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Semi-Heyting Algebras Term-equivalent to Gödel Algebras

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Abstract In this paper we investigate those subvarieties of the variety $\mathcal{SH}$ of semi-Heyting algebras which are term-equivalent to the variety $\mathcal{L}_{\mathcal{H}}$ of Gödel algebras (linear Heyting algebras). We prove that the only other subvarieties with this property are the variety $\mathcal{L}^{\text{Com}}$ of commutative semi-Heyting algebras and the variety $\mathcal{L}^{\vee}$ generated by the chains in which $a < b$ implies $a \to b = b$. We also study the variety $\mathcal{C}$ generated within $\mathcal{SH}$ by $\mathcal{L}_{\mathcal{H}}$, $\mathcal{L}_{\vee}$ and $\mathcal{L}_{\text{Com}}$. In particular we prove that $\mathcal{C}$ is locally finite and we obtain a construction of the finitely generated free algebra in $\mathcal{C}$.

Keywords Semi-Heyting algebra · Heyting algebra · Linear Heyting algebra · Term-equivalent varieties

1 Introduction and Preliminaries

In [7], Sankappanavar considered the following conjecture: there exists a variety $\mathcal{V}$ of algebras (of the same type as that of Heyting algebras) that possesses the following well known properties and includes Heyting algebras: (1) the algebras in $\mathcal{V}$ are pseudocomplemented, (2) they are distributive, and congruences on them are determined by filters. He settled this conjecture with the discovery of semi-Heyting algebras.

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Definition 1.1 An algebra $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a semi-Heyting algebra if the following conditions hold:

($SH1$) $\langle A, \vee, \wedge, 0, 1\rangle$ is a lattice with 0 and 1.
($SH2$) $x \wedge (x \rightarrow y) \approx x \wedge y$.
($SH3$) $x \wedge (y \rightarrow z) \approx x \wedge [(x \wedge y) \rightarrow (x \wedge z)]$.
($SH4$) $x \rightarrow x \approx 1$.

We will denote by $\mathcal{SH}$ the variety of semi-Heyting algebras. The variety $\mathcal{H}$ of Heyting algebras is the subvariety of $\mathcal{SH}$ characterized by the equation $(x \wedge y) \rightarrow x \approx 1$ [7].

Any semi-Heyting algebra $\mathbf{A}$ is a pseudocomplemented distributive lattice, congruences on $\mathbf{A}$ are determined by filters and the variety of semi-Heyting algebras is arithmetic, thus extending the corresponding results of Heyting algebras. Besides, semi-Heyting algebras share with Heyting algebras some other strong properties.

In the next lemma we include some useful elementary properties of semi-Heyting algebras.

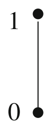
Lemma 1.2 [7] *Let* $\mathbf{A} \in \mathcal{SH}$ *and* $a, b \in A$.

(a) *If* $a \rightarrow b = 1$ *then* $a \leq b$.
(b) *If* $a \leq b$ *then* $a \leq a \rightarrow b$.
(c) $a = b$ *if and only if* $a \rightarrow b = b \rightarrow a = 1$.
(d) $1 \rightarrow a = a$.

Proof From $a \rightarrow b = 1$ and (SH3), we get $a \wedge 1 = a \wedge b$, that is $a = a \wedge b$, and we have (a). For (b), by (SH3) and since $a \leq b$ it follows that $a = a \wedge (a \rightarrow b) \leq a \rightarrow b$. Property (c) is clear. To prove (d), observe that $a = 1 \wedge a = 1 \wedge (1 \rightarrow a) = 1 \rightarrow a$. □

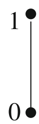
The algebras **2** and $\bar{\mathbf{2}}$, which have the two-element chain as their lattice reduct and whose $\rightarrow$ operation is defined in the following figure, are the most elementary examples of semi-Heyting algebras. One easily verifies that **2** is a Heyting algebra while $\bar{\mathbf{2}}$ is not.

2 :



$\rightarrow$	0	1
0	1	1
1	0	1

$\bar{\mathbf{2}}$:



$\rightarrow$	0	1
0	1	0
1	0	1

Observe that, in $\mathcal{SH}$, the (Heyting) property "if $a \leq b$ then $a \rightarrow b = 1$" does not hold.

We have the following characterization of subdirectly irreducible algebras in $\mathcal{SH}$ (see [7, Theorem 7.5]).

Theorem 1.3 *Let* $\mathbf{A} \in \mathcal{SH}$ *with* $|\mathbf{A}| \geq 2$. *The following are equivalent:*

(a) $\mathbf{A}$ *is subdirectly irreducible.*
(b) $\mathbf{A}$ *has a unique coatom.*

Observe that as a consequence of this theorem, if $\mathbf{A}$ is subdirectly irreducible, then $1 \in A$ is $\vee$-irreducible.

We say that $\mathbf{A} \in \mathcal{SH}$ is a *semi-Heyting chain* if the lattice reduct of $\mathbf{A}$ is totally ordered. We say that $\mathbf{A}$ is a *linear semi-Heyting algebra* if $\mathbf{A}$ belongs to the subvariety of $\mathcal{SH}$ generated by semi-Heyting chains. This subvariety will be denoted by $\mathcal{L}$.

Theorem 1.4 [1] *An equational basis for* $\mathcal{L}$ *relative to* $\mathcal{SH}$ *is given by*

$$((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1 \tag{Ch}$$

Proof Let us prove that if $\mathbf{A}$ is a semi-Heyting chain, then $\mathbf{A}$ satisfies the identity (Ch). Let $a, b \in A$. If $a \leq b$, by Lemma 1.2, $a \leq a \to b$. Hence $a \vee (a \to b) = a \to b$, so $(a \vee (a \to b)) \to (a \to b) = (a \to b) \to (a \to b) = 1$. If $b < a$, $b = b \wedge a$ and then $b \to (a \wedge b) = b \to b = 1$.

Conversely, let us prove that if $\mathbb{V}$ is the subvariety of $\mathcal{SH}$ defined by (Ch) then every subdirectly irreducible algebra in $\mathbb{V}$ is a chain. Let $\mathbf{A} \in \mathbb{V}$ subdirectly irreducible and let $a, b \in A$. Since A satisfies (Ch), $((a \vee (a \to b)) \to (a \to b)) \vee (b \to (a \wedge b)) = 1$. As $\mathbf{A}$ is subdirectly irreducible, 1 is $\vee$-irreducible. Thus $(a \vee (a \to b)) \to (a \to b) = 1$ or $b \to (a \wedge b) = 1$.

Suppose that $(a \vee (a \to b)) \to (a \to b) = 1$. Then $a \vee (a \to b) \leq a \to b$, and thus, $a \leq a \vee (a \to b) \leq a \to b$. Hence $a \wedge b = a \wedge (a \to b) = a$, so $a \leq b$.

If $b \to (a \wedge b) = 1$, then $b \leq a \wedge b$, and thus $b \leq a$.

Hence $\mathbf{A}$ is a chain. □

Recall that the lower half, including the main diagonal, of the operation table of $\to$ in a semi-Heyting chain is uniquely determined [7, Theorem 4.3], that is, if $\mathbf{A}$ is a semi-Heyting chain, $a, b \in A$, and $a < b$ then $b \to a = a$, and consequently we have just to consider the cases $a \to b$ whenever $a < b$ in order to complete the operation table of $\to$. In this paper we are going to consider those semi-Heyting chains in which $a \to b \in \{1, a, b\}$ for $a < b$, and this will led us to consider three subvarieties of $\mathcal{L}$.

1. The first one is the subvariety generated be the semi-Heyting chains that satisfy that $a < b$ implies $a \to b = 1$, and this is the variety $\mathcal{L}_{\mathcal{H}}$ of *Gödel algebras* (also known as the variety of *linear Heyting algebras*).
2. In the second place, we will consider the subvariety generated by the semi-Heyting chains that satisfy that $a < b$ implies $a \to b = a$, and this is the subvariety $\mathcal{L}_{\text{Com}}$ of commutative linear semi-Heyting algebras.
3. Finally, we have the subvariety of $\mathcal{L}$ generated by the semi-Heyting chains in which $a < b$ implies $a \to b = b$. This subvariety will be denoted by $\mathcal{L}_{\vee}$ since it satisfies that for $a < b$, $a \to b = a \vee b$.

The main objective of this paper is to prove that the subvarieties $\mathcal{L}_{\text{Com}}$ and $\mathcal{L}_{\vee}$ are both term-equivalent to the variety $\mathcal{L}_{\mathcal{H}}$ of Gödel algebras, and that they are the only other subvarieties of $\mathcal{L}$ term-equivalent to $\mathcal{L}_{\mathcal{H}}$.

As the second objective of the paper, we investigate the variety $\mathcal{C}$ generated within $\mathcal{SH}$ by $\mathcal{L}_{\mathcal{H}}$, $\mathcal{L}_{\vee}$ and $\mathcal{L}_{\mathrm{Com}}$. In particular we prove that $\mathcal{C}$ is locally finite and we obtain a construction of the finitely generated free algebra in $\mathcal{C}$.

2 The Subvariety $\mathcal{L}_{\mathrm{Com}}$

In this section we consider the subvariety generated by the semi-Heyting chains that satisfy that $a < b$ implies $a \to b = a$. This is the subvariety $\mathcal{L}_{\mathrm{Com}}$ of *commutative linear semi-Heyting algebras* characterized within $\mathcal{L}$ by the equation $x \to y \approx y \to x$. We prove that $\mathcal{L}_{\mathrm{Com}}$ is locally finite, we determine its lattice of subvarieties and we find an equational basis for each subvariety.

The following lemma is clear.

Lemma 2.1 *Given a totally ordered lattice* $\mathbf{A}$ *with least element* 0 *and last element* 1*, if for every* $a, b \in A$ *we define*

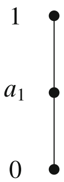
$$a \to b = \begin{cases} 1 & \text{for } a = b \\ a \wedge b & \text{for } a \neq b \end{cases}$$

then $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ *is a commutative linear semi-Heyting algebra.*

It is easy to see that the only structure of a commutative semi-Heyting algebra that can be defined on a chain is the one defined above.

For each integer $n \geq 0$, let L_n and L_ω respectively denote the chains $0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1$ and $0 = b_0 < b_1 < \ldots < b_{n-1} < b_n < \ldots < 1$, and let $\mathbf{L}_n^{\mathrm{Com}}$ and $\mathbf{L}_\omega^{\mathrm{Com}}$ denote the corresponding algebras in $\mathcal{L}_{\mathrm{Com}}$ with universes L_n and L_ω. We have that $\mathbf{L}_0^{\mathrm{Com}}$ is the algebra $\bar{\mathbf{2}}$ and $\mathbf{L}_1^{\mathrm{Com}}$ is the chain

$\mathbf{L}_1^c$



$\to$	0	a_1	1
0	1	0	0
a	0	1	a_1
1	0	a_1	1

Since the class of linear commutative semi-Heyting algebras is a subvariety of $\mathcal{L}$, every linear commutative semi-Heyting algebra can be represented as a subdirect product of commutative semi-Heyting chains.

We are going to prove that $\mathcal{L}_{\mathrm{Com}}$ is generated by the chains $\mathbf{L}_n^{\mathrm{Com}}$, $n \geq 0$.

Theorem 2.2 *A subvariety* $\mathcal{V}$ *of* $\mathcal{L}_{\mathrm{Com}}$ *is proper if and only if* $\mathbf{L}_n^{\mathrm{Com}} \notin \mathcal{V}$ *for some* $n \geq 0$.

Proof Suppose that $\mathcal{V}$ is a proper subvariety of $\mathcal{L}_{\mathrm{Com}}$, that is, $\mathcal{V} \neq \mathcal{L}_{\mathrm{Com}}$. Then there exists an identity $\epsilon \approx \lambda$ such that $\mathcal{V} \models \epsilon \approx \lambda$ and $\mathcal{L}_{\mathrm{Com}} \not\models \epsilon \approx \lambda$. Then there exists a chain $\mathbf{A} \in \mathcal{L}_{\mathrm{Com}}$ such that $\mathbf{A} \not\models \epsilon \approx \lambda$. Let $a_1, a_2, \ldots, a_m \in A$ such that $\epsilon(a_1, a_2, \ldots, a_m) \neq \lambda(a_1, a_2, \ldots, a_m)$, and consider the subalgebra $\mathbf{B}$ of $\mathbf{A}$ generated by the elements $a_1, a_2, \ldots, a_m$. Then $\mathbf{B}$ is a finite chain in $\mathcal{L}_{\mathrm{Com}}$, and consequently $\mathbf{B} \cong \mathbf{L}_n^{\mathrm{Com}}$ for some $n \geq 0$. So $\mathbf{L}_n^{\mathrm{Com}} \notin \mathcal{V}$. □

Corollary 2.3 *Every subvariety of* $\mathcal{L}_{\mathrm{Com}}$ *is generated by its finite chains.*

The following lemma is immediate.

Lemma 2.4

(a) $\mathbf{L}_n^{\mathrm{Com}}$ *is isomorphic to a subalgebra of* $\mathbf{L}_\omega^{\mathrm{Com}}$*, for every* $n \geq 0$.
(b) *The variety of linear commutative semi-Heyting algebras is generated by* $\mathbf{L}_\omega^{\mathrm{Com}}$.
(c) *If* $n \leq n'$, $\mathbf{L}_n^{\mathrm{Com}}$ *is isomorphic to a subalgebra of* $\mathbf{L}_{n'}^{\mathrm{Com}}$.

Now we will determine the lattice of subvarieties of the A variety $\mathcal{L}_{\mathrm{Com}}$. We also give an equational base for each one of these subvarieties.

Theorem 2.5 *The only subvarieties of* $\mathcal{L}_{\mathrm{Com}}$ *are* $\mathcal{V}(\mathbf{L}_\omega^{\mathrm{Com}})$ *and* $\mathcal{V}(\mathbf{L}_n^{\mathrm{Com}})$, $n \geq 0$.

Proof Let $\mathcal{V}$ be a subvariety of $\mathcal{L}_{\mathrm{Com}}$. If $\mathcal{V}$ is the whole variety $\mathcal{L}_{\mathrm{Com}}$, then, by Lemma 2.4, $\mathcal{V} = \mathcal{V}(\mathbf{L}_\omega^{\mathrm{Com}})$. Suppose that $\mathcal{V}$ is a proper subvariety of $\mathcal{L}_{\mathrm{Com}}$. From Theorem 2.2, there exists an integer $n \geq 0$ such that $\mathbf{L}_n^{\mathrm{Com}} \notin \mathcal{V}$. Let $t = \max\{n \in \mathbb{N} \cup \{0\} : \mathbf{L}_n^{\mathrm{Com}} \in \mathcal{V}\}$. Then, by Lemma 2.4, every finite chain in $\mathcal{V}$ is isomorphic to a subalgebra of $\mathbf{L}_t^{\mathrm{Com}}$. Thus $\mathcal{V} = \mathcal{V}(\mathbf{L}_t^{\mathrm{Com}})$. □

Hence we have that the lattice of subvarieties of $\mathcal{L}_{\mathrm{Com}}$ is an $(\omega+1)$-chain:

$$\mathcal{T} \subseteq \mathcal{V}\left(\mathbf{L}_0^{\mathrm{Com}}\right) \subseteq \mathcal{V}\left(\mathbf{L}_1^{\mathrm{Com}}\right) \subseteq \ldots \subseteq \mathcal{V}\left(\mathbf{L}_n^{\mathrm{Com}}\right) \subseteq \ldots \subseteq \mathcal{L}_{\mathrm{Com}} = \mathcal{V}\left(\mathbf{L}_\omega^{\mathrm{Com}}\right),$$

$\mathcal{T}$ the trivial variety, and, consequently, is isomorphic to the lattice of subvarieties of the variety of Gödel algebras (see [4]).

Lemma 2.6 [1] *Let* $\mathbf{L}$ *be a chain in* $\mathcal{L}$. *If* $\mathbf{L}$ *satisfies the identity* ($\mathrm{H_n}$)*, then* $|L| \leq n$.

$$\bigvee_{i=1}^{n-1}\left(x_i \vee x_i^*\right) \vee \bigvee_{j=1;\, j<i}^{n-1}\left(x_i \to x_j\right) \approx 1 \tag{H$_\mathrm{n}$}$$

Corollary 2.7 ($\mathrm{H_n}$) *is, within* $\mathcal{L}_{\mathrm{Com}}$*, a basis for* $\mathcal{V}(\mathbf{L}_{n-2}^{\mathrm{Com}})$*, for each integer* $n \geq 2$.

3 The Subvariety $\mathcal{L}_\vee$

In this section we consider the subvariety $\mathcal{L}_\vee$ of $\mathcal{L}$ generated by the semi-Heyting chains in which $a < b$ implies $a \to b = b$.

In the following theorem we abbreviate $x \leftrightarrow y$ to denote the term $(x \to y) \wedge (y \to x)$. It is clear that $x \leftrightarrow y \approx 1$ if and only if $x \approx y$.

Theorem 3.1 $\mathcal{L}_\vee$ *is characterized within* $\mathcal{L}$ *by the equation*

$$((x \wedge y) \leftrightarrow y) \vee ((x \to y) \leftrightarrow y) \approx 1.$$

Proof Let $\mathbf{A}$ be a chain in $\mathcal{L}_\vee$ and $a, b \in A$. If $a = b$, then $((a \wedge a) \leftrightarrow a) \vee ((a \to a) \leftrightarrow a) = 1$. If $a < b$, $((a \wedge b) \leftrightarrow b) \vee ((a \to b) \leftrightarrow b) = (a \leftrightarrow b) \vee (b \leftrightarrow b) = 1$. If

$a > b$, $((a \wedge b) \leftrightarrow b) \vee ((a \to b) \leftrightarrow b) = (b \leftrightarrow b) \vee (b \leftrightarrow b) = 1$. So **A** satisfies the desired equation.

Suppose now that **A** is a semi-Heyting chain that satisfies $((x \wedge y) \leftrightarrow y) \vee ((x \to y) \leftrightarrow y) \approx 1$ and let $a, b \in A$, $a < b$. Then $((a \wedge b) \leftrightarrow b) \vee ((a \to b) \leftrightarrow b) = 1$, that is, $(a \leftrightarrow b) \vee ((a \to b) \leftrightarrow b) = 1$. So $a \leftrightarrow b = 1$ or $(a \to b) \leftrightarrow b = 1$. If $a \leftrightarrow b = 1$ then $a = b$, which contradicts the assumption. So $((a \to b) \leftrightarrow b) = 1$, that is, $a \to b = b$. So **A** belongs to the variety $\mathcal{L}_\vee$. □

The following result follows immediately.

Lemma 3.2 *Given a totally ordered lattice* **A** *with least element* 0 *and last element* 1*, if for every* $a, b \in A$ *we define*

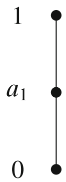
$$a \to b = \begin{cases} 1 & \text{if } a = b \\ b & \text{if } a \neq b \end{cases}$$

then $\mathbf{A} = \langle A, \wedge, \vee, \to, 0, 1\rangle$ *is an algebra in* $\mathcal{L}_\vee$.

It is easy to see that the only structure of an algebra in $\mathcal{L}_\vee$ that can be defined on a chain is the one defined above.

Let $\mathbf{L}_n^\vee$ and $\mathbf{L}_\omega^\vee$ respectively denote the (uniquely determined) algebras in $\mathcal{L}_\vee$ with underlying lattice $\langle L_n, \vee, \wedge, 0, 1\rangle$ and $\langle L_\omega, \vee, \wedge, 0, 1\rangle$ respectively, that is, where the operation $\to$ is defined by $a \to b = 1$ if $a = b$ and $a \to b = b$ if $a \neq b$. We have that $\mathbf{L}_0^\vee$ is the Heyting algebra **2** and $\mathbf{L}_1^\vee$ is the algebra

$\mathbf{L}_1^\vee$



$\to$	0	a_1	1
0	1	a_1	1
a_1	0	1	1
1	0	a_1	1

and every finite chain in $\mathcal{L}_\vee$ is isomorphic to $\mathbf{L}_n^\vee$ for some $n \geq 0$.

The proof of the following theorem is similar to that of Theorem 2.2.

Theorem 3.3 *A subvariety* $\mathcal{V}$ *of* $\mathcal{L}_\vee$ *is proper if and only if* $\mathbf{L}_n^\vee \notin \mathcal{V}$ *for some* $n \geq 0$.

Remark 3.4 Arguing as in the proof of Theorem 2.2, we can prove that every subvariety of $\mathcal{L}_\vee$ is generated by its finite chains, that is is generated by the algebras $\{\mathbf{L}_n^\vee, n \in I\}$ for some $I \subseteq \omega$.

Lemma 3.5

(a) $\mathbf{L}_n^\vee$ *is isomorphic to a subalgebra of* $\mathbf{L}_\omega^\vee$*, for every* $n \geq 0$.
(b) $\mathcal{L}_\vee$ *is generated by* $\mathbf{L}_\omega^\vee$.
(c) *If* $n \leq n'$, $\mathbf{L}_n^\vee$ *is isomorphic to a subalgebra of* $\mathbf{L}_{n'}^\vee$.

The following theorem characterizes the lattice of subvarieties of $\mathcal{L}_\vee$.

Theorem 3.6 *The only subvarieties of $\mathcal{L}_\vee$ are $\mathcal{V}(\mathbf{L}_\omega^\vee)$ and $\mathcal{V}(\mathbf{L}_n^\vee)$, $n \geq 0$.*

Proof Let $\mathcal{V}$ be a subvariety of $\mathcal{L}_\vee$. If $\mathcal{V} = \mathcal{L}_\vee$, by Lemma 3.5, $\mathcal{V} = \mathcal{V}(\mathbf{L}_\omega^\vee)$. Suppose that $\mathcal{V}$ is a proper subvariety. From Theorem 3.3 there exists an integer $n \geq 0$ such that $\mathbf{L}_n^\vee \notin \mathcal{V}$. Let $t = \max\{n \in \mathbb{N} \cup \{0\} : \mathbf{L}_n^\vee \in \mathcal{V}\}$. Then every finite chain in $\mathcal{L}_\vee$ is isomorphic to a subalgebra of $\mathbf{L}_t^\vee$, by Lemma 3.5. Hence $\mathcal{V} = \mathcal{V}(\mathbf{L}_t^\vee)$. □

So, the lattice of subvarieties of $\mathcal{L}_\vee$ is an $(\omega + 1)$-chain:

$$\mathcal{T} \subseteq \mathcal{V}\left(\mathbf{L}_0^\vee\right) \subseteq \mathcal{V}\left(\mathbf{L}_1^\vee\right) \subseteq \ldots \subseteq \mathcal{V}\left(\mathbf{L}_n^\vee\right) \subseteq \ldots \subseteq \mathcal{L}_\vee = \mathcal{V}\left(\mathbf{L}_\omega^\vee\right).$$

As a corollary, we get, as in Section 2, an equational characterization for each subvariety of $\mathcal{L}_\vee$.

Corollary 3.7 $(\mathrm{H_n})$ *is, modulo $\mathcal{L}_\vee$, a base for $\mathcal{V}(\mathbf{L}_{n-2}^\vee)$.*

4 Subvarieties Term-equivalent to the Variety of Gödel Algebras

In this section we prove that the varieties $\mathcal{L}_{\mathrm{Com}}$ and $\mathcal{L}_\vee$ are term-equivalent to the variety $\mathcal{L}_\mathcal{H}$ of Gödel algebras (linear Heyting algebras), and that they are the only other subvarieties in $\mathcal{L}$ with this property.

The following lemma states that we can always define a Heyting implication in any semi-Heyting algebra. Moreover, among all the semi-Heyting implication operations that can be defined in a given distributive lattice, the Heyting implication is the greatest one.

Lemma 4.1 *Let $\langle A, \vee, \wedge, \to, 0, 1\rangle$ be a semi-Heyting algebra. If we define $a \to_H b = a \to (a \wedge b)$ for every $a, b \in A$, then*

(a) $\langle A, \vee, \wedge, \to_H, 0, 1\rangle$ *is a Heyting algebra.*
(b) $a \to b \leq a \to_H b$ *for every $a, b \in A$.*

Proof Let us prove that $\to_H$ is a Heyting implication. Let $a, b, c \in A$. Then $a \to_H a = a \to (a \wedge a) = 1$, so we have (SH4). Now, $a \wedge (a \to_H b) = a \wedge (a \to (a \wedge b)) = a \wedge a \wedge b = a \wedge b$, and we get (SH2). For (SH3), $a \wedge (b \to_H c) = a \wedge (b \to (b \wedge c)) = a \wedge [(a \wedge b) \to (a \wedge b \wedge c)] = a \wedge [(a \wedge b) \to (a \wedge b \wedge a \wedge c)] = a \wedge [(a \wedge b) \to_H (a \wedge c)]$. Finally, $(a \wedge b) \to_H a = (a \wedge b) \to (a \wedge b \wedge a) = 1$. So $\to_H$ is a Heyting implication, and we have proved (a).

For (b), $(a \to b) \wedge (a \to_H b) = (a \to b) \wedge [a \to (a \wedge b)] = (a \to b) \wedge [(a \wedge (a \to b)) \to (a \wedge b \wedge (a \to b))] = (a \to b) \wedge ((a \wedge b) \to (a \wedge b)) = (a \to b) \wedge 1 = a \to b$. Thus $a \to b \leq a \to_H b$. □

Similarly, we have the following.

Lemma 4.2 *Let $\langle A, \vee, \wedge, \to, 0, 1\rangle$ be a semi-Heyting chain. If we define*

$$a \to_J b = b \vee [(a \to_H b) \wedge (b \to_H a)], \quad \textit{for } a, b \in A,$$

then $\langle A, \vee, \wedge, \to_J, 0, 1\rangle \in \mathcal{L}_\vee$.

Proof Let $a, b \in A$. If $a \le b$, $a \to_J b = b \vee (a \leftrightarrow_H b) = b$. So $\langle A, \vee, \wedge, \to_J, 0, 1\rangle \in \mathcal{L}_\vee$. □

Since every linear Heyting algebra is a subdirect product of Heyting chains, every linear Heyting algebra can be transformed into an algebra in $\mathcal{L}_\vee$.

The following lemma states that we can define an operation $\to_c$ on a semi-Heyting algebra to obtain an algebra in $\mathcal{L}_{\text{Com}}$. Its proof is easy.

Lemma 4.3 *Let $\langle A, \vee, \wedge, \to, 0, 1\rangle$ be a semi-Heyting chain. If we define*

$$a \to_c b = (a \to_H b) \wedge (b \to_H a), \quad \textit{for } a, b \in A,$$

then $\langle A, \vee, \wedge, \to_c, 0, 1\rangle \in \mathcal{L}_{\text{Com}}$.

Thus, from Lemmas 4.1, 4.2 and 4.3, the varieties $\mathcal{L}_{\text{Com}}$, $\mathcal{L}_\vee$ are both term-equivalent to the variety $\mathcal{L}_\mathcal{H}$ of Gödel algebras.

If $\mathfrak{F}_\mathcal{K}(X)$ denotes the free algebra over a set X of free generators in a given class $\mathcal{K}$, we have the following result.

Corollary 4.4 *The lattice reduct of the algebras $\mathfrak{F}_{\mathcal{L}_H}(X)$, $\mathfrak{F}_{\mathcal{L}_{\text{Com}}}(X)$ and $\mathfrak{F}_{\mathcal{L}_\vee}(X)$ are pairwise isomorphic.*

Let us see now that $\mathcal{L}_{\text{Com}}$ and $\mathcal{L}_\vee$ are the only subvarieties of $\mathcal{L}$ term equivalent to $\mathcal{L}_\mathcal{H}$.

The proof of the following lemma can be done by induction on the length of the Heyting term $t(x, y)$, where by a Heyting term we understand a term in the language $\{\wedge, \vee, \to_H, 0, 1\}$.

Lemma 4.5 *Let $\langle \mathbf{L}, \wedge, \vee, \to, 0, 1\rangle$ be a semi-Heyting chain such that $x \to y = t(x, y)$, where $t(x, y)$ is a Heyting term. Then for every $a, b \in L$ such that $a < b$, $a \to b = t(a, b) \in \{a, b, 1\}$.*

As a consequence of the previous lemma we obtain the following.

Corollary 4.6 *Let $t(x, y)$ be a Heyting term, with $t(x, y) \neq 0, 1$, and let $\mathbf{L}_1$ and $\mathbf{L}_2$ be semi-Heyting chains. Let $a, b \in L_1$ and $c, d \in L_2$ such that $a < b$ and $c < d$. Then the following conditions hold:*

(a) *If $t^{\mathbf{L}_1}(a, b) = a$ then $t^{\mathbf{L}_2}(c, d) = c$.*
(b) *If $t^{\mathbf{L}_1}(a, b) = b$ then $t^{\mathbf{L}_2}(c, d) = d$.*
(c) *If $t^{\mathbf{L}_1}(a, b) = 1$ then $t^{\mathbf{L}_2}(c, d) = 1$.*

Theorem 4.7 *The varieties $\mathcal{L}_{\text{Com}}$ and $\mathcal{L}_\vee$ are the only subvarieties of $\mathcal{L}$ term equivalent to the variety of linear Heyting algebras.*

Proof Let $\mathcal{V}$ be a subvariety of $\mathcal{L}$ term equivalent to the variety of linear Heyting algebras. Then there exists a Heyting term $t(x, y)$ such that for every algebra $\mathbf{A} \in \mathcal{V}$, $x \to y = t(x, y)$. Let $\mathbf{L}$ be a chain in $\mathcal{V}$. Suppose that $a \to b = a$ for every

$a, b \in \mathbf{L}$ such that $a < b$. Then $\mathbf{L} \in \mathcal{L}_{\mathrm{Com}}$. In addition, by Corollary 4.6, every chain in $\mathcal{V}$ belongs to $\mathcal{L}_{\mathrm{Com}}$. So $\mathcal{V} \subseteq \mathcal{L}_{\mathrm{Com}}$. The other cases are similar. □

5 The Variety Generated by $\mathcal{L}_{\mathbf{Com}}$, $\mathcal{L}_{\vee}$ and $\mathcal{L}_{\mathcal{H}}$

The aim of this section is to investigate the variety generated by the varieties $\mathcal{L}_{\mathrm{Com}}$, $\mathcal{L}_{\vee}$ and the variety of linear Heyting algebras. We will find an equational basis for this variety, we will determine its lattice of subvarieties and we will study free algebras of this variety.

Recall that the lattice of subvarieties of $\mathcal{L}_{\mathcal{H}}$ is an $(\omega + 1)$-chain

$$\mathcal{T} \subseteq \mathcal{V}\left(\mathbf{L}_0^{\mathcal{H}}\right) \subseteq \mathcal{V}\left(\mathbf{L}_1^{\mathcal{H}}\right) \subseteq \ldots \subseteq \mathcal{V}\left(\mathbf{L}_n^{\mathcal{H}}\right) \subseteq \ldots \subseteq \mathcal{L}_{\mathcal{H}} = \mathcal{V}\left(\mathbf{L}_\omega^{\mathcal{H}}\right),$$

where $\mathbf{L}_n^{\mathcal{H}}$ is the unique Heyting algebra having as a lattice reduct the $(n+2)$-element chain $\mathbf{L}_n$, $n \in \{0\} \cup \mathbb{N} \cup \{\omega\}$.

Also observe that $\mathbf{L}_0^{\mathcal{H}} = \mathbf{L}_0^{\vee}$.

Consider the following terms.

$$\begin{aligned} \gamma_H(x, y) &= (x \wedge y) \to x \\ \gamma_{\mathrm{Com}}(x, y) &= (x \to y) \leftrightarrow (y \to x) \\ \gamma_{\vee}(x, y) &= ((x \wedge y) \leftrightarrow y) \vee ((x \to y) \leftrightarrow y) \end{aligned} \quad .$$

Observe that $\gamma_H(x, y) \approx 1$, $\gamma_{\mathrm{Com}}(x, y) \approx 1$ and $\gamma_{\vee}(x, y) \approx 1$ respectively represent an equational basis, modulo $\mathcal{L}$, for the subvarieties $\mathcal{L}_{\mathcal{H}}$, $\mathcal{L}_{\mathrm{Com}}$ and $\mathcal{L}_{\vee}$.

Definition 5.1 Let $\mathcal{C}$ be the subvariety of $\mathcal{L}$ characterized by the equation:

$$\gamma_H(x_1, x_2) \vee \gamma_{\mathrm{Com}}(y_1, y_2) \vee \gamma_{\vee}(z_1, z_2) \approx 1$$

We are going to prove that $\mathcal{C}$ is the variety generated by the varieties $\mathcal{L}_{\mathrm{Com}}$, $\mathcal{L}_{\vee}$ and $\mathcal{L}_{\mathcal{H}}$.

Theorem 5.2 $\mathcal{V}(\mathcal{L}_{\mathcal{H}}, \mathcal{L}_{\mathrm{Com}}, \mathcal{L}_{\vee}) = \mathcal{C}$.

Proof It is clear that $\mathcal{V}(\mathcal{L}_{\mathcal{H}}, \mathcal{L}_{\mathrm{Com}}, \mathcal{L}_{\vee}) \subseteq \mathcal{C}$. For the other inclusion, consider a semi-Heyting chain $\mathbf{L} \in \mathcal{C}$. If we suppose that $\mathbf{L} \notin \mathcal{L}_{\mathcal{H}} \cup \mathcal{L}_{\mathrm{Com}} \cup \mathcal{L}_{\vee}$ then there exist $a_1, a_2, b_1, b_2, c_1, c_2 \in L$ such that $\gamma_H(a_1, a_2) \neq 1$, $\gamma_{\mathrm{Com}}(b_1, b_2) \neq 1$ and $\gamma_{\vee}(c_1, c_2) \neq 1$. Thus, by Theorem 1.3, $\gamma_H(a_1, a_2) \vee \gamma_{\mathrm{Com}}(b_1, b_2) \vee \gamma_{\vee}(c_1, c_2) \neq 1$, a contradiction. □

Let V be a subvariety of $\mathcal{C}$. As in the proof of Theorem 5.2, $Si(V) = Si(\mathcal{L}_{\mathcal{H}}) \cup Si(\mathcal{L}_{\vee}) \cup Si(\mathcal{L}_{\mathrm{Com}})$, where with $Si(\mathcal{K})$ we denote the collection of subdirectly irreducible algebras in a class $\mathcal{K}$. Then it is easy to see the following theorem.

Theorem 5.3 *Every subvariety of* $\mathcal{C}$ *is of the form* $\mathcal{V}(\mathbf{L}_i^{\mathcal{H}}, \mathbf{L}_j^{\mathrm{Com}}, \mathbf{L}_k^{\vee})$, *for* $i, j, k \in \{0\} \cup \mathbb{N} \cup \{\omega\}$.

Recall (see Lemma 2.6) that if

$$\psi_n(x_1,\ldots,x_{n-1}) = \bigvee_{i=1}^{n-1}\left(x_i \vee x_i^*\right) \vee \bigvee_{i,j=1;\, j<i}^{n-1}\left(x_i \to x_j\right),$$

then $\psi_n(x_1,\ldots,x_{n-1}) \approx 1$ characterizes the height of a chain $\mathbf{L}$. So the following corollary follows immediately.

Corollary 5.4 *An equational basis for the subvariety* $\mathcal{V}(\mathbf{L}_i^{\mathcal{H}}, \mathbf{L}_j^{\mathrm{Com}}, \mathbf{L}_k^{\vee})$, $i, j, k \in \{0\} \cup \mathbb{N} \cup \{\omega\}$, *within* $\mathcal{L}$, *is the following:*

$$\delta_H^i(x_1,\ldots,x_{n-1},x_n,x_{n+1}) \vee \delta_C^j(y_1,\ldots,y_{n-1},y_n,y_{n+1}) \vee \delta_\vee^k(z_1,\ldots,z_{n-1},z_n,z_{n+1}) \approx 1, \textit{ where}$$

$$\delta_H^i(x_1,\ldots,x_{n-1},x_n,x_{n+1}) = \psi_i(x_1,\ldots,x_{n-1}) \wedge \gamma_H(x_n,x_{n+1})$$
$$\delta_C^j(y_1,\ldots,y_{n-1},y_n,y_{n+1}) = \psi_j(y_1,\ldots,y_{n-1}) \wedge \gamma_{\mathrm{Com}}(y_n,y_{n+1})$$
$$\delta_\vee^k(z_1,\ldots,z_{n-1},z_n,z_{n+1}) = \psi_k(z_1,\ldots,z_{n-1}) \wedge \gamma_\vee(z_n,z_{n+1}).$$

6 Free Algebras in the Variety $\mathcal{C}$

Our next objective is to obtain a construction of the finitely generated free algebra of the variety $\mathcal{C}$. Free algebras in $\mathcal{L}_{\mathcal{H}}$ have been studied in [2, 3] and [5]. We shall follow a technique similar to that of Abad and Monteiro in [2].

For each $n \geq 0$, let $\mathbf{L}_i^{\mathcal{H}}(n)$, $\mathbf{L}_i^{\mathrm{Com}}(n)$ and $\mathbf{L}_i^{\vee}(n)$ respectively denote the subalgebra of $\mathbf{L}_n^{\mathcal{H}}$, $\mathbf{L}_n^{\mathrm{Com}}$ and $\mathbf{L}_n^{\vee}$ with universe $L_i(n) = \{0, a_1, \ldots, a_i, 1\}$, $0 \leq i \leq n$.

For a given semi-Heyting algebra $\mathbf{A}$ and $X \subseteq A$, let $S(X)$ denote the subalgebra of $\mathbf{A}$ generated by X. The following lemma is clear.

Lemma 6.1 *Let* $\mathbf{L}$ *be a chain in* $\mathcal{C}$ *and* $X \subseteq L$. *Then* $S(X) = X \cup \{0, 1\}$.

The proof of the following theorem can be found in [6, Theorem V.1.4].

Theorem 6.2 *Let* $\mathbf{A}$ *be an algebra in* $\mathcal{C}$ *and* P *a prime filter of* $\mathbf{A}$. *Then* $\mathbf{A}/P$ *is a chain if and only if the family of all proper filters of* $\mathbf{A}$ *containing* P *is a chain.*

Theorem 6.3 *Let* $A \in \mathcal{L}$ *and* P *a prime filter of* $\mathbf{A}$. *Then the family of all proper filters of* A *containing* P *is a chain.*

Proof Let $\mathfrak{F} = \{F \subseteq A : F \neq A,\ F \text{ is a filter of } A \text{ and } P \subseteq F\}$. Let $F_0, F_1 \in \mathfrak{F}$ and suppose that $F_0 \not\subseteq F_1$ and $F_1 \not\subseteq F_0$. Then there exist $a, b \in A$ such that $a \in F_0 \setminus F_1$ and $b \in F_1 \setminus F_0$. Since $\mathbf{A} \in \mathcal{L}$, $\mathbf{A}$ satisfies the identity (Ch). Since $1 \in P$ and P is a prime filter, $(a \vee (a \to b)) \to (a \to b) \in P$ or $b \to (a \wedge b) \in P$.

If $(a \vee (a \to b)) \to (a \to b) \in P$, then $a \wedge [(a \vee (a \to b)) \to (a \to b)] = a \wedge b$. Since $(a \vee (a \to b)) \to (a \to b) \in F_0$ and $a \in F_0$ (being that $P \subseteq F_0$), we have that $a \wedge b \in F_0$. Consequently $b \in F_0$.

If $b \to (a \wedge b) \in P$, then $b \wedge [b \to (a \wedge b)] = a \wedge b$, and as in the previous case, we obtain that $a \in F_1$.

As a consequence, $F_0 \subseteq F_1$ or $F_1 \subseteq F_0$, that is, $\mathfrak{F}$ is a chain. □

From Theorems 6.2 and 6.3 the following corollary follows.

Corollary 6.4 *If* $\mathbf{A} \in \mathcal{L}$ *and* P *is a prime filter of* A*, then* $\mathbf{A}/P$ *is a chain.*

For a given semi-Heyting algebra $\mathbf{A}$, let $\mathbb{P}(\mathbf{A})$ denote the collection of prime filters of $\mathbf{A}$, and for finite $\mathbf{A}$, let $\Pi(\mathbf{A})$ be the set of its prime elements.

Remark 6.5 It is known that a non-trivial semi-Heyting algebra $\mathbf{A}$ belongs to $\mathcal{L}$ if and only if $\mathbf{A}$ is isomorphic to a subdirect product of semi-Heyting chains. This result can be rephrased by saying that $\mathbf{A}$ belongs to $\mathcal{L}$ if and only if $\mathbf{A}$ is isomorphic to a subdirect product $\Pi_{P \in \mathbb{P}(\mathbf{A})} \mathbf{A}/P$.

The following lemma is a consequence of Theorem 6.3.

Lemma 6.6 *If* $\mathbf{A} \in \mathcal{L}$ *is finite and* $p \in \Pi(\mathbf{A})$*, then the set* $I(p) = \{q \in \Pi(\mathbf{A}) : q \leq p\}$ *is a chain.*

Let $\mathbf{A} \in \mathcal{L}$, $\mathbf{A}$ finite. We say that $p \in \Pi(A)$ is of *level* i in $\Pi(A)$, i a positive integer, if $|I(p)| = i$. It is clear that if $p \in \Pi(A)$ is of level i and $Fg(p)$ is the filter generated by p, then $\mathbf{A}/Fg(p) \cong \mathbf{L}_{i-1}(n)$.

Let $\mathfrak{F}_{\mathcal{C}}(n)$ be the free algebra in the variety $\mathcal{C}$ over n free generators, $n > 0$. In what follows we denote $\mathbb{P}(n)$ the collection of prime filters of $\mathfrak{F}_{\mathcal{C}}(n)$ and $\Pi(n)$ the set of prime elements of $\mathfrak{F}_{\mathcal{C}}(n)$. We have that $\mathfrak{F}_{\mathcal{C}}(n)$ is isomorphic to a subalgebra of the direct product $\Pi_{P \in \mathbb{P}(n)} \mathfrak{F}_{\mathcal{C}}(n)/P$.

We are going to prove that $\mathfrak{F}_{\mathcal{C}}(n)$ is finite, and in that sense, we shall prove that $\mathbb{P}(n)$ is finite and that $\mathfrak{F}_{\mathcal{C}}(n)/P$ is finite for every $P \in \mathbb{P}(n)$.

Lemma 6.7 *Let* $\mathbf{A}$ *be an algebra in* $\mathcal{C}$*,* G *a finite set of generators of* $\mathbf{A}$ *with* $|G| = n$ *and* $P \in \mathbb{P}(\mathbf{A})$*. Then* $|\mathbf{A}/P| \leq n + 2$*.*

Proof Let $h : \mathbf{A} \to \mathbf{A}/P$ be the natural epimorphism. Since $S(G) = \mathbf{A}$, $S(h(G)) = \mathbf{A}/P$. By Corollary 6.4, $\mathbf{A}/P$ is a chain. So $\mathbf{A}/P = h(\mathbf{A}) = S(h(G)) = h(G) \cup \{0, 1\}$ by Lemma 6.1. Thus $|\mathbf{A}/P| \leq |h(G)| + 2 \leq n + 2$. □

Corollary 6.8 *If* $P \in \mathbb{P}(n)$*, then* $\mathfrak{F}_{\mathcal{C}}(n)/P$ *is a finite chain.*

By Lemma 6.7, if $P \in \mathbb{P}(n)$, the family of filters of $\mathfrak{F}_{\mathcal{C}}(n)$ containing P has at most $n + 2$ elements, and the family of prime filters of $\mathfrak{F}_{\mathcal{C}}(n)$ containing P has at most $n + 1$ elements. Then $\mathfrak{F}_{\mathcal{C}}(n)/P$ is isomorphic to either $\mathbf{L}_i^{\mathcal{H}}(n)$ or $\mathbf{L}_i^{\mathrm{Com}}(n)$ or $\mathbf{L}_i^{\vee}(n)$, with $0 \leq i \leq n$,

and $h : \mathfrak{F}_{\mathcal{C}}(n) \to \mathfrak{F}_{\mathcal{C}}(n)/P$ is defined by $h(x) = \begin{cases} 1 & if\ x \in P \\ a_i & if\ x \in P_i \setminus P_{i+1},\ 0 \leq i \leq t \end{cases}$.

Consider the following sets:

$$P_i^{\mathcal{H}}(n) = \left\{P \in \mathbb{P}(n) :\ \mathfrak{F}_{\mathcal{C}}(n)/P \simeq \mathbf{L}_i^{\mathcal{H}}(n)\right\},$$

$$F_i^{\mathcal{H}}(n) = \left\{f : G \to \mathbf{L}_i(n) :\ S(f(G)) \simeq \mathbf{L}_i^{\mathcal{H}}(n)\right\}.$$

Similarly, we define the sets $P_i^{\vee}(n)$, $P_i^{\mathrm{Com}}(n)$, $F_i^{\vee}(n)$ and $F_i^{\mathrm{Com}}(n)$.

Finally, consider

$$P_i(n) = P_i^{\mathcal{H}}(n) \cup P_i^{\vee}(n) \cup P_i^{\mathrm{Com}}(n),$$

$$F_i(n) = F_i^{\mathcal{H}}(n) \cup F_i^{\vee}(n) \cup F_i^{\mathrm{Com}}(n),\ \ \mathbb{F}(n) = \bigcup_{i=1}^{n} F_i(n).$$

Clearly $P_i(n) \cap P_j(n) = \emptyset$ for $i \neq j$, and $F_i(n) \neq \emptyset$ with $0 \leq i, j \leq n$.

Let $f \in F_i^{\mathcal{H}}(n)$ and $\overline{f} : \mathfrak{F}_{\mathcal{C}}(n) \to \mathbf{L}_i^{\mathcal{H}}(n)$ the extension of f. $\mathbf{L}_i^{\mathcal{H}}(n) = S(f(G)) = S(\overline{f}(G)) = \overline{f}(\mathfrak{F}_{\mathcal{C}}(n))$. Thus $\overline{f} : \mathfrak{F}_{\mathcal{C}}(n) \to \mathbf{L}_i^{\mathcal{H}}(n)$ is an epimorphism, and $Ker\overline{f}$ is a prime filter of $\mathfrak{F}_{\mathcal{C}}(n)$. Similarly, for $F_i^{\vee}(n)$ and $F_i^{\mathcal{H}}(n)$.

Lemma 6.9 *If* $f \in F_i^{\mathcal{H}}(n)$, *then* $\mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f} \simeq \mathbf{L}_i^{\mathcal{H}}(n)$. *If* $f \in F_i^{\vee}(n)$, *then* $\mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f} \simeq \mathbf{L}_i^{\vee}(n)$. *If* $f \in F_i^{\mathrm{Com}}(n)$, *then* $\mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f} \simeq \mathbf{L}_i^{\mathrm{Com}}(n)$.

Proof Suppose that $f \in F_i^{\mathcal{H}}(n)$. Consider the natural homomorphism $\gamma : \mathfrak{F}_{\mathcal{C}}(n) \to \mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f}$. Let $g : \mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f} \to \mathbf{L}_i^{\mathcal{H}}(n)$ be defined by $g(a/Ker\overline{f}) = \overline{f}(a)$ for $a \in \mathfrak{F}_{\mathcal{C}}(n)$.

Let us see that g is well defined. If $b \in a/Ker\overline{f}$, there exists $c \in Ker\overline{f}$ such that $ax \wedge c = b \wedge c$. So $\overline{f}(a) = \overline{f}(a \wedge c) = \overline{f}(b \wedge c) = \overline{f}(b)$.

Let $a/Ker\overline{f}, b/Ker\overline{f} \in \mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f}$ be such that $g(a/Ker\overline{f}) = g(b/Ker\overline{f})$. Then $\overline{f}(a) = \overline{f}(b)$. So $\overline{f}((a \to b) \wedge (b \to a)) = 1$ and consequently, $(a \to b) \wedge (b \to a) \in Ker\overline{f}$. In addition, $a \wedge (a \to b) \wedge (b \to a) = a \wedge b \wedge (b \to a) = a \wedge b \wedge a = a \wedge b = b \wedge a \wedge b = b \wedge a \wedge (a \to b) = b \wedge (b \to a) \wedge (a \to b)$. Then $a/Ker\overline{f} = b/Ker\overline{f}$. So g is injective.

Now, for every $a \in \mathfrak{F}_{\mathcal{C}}(n)$, $(g \circ \gamma)(a) = g(\gamma(a)) = g(a/Ker\overline{f}) = \overline{f}(a)$, so $g \circ \gamma = \overline{f}$.

Let $a \in \mathbf{L}_i^{\mathcal{H}}(n)$. Then there exists $c \in \mathfrak{F}_{\mathcal{C}}(n)$ such that $\overline{f}(c) = a$, that is $g(\gamma(c)) = a$. So g is onto.

The same proof applies to the cases in which $f \in F_i^{\vee}(n)$ or $f \in F_i^{\mathrm{Com}}(n)$. □

In order to avoid unnecessary repetitions, we shall use in what follows the symbol $*$ to replace the superscripts $\mathcal{H}$, $\vee$ or *Com*. For instance, $F_i^*(n)$ will stand for any of the sets $F_i^{\mathcal{H}}(n)$, $F_i^{\vee}(n)$ and $F_i^{\mathrm{Com}}(n)$, $\mathbf{L}_i^*$ will denote any of the chains $\mathbf{L}_i^{\mathcal{H}}$, $\mathbf{L}_i^{\vee}$ or $\mathbf{L}_i^{\mathrm{Com}}$, and so on.

Lemma 6.10 *The function* $\psi_i^* : F_i^*(n) \to P_i^*(n)$ *defined by* $\psi_i^*(f) = Ker\overline{f}$ *is onto.*

Proof For $P \in P_i^*(n)$ (that is, $\mathfrak{F}_{\mathcal{C}}(n)/P \simeq \mathbf{L}_i^*(n)$), consider the natural homomorphism $\lambda : \mathfrak{F}_{\mathcal{C}}(n) \to \mathfrak{F}_{\mathcal{C}}(n)/P$, and let $f = \lambda|_G$ the restriction of λ to G. Then $S(f(G)) = S(\lambda(G)) = \lambda(S(G)) = \lambda(\mathfrak{F}_{\mathcal{C}}(n)) \simeq \mathbf{L}_i^*(n)$, so $f \in F_i^*(n)$. Now let $\overline{f}$ be the extension of f. Clearly $\overline{f}|_G = f = \lambda|_G$ and consequently, $\overline{f} = \lambda$ and $\psi_i^*(f) = Ker\overline{f} = Ker\lambda = P$. □

Lemma 6.11 $\mathbb{P}(n)$ *is a finite set.*

Proof Since G and $\mathbf{L}_i^*(n)$ are finite, $F_i^*(n)$ is finite for every $0 \leq i \leq n$. By Lemma 6.10, ψ_i^* is onto and consequently, $P_i^*(n)$ is finite for every $0 \leq i \leq n$. Since $\mathbb{P}(n) = \bigcup_{i=1}^n P_i(n) = \bigcup_{i=1}^n (P_i^{\mathcal{H}}(n) \cup P_i^{\vee}(n) \cup P_i^{\mathrm{Com}}(n))$, $\mathbb{P}(n)$ is finite. □

Theorem 6.12 $\mathfrak{F}_{\mathcal{C}}(n)$ *is finite.*

Proof From Remark 6.5, $\mathfrak{F}_{\mathcal{C}}(n) \in \mathbb{IS}\left(\Pi_{P\in\mathbb{P}(n)}\mathfrak{F}_{\mathcal{C}}(n)/P\right)$. By Lemma 6.11, $\mathbb{P}(n)$ is finite and, by Corollary 6.8, $\mathfrak{F}_{\mathcal{C}}(n)/P$ is finite for every $P \in \mathbb{P}(n)$. Then $\mathfrak{F}_{\mathcal{C}}(n)$ is finite. □

Since every finite distributive lattice is determined, up to isomorphism, by the ordered set of its prime elements, our next objective is to obtain a description of $\Pi(n)$. For this, we are going to represent each element of $\Pi(n)$ by an element of $\mathbb{F}(n)$, that is, a function f from G to $\mathbf{L}_i^*(n)$, $0 \leq i \leq n$, such that $S(f(G)) \simeq \mathbf{L}_i^*(n)$, for $* \in \{\mathcal{H}, \vee, Com\}$.

For each $* \in \{\mathcal{H}, \vee, Com\}$, consider the sets

$$\mathbb{F}^*(n) = \bigcup_{i=0}^{n} F_i^*(n) \text{ and } \mathbb{P}^*(n) = \bigcup_{i=0}^{n} P_i^*(n),$$

and define $\psi^* : \mathbb{F}^*(n) \to \mathbb{P}^*(n)$ by $\psi^*(f) = \psi_i^*(f) = Ker(\overline{f})$ with $f \in \mathbb{F}^*(n)$.

By Lemma 6.9, ψ^* is well defined and is injective.

If $f \in \mathbb{F}(n)$, $f \in F_i(n)$ for some $0 \leq i \leq n$, then $f \in F_i^*(n)$ for some $* \in \{\mathcal{H}, \vee, Com\}$. Thus $Ker\overline{f} \in \mathbb{P}(n)$ and $\mathfrak{F}_{\mathcal{C}}(n)/Ker\overline{f} \simeq \mathbf{L}_i^*$. Since $\mathfrak{F}_{\mathcal{C}}(n)$ is finite, then there exists $p_f \in \Pi(n)$ such that $Ker\overline{f} = F_g(p_f)$, where $Fg(p_f)$ is the filter generated by p_f.

Lemma 6.13 *The function* $\Phi : \mathbb{F}(n) \to \Pi(n)$ *defined by* $\phi(f) = p_f$ *is a bijection.*

Proof Let $P \in \Pi(n)$ and consider $F_g(p) \in \mathbb{P}(n)$. By Corollary 6.4, $\mathfrak{F}_{\mathcal{C}}(n)/F_g(p)$ is a chain and so, $\mathfrak{F}_{\mathcal{C}}(n)/F_g(p) \simeq \mathbf{L}_j^*(n)$ for some $0 \leq j \leq n$ and $* \in \{\mathcal{H}, \vee, Com\}$. Let $\lambda : \mathfrak{F}_{\mathcal{C}}(n) \to \mathfrak{F}_{\mathcal{C}}(n)/F_g(p)$ the natural epimorphism and $\lambda' = \lambda|_G$. Then $\lambda' : G \to \mathbf{L}_j^*(n)$ and $\lambda' \in F_j^*(n) \subseteq \mathbb{F}(n)$. Thus $Ker(\overline{\lambda'}) = Ker(\lambda) = Fg(p)$ and $\Phi(\lambda') = p$.

Let $f_1, f_2 \in \mathbb{F}(n)$ such that $\Phi(f_1) = \Phi(f_2)$. Then $p_{f_1} = p_{f_2}$ with $Ker(\overline{f_1}) = Fg(p_{f_1})$ and $Ker(\overline{f_2}) = Fg(p_{f_2})$. Since $p_{f_1} = p_{f_2}$, $Ker(\overline{f_1}) = Fg(p_{f_1}) = Fg(p_{f_2}) = Ker(\overline{f_2})$. Consequently $\mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f_1}) \simeq \mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f_2})$. From Theorem 5.2 and Corollary 6.4, $\mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f_1}) \simeq \mathbf{L}_j^*(n)$ for some $0 \leq j \leq n$ and $* \in \{\mathcal{H}, \vee, Com\}$. Then $Ker(\overline{f_1}), Ker(\overline{f_2}) \in P_i^*(n)$ and $\psi^*(f_1) = Ker(\overline{f_1}) = Ker(\overline{f_2}) = \psi^*(f_2)$. Since ψ^* is injective, then $f_1 = f_2$. □

Remark 6.14 If $p_f \in \Pi(n)$ is of level i, $1 \le i \le n+1$, then $\mathfrak{F}_{\mathcal{C}}(n)/F_g(p_f) \simeq \mathbf{L}^*_{i-1}(n)$, with $** \in \{\mathcal{H}, \vee, Com\}$.

Lemma 6.15 *$p_f \in \Pi(n)$ is of level 1 if and only if $f(g) \in \{0, 1\}$ for every $g \in G$.*

Proof If $p_f \in \Pi(n)$ is of level 1, $\mathfrak{F}_{\mathcal{C}}(n)/F_g(p_f) \simeq \mathbf{L}^*_0(n)$, with $* \in \{\mathcal{H}, \vee, Com\}$, and this is equinalent to say that $Fg(p_f) \in P^*_0(n)$. Consider $\psi^*_0 : F^*_0(n) \to P^*_0(n)$. By Lemma 6.10, ψ^*_0 is bijective, and then, there exists $f \in F^*_0(n) :\ \psi^*_0(f) = Fg(p_f) = Ker(\overline{f})$. Thus $f \in \left(\psi^*_0\right)^{-1}\left(Fg(p_f)\right)$. Since $f \in F^*_0(n)$, then $S(f(G)) = \mathbf{L}^*_0(n)$. So $f(G) \cup \{0, 1\} = \mathbf{L}^*_0(n)$. Then $f(G) \subseteq \{0, 1\}$. □

From Lemma 6.13, there is a one-to-one correspondence between the set of prime elements of $\mathfrak{F}_{\mathcal{C}}(n)$ and the set $\mathbb{F}(n)$. By means of this relation, an element $p \in \Pi(n)$ of level i corresponds to a function $f \in F_{i-1}(n)$, that is, a function $f \in F^{\mathcal{H}}_{i-1}(n) \cup F^{\vee}_{i-1}(n) \cup F^{\mathrm{Com}}_{i-1}(n)$. In particular, $p \in \Pi(n)$ is of level 1 if and only if the corresponding function $f \in F^{\mathcal{H}}_0(n) \cup F^{\vee}_0(n) \cup F^{\mathrm{Com}}_0(n) = F^{\mathcal{H}}_0(n) \cup F^{\mathrm{Com}}_0(n)$ being that $F^{\mathcal{H}}_0(n) = F^{\vee}_0(n)$. Besides, $|F^{\mathcal{H}}_0(n) \cup F^{\mathrm{Com}}_0(n)| = |F^{\mathcal{H}}_0(n)| + |F^{\mathrm{Com}}_0(n)|$ since $F^{\mathcal{H}}_0(n) \cap F^{\mathrm{Com}}_0(n) = \emptyset$. Thus $\Pi(n)$ has $2^n + 2^n = 2^{n+1}$ minimal elements, that is, $\Pi(n)$ has 2^{n+1} elements of level 1.

The following lemma can be proved in a similar way to Lemma 6.15.

Lemma 6.16 *$p_f \in \Pi(n)$ is of level i, $2 \le i \le n+1$, if and only if $f(G) \subseteq \mathbf{L}_{i-1}(n)$ and $a_1, a_2, \dots, a_{n-1} \in f(G)$.*

Lemma 6.17 *Let $p, q \in \Pi(n)$. Then q covers p in $\Pi(n)$ if and only if the following conditions hold:*

(1) *$Fg(q) \subset Fg(p)$,*
(2) *$Fg(p) \in P^*_t(n)$,*
(3) *$Fg(q) \in P^*_{t+1}(n)$ for some $0 \le t \le n-1$ and $* \in \{\mathcal{H}, \vee, Com\}$.*

Proof Since q covers p, $Fg(q) \subset Fg(p)$ and there is no $P \in \mathbb{P}(n)$ such that $Fg(q) \subset P \subset Fg(p)$. Since $Fg(p) \in \mathbb{P}(n)$, $Fg(p) \in P^*_t(n)$ for some $0 \le t \le n$ and $* \in \{\mathcal{H}, \vee, Com\}$. besides, $t \ne n$ being that the family of prime filters containing $Fg(q)$ has at most n+1 elements. Since there is no $P \in \mathbb{P}(n)$ such that $Fg(q) \subset P \subset Fg(p)$, then $Fg(q) \in P^*_{t+1}(n)$.

For the converse, consider $p, q \in \Pi(n)$ satisfying conditions (1), (2) and (3). From (1), $p < q$. Suppose that there exists $p' \in \Pi(n)$ such that $p < p' < q$. Then $Fg(q) \subset Fg(p') \subset Fg(p)$. By (2), $Fg(p) \in P^*_t(n)$ and consequently, $Fg(q) \in P^*_{t+2}(n)$ which contradicts the hypothesis (3). □

Theorem 6.18 *Let $f, h \in \mathbb{F}(n)$. then $\Phi(h) = p_h = q$ covers $\Phi(f) = p_f = p$ if and only if $f \in F^*_t(n)$, $h \in F^*_{t+1}(n)$ for some $0 \le t \le n-1$ and $* \in \{\mathcal{H}, \vee, Com\}$, and the following conditions hold:*

(I) *$f(g) = a_j$ if and only if $h(g) = a_j$ for every $0 \le j \le t$.*
(II) *$f(g) = 1$ if and only if $h(g) = 1$ ó $h(g) = a_{t+1}$.*
(III) *There exists $g \in G :\ f(g) \ne h(g)$.*

Proof Suppose that $\Phi(h) = p_h = q$ covers $\Phi(f) = p_f = p$. By Lemma 6.17, $Fg(q) \subset Fg(p)$, $Fg(p) \in P_t^*(n)$ and $Fg(q) \in P_{t+1}^*(n)$ for some $0 \leq t \leq n-1$ and $* \in \{\mathcal{H}, \vee, Com\}$. From $Ker(\overline{f}) = Fg(p) \in P_t^*(n)$ we have that $\mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f}) \simeq \mathbf{L}_t^*(n)$. Consider the natural homomorphism $\lambda : \mathfrak{F}_{\mathcal{C}}(n) \to \mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f})$. So $\lambda = \overline{f}$ and thus $f \in F_t^*(n)$. Similarly $h \in F_{t+1}^*(n)$.

Since $f \in F_t^*(n)$, $S(f(G)) = f(G) \cup \{0,1\} = \mathbf{L}_t(n)$. If $t = 0$, $f(G) \subseteq \{0,1\}$ and if $t \neq 0$, $a_1, \ldots, a_t \in f(G)$. In a similar way $a_1, \ldots, a_t, a_{t+1} \in h(G)$. Since $P_{t+2} = Fg(p_h) \subseteq P_{t+1} = Fg(p_f) \subset P_t \subset \ldots \subset P_1 \subset P_0 = \mathfrak{F}_{\mathcal{C}}(n)$ it follows that

$$\overline{f}(x) = \begin{cases} 1 \;\; if \; x \in F_g(p_f) \\ a_j \; if \; x \in P_j \setminus P_{j+1}, \; 0 \leq j \leq t \end{cases}$$

and

$$\overline{h}(x) = \begin{cases} 1 \;\; if \; x \in F_g(p_h) \\ a_j \; if \; x \in P_j \setminus P_{j+1}, \; 0 \leq j \leq t+1 \end{cases}$$

We have that $f(g) = a_j \Leftrightarrow \overline{f}(g) = a_j \Leftrightarrow g \in P_j \setminus P_{j+1} \Leftrightarrow \overline{h}(g) = a_j \Leftrightarrow h(g) = a_j$.

In addition, $f(g) = 1 \Leftrightarrow \overline{f}(g) = 1 \Leftrightarrow g \in Fg(p_f) \Leftrightarrow g \in \big(Fg(p_f) \setminus Fg(p_h)\big) \cup Fg(p_h) \Leftrightarrow g \in Fg(p_f) \setminus Fg(p_h)$ or $g \in Fg(p_h) \Leftrightarrow g \in P_{t+1} \setminus P_{t+2}$ or $g \in Fg(p_h) \Leftrightarrow \overline{h}(g) = a_{t+1}$ or $\overline{h}(g) = 1 \Leftrightarrow h(g) = a_{t+1}$ or $h(g) = 1$.

Clearly there exists $g \in G$ such that $f(g) \neq h(g)$.

For the converse, let $f \in F_t^*(n)$, $h \in F_{t+1}^*(n)$ for some $0 \leq t \leq n-1$ and $* \in \{\mathcal{H}, \vee, Com\}$, satisfying conditions (I), (II) and (III). Since $f \in F_t^*(n)$, $S(f(G)) = \mathbf{L}_t^*(n)$. Thus $\mathfrak{F}_{\mathcal{C}}(n)/Fg(p_f) = \mathfrak{F}_{\mathcal{C}}(n)/Ker(\overline{f}) = \mathbf{L}_t^*(n)$, and so $Fg(p_f) \in P_t^*(n)$. Similarly $Fg(p_h) \in P_{t+1}^*(n)$.

By Lemma 6.17, it is enough to prove that $Fg(p_h) \subset Fg(p_f)$.

Consider

$$Fg(p_f) = P_{t+1} \subset P_t \subset \ldots \subset P_1 \subset P_0 = \mathfrak{F}_{\mathcal{C}}(n)$$

and

$$Fg(p_h) = Q_{t+2} \subset Q_{t+1} \subset \ldots \subset Q_1 \subset Q_0 = \mathfrak{F}_{\mathcal{C}}(n)$$

the chains of prime filters containing $Fg(p_f)$ and $Fg(p_h)$ respectively. Let

$$C_{t+2} = Q_{t+2} \cap P_{t+1}, \; C_{t+1} = (Q_{t+1} \setminus Q_{t+2}) \cap P_{t+1} \text{ and}$$

$$C_j = (Q_j \setminus Q_{j+1}) \cap (P_j \setminus P_{j+1}), \; 0 \leq j \leq t.$$

We have that

$$z \in C_{t+2} \Leftrightarrow \overline{h}(z) = 1 \text{ and } \overline{f}(z) = 1;$$

$$z \in C_{t+1} \Leftrightarrow \overline{h}(z) = a_{t+1} \text{ and } \overline{f}(z) = 1;$$

$$z \in C_j \Leftrightarrow \overline{h}(z) = a_j \text{ and } \overline{f}(z) = a_j, 0 \leq j \leq t.$$

Observe that the sets C_j are pairwise disjoint, $0 \leq j \leq t+2$, and it is long but computational to verify that $S = \bigcup_{j=0}^{t+2} C_j$ is a subalgebra of $\mathfrak{F}_{\mathcal{C}}(n)$.

Let $g \in G$. Then $h(g) \in \{0 = a_0, a_1, \ldots, a_t, a_{t+1}, 1\}$. If $h(g) = 1$ then $g \in Q_{t+2}$. By (II), $f(g) = 1$ and consequently $g \in P_{t+1}$ and $g \in C_{t+2} \subseteq S$. If $h(g) = a_{t+1}$ then $g \in Q_{t+1} \setminus Q_{t+2}$ and $g \in P_{t+1}$ by (II). Thus $g \in C_{t+1} \subseteq S$. If $h(g) = a_j$, $0 \leq j \leq t$, then $g \in Q_j \setminus Q_{j+1}$ and $f(g) = a_j$ by (I). So $g \in C_j \subseteq S$. Therefore $G \subseteq S$ and consequently $\mathfrak{F}_{\mathcal{C}}(n) = S$.

Then $F_g(p_h) = Q_{t+2} = Q_{t+2} \cap \mathfrak{F}_{\mathcal{C}}(n) = Q_{t+2} \cap \left(\bigcup_{j=0}^{t+2} C_j\right) = \bigcup_{j=0}^{t+2} \left(Q_{t+2} \cap C_j\right) = Q_{t+2} \cap C_{t+2} = Q_{t+2} \cap P_{t+1} = Fg(p_h) \cap Fg(p_f)$. Then $Fg(p_h) \subseteq Fg(p_f)$. If $Fg(p_h) = Fg(p_f)$, $Ker\overline{h} = Ker\overline{f}$. So $\overline{h} = \overline{f}$ and then $h = f$. □

As an example, suppose that $G = \{g_1, g_2\}$ and let us determine the ordered set $\Pi(2)$. By Lemma 6.15, the minimal elements of $\Pi(2)$ are determined by the set $\mathbb{F}_0(2)$.

Recall that $\mathbf{L}_0^{\mathcal{H}} = \mathbf{L}_0^{\vee}$, and consequently, $\mathbb{F}_0^{\mathcal{H}}(2) = \mathbb{F}_0^{\vee}(2)$. So $\mathbb{F}_0(2) = \mathbb{F}_0^{\mathcal{H}}(2) \cup \mathbb{F}_0^{\mathrm{Com}}(2)$.

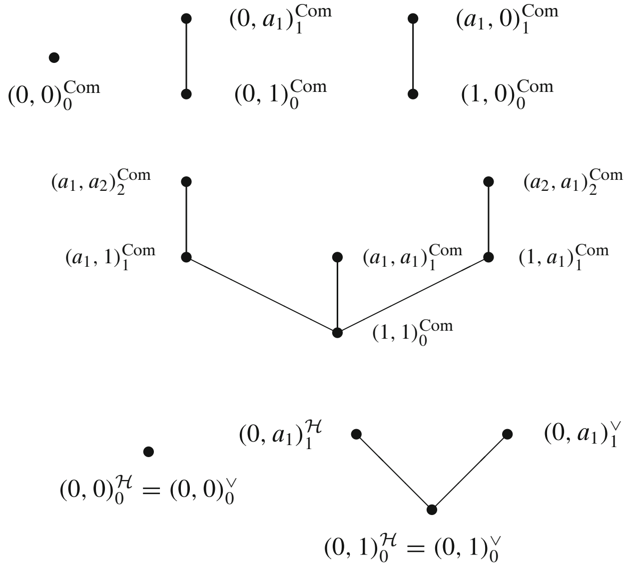
Let $(x, y)_i^*$ denote the function $f : G \to \mathbf{L}_i^*(2)$ such that $f(g_1) = x$, $f(g_2) = y$ and $S(f(g_1), f(g_2)) \cong \mathbf{L}_i^*(2)$ with $* \in \{\mathcal{H}, Com, \vee\}$, $0 \leq i \leq 2$. Then the minimal elements of $\Pi(2)$ are represented by the functions

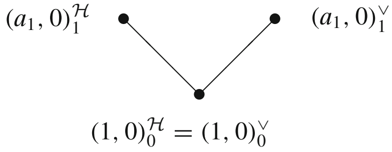
$$\mathbb{F}_0^{\mathcal{H}}(2) = \left\{(0,0)_0^{\mathcal{H}}, (0,1)_0^{\mathcal{H}}, (1,0)_0^{\mathcal{H}}, (1,1)_0^{\mathcal{H}}\right\}$$

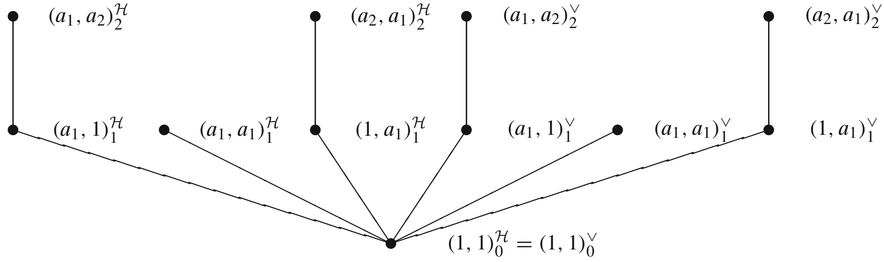
and

$$\mathbb{F}_0^{\mathrm{Com}}(2) = \left\{(0,0)_0^{\mathrm{Com}}, (0,1)_0^{\mathrm{Com}}, (1,0)_0^{\mathrm{Com}}, (1,1)_0^{\mathrm{Com}}\right\}.$$

By using the conditions of Theorem 6.18 we construct the corresponding connected components of $\Pi(2)$.







The free algebra $\mathfrak{F}_{\mathcal{C}}(n)$ can be constructed from the ordered set $\Pi(n)$. Let $[f) = \{g \in \mathbb{F}(n) : \ f \leq g\}$, and for $0 \leq j \leq n$ consider the sets

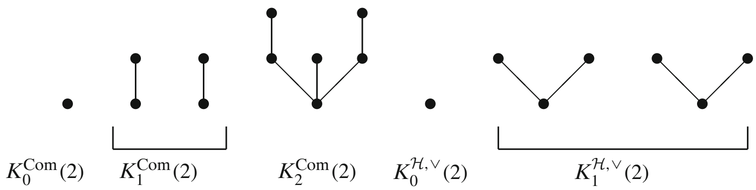
$$K_j^{\text{Com}}(n) = \left\{[f) : \ \left|f^{-1}(1)\right| = j \text{ and } f \in \mathbb{F}_0^{\text{Com}}(n)\right\},$$

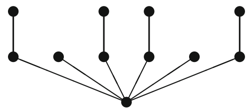
$$K_j^{\mathcal{H},\vee}(n) = \left\{[f) : \ \left|f^{-1}(1)\right| = j \text{ and } f \in \mathbb{F}_0^{\mathcal{H}}(n)\right\}.$$

Let $R_j^{\text{Com}}(n)$ and $R_j^{\mathcal{H},\vee}(n)$ respectively denote the distributive lattice such that $\Pi(R_j^{\text{Com}}(n)) \in K_j^{\text{Com}}(n)$ and $\Pi(R_j^{\mathcal{H},\vee}(n)) \in K_j^{\mathcal{H},\vee}(n)$. Then

$$\mathfrak{F}_{\mathcal{C}}(n) = \prod_{j=0}^{n} \left(R_j^{\mathcal{H},\vee}(n)\right)^{\binom{n}{j}} \times \prod_{j=0}^{n} \left(R_j^{\text{Com}}(n)\right)^{\binom{n}{j}}.$$

In the above example, $\Pi(2)$ is



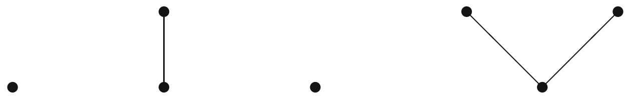


$K_2^{\mathcal{H},\vee}(2)$

and consequently,

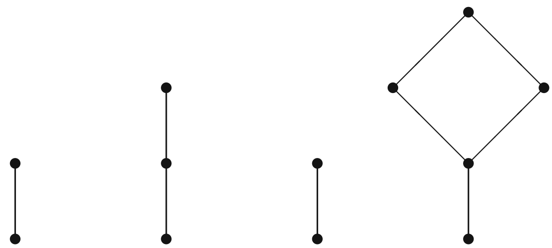
$$\mathfrak{F}_{\mathcal{C}}(2) = R_0^{\mathrm{Com}}(2) \times R_1^{\mathrm{Com}}(2) \times R_2^{\mathrm{Com}}(2) \times R_0^{\mathcal{H},\vee}(2) \times R_1^{\mathcal{H},\vee}(2) \times R_2^{\mathcal{H},\vee}(2).$$

In the case $n = 1$, the ordered set $\Pi(1)$ is:



$K_0^{\mathrm{Com}}(1)$ $K_1^{\mathrm{Com}}(1)$ $K_0^{\mathcal{H},\vee}(1)$ $K_1^{\mathcal{H},\vee}(1)$

and thus, $\mathfrak{F}_{\mathcal{C}}(1) = R_0^{\mathrm{Com}}(1) \times R_1^{\mathrm{Com}}(1) \times R_0^{\mathcal{H},\vee}(1) \times R_1^{\mathcal{H},\vee}(1)$, where



$R_0^{\mathrm{Com}}(1)$ $R_1^{\mathrm{Com}}(1)$ $R_0^{\mathcal{H},\vee}(1)$ $R_1^{\mathcal{H},\vee}(1)$

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