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A GENERATING FUNCTION RELATED TO Π

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Abstract. The generating function (in the variable t) given by

$$\int_0^1\int_0^1\int_0^1 \frac{v\sqrt{d}}{(d+x^2)\sqrt{1-v^2}\sqrt{1-u^2}}\frac{(1+tu^2)^2}{\sqrt{\{1+tu^2(2-4x^2u^2v^2+tu^2)\}^3}}dudvdx$$

is shown to be related to certain algebraic approximations to π.

1. Introduction and main result.

The aim of this note is to present a new generating function given by formula (2) which (in some sense) is related to certain algebraic approximations to π. For a source book on π we refer the reader to [3]. Our result is not meant to compete with other existing results. We prove the following theorem.

Theorem 1.1. *Let $0<d,\sqrt{d}$. For $n=1,2,3,\dots,$ set*

$$A_n(d) :=$$

$$\frac{(-1)^n}{4^n}\sum_{j=0}^{n}(-1)^j\binom{2n+2j}{2n}\binom{2n}{n-j}\Big\{\frac{1}{2j-1}-\frac{d}{2j-3}+\frac{d^2}{2j-5}-\cdots+(-d)^{j-1}\Big\},$$

$$B_n(d) := \frac{(-1)^n}{4^n}\sum_{j=0}^{n}\binom{2n+2j}{2n}\binom{2n}{n-j}d^j, \tag{1}$$

and $A_0(d)=0, B_0(d)=1$. Note that in $A_n(d)$, the bracket $\{\}=0$ if $j=0$.

i) *For t in a neighbourhood of zero one has*

$$\sum_{n=0}^{\infty}\Big\{\sqrt{d}A_n(d)+B_n(d)\arctan\Big(\frac{1}{\sqrt{d}}\Big)\Big\}t^n = \tag{2}$$

$$\frac{2}{\pi}\int_0^1\int_0^1\int_0^1 \frac{v\sqrt{d}}{(d+x^2)\sqrt{1-v^2}\sqrt{1-u^2}}\frac{(1+tu^2)^2}{\sqrt{\{1+tu^2(2-4x^2u^2v^2+tu^2)\}^3}}dudvdx.$$

ii) *If $d\geq 3$ then $B_n(d), A_n(d)\neq 0$ for all $n\geq 1$. Also $A_n(d)4^n lcm\{1,\dots,2n\}$*

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and $B_n(d)4^n$ *are polynomials of degree* $n-1$ *and* n *respectively in* $Z(d)$. *Let* d *be any of the numbers* $d_0 = 3$, $d_1 = 7+4\sqrt{3}$, $d_2 = 15+8\sqrt{3}+2\sqrt{104+60\sqrt{3}}, \ldots$, *where* d_k *is obtained recursively from the formula*

$$d_k = 2d_{k-1} + 1 + 2\sqrt{d_{k-1}(d_{k-1}+1)}.$$

Then for $n \geq 1, k \geq 0$ *one has*

$$0 < |\sqrt{d_k}A_n(d_k) + B_n(d_k)\frac{\pi}{6.2^k}| \leq \frac{(\sqrt{d_k+1}-\sqrt{d_k})^{2n}}{\sqrt{d_k}}, \tag{3}$$

and $\frac{\pi}{6.2^k} = \arctan(\frac{1}{\sqrt{d_k}})$.

2. Proof.

i) Set

$$F_n(x) := \sum_{j=0}^{n} \frac{(-1)^{n+j}}{4^n} \binom{2n+2j}{2n} \binom{2n}{n-j} x^{2j},$$

and considering $i = +\sqrt{-1}$ observe that

$$\begin{aligned} A_n(d) &= \int_0^1 \frac{F_n(x) - F_n(i\sqrt{d})}{d+x^2} dx, \\ B_n(d) &= F_n(i\sqrt{d}), \\ \arctan\left(\frac{1}{\sqrt{d}}\right) &= \sqrt{d}\int_0^1 \frac{dx}{d+x^2} \end{aligned}$$

$$\textit{and} \qquad \sqrt{d}\int_0^1 \frac{F_n(x)dx}{d+x^2} = A_n(d)\sqrt{d} + B_n(d)\arctan\left(\frac{1}{\sqrt{d}}\right). \tag{4}$$

To prove the formula (2), recall the notations $(2m-1)!! = 1.3.5\cdots(2m-1)$, $(2m)!! = 2.4\cdots(2m)$ (see [1]) and notice that

$$\frac{\pi}{2}\frac{(2m-1)!!}{(2m)!!} = \int_0^1 \frac{x^{2m}}{\sqrt{1-x^2}} dx,$$

and

$$\frac{(2m)!!}{(2m+1)!!} = \int_0^1 \frac{x^{2m+1}}{\sqrt{1-x^2}} dx,$$

so that for $0 \leq j, 1 \leq n$

$$\frac{[2(j+n)-1][2(j+n)-3]\cdots(2j+3)(2j+1)}{(j+n)(j+n-1)\cdots(j+1)}$$
$$= racu^{2(j+n)}\sqrt{1-u^2}du \int_0^1 \frac{v^{2j+1}}{\sqrt{1-v^2}} dv,$$

and

$$\frac{[2(j+n)-1][2(j+n)-3]\cdots(2j+3)(2j+1)}{(j+n)(j+n-1)\cdots(j+1)} \binom{n}{j}\binom{n+j}{j}$$
$$= \frac{1}{2^n}\binom{2n+2j}{2n}\binom{2n}{n-j}.$$

Thus

$$F_n(x) = \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\binom{n+j}{j}(2j+1)\Big(\frac{2}{\pi}\int_0^1 \frac{u^{2(j+n)}}{\sqrt{1-u^2}}du \int_0^1 \frac{v^{2j+1}}{\sqrt{1-v^2}}dv\Big)x^{2j}$$

$$= \frac{2}{\pi}\int_0^1\int_0^1 G_n(xuv)\frac{u^{2n}}{\sqrt{1-u^2}}\frac{v}{\sqrt{1-v^2}}dudv, \tag{5}$$

where

$$G_n(w) := \sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\binom{n+j}{j}(2j+1)w^{2j}.$$

The definition of Legendre polynomials (see [2]) gives

$$\frac{1}{\sqrt{1+2y(1-2w^2)+y^2}} = \sum_{n=0}^{\infty} y^n\Big\{\sum_{j=0}^{n}(-1)^{n+j}\binom{n}{j}\binom{n+j}{j}w^{2j}\Big\},$$

and therefore

$$\sum_{n=0}^{\infty} G_n(w)y^n = \frac{d}{dw}\Big\{\frac{w}{\sqrt{1+2y(1-2w^2)+y^2}}\Big\} = \frac{(1+y)^2}{(1+y(2-4w^2+y))^{3/2}}. \tag{6}$$

The formula (2) now follows from (4), (5) and (6).

ii) From formula (1) it follows that $B_n(d) \neq 0$ if $d > 0$, $n = 1, 2, 3, \ldots$. Also notice that $\frac{1}{2j-1} - \frac{d}{2j-3} + \frac{d^2}{2j-5} - \ldots + (-d)^{j-1}$ is equal to $(-d)^{j-1}$ multiplied by something positive if $d \geq 3$; which gives, using formula (1), that $A_n(d) \neq 0$ as stated.

That $A_n(d)4^n lcm\{1, \ldots, 2n\}$ and $B_n(d)4^n$ are polynomials of degree $n-1$ and n respectively in $Z(d)$ is trivial from the definitions.

The following formula holds: if $1 \leq d, n$; $d \in \mathbb{R}$ then

$$0 < |\sqrt{d}A_n(d) + B_n(d)\arctan\Big(\frac{1}{\sqrt{d}}\Big)| \leq \frac{(\sqrt{d+1}-\sqrt{d})^{2n}}{\sqrt{d}}.$$

This yields the inequality (3). The above formula is a special case of Proposition 1 in [4] and we refer the reader to that paper.

Finally the recursive formula for the d_k is obtained from the relation

$$\arctan(z) = 2\arctan\Big(\frac{z}{\sqrt{z^2+1}+1}\Big).$$

Thus putting $z = \frac{1}{\sqrt{d}}$ in this last formula gives

$$\arctan\frac{1}{\sqrt{d}} = 2\arctan\frac{1}{\sqrt{d'}}, \qquad \text{if} \qquad d' = 2d+1+2\sqrt{d(d+1)}.$$

It follows that

$$\arctan\frac{1}{\sqrt{d_0}} = 2\arctan\frac{1}{\sqrt{d_1}} = \cdots = 2^k\arctan\frac{1}{\sqrt{d_k}}$$

For the sequence starting with $d_0 = 3$, notice that $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. This completes the proof of the formula (3). □

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