 2.1. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation such that $h_R(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$. The following conditions are equivalent:*

(1) *For every $(x, y) \notin R$, there exists $U \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U$ and $y \notin U$.*

(2) *$R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$ for all $x \in X_1$.*

(3) *For every $(x, y) \in X_1 \times X_2$,*

$$(x, y) \in R \quad \text{iff} \quad H_{X_2}(y) \subseteq h_R^{-1}(H_{X_1}(x)).$$

Proof. (1) $\Rightarrow$ (2) It follows that $R(x) \subseteq \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$. We show the other inclusion. Suppose there is $y \in \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$ such that $(x, y) \notin R$. So, by hypothesis, there exists $U_0 \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U_0$ and $y \notin U_0$ which is a contradiction. Therefore,

$$R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}.$$

(2) $\Rightarrow$ (3) Let $(x, y) \notin R$. Then $y \notin R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$. So, there exists $U_0 \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U_0$ and $y \notin U_0$. Thus,

$$U_0 \notin h_R^{-1}(H_{X_1}(x)) \quad \text{and} \quad U_0 \in H_{X_2}(y),$$

i.e.,

$$H_{X_2}(y) \nsubseteq h_R^{-1}(H_{X_1}(x)).$$

The other direction is similar.

(3) $\Rightarrow$ (1) Let $(x, y) \notin R$. Then $H_{X_2}(y) \nsubseteq h_R^{-1}(H_{X_1}(x))$. It follows that there is $U \in D_{\mathcal{K}_2}(X_2)$ such that $U \in H_{X_2}(y)$ and $U \notin h_R^{-1}(H_{X_1}(x))$. Since $h_R(U) \notin H_{X_1}(x)$, $x \in h_R(U)$ and $R(x) \subseteq U$. Thus, there exists $U \in D_{\mathcal{K}_2}(X_2)$ such that

$$R(x) \subseteq U \quad \text{and} \quad y \notin U. \qquad \square$$

Proposition 2.2. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $R \in \mathcal{NR}_\wedge[X_1, X_2]$. Then $R[C]$ is a closed subset of X_2 for all closed subset C of X_1.*

P r o o f. Let C be a closed subset of X_1. Since $\mathcal{K}_1$ is a basis of X_1, there exists a family $\{V_i \colon i \in I\} \subseteq \mathcal{K}_1$ such that $C = \bigcap \{V_i^c \colon i \in I\}$. It is enough to show that for each $y \notin R[C]$ there exists $U \in \mathcal{K}_2$ such that $R[C] \subseteq U^c$ and $y \in U$. If $y \notin R[C]$ then for every $x \in C$, $y \notin R(x)$. As $R(x)$ is a closed subset of X_2, there exists $U_x \in \mathcal{K}_2$ such that $R[C] \subseteq U_x^c$ and $y \in U_x$. So, $x \in h_R(U_x^c)$. We consider the families $A = \{V_i^c \colon i \in I\}$ and $B = \{h_R(U_x^c) \colon x \in C$ and $U_x \in \mathcal{K}_2\}$ of $D_{\mathcal{K}_1}(X_1)$. Then $C \subseteq \bigcup \{h_R(U_x^c) \colon x \in C$ and $U_x \in \mathcal{K}_2\}$, i.e.,

$$\bigcap \{V_i^c \colon i \in I\} \subseteq \bigcup \{h_R(U_x^c) \colon x \in C \text{ and } U_x \in \mathcal{K}_2\}.$$

As X_1 is an N-space, by Theorem 1.3, there exist $V_1^c, \dots, V_n^c \in [A)$ and $x_1, \dots, x_m \in C$ such that $V_1^c \cap \dots \cap V_n^c \in D_{\mathcal{K}_1}(X_1)$ and $V_1^c \cap \dots \cap V_n^c \subseteq h_R(U_{x_1}^c) \cup \dots \cup h_R(U_{x_m}^c)$. Then $C \subseteq h_R(U^c)$ where $U^c = U_{x_1}^c \cup \dots \cup U_{x_m}^c \in D_{\mathcal{K}_2}(X_2)$. So, there exists $U \in \mathcal{K}_2$ such that $R[C] \subseteq U^c$ and $y \in U$. Therefore, $R[C]$ is a closed subset of X_2. □

Lemma 2.2. *Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be two N-spaces. Let $f \colon X_1 \to X_2$ be a map such that $f^{-1}(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$. Then the relation $f^* \subseteq X_1 \times X_2$ defined by*

$$(x, y) \in f^* \text{ iff } f(x) \preceq_2 y$$

is an $N_\wedge$-relation.

P r o o f. By definition, $f^*(x) = \mathrm{Cl}(f(x))$ for all $x \in X_1$. We show that $h_{f^*}(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$. Hence,

$$\begin{aligned} h_{f^*}(U) &= \{x \in X_1 \colon f^*(x) \subseteq U\} = \{x \in X_1 \colon \mathrm{Cl}(f(x)) \cap U^c = \emptyset\} \\ &= \{x \in X_1 \colon \{f(x)\} \cap U^c = \emptyset\} = \{x \in X_1 \colon f(x) \in U\} \\ &= \left\{x \in X_1 \colon x \in f^{-1}(U)\right\} = f^{-1}(U). \end{aligned}$$

Since $f^{-1}(U) \in D_{\mathcal{K}_1}(X_1)$, $h_{f^*}(U) \in D_{\mathcal{K}_1}(X_1)$. Then f^* is an $N_\wedge$-relation. □

The next result affirm that the N-spaces with $N_\wedge$-relations form a category.

Theorem 2.3. *Let $\langle X_1, \mathcal{K}_1\rangle$, $\langle X_2, \mathcal{K}_2\rangle$ and $\langle X_3, \mathcal{K}_3\rangle$ be three N-spaces. Let $R \in \mathcal{NR}_\wedge[X_1, X_2]$ and $S \in \mathcal{NR}_\wedge[X_2, X_3]$. Then:*

(1) $\preceq_1 \in \mathcal{NR}_\wedge[X_1, X_1]$.

(2) $\preceq_2 \circ R = R$ *and* $S \circ \preceq_2 = S$.

(3) $S \circ R \in \mathcal{NR}_\wedge[X_1, X_3]$.

P r o o f. (1) It is clear that $\preceq_1(x) = \mathrm{Cl}(x)$ for all $x \in X_1$. We see that $h_{\preceq_1}(U) = U$ for all $U \in D_{\mathcal{K}_1}(X_1)$. If $x \in h_{\preceq_1}(U)$ then $\preceq_1(x) \subseteq U$, i.e., $\mathrm{Cl}(x) \subseteq U$ and $x \in U$. Conversely, let $x \in U$. As U is a closed subset, $\mathrm{Cl}(x) \subseteq U$ and $\preceq_1(x) \subseteq U$. So,

$$x \in h_{\preceq_1}(U) \quad \text{and} \quad \preceq_1 \in \mathcal{NR}_\wedge[X_1, X_1].$$

(2) It is easy to prove that $R \subseteq \preceq_2 \circ R$ and $S \subseteq S \circ \preceq_2$. Let $(x, z) \in S \circ \preceq_2$. Then there is $y \in X_2$ such that $(x, y) \in \preceq_2$ and $(y, z) \in S$, i.e., $x \preceq_2 y$ and $z \in S(y)$. By Remark 2, $S(y) \subseteq S(x)$ and $z \in S(x)$. So, $(x, z) \in S$ and $S \circ \preceq_2 = S$. Similarly, if $(x, z) \in \preceq_2 \circ R$ then there is $y \in X_2$ such that $(x, y) \in R$ and $(y, z) \in \preceq_2$. Thus, $y \in R(x)$ and $z \in \mathrm{Cl}(y)$. As $R(x)$ is a closed subset of X_1, $\mathrm{Cl}(y) \subseteq R(x)$ and $z \in R(x)$. We deduce that

$$(x, z) \in R \quad \text{and} \quad \preceq_2 \circ R = R.$$

(3) Let $U \in D_{\mathcal{K}_3}(X_3)$. Then $h_{(S \circ R)}(U) \in D_{\mathcal{K}_1}(X_1)$. Indeed,

$$\begin{aligned} h_{(S \circ R)}(U) &= \{x \in X_1 \colon (S \circ R)(x) \subseteq U\} = \{x \in X_1 \colon S(R(x)) \subseteq U\} \\ &= \{x \in X_1 \colon R(x) \subseteq h_S(U)\} = h_R(h_S(U)). \end{aligned}$$

By Proposition 2.2, $(S \circ R)(x)$ is a closed subset of X_3 for all $x \in X_1$ and $S \circ R \in \mathcal{NR}_\wedge [X_1, X_3]$. $\square$

We denote by $\mathcal{NR}_\wedge$ the category whose objects are N-spaces and morphisms $N_\wedge$-relations between them.

Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$ and $h \in \mathcal{SDN}_\wedge [\mathbf{A}, \mathbf{B}]$. Let $R_h \subseteq X(\mathbf{B}) \times X(\mathbf{A})$ be the binary relation defined by

$$(P, Q) \in R_h \text{ iff } Q \subseteq h^{-1}(P).$$

Proposition 2.4. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$ and $h \in \mathcal{SDN}_\wedge [\mathbf{A}, \mathbf{B}]$. Then:*

1. *For every $P \in X(\mathbf{A})$ and for every $a \in A$, $h(a) \in P$ if and only if there exists $Q \in X(\mathbf{A})$ such that $(P, Q) \in R_h$ and $a \in Q$.*
2. *$R_h \in \mathcal{NR}_\wedge [X(\mathbf{B}), X(\mathbf{A})]$.*
3. *If $\mathbf{C} \in \mathcal{DN}$ and $k \in \mathcal{SDN}_\wedge [\mathbf{B}, \mathbf{C}]$ then $R_{k \circ h} = R_h \circ R_k$.*
4. *$h_{R_h}(\varphi_{\mathbf{A}}(a)) = \varphi_{\mathbf{B}}(h(a))$ for all $a \in A$.*

Proof. (1) Let $a \in A$ and $P \in X(\mathbf{A})$. If $h(a) \in P$ then $a \notin h^{-1}(P)^c$. By Lemma 2.1, we have $h^{-1}(P)^c \in \mathrm{Fi}(\mathbf{A})$. Thus, $(a] \cap h^{-1}(P)^c = \emptyset$ and by Theorem 1.1 there exists $Q \in X(\mathbf{A})$ such that $(a] \subseteq Q$ and $Q \cap h^{-1}(P)^c = \emptyset$. Therefore,

$$a \in Q \quad \text{and} \quad (P, Q) \in R_h.$$

The reciprocal is immediate.

(2) By item (1), $h_{R_h}(\varphi_{\mathbf{A}}(a)) \in D_{\mathcal{K}_{\mathbf{B}}}(X(\mathbf{B}))$ for all $\varphi_{\mathbf{A}}(a) \in D_{\mathcal{K}_{\mathbf{A}}}(X(\mathbf{A}))$. Let $P \in X(\mathbf{B})$. We show that $R_h(P) = \bigcap \{\varphi_{\mathbf{A}}(a) : h(a) \notin P\}$. Then

$$\begin{aligned} Q \in R_h(P) \quad &\text{iff} \quad Q \subseteq h^{-1}(P) \\ &\text{iff} \quad \text{for all } a \in A \left(a \notin h^{-1}(P) \Rightarrow a \notin Q\right) \\ &\text{iff} \quad \text{for all } a \in A \left(h(a) \notin P \Rightarrow Q \in \varphi_{\mathbf{A}}(a)\right) \\ &\text{iff} \quad Q \in \bigcap \{\varphi_{\mathbf{A}}(a) : h(a) \notin P\}. \end{aligned}$$

So, $R_h(P)$ is a closed subset of $X(\mathbf{A})$ and $R_h \in \mathcal{NR}_\wedge [X(\mathbf{B}), X(\mathbf{A})]$.

(3) Let $P \in X(\mathbf{B})$ and $Q \in X(\mathbf{A})$. If $(P, Q) \in R_{k \circ h}$ then $Q \subseteq h^{-1}\left(k^{-1}(P)\right)$. By Lemma 2.1, $k^{-1}(P)^c \in \mathrm{Fi}(\mathbf{B})$. We consider the set $h(Q) = \{h(q) : q \in Q\} \subseteq B$. We note that

$$I(h(Q)) \cap k^{-1}(P)^c = \emptyset.$$

Otherwise, if there is $a \in A$ such that $a \in I(h(Q)) \cap k^{-1}(P)^c$ then there exists $q \in Q$ such that $a \leq h(q)$ and $a \in k^{-1}(P)^c$. Since $k^{-1}(P)^c \in \mathrm{Fi}(\mathbf{B})$, $h(q) \in k^{-1}(P)^c$. Thus, $q \notin h^{-1}\left(k^{-1}(P)\right)$ and $q \notin Q$ which is a contradiction. So, $I(h(Q)) \cap k^{-1}(P)^c = \emptyset$ and by Theorem 1.1 there exists $D \in X(\mathbf{B})$ such that $h(Q) \subseteq D$ and $D \cap k^{-1}(P)^c = \emptyset$. Then $(P, D) \in R_k$ and $(D, Q) \in R_h$, i.e., $(P, Q) \in R_h \circ R_k$. It is easy to check that $R_h \circ R_k \subseteq R_{k \circ h}$.

(4) It follows from (1). $\square$

Proposition 2.5. *Let $\langle X_1, \mathcal{K}_1 \rangle$, $\langle X_2, \mathcal{K}_2 \rangle$ and $\langle X_3, \mathcal{K}_3 \rangle$ be three N-spaces. Let $R \in \mathcal{NR}_\wedge [X_1, X_2]$ and $S \in \mathcal{NR}_\wedge [X_2, X_3]$. Then:*

1. $h_R \in \mathcal{SDN}_\wedge [D_{\mathcal{K}_2}(X_2), D_{\mathcal{K}_1}(X_1)]$.
2. $h_{S \circ R} = h_R \circ h_S$.
3. *For every $(x, y) \in X_1 \times X_2$,*

$$(x, y) \in R \text{ iff } (H_{X_1}(x), H_{X_2}(y)) \in R_{h_R}.$$

4. $R_{h_R} \circ H_{X_1} = H_{X_2} \circ R$.

Proof. (1) Since R is an $N_\wedge$-relations, $h_R(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_2}(X_2)$ and h_R is well defined. Clearly $h_R(X_2) = X_1$ and if there is $U \cap V \in D_{\mathcal{K}_2}(X_2)$ then $h_R(U \cap V) = h_R(U) \cap h_R(V)$. So, $h_R \in \mathcal{SDN}_\wedge[D_{\mathcal{K}_2}(X_2), D_{\mathcal{K}_1}(X_1)]$.

(2) – (3) It follows from Theorem 2.3 and Proposition 2.1, respectively.

(4) Let $(x, U) \in R_{h_R} \circ H_{X_1}$. Then there exists $V \in X(D_{\mathcal{K}_1}(X_1))$ such that $(x, V) \in H_{X_1}$ and $(V, U) \in R_{h_R}$. As H_{X_1} and H_{X_2} are homeomorphisms, $H_{X_1}(x) = V$ and there exists $y \in X_2$ such that $H_{X_2}(y) = U$. Thus, $(H_{X_1}(x), H_{X_2}(y)) \in R_{h_R}$ and by (3) we have $(x, y) \in R$. Therefore, $(x, U) \in H_{X_2} \circ R$. The reciprocal is similar. $\square$

From the results developed in [9] and Proposition 2.5 we conclude that the functor

$$\mathbb{D}\colon \mathcal{NR}_\wedge \to \mathcal{SDN}_\wedge$$

defined by

$$\begin{aligned} &\mathbb{D}(X) = D_{\mathcal{K}}(X) && \text{if } \langle X, \mathcal{K}\rangle \text{ is an } N\text{-space,} \\ &\mathbb{D}(R) = h_R && \text{if } R \text{ is an } N_\wedge\text{-relation} \end{aligned} \tag{2.1}$$

is a contravariant functor. By [9] and Proposition 2.4 we have that the functor

$$\mathbb{X}\colon \mathcal{SDN}_\wedge \to \mathcal{NR}_\wedge$$

given by

$$\begin{aligned} &\mathbb{X}(\mathbf{A}) = \langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle && \text{if } \mathbf{A} \text{ is a distributive nearlattice,} \\ &\mathbb{X}(h) = R_h && \text{if } h \text{ is a } \wedge\text{-semi-homomorphism} \end{aligned} \tag{2.2}$$

is a contravariant functor.

Theorem 2.6. *The contravariant functors $\mathbb{X}$ and $\mathbb{D}$ define a dual equivalence between the categories $\mathcal{SDN}_\wedge$ and $\mathcal{NR}_\wedge$.*

3. $\square$-distributive nearlattices

A modal algebra is a pair $\mathbf{B} = \langle B, \square\rangle$ where B is a Boolean algebra and $\square$ is a $\wedge$-semi-homomorphism. Recall that the modal algebras are the algebraic semantics of the modal propositional logic in the same way that Boolean algebras are the algebraic counterpart of the classical propositional logic [11]. Analogously, a modal lattice is a pair $\mathbf{L} = \langle L, \square\rangle$ where L is a bounded distributive lattice and $\square$ is a $\wedge$-semi-homomorphism [7]. We give an extension of this notion.

Definition 6. We say that a pair $\mathbf{A}_\square = \langle A, \square\rangle$ is a *distributive nearlattice with a necessity modal operator*, or *$\square$-distributive nearlattice*, if $\mathbf{A} \in \mathcal{DN}$ and $\square$ is a $\wedge$-semi-homomorphism defined on $\mathbf{A}$.

Let us denote by $\mathcal{DN}_\square$ the class of $\square$-distributive nearlattices. Since a $\square$-distributive nearlattice is a distributive nearlattice endowed with a $\wedge$-semi-homomorphism, we have that all results on representation and duality given in the previous section can be applied to these class of structures.

Definition 7. Let $\mathbf{A}_{\square_1}, \mathbf{B}_{\square_2} \in \mathcal{DN}_\square$. Let $h \in \mathcal{SDN}_\wedge[\mathbf{A}, \mathbf{B}]$. We say that h is a *$\square$-homomorphism* if $h(\square_1 a) = \square_2 h(a)$ for all $a \in A$. If h is an isomorphism then we say that h is a *$\square$-isomorphism.*

Denote by $\mathcal{SDN}_\square\left[\mathbf{A}_{\square_1}, \mathbf{B}_{\square_2}\right]$ the set of all $\square$-homomorphisms from $\mathbf{A}_{\square_1}$ into $\mathbf{B}_{\square_2}$. The category of $\square$-distributive nearlattices and their $\square$-homomorphisms is denoted by $\mathcal{SDN}_\square$.

***Example* 2.** In [7] the author studied the bounded distribuitive lattices with a necessity modal operator $\square$. It is immediate that every bounded distribuitive lattice with a necessity modal operator is a $\square$-distributive nearlattice.

***Example* 3.** In [6] the variety of modal Tarski algebras is introduced. A modal Tarski algebra is an algebra $\mathbf{A}_\Box = \langle \mathbf{A}, \Box \rangle$ where $\mathbf{A} = \langle A, \to, 1 \rangle$ is a Tarski algebra and $\Box$ is a unary operator defined on $\mathbf{A}$ such that it verifies the following conditions:

(1) $\Box 1 = 1$,

(2) $\Box(a \to b) \leq \Box a \to \Box b$ for all $a, b \in A$.

Then $\mathbf{A}_\Box$ is a $\Box$-distributive nearlattice where the join of two elements $a, b \in A$ is defined by $a \vee b = (a \to b) \to b$. First, we see that $\Box$ preserves the order. Let $a, b \in A$ such that $a \leq b$. Then $a \to b = 1$ and $\Box(a \to b) = \Box 1 = 1$. By (2), $\Box a \to \Box b = 1$, i.e., $\Box a \leq \Box b$. Suppose that $a \wedge b$ exists. As $\Box$ preserves the order, $\Box(a \wedge b) \leq \Box a \wedge \Box b$. On the other hand, since $a \wedge b \leq a \wedge b$ we have $a \leq b \to (a \wedge b)$. Thus, $\Box a \leq \Box(b \to (a \wedge b)) \leq \Box b \to \Box(a \wedge b)$ by (2) and $\Box a \wedge \Box b \leq \Box(a \wedge b)$. Therefore, $\Box$ is a $\wedge$-semi-homomorphism and $\mathbf{A}_\Box$ is a $\Box$-distributive nearlattice.

In [9] a topological duality was developed for distributive nearlattices. Using the results of the Section 2, we extend this representation and duality to the class of distributive nearlattices with a necessity modal operator.

Definition 8. We say that the structure $\langle X, \mathcal{K}, Q \rangle$ is an *$N\Box$-space* if $\langle X, \mathcal{K} \rangle$ is an N-space and $Q \subseteq X \times X$ is an $N_\wedge$-relation.

If $\langle X, \mathcal{K}, Q \rangle$ is an $N\Box$-space then the map $\Box_Q \colon D_\mathcal{K}(X) \to D_\mathcal{K}(X)$ given by

$$\Box_Q(U) = \{x \in X \colon Q(x) \subseteq U\}$$

is a $\wedge$-semi-homomorphism. Then $\langle D_\mathcal{K}(X), \Box_Q \rangle \in \mathcal{DN}_\Box$.

Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. By the results given in the Section 2, the binary relation $Q_\Box \subseteq X(\mathbf{A}) \times X(\mathbf{A})$ given by

$$(P, R) \in Q_\Box \text{ iff } R \subseteq \Box^{-1}(P),$$

is a $N_\wedge$-relation. We have the following result.

Theorem 3.1. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. Then $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}, Q_\Box \rangle$ is an $N\Box$-space.*

Definition 9. Let $\langle X_1, \mathcal{K}_1, Q_1 \rangle$ and $\langle X_2, \mathcal{K}_2, Q_2 \rangle$ be two $N\Box$-spaces. Let $R \in \mathcal{NR}_\wedge[X_1, X_2]$. We say that R is an *$N_\wedge\Box$-relation* if $R \circ Q_1 = Q_2 \circ R$.

Denote by $\mathcal{NR}_\Box[X_1, X_2]$ the set of all $N_\wedge\Box$-relations between $\langle X_1, \mathcal{K}_1, Q_1 \rangle$ and $\langle X_2, \mathcal{K}_2, Q_2 \rangle$. The category whose objects are $N\Box$-spaces and morphisms $N_\wedge\Box$-relations between them will be denoted by $\mathcal{NR}_\Box$.

Theorem 3.2. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. Then there exists an $N\Box$-space $\langle X, \mathcal{K}, Q \rangle$ such that $\mathbf{A}_\Box$ is $\Box$-isomorphic to $\langle D_\mathcal{K}(X), \Box_Q \rangle$.*

Proof. Since $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A} \rangle$ is an N-space and $Q_\Box$ an $N_\wedge$-relation, the structure $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}, Q_\Box \rangle$ is an $N\Box$-space. So, $\left\langle D_{\mathcal{K}_\mathbf{A}}(X(\mathbf{A})), \Box_{Q_\Box} \right\rangle$ is a $\Box$-distributive nearlattice. By Proposition 2.4, $\varphi_\mathbf{A}(\Box a) = \Box_{Q_\Box}(\varphi_\mathbf{A}(a))$ for all $a \in A$. Then $\varphi_\mathbf{A} \in \mathcal{SDN}_\Box[\mathbf{A}_\Box, D_{\mathcal{K}_\mathbf{A}}(X(\mathbf{A}))]$ and by Theorem 1.2 we have that $\mathbf{A}_\Box$ is $\Box$-isomorphic to $\left\langle D_{\mathcal{K}_\mathbf{A}}(X(\mathbf{A})), \Box_{Q_\Box} \right\rangle$. $\Box$

Let $\langle X, \mathcal{K} \rangle$ be an N-space. Note that $P \preceq Q$ if and only if $Q \subseteq P$ for all $P, Q \in X(D_\mathcal{K}(X))$. By the results developed in [9] and Lemma 2.2, the map $H_X \colon X \to X(D_\mathcal{K}(X))$ induces an $N_\wedge$-relation $H_X^* \subseteq X \times X(D_\mathcal{K}(X))$ given by

$$(x, P) \in H_X^* \text{ iff } P \subseteq H_X(x).$$

Theorem 3.3. *Let $\langle X, \mathcal{K}, Q \rangle$ be an $N\Box$-space. Then $H_X^* \in \mathcal{NR}_\Box[X, X(D_\mathcal{K}(X))]$.*

P r o o f. We know that $H_X\colon X \to X(D_{\mathcal{K}}(X))$ is an homeomorphism between the $N\Box$-spaces $\langle X, \mathcal{K}, Q\rangle$ and $\left\langle X(D_{\mathcal{K}}(X)), \mathcal{K}_{D_{\mathcal{K}}(X)}, Q_{\Box_Q}\right\rangle$ such that

$$(x,y) \in Q \text{ iff } (H_X(x), H_X(y)) \in Q_{\Box_Q}.$$

We only see that $H_X^* \circ Q = Q_{\Box_Q} \circ H_X^*$. Let $(x,P) \in H_X^* \circ Q$. Then there is $y \in X$ such that $(x,y) \in Q$ and $(y,P) \in H_X^*$. Thus, $(H_X(x), H_X(y)) \in Q_{\Box_Q}$ and $P \subseteq H_X(y)$. It follows that $P \subseteq \Box_Q^{-1}(H_X(x))$ and $(H_X(x), P) \in Q_{\Box_Q}$. On the other hand, $(x, H_X(x)) \in H_X^*$. Hence, $(x,P) \in Q_{\Box_Q} \circ H_X^*$ and $H_X^* \circ Q \subseteq Q_{\Box_Q} \circ H_X^*$. Reciprocally, let $(x,P) \in Q_{\Box_Q} \circ H_X^*$. Then there exists $F \in X(D_{\mathcal{K}}(X))$ such that $(x,F) \in H_X^*$ and $(F,P) \in Q_{\Box_Q}$, i.e., $F \subseteq H_X(x)$ and $P \subseteq \Box_Q^{-1}(F)$. As H_X is onto, there exist $y,z \in X$ such that $H_X(y) = F$ and $H_X(z) = P$. So, $H_X(z) \subseteq \Box_Q^{-1}(H_X(y)) \subseteq \Box_Q^{-1}(H_X(x))$ and $(H_X(x), H_X(z)) \in Q_{\Box_Q}$. Then $(x,z) \in Q$. Since $(z,P) \in H_X^*$, we have $(x,P) \in H_X^* \circ Q$. Therefore, $H_X^* \circ Q = Q_{\Box_Q} \circ H_X^*$. $\Box$

In Section 2 it was proved that if $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ are N-spaces and $R \subseteq X_1 \times X_2$ is an $N_\wedge$-relation then the map $h_R\colon D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ is a $\wedge$-semi-homomorphism.

Theorem 3.4. *Let $\langle X_1, \mathcal{K}_1, Q_1\rangle$ and $\langle X_2, \mathcal{K}_2, Q_2\rangle$ be two $N\Box$-spaces. If $R \in \mathcal{NR}_\Box[X_1, X_2]$ then $h_R \in \mathcal{SDN}_\Box[D_{\mathcal{K}_2}(X_2), D_{\mathcal{K}_1}(X_1)]$.*

P r o o f. Let $x \in X_1$ and $U \in D_{\mathcal{K}_2}(X_2)$. Since R is an $N_\wedge\Box$-relation,

$$\begin{array}{lll} x \in h_R(\Box_{Q_2}(U)) & \text{iff } R(x) \subseteq \Box_{Q_2}(U) & \text{iff for all } y \in R(x)\,(Q_2(y) \subseteq U) \\ & \text{iff } Q_2(R(x)) \subseteq U & \text{iff } R(Q_1(x)) \subseteq U \\ & \text{iff for all } z \in Q_1(x)\,(R(z) \subseteq U) & \text{iff } Q_1(x) \subseteq h_R(U) \\ & \text{iff } x \in \Box_{Q_1}(h_R(U)). & \end{array}$$

Thus, $h_R(\Box_{Q_2}(U)) = \Box_{Q_1}(h_R(U))$ and h_R is a $\Box$-homomorphism. $\Box$

Let $\mathbf{A}, \mathbf{B} \in \mathcal{DN}$ and $h : A \to B$ be a $\wedge$-semi-homomorphism. In Section 2 it was proved that the relation $R_h \subseteq X(\mathbf{B}) \times X(\mathbf{A})$ is an $N_\wedge$-relation. We study R_h when h is a $\Box$-homomorphism.

Theorem 3.5. *Let $\mathbf{A}_{\Box_1}, \mathbf{B}_{\Box_2} \in \mathcal{DN}_\Box$. If $h \in \mathcal{SDN}_\Box\left[\mathbf{A}_{\Box_1}, \mathbf{B}_{\Box_2}\right]$ then $R_h \in \mathcal{NR}_\Box[X(\mathbf{B}), X(\mathbf{A})]$.*

P r o o f. It is enough to show that $R_h \circ Q_{\Box_2} = Q_{\Box_1} \circ R_h$. Let $P \in X(\mathbf{B})$ and $Q \in X(\mathbf{A})$ such that $(P,D) \in Q_{\Box_1} \circ R_h$. Then there is $F \in X(\mathbf{A})$ such that $(P,F) \in R_h$ and $(F,D) \in Q_{\Box_1}$, i.e., $F \subseteq h^{-1}(P)$ and $D \subseteq \Box_1^{-1}(F)$. By Lemma 2.1 we have $\Box_2^{-1}(P)^c \in \mathrm{Fi}(\mathbf{B})$. We prove that $I(h(D)) \cap \Box_2^{-1}(P)^c = \emptyset$. If $b \in I(h(D)) \cap \Box_2^{-1}(P)^c$ then there exists $a \in D$ such that $b \leq h(a)$ and $\Box_2 b \notin P$. Since h is a $\Box$-homomorphism, $\Box_2 b \leq \Box_2(h(a)) = h(\Box_1 a)$. It follows that $h(\Box_1 a) \notin P$. So, $\Box_1 a \notin h^{-1}(P)$ and $a \notin \Box_1^{-1}(F)$. Then $a \notin D$ which is a contradiction. Since $I(h(D)) \cap \Box_2^{-1}(P)^c = \emptyset$, by Theorem 1.1, there exists $H \in X(\mathbf{B})$ such that $I(h(D)) \subseteq H$ and $H \cap \Box_2^{-1}(P)^c = \emptyset$. Thus, $D \subseteq h^{-1}(H)$ and $H \subseteq \Box_2^{-1}(P)$, i.e., $(H,D) \in R_h$ and $(P,H) \in Q_{\Box_2}$. Therefore, $(P,D) \in R_h \circ Q_{\Box_2}$ and $Q_{\Box_1} \circ R_h \subseteq R_h \circ Q_{\Box_2}$. The other inclusion is similar. $\Box$

From the functors $\mathbb{D}$ and $\mathbb{X}$ given by 2.1 and 2.2, respectively, and the Theorems 3.4 and 3.5, we give the following result.

Theorem 3.6. *The contravariant functors $\mathbb{X}|_{\mathcal{SDN}_\Box}$ and $\mathbb{D}|_{\mathcal{NR}_\Box}$ define a dual equivalence between the categories $\mathcal{SDN}_\Box$ and $\mathcal{NR}_\Box$.*

4. Some classes of $\Box$-distributive nearlattices

There exist many important classes of Boolean algebras endowed with a modal operator. For example, interior algebras, also know as topological Boolean algebras or closure algebras, are modal algebras $\mathbf{B} = \langle B, \Box \rangle$ where the modal operator satisfies the additional conditions $\Box a \leq a$ and $\Box a \leq \Box^2 a$ for all $a \in B$. Some classes of modal algebras can be characterized by means of first order conditions defined in the associated modal space, for example, a modal algebra $\mathbf{B} = \langle B, \Box \rangle$ is an interior algebra if and only if the relation $Q_\Box$ associated with the modal operator $\Box$ is reflexive and transitive. In this section we give similar results for some classes of $\Box$-distributive nearlattices.

Let Q be a binary relation defined on a set X. For each $n \geq 0$ we define inductively the relation Q^n as follows: $(x, y) \in Q^0$ if and only if $x = y$ and $(x, y) \in Q^{n+1} = Q \circ Q^n$ where $\circ$ is the usual composition of relations.

Lemma 4.1. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. Let $P \in X(\mathbf{A})$ and $a \in A$. For each $n \in \mathbb{N}$, $\Box^n a \in P$ if and only if there exists $R \in X(\mathbf{A})$ such that $(P, R) \in Q_\Box^n$ and $a \in R$.*

P r o o f. The proof is by induction on n. By Proposition 2.4, it is immediately for $n = 0$. Assume that $\Box^n a \in P$ implies that there exists $R \in X(\mathbf{A})$ such that $(P, R) \in Q_\Box^n$ and $a \in R$. Suppose that $\Box^{n+1} a \in P$, i.e., $\Box(\Box^n a) \in P$. By Proposition 2.4, there is $R \in X(\mathbf{A})$ such that $R \subseteq \Box^{-1}(P)$ and $\Box^n a \in R$. By assumption, there exists $D \in X(\mathbf{A})$ such that $(R, D) \in Q_\Box^n$ and $a \in R$. Since $(P, R) \in Q_\Box$ and $(R, D) \in Q_\Box^n$ we get that $(P, D) \in Q_\Box^{n+1}$. Conversely, suppose that $(P, R) \in Q_\Box^{n+1}$ and $a \in R$. So, there exists $D \in X(\mathbf{A})$ such that $(P, D) \in Q_\Box^n$ and $(D, R) \in Q_\Box$. Therefore, $R \subseteq \Box^{-1}(D)$ and as $a \in R$, $a \in \Box^{-1}(D)$. Thus, $(P, D) \in Q_\Box^n$ and $\Box a \in D$. It follows by assumption that $\Box^{n+1} a \in P$. $\Box$

Theorem 4.1. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. Then:*

(1) *$a \leq \Box a$ iff for all P and R $((P, R) \in Q_\Box \Rightarrow R \subseteq P)$.*

(2) *$a \leq \Box^n a$ iff for all P and R $\big((P, R) \in Q_\Box^n \Rightarrow R \subseteq P\big)$ with $n \in \mathbb{N}$.*

(3) *$\Box a \leq a$ iff $Q_\Box$ is reflexive.*

(4) *$\Box a \leq \Box^2 a$ iff $Q_\Box$ is transitive.*

(5) *$\Box^2 a \leq \Box a$ iff for all P and R $((P, R) \in Q_\Box \Rightarrow$ there exists T $((P, T) \in Q_\Box$ and $(T, R) \in Q_\Box))$.*

P r o o f. We prove only the assertions (2), (3) and (5).

(2) Let $n \in \mathbb{N}$. Suppose that there exist $P, R \in X(\mathbf{A})$ such that $(P, R) \in Q_\Box^n$ and $R \nsubseteq P$. Hence, there is $a \in R$ such that $a \notin P$. Since $(P, R) \in Q_\Box^n$ and $a \in R$, by Lemma 4.1, $\Box^n a \in P$. As $a \leq \Box^n a$ we have $a \in P$, which is a contradiction. Conversely, suppose that there exists $a \in A$ such that $a \nleq \Box^n a$. By Theorem 1.1 it follows that there is $P \in X(\mathbf{A})$ such that $\Box^n a \in P$ and $a \notin P$. So, by Lemma 4.1, there exists $R \in X(\mathbf{A})$ such that $(P, R) \in Q_\Box^n$ and $a \in R$. By assumption, $R \subseteq P$ and $a \in P$ which is a contradiction.

(3) Let $P \in X(\mathbf{A})$ and $a \in A$ such that $a \in P$. Since $\Box a \leq a$, $\Box a \in P$. So, $a \in \Box^{-1}(P)$ and $P \subseteq \Box^{-1}(P)$. Then $(P, P) \in Q_\Box$ and $Q_\Box$ is reflexive. Reciprocally, suppose that there is $a \in A$ such that $\Box a \nleq a$. So, by Theorem 1.1 there exists $P \in X(\mathbf{A})$ such that $a \in P$ and $\Box a \notin P$, i.e., $a \notin \Box^{-1}(P)$. As $Q_\Box$ is reflexive, $P \subseteq \Box^{-1}(P)$ and $a \in \Box^{-1}(P)$ which is impossible.

(5) Assume that $\Box^2 a \leq \Box a$ for all $a \in A$ and let $(P, R) \in Q_\Box$. Consider the ideal $I(\Box(R))$ and the filter $\Box^{-1}(P)^c$. Suppose that there exists $c \in A$ such that $c \in I(\Box(R)) \cap \Box^{-1}(P)^c$. Thus, there exist $r_1, \ldots, r_n \in R$ such that $c \leq \Box r_1 \vee \cdots \vee \Box r_n$. Since $r_1, \ldots, r_n \in R$ and R is a ideal, $r_1 \vee \cdots \vee r_n = r \in R$. Then $\Box r_i \leq \Box r$ for all $1 \leq i \leq n$ and $c \leq \Box r$. It follows that $r \in R \subseteq \Box^{-1}(P)$, i.e., $\Box r \in P$. As $\Box^2 r \leq \Box r$, $\Box^2 r \in P$. On the other hand, $c \leq \Box r$ and $\Box r \in \Box^{-1}(P)^c$. Then $\Box^2 r \notin P$, which is impossible. So, $I(\Box(R)) \cap \Box^{-1}(P)^c = \emptyset$ and by Theorem 1.1 there exists

$T \in X(\mathbf{A})$ such that $\Box(R) \subseteq T$ and $T \cap \Box^{-1}(P)^c = \emptyset$. This is, $(P,T) \in Q_\Box$ and $(T,R) \in Q_\Box$. Reciprocally, suppose that there is $a \in A$ such that $\Box^2 a \nleq \Box a$. Thus, there exists $P \in X(\mathbf{A})$ such that $\Box a \in P$ and $\Box^2 a \notin P$. By Proposition 2.4, there exists $R \in X(\mathbf{A})$ such that $(P,R) \in Q_\Box$ and $a \in R$. By assumption, there is $T \in X(\mathbf{A})$ such that $(P,T) \in Q_\Box$ and $(T,R) \in Q_\Box$, i.e., $(P,R) \in Q_\Box^2$ and since $a \in R$, by Lemma 4.1, we have $\Box^2 a \in P$ which is a contradiction. $\Box$

Each equation of the Theorem 4.1 defines a subclass of $\Box$-distributive nearlattices where its dual space satisfies the correspondent condition.

4.1. S5-nearlattices

Recall that a monadic Boolean algebra, in the sense of Halmos [19], is a Boolean algebra $\mathbf{B}$ with a modal operator $\forall$ satisfying the equations $\forall 1 = 1$, $\forall(a \vee \forall b) = \forall a \vee \forall b$ and $\forall a \leq a$ for all $a, b \in B$. The name "monadic" comes from the connection with predicate logics for languages having one-placed predicates and a single quantifier. There are different generalizations of the monadic algebras such as the variety of Q-distributive lattices introduced in [16], the monadic Heyting algebras studied in [4], the lattices with an antitone involution endowed with a quantifier studied in [15] or the orthoposets with a quantifier studied in [17]. In this subsection we introduce the class of **S5**-nearlattices.

Definition 10. Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. We say that $\mathbf{A}_\Box$ is an **S4**-*nearlattice* if it verifies the following conditions:

(1) $\Box a \leq a$,

(2) $\Box a \leq \Box^2 a$ for all $a \in A$.

We say that $\mathbf{A}_\Box$ is an **S5**-*nearlattice* if $\mathbf{A}_\Box$ is an **S4**-nearlattice and satisfies the additional condition:

$$\Box(\Box a \vee b) \leq \Box a \vee \Box b \text{ for all } a, b \in A.$$

By Theorem 4.1, $\mathbf{A}_\Box$ is an **S4**-nearlattice if and only if $Q_\Box$ is reflexive and transitive. Thus, the dual space of an **S4**-nearlattice $\mathbf{A}_\Box$ is an $N\Box$-space $\langle X, \mathcal{K}, Q\rangle$ such that $Q_\Box$ is a quasi-order. The next objective is to characterize the dual space of an **S5**-nearlattice.

Lemma 4.2. *Let $\mathbf{A}_\Box$ be an **S4**-nearlattice. Then:*

(1) $\Box a \vee \Box b \leq \Box(\Box a \vee \Box b)$,

(2) $\Box a \vee \Box b \leq \Box(\Box a \vee b)$ *for all $a, b \in A$.*

Proof. (1) Let $a, b \in A$. As $\Box a \leq \Box a \vee \Box b$ and $\Box$ preserves the order, we have $\Box^2 a \leq \Box(\Box a \vee \Box b)$. Since $\Box a \leq \Box^2 a$, $\Box a \leq \Box(\Box a \vee \Box b)$. Analogously, $\Box b \leq \Box(\Box a \vee \Box b)$ and $\Box a \vee \Box b \leq \Box(\Box a \vee \Box b)$.

(2) Let $a, b \in A$. Since $\Box a \leq \Box a \vee b$, it follows that $\Box a \leq \Box^2 a \leq \Box(\Box a \vee b)$. Similarly, $\Box b \leq \Box(\Box a \vee b)$. Therefore $\Box a \vee \Box b \leq \Box(\Box a \vee b)$. $\Box$

Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. We consider the relation

$$E_\Box = Q_\Box \cap Q_\Box^{-1}.$$

Note that if $Q_\Box$ is reflexive and transitive then $E_\Box$ is an equivalence relation. For $U \subseteq X(\mathbf{A})$, let

$$h_{E_\Box}(U) = \{P \in X(\mathbf{A}) : E_\Box(P) \subseteq U\}.$$

Lemma 4.3. *Let $\mathbf{A}_\Box$ be an **S4**-nearlattice. The following conditions are equivalents:*

(1) $(P, D) \in E_\Box$,

(2) $\Box^{-1}(P) = \Box^{-1}(D)$ *for all $P, D \in X(\mathbf{A})$.*

Proof. (1) $\Rightarrow$ (2) Let $P, D \in X(\mathbf{A})$ such that $(P,D) \in E_{\Box}$. Then $D \subseteq \Box^{-1}(P)$ and $P \subseteq \Box^{-1}(D)$. Since $\Box a \leq \Box^2 a$, $\Box^{-1}(D) \subseteq \Box^{-1}\left(\Box^{-1}(P)\right) \subseteq \Box^{-1}(P)$ and $\Box^{-1}(P) \subseteq \Box^{-1}\left(\Box^{-1}(D)\right) \subseteq \Box^{-1}(D)$. Thus, $\Box^{-1}(D) \subseteq \Box^{-1}(P)$ and $\Box^{-1}(P) \subseteq \Box^{-1}(D)$, i.e., $\Box^{-1}(P) = \Box^{-1}(D)$.

(2) $\Rightarrow$ (1) Let $P, D \in X(\mathbf{A})$ such that $\Box^{-1}(P) = \Box^{-1}(D)$. As $Q_{\Box}$ is reflexive, $P \subseteq \Box^{-1}(P) = \Box^{-1}(D)$ and $D \subseteq \Box^{-1}(D) = \Box^{-1}(P)$, i.e., $(D,P) \in Q_{\Box}$ and $(P,D) \in Q_{\Box}$. So, $(P,D) \in E_{\Box}$. $\Box$

Remark 3. Let $\mathbf{A}_{\Box} \in \mathcal{DN}_{\Box}$ and $P \in X(\mathbf{A})$. It is easy to see that $(\Box(P)] \in \mathrm{Id}(\mathbf{A})$.

Remark 4. Let X be a set and let $R \subseteq X \times X$ be an equivalence relation. So, $\Box_R(U \cup \Box_R(V)) = \Box_R(U) \cup \Box_R(V)$ for all $U, V \in \mathcal{P}(X)$. Let $x \in \Box_R(U \cup \Box_R(V))$. If $x \notin \Box_R(U) \cup \Box_R(V)$ then there exist $y, z \in X$ such that $y \in R(x) - U$ and $z \in R(x) - V$. Since $R(x) \subseteq U \cup \Box_R(V)$, we have $y \in \Box_R(V)$. As R is an equivalence relation, $(y,z) \in R$ and consequently $z \in V$ which is a contradiction. Thus, $\Box_R(U \cup \Box_R(V)) \subseteq \Box_R(U) \cup \Box_R(V)$. On the other hand, $\Box_R(U) \subseteq \Box_R(U \cup \Box_R(V))$ and $\Box_R(V) \subseteq \Box_R(U \cup \Box_R(V))$. Therefore $\Box_R(U) \cup \Box_R(V) \subseteq \Box_R(U \cup \Box_R(V))$.

Proposition 4.2. *Let $\mathbf{A}_{\Box}$ be an* **S4**-*nearlattice. The following conditions are equivalents:*

(1) *$\mathbf{A}_{\Box}$ is an* **S5**-*nearlattice.*
(2) *For every $P \in X(\mathbf{A})$ and for every $a \in A$, $\Box a \in P$ if and only if there exists $Q \in X(\mathbf{A})$ such that $(P,Q) \in E_{\Box}$ and $a \in Q$.*
(3) *$\varphi_{\mathbf{A}}(\Box a) = h_{E_{\Box}}(\varphi_{\mathbf{A}}(a))$ for all $a \in A$.*
(4) *$Q_{\Box} \subseteq \left(E_{\Box} \circ \subseteq^{-1}\right)$.*

Proof. (1) $\Rightarrow$ (2) Let $P \in X(\mathbf{A})$ and $a \in A$ such that $\Box a \in P$. By Lemma 2.1, $\Box^{-1}(P)^c \in \mathrm{Fi}(\mathbf{A})$. We consider the ideal $I = I((\Box(P)] \cup \{a\})$. Then $I \cap \Box^{-1}(P)^c = \emptyset$. Otherwise, there exists $b \in A$ such that $b \in \Box^{-1}(P)^c$ and there exists $c \in P$ such that $b \leq \Box c \vee a$. So, $\Box b \leq \Box(\Box c \vee a) \leq \Box c \vee \Box a$. Since $\Box b \notin P$, $\Box a \vee \Box c \notin P$. Also, as $\Box a \in P$, we have $\Box c \notin P$. On the other hand, $P \subseteq \Box^{-1}(P)$ and $c \in P$. Then $\Box c \in P$ which is a contradiction. Therefore, $I \cap \Box^{-1}(P)^c = \emptyset$ and by Theorem 1.1 there exists $Q \in X(\mathbf{A})$ such that $I \subseteq Q$ and $Q \cap \Box^{-1}(P)^c = \emptyset$. Then $a \in Q$, $\Box(P) \subseteq Q$ and $Q \subseteq \Box^{-1}(P)$. Since $\Box a \leq \Box^2 a$, we have $\Box^{-1}(Q) \subseteq \Box^{-1}(P)$ and $\Box^{-1}(P) \subseteq \Box^{-1}(Q)$, i.e, $\Box^{-1}(P) = \Box^{-1}(Q)$. Thus, by Lemma 4.3, $(P,Q) \in E_{\Box}$ and $a \in Q$.

Conversely, if there is $Q \in X(\mathbf{A})$ such that $(P,Q) \in E_{\Box}$ and $a \in Q$ then, by Lemma 4.3, $\Box^{-1}(P) = \Box^{-1}(Q)$. Since $\Box a \leq a$, $\Box a \in Q$ and $a \in \Box^{-1}(Q)$. So, $\Box a \in P$.

(2) $\Rightarrow$ (3) It is immediate.

(3) $\Rightarrow$ (4). Let $P, D \in X(\mathbf{A})$ such that $(P,D) \in Q_{\Box}$. We consider the filter $\Box^{-1}(P)^c$ and the ideal $I = I(D \cup (\Box(P)])$. Then $I \cap \Box^{-1}(P)^c = \emptyset$. If there exists $b \in A$ such that $b \in \Box^{-1}(P)^c$ and $b \in I$ then there exists $d \in D$ and $c \in P$ such that $b \leq d \vee \Box c$. So, $\Box b \leq \Box(d \vee \Box c)$. Since $\Box b \notin P$, we have $\Box(d \vee \Box c) \notin P$. By hypothesis and Remark 4 it follows that

$$
\begin{aligned}
P \in \varphi_{\mathbf{A}}(\Box(d \vee \Box c)) &= h_{E_{\Box}}(\varphi_{\mathbf{A}}(d \vee \Box c)) \\
&= h_{E_{\Box}}(\varphi_{\mathbf{A}}(d) \cup \varphi_{\mathbf{A}}(\Box c)) \\
&= h_{E_{\Box}}(\varphi_{\mathbf{A}}(d) \cup h_{E_{\Box}}(\varphi_{\mathbf{A}}(c))) \\
&= h_{E_{\Box}}(\varphi_{\mathbf{A}}(d)) \cup h_{E_{\Box}}(\varphi_{\mathbf{A}}(c)) \\
&= \varphi_{\mathbf{A}}(\Box d) \cup \varphi_{\mathbf{A}}(\Box c) \\
&= \varphi_{\mathbf{A}}(\Box d \vee \Box c).
\end{aligned}
$$

Then $P \in \varphi_{\mathbf{A}}(\Box d \vee \Box c)$, i.e., $\Box d \vee \Box c \notin P$. As $d \in D \subseteq \Box^{-1}(P)$, $\Box d \in P$. Thus, $\Box c \notin P$ which is a contradiction because $c \in P \subseteq \Box^{-1}(P)$. Therefore $I \cap \Box^{-1}(P)^c = \emptyset$ and by Theorem 1.1 there exists $S \in X(\mathbf{A})$ such that $I \subseteq S$ and $S \cap \Box^{-1}(P)^c = \emptyset$. So, $D \subseteq S$, $\Box(P) \subseteq S$ and $S \subseteq \Box^{-1}(P)$. It is easy to see that $(P,S) \in E_{\Box}$ and $(P,D) \in E_{\Box} \circ \subseteq^{-1}$. Then $Q_{\Box} \subseteq \left(E_{\Box} \circ \subseteq^{-1}\right)$.

(4) $\Rightarrow$ (1). Let $a, b \in A$ such that $\Box(\Box a \vee b) \not\leq \Box a \vee \Box b$. By Theorem 1.1, there exists $P \in X(\mathbf{A})$ such that $\Box a \vee \Box b \in P$ and $\Box(\Box a \vee b) \notin P$. Then $\Box a \in P$ and $\Box b \in P$. So, by Proposition 2.4, there exists $R \in X(\mathbf{A})$ such that $(P, R) \in Q_{\Box}$ and $b \in R$. Since $Q_{\Box} \subseteq (E_{\Box} \circ \subseteq^{-1})$, there is $Z \in X(\mathbf{A})$ such that $(Z, R) \in \subseteq^{-1}$ and $(P, Z) \in E_{\Box}$, i.e., $R \subseteq Z$ and $\Box^{-1}(P) = \Box^{-1}(Z)$. So, $b \in Z$. On the other hand, $a \in \Box^{-1}(P) = \Box^{-1}(Z)$ and $\Box a \in Z$. It follows that $\Box a \vee b \in Z$. As $\Box a \vee b \in Z \subseteq \Box^{-1}(Z) = \Box^{-1}(P)$, we have $\Box a \vee b \in \Box^{-1}(P)$ and $\Box(\Box a \vee b) \in P$ which is a contradiction. Therefore, $\mathbf{A}_{\Box}$ is an **S5**-nearlattice. $\Box$

The previous result suggests the following definition.

Definition 11. We say that the structure $\langle X, Q, \mathcal{K}\rangle$ is an **S5**-*space* if $\langle X, Q, \mathcal{K}\rangle$ is an $N\Box$-space and $Q \subseteq (Q \cap Q^{-1}) \circ \leq^{-1}$.

If $\langle X, Q, \mathcal{K}\rangle$ is an **S5**-space then $\langle D_{\mathcal{K}}(X), \Box_Q\rangle$ is an **S5**-nearlattice.

Remark 5. Let $\langle X, Q, \mathcal{K}\rangle$ be an **S5**-space. We consider the equivalence relation $E_Q = Q \cap Q^{-1}$. Then the structure $\langle X, E_Q, \mathcal{K}\rangle$ verifies the following conditions:

(1) $h_{E_Q}(U) = \{x \in X : E_Q(x) \subseteq U\} \in D_{\mathcal{K}}(X)$ for all $U \in D_{\mathcal{K}}(X)$.

(2) $E_Q(x)$ is a closed subset of X for all $x \in X$.

Conversely, if $\langle X, E, \mathcal{K}\rangle$ is a structure such that $\langle X, \mathcal{K}\rangle$ is an N-space and E is an equivalence relation on X satisfying the conditions (1) and (2) above then $\langle X, Q_E, \mathcal{K}\rangle$, where Q_E is a reflexive and transitive relation defined by $Q_E = E \circ \leq^{-1}$, is an **S5**-space. Thus, we can identify the **S5**-spaces with triples $\langle X, E, \mathcal{K}\rangle$ where $\langle X, \mathcal{K}\rangle$ is an N-space and E is an equivalence relation on X satisfying the conditions (1) and (2).

5. Application of the duality

In this section we give an application of the topological duality developed in Section 3.

5.1. $\Box$-congruences

Let $\mathbf{A} \in \mathcal{DN}$. In [9] it was shown that the distributive lattice of the congruences $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$ is dually isomorphic to certain subsets, called N-*subspaces*, of the dual space $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle$. We extend this result to the congruences of a $\Box$-distributive nearlattice.

Recall that if $\langle X, \mathcal{K}\rangle$ is a topological space with a base $\mathcal{K}$ and Y is a subset of X then the family $\mathcal{K}_Y = \{U \cap Y : U \in \mathcal{K}\}$ is a basis for a topology on Y and the pair $\langle Y, \mathcal{K}_Y\rangle$ is a topological space.

Definition 12. Let $\langle X, \mathcal{K}\rangle$ be an N-space and let Y be a subset of X. We say that Y is an N-*subspace* if $\langle Y, \mathcal{K}_Y\rangle$ is an N-space.

Denote by $\mathcal{S}(X)$ the set of all N-subspaces of an N-space $\langle X, \mathcal{K}\rangle$.

Let $\mathbf{A} \in \mathcal{DN}$ and let $\langle X(\mathbf{A}), \mathcal{K}_{\mathbf{A}}\rangle$ be the dual space of $\mathbf{A}$. By [9], there exists a dual correspondence $Y \to \theta(Y)$ from the N-subspaces $\mathcal{S}(X(\mathbf{A}))$ of $X(\mathbf{A})$ onto the congruences $\mathrm{Con}(\mathbf{A})$ of $\mathbf{A}$. If Y is a subset of $X(\mathbf{A})$ then the binary relation $\theta(Y) \subseteq A \times A$ given by $(a, b) \in \theta(Y)$ if and only if $\varphi_{\mathbf{A}}(a)^c \cap Y = \varphi_{\mathbf{A}}(b)^c \cap Y$ is a congruence on $\mathbf{A}$. Conversely, if $\theta \in \mathrm{Con}(\mathbf{A})$ then we have the canonical map $q_\theta \colon A \to A/\theta$ where each element $a \in A$ is assigned the equivalence class $q_\theta(a) = a/\theta$. We consider the set

$$Y_\theta = \left\{q_\theta^{-1}(P) : P \in X(\mathbf{A}/\theta)\right\}.$$

Then $Y_\theta \subseteq X(\mathbf{A})$ and the pair $\langle Y_\theta, \mathcal{K}_{Y_\theta}\rangle$ is an N-subspace. For more details see [9].

Definition 13. Let $\langle X, \mathcal{K}, Q\rangle$ be an $N\Box$-space and let Y be a subset of X. We say that Y is *Q-saturated* if $\langle Y, \mathcal{K}_Y\rangle$ is an N-subspace and $\max Q(x) \subseteq Y$ for all $x \in Y$.

We denote by $\mathcal{S}_Q(X)$ the set of all Q-saturated subsets of an $N\Box$-space $\langle X, \mathcal{K}, Q\rangle$.

Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. The congruences of $\mathbf{A}_\Box$ are congruences of $\mathbf{A}$ compatible with the operator $\Box$, i.e., if $(a, b) \in \theta$ then $(\Box a, \Box b) \in \theta$. Denote by $\mathrm{Con}_\Box(\mathbf{A})$ the lattice of all congruences of $\mathbf{A}_\Box$.

Theorem 5.1. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$ and let $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}, Q_\Box\rangle$ be the dual space of $\mathbf{A}_\Box$. Then the map*

$$F\colon \mathcal{S}_{Q_\Box}(X(\mathbf{A})) \to \mathrm{Con}_\Box(\mathbf{A})$$

defined by $F(Y) = \theta(Y)$ is a dual isomorphism.

Proof. Let Y be a $Q_\Box$-saturated subset of $X(\mathbf{A})$. We prove that $\theta(Y)$ is compatible with the operator $\Box$. Let $a, b \in A$ such that $(a, b) \in \theta(Y)$ and suppose that $\varphi_\mathbf{A}(\Box a)^c \cap Y \neq \varphi_\mathbf{A}(\Box b)^c \cap Y$, i.e., there is $P \in \varphi_\mathbf{A}(\Box a)^c$ such that $P \notin \varphi_\mathbf{A}(\Box b)^c$. Then $P \in Y$, $\Box a \in P$ and $\Box b \notin P$. Thus, by Proposition 2.4, there exists $R \in X(\mathbf{A})$ such that $R \subseteq \Box^{-1}(P)$ and $a \in R$. Consider the family

$$\mathcal{G} = \left\{H \in X(\mathbf{A}) : R \subseteq H \subseteq \Box^{-1}(P)\right\}.$$

Then $\mathcal{G}$ is non-empty and by Zorn's Lemma there exists a maximal element $D \in \mathcal{G}$. As $a \in R \subseteq D$, $D \in \varphi_\mathbf{A}(a)^c$. On the other hand, $D \in \max Q_\Box(P) \subseteq Y$. So, $D \in \varphi_\mathbf{A}(a)^c \cap Y = \varphi_\mathbf{A}(b)^c \cap Y$ and $D \in \varphi_\mathbf{A}(b)^c$, i.e., $b \in D \subseteq \Box^{-1}(P)$. Then $\Box b \in P$ which is a contradiction. Thus, $(\Box a, \Box b) \in \theta(Y)$.

Reciprocally, let $\theta \in \mathrm{Con}_\Box(\mathbf{A})$. We know that the pair $\langle Y_\theta, \mathcal{K}_\theta\rangle$ is an N-space such that $F(Y_\theta) = \theta$. We see that Y_θ is a $Q_\Box$-saturated subset. Let $P \in Y_\theta$ and $R \in \max Q_\Box(P)$ such that $R \notin Y_\theta$. Then $R \subseteq \Box^{-1}(P)$ and we consider

$$\bigcap\left\{\varphi_\mathbf{A}(b) \cap Y_\theta \colon \varphi_\mathbf{A}(b) \notin H_{X(\mathbf{A})}(R)\right\} \cap \bigcap\left\{\varphi_\mathbf{A}(c)^c \cap Y_\theta \colon \varphi_\mathbf{A}(c) \in H_{X(\mathbf{A})}(R)\right\}.$$

If this intersection is non-empty then there exists $D \in X(\mathbf{A})$ such that $H_{X(\mathbf{A})}(D) = H_{X(\mathbf{A})}(R)$ and since $H_{X(\mathbf{A})}$ is 1-1, $D = R \in Y_\theta$ which is a contradiction. Then

$$\bigcap\left\{\varphi_\mathbf{A}(b) \cap Y_\theta \colon \varphi_\mathbf{A}(b) \notin H_{X(\mathbf{A})}(R)\right\} \subseteq \bigcup\left\{\varphi_\mathbf{A}(c) \cap Y_\theta \colon \varphi_\mathbf{A}(c) \in H_{X(\mathbf{A})}(R)\right\}.$$

Let $B = \left\{b \colon \varphi_\mathbf{A}(b) \notin H_{X(\mathbf{A})}(R)\right\}$ and $C = \left\{c \colon \varphi_\mathbf{A}(c) \in H_{X(\mathbf{A})}(R)\right\}$. Since Y_θ is an N-space, by Theorem 1.3, there exist $b_1, \ldots, b_n \in [B)$ and $c_1, \ldots, c_m \in C$ such that $b_1 \wedge \cdots \wedge b_n$ exists and

$$[\varphi_\mathbf{A}(b_1) \cap Y_\theta] \cap \cdots \cap [\varphi_\mathbf{A}(b_n) \cap Y_\theta] \subseteq [\varphi_\mathbf{A}(c_1) \cap Y_\theta] \cup \cdots \cup [\varphi_\mathbf{A}(c_m) \cap Y_\theta].$$

As $b_1, \ldots, b_n \in [B)$, there exist $\overline{b_1}, \ldots, \overline{b_n} \in B$ such that $\overline{b_i} \leq b_i$ for all $i = 1, \ldots, n$. Let $b = b_1 \wedge \cdots \wedge b_n$ and $c = c_1 \vee \cdots \vee c_m$. Then $\varphi_\mathbf{A}(b) \cap Y_\theta \subseteq \varphi_\mathbf{A}(c) \cap Y_\theta$, or equivalently, $\varphi_\mathbf{A}(c)^c \cap Y_\theta \subseteq \varphi_\mathbf{A}(b)^c \cap Y_\theta$. So, the pair $(b \vee c, c) \in \theta(Y_\theta)$ and as $\theta(Y_\theta)$ is a congruence, $(\Box(b \vee c), \Box c) \in \theta(Y_\theta)$, i.e., $\varphi_\mathbf{A}(\Box(b \vee c))^c \cap Y_\theta = \varphi_\mathbf{A}(\Box c)^c \cap Y_\theta$. It is clear that $P \in \varphi_\mathbf{A}(\Box c)^c$ because if $P \in \varphi_\mathbf{A}(\Box c)$ then $c \notin \Box^{-1}(P)$ and $c \notin R$. Also, since $\varphi_\mathbf{A}(c) \in H_{X(\mathbf{A})}(R)$ we have $R \notin \varphi_\mathbf{A}(c)$. So, $c \in R$ which is a contradiction. Then $P \in \varphi_\mathbf{A}(\Box c)^c \cap Y_\theta$ and $P \in \varphi_\mathbf{A}(\Box(b \vee c))^c \cap Y_\theta$. Thus, $b \vee c \notin \Box^{-1}(P)^c$ and as $\Box^{-1}(P)^c$ is a filter it follows that $\Box b \in P$. Then $\Box b = \Box(b_1 \wedge \cdots \wedge b_n) = \Box b_1 \wedge \cdots \wedge \Box b_n \in P$ and for primality of P there exists $i \in \{1, \ldots, n\}$ such that $\Box b_i \in P$. So, $\Box\overline{b_i} \leq \Box b_i$ and $\Box\overline{b_i} \in P$. By Proposition 2.4, there is $D \in X(\mathbf{A})$ such that $D \subseteq \Box^{-1}(P)$ and $\overline{b_i} \in D$. Since R is maximal, we have $D \subseteq R$ and $\overline{b_i} \in R$. On the other hand, as $\varphi_\mathbf{A}(\overline{b_i}) \notin H_{X(\mathbf{A})}(R)$, $\overline{b_i} \notin R$ which is a contradiction. Therefore, Y_θ is a $Q_\Box$-saturated subset of $X(\mathbf{A})$ and F is a dual isomorphism. $\Box$

5.2. □-subalgebras

In this subsection we characterize the subalgebras of a □-distributive nearlattice $\mathbf{A}_\Box$. The set of all subalgebras of $\mathbf{A}_\Box$ is denoted by $\mathrm{Sub}_\Box(\mathbf{A})$.

Definition 14. Let $\langle X, \mathcal{K}\rangle$ be an N-space and let $\mathcal{L}$ be a non-empty subset of $\mathcal{K}$. We say that $\mathcal{L}$ is *N-basic* if $(U \cap W) \cup (V \cap W) \in \mathcal{L}$ for all $U, V, W \in \mathcal{L}$.

Let $\mathbf{A} \in \mathcal{DN}$. By the results developed in [9], there is a correspondence between subalgebras of $\mathbf{A}$ and N-basic subsets of the dual space $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}\rangle$. We have the following definition.

Definition 15. Let $\langle X, \mathcal{K}, Q\rangle$ be an $N\Box$-space and let $\mathcal{L}$ be a non-empty subset of $\mathcal{K}$. We say that $\mathcal{L}$ is *$N\Box$-basic* if it is N-basic and $\Box_Q(U^c)^c \in \mathcal{L}$ for all $U \in \mathcal{L}$.

We consider $\mathrm{Nb}_\Box(X) = \{\mathcal{L} \subseteq \mathcal{K} \colon \mathcal{L} \text{ is } N\Box\text{-basic}\}$.

Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. We define $T(\mathbf{B}) = \{\varphi_\mathbf{A}(b)^c : b \in B\}$ for each $\mathbf{B} \in \mathrm{Sub}(\mathbf{A})$.

Lemma 5.1. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. If $\mathbf{B} \in \mathrm{Sub}_\Box(\mathbf{A})$ then $T(\mathbf{B}) \in \mathrm{Nb}_\Box(X(\mathbf{A}))$.*

Proof. Let $U \in T(\mathbf{B})$. Then there is $b \in B$ such that $U = \varphi_\mathbf{A}(b)^c$. Since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, $\Box b \in B$ and $\varphi_\mathbf{A}(\Box b)^c \in T(\mathbf{B})$. On the other hand, by Proposition 2.4, $\Box_{Q_\Box}(\varphi_\mathbf{A}(b)) = \varphi_\mathbf{A}(\Box b)$. Then $\Box_{Q_\Box}(\varphi_\mathbf{A}(b))^c = \Box_{Q_\Box}(U^c)^c \in T(\mathbf{B})$ and $T(\mathbf{B}) \in \mathrm{Nb}_\Box(X(\mathbf{A}))$. □

Let $\mathcal{L} \in \mathrm{Nb}_\Box(X(\mathbf{A}))$. We consider $S(\mathcal{L}) = \{a \in A \colon \varphi_\mathbf{A}(a)^c \in \mathcal{L}\}$.

Lemma 5.2. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. If $\mathcal{L} \in \mathrm{Nb}_\Box(X(\mathbf{A}))$ then $S(\mathcal{L}) \in \mathrm{Sub}_\Box(\mathbf{A})$.*

Proof. We prove that $S(\mathcal{L})$ is closed under the modal operator □. If $a \in S(\mathcal{L})$ then $\varphi_\mathbf{A}(a)^c \in \mathcal{L}$. As $\mathcal{L}$ is $N\Box$-basic and by Proposition 2.4, we have $\Box_{Q_\Box}(\varphi_\mathbf{A}(a))^c = \varphi_\mathbf{A}(\Box a)^c \in \mathcal{L}$. So, $\Box a \in S(\mathcal{L})$. Therefore $S(\mathcal{L}) \in \mathrm{Sub}_\Box(\mathbf{A})$. □

Theorem 5.2. *Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$ and let $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}, Q_\Box\rangle$ be the dual space of $\mathbf{A}_\Box$. Then the lattices $\mathrm{Sub}_\Box(\mathbf{A})$ and $\mathrm{Nb}_\Box(X(\mathbf{A}))$ are isomorphic.*

Proof. By definition of T and S it follows that $S(T(\mathbf{B})) = \mathbf{B}$ for all $\mathbf{B} \in \mathrm{Sub}_\Box(\mathbf{A})$ and $T(S(\mathcal{L})) = \mathcal{L}$ for all $\mathcal{L} \in \mathrm{Nb}_\Box(X(\mathbf{A}))$. The result is immediate. □

5.3. The free □-distributive lattice extension

Let $\mathbf{A} \in \mathcal{DN}$ and let $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}\rangle$ be the dual space of $\mathbf{A}$. Let $\mathcal{KO}(X(\mathbf{A}))$ be the family of all open and compact subsets of $\langle X(\mathbf{A}), \mathcal{K}_\mathbf{A}\rangle$. We consider $D_{\mathcal{KO}}[X(\mathbf{A})] = \{U \colon U^c \in \mathcal{KO}(X(\mathbf{A}))\}$. Thus, $U \in D_{\mathcal{KO}}[X(\mathbf{A})]$ if and only if there exist $a_1, \ldots, a_n \in A$ such that $U = \varphi_\mathbf{A}(a_1) \cap \cdots \cap \varphi_\mathbf{A}(a_n)$. In [10] it was shown that the structure $\langle D_{\mathcal{KO}}[X(\mathbf{A})], \cup, \cap, \emptyset, X(\mathbf{A})\rangle$ is a bounded distributive lattice and the pair $\langle D_{\mathcal{KO}}[X(\mathbf{A})], \varphi_\mathbf{A}\rangle$ is the free distributive lattice extension where the map $\varphi_\mathbf{A} \colon A \to D_{\mathcal{KO}}[X(\mathbf{A})]$ is given by $\varphi_\mathbf{A}(a) = \{P \in X(\mathbf{A}) : a \notin P\}$. Now, we extend these results to the class of □-distributive nearlattices.

Definition 16. Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. A pair $\mathbf{L}_\Box = \langle\langle L, \Box\rangle, e\rangle$, where $\langle L, \Box\rangle$ is a bounded distributive lattice with a necessity modal operator and $e \colon A \to L$ a 1-1 □-homomorphism, is a *□-free distributive lattice extension of* $\mathbf{A}_\Box$ if the following universal property holds: for every bounded distributive lattice with a necessity modal operator $\langle \overline{L}, \Delta\rangle$ and every □-homomorphism $h \colon A \to \overline{L}$, there exists a unique □-homomorphism $\overline{h} \colon L \to \overline{L}$ such that $h = \overline{h} \circ e$.

Let $\mathbf{A}_\Box \in \mathcal{DN}_\Box$. We consider the map $\overline{\Box} \colon D_{\mathcal{KO}}[X(\mathbf{A})] \to D_{\mathcal{KO}}[X(\mathbf{A})]$ given by

$$\overline{\Box}(U) = \varphi_\mathbf{A}(\Box a_1) \cap \cdots \cap \varphi_\mathbf{A}(\Box a_n),$$

where $U = \varphi_\mathbf{A}(a_1) \cap \cdots \cap \varphi_\mathbf{A}(a_n)$ for some $a_1, \ldots, a_n \in A$. Note that:

(1) $\overline{\Box}\left(X\left(\mathbf{A}\right)\right) = X\left(\mathbf{A}\right)$,

(2) $\overline{\Box}\left(U \cap V\right) = \overline{\Box}\left(U\right) \cap \overline{\Box}\left(V\right)$ for all $U, V \in D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right]$.

So, $\overline{\Box}$ is a modal operator on $D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right]$, i.e., the structure $\left\langle D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right], \cup, \cap, \overline{\Box}, \emptyset, X\left(\mathbf{A}\right)\right\rangle$ is a bounded distributive lattice with a necessity modal operator.

Theorem 5.3. *Let $\mathbf{A}_{\Box} \in \mathcal{DN}_{\Box}$ and let $\left\langle X\left(\mathbf{A}\right), \mathcal{K}_{\mathbf{A}}, Q_{\Box}\right\rangle$ be the dual space of $\mathbf{A}_{\Box}$. Then the pair $\left\langle\left\langle D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right], \overline{\Box}\right\rangle, \varphi_{\mathbf{A}}\right\rangle$ is the $\Box$-free distributive lattice extension of $\mathbf{A}_{\Box}$.*

Proof. Let $\left\langle \overline{L}, \Delta\right\rangle$ be a bounded distributive lattice with a necessity modal operator and let $h\colon A \to \overline{L}$ be a $\Box$-homomorphism. We define $\overline{h}\colon D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right] \to \overline{L}$ by

$$\overline{h}\left(\varphi_{\mathbf{A}}\left(a_1\right) \cap \dots \cap \varphi_{\mathbf{A}}\left(a_n\right)\right) = h\left(a_1\right) \wedge \dots \wedge h\left(a_n\right).$$

By [10], we have that $\overline{h}$ is a unique homomorphism such that $h = \overline{h} \circ \varphi$. We only prove that $\overline{h}\left(\overline{\Box}\left(U\right)\right) = \Delta\left(\overline{h}\left(U\right)\right)$ for all $U \in D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right]$. Thus, if $U \in D_{\mathcal{KO}}\left[X\left(\mathbf{A}\right)\right]$ then there exist $a_1, \dots, a_n \in A$ such that $U = \varphi_{\mathbf{A}}\left(a_1\right) \cap \dots \cap \varphi_{\mathbf{A}}\left(a_n\right)$. Then

$$\begin{aligned}
\overline{h}\left(\overline{\Box}\left(U\right)\right) &= \overline{h}\left(\overline{\Box}\left(\varphi_{\mathbf{A}}\left(a_1\right) \cap \dots \cap \varphi_{\mathbf{A}}\left(a_n\right)\right)\right) = \overline{h}\left(\varphi_{\mathbf{A}}\left(\Box a_1\right) \cap \dots \cap \varphi_{\mathbf{A}}\left(\Box a_n\right)\right) \\
&= h\left(\Box a_1\right) \wedge \dots \wedge h\left(\Box a_n\right) = \Delta\left(h\left(a_1\right)\right) \wedge \dots \wedge \Delta\left(h\left(a_n\right)\right) \\
&= \Delta\left(h\left(a_1\right) \wedge \dots \wedge h\left(a_n\right)\right) = \Delta\left(\overline{h}\left(\varphi_{\mathbf{A}}\left(a_1\right) \cap \dots \cap \varphi_{\mathbf{A}}\left(a_n\right)\right)\right) \\
&= \Delta\left(\overline{h}\left(U\right)\right).
\end{aligned}$$

Therefore, $\overline{h}$ is a $\Box$-homomorphism. $\Box$

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