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Kripke Style Semantic for the Logic of Two Valued-States

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Abstract In this paper we develope a Kripke style semantic for the logic of two-valued states on orthomodular lattices. Kripke models are built from Baer* semigroups enriched with an unary operation. A completeness theorem with respect to this Kripkean semantic is established.

Keywords Orthomodular logic · Two-valued states · Kripke frames

1 Introduction

The notion of state is useful to model probabilities in different algebraic structures. In the last decades, several authors have paid attention to the study of states on algebraic structures, directly or indirectly related to quantum mechanics, as orthomodular posets

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[3, 17], MV-algebras [4, 18, 20] or effect algebras [21]. In this context, several families of states are investigated within the quantum logical program because they provide different representations of the event structure of quantum systems [9, 10, 12, 15, 16, 23, 24].

We focus our attention in the family of two-valued states on orthomodular lattices. Let us recall that Hilbert lattices based on Hilbert spaces of dimension >2 do not admit any two-valued state. However, there are algebras in the variety of orthomodular lattices in which a two-valued state can be defined (see Example 1). Thus, from the algebraic-logic point of view we can study the subclass of orthomodular lattices admitting two-valued states [5, 7]. In this work we introduce a Kripke style semantic related to a logical calculus for orthomodular lattices admiting two-valued states. A completeness theorem for this logic system is obtained.

The paper is organized as follows. In Section 2 we recall some basic notions of universal algebra and orthomodular lattices. In Section 3, by adding an unary operation to the orthomodular structure, an equational class called IE_2-lattices capturing the notion two-valued states, is introduced. In Section 4, a Hilbert-style calculus, algebrizable in the variety of IE_2-lattices, is defined. In Section 5, a Kripke style semantic based on Baer*-semigroups is developed. In this framework, a strong completeness theorem for the mentioned Hilbert-style calculus is obtained. Finally, Section 6 is devoted to the conclusions.

2 Basic Notions

We first introduce some basic notions about universal algebra and orthomodular lattices that will play an important role in what follows. A *variety* is a class of algebras of the same type defined by a set of equations. Let $\mathcal{A}$ be a variety of algebras of type σ. We denote by $Term_{\mathcal{A}}$ the *absolutely free algebra* of type σ built from the set of variables $V = \{x_1, x_2, ...\}$. Each element of $Term_{\mathcal{A}}$ is referred to as a *term*. We denote by $t = s$ the equations of $Term_{\mathcal{A}}$. Let $A \in \mathcal{A}$. If $t \in Term_{\mathcal{A}}$ and $a_1, \ldots, a_n \in A$, by $t^A(a_1, \ldots, a_n)$ we denote the result of the application of the term operation t^A to the elements $a_1, \ldots, a_n$. A *valuation* in A is a map of the form $v : V \to A$. Note that any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : Term_{\mathcal{A}} \to A$ in the usual way, i.e., if $t_1, \ldots, t_n \in Term_{\mathcal{A}}$ then $v(t(t_1, \ldots, t_n)) = t^A(v(t_1), \ldots, v(t_n))$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in Term_{\mathcal{A}}$, $\models_A t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\models_{\mathcal{A}} t = s$ means that for each $A \in \mathcal{A}$, $\models_A t = s$. A is *directly indecomposable* if A is not isomorphic to a product of two non trivial algebras.

A *lattice with involution* is an algebra $\langle L, \vee, \wedge, \neg \rangle$ such that $\langle L, \vee, \wedge \rangle$ is a lattice and $\neg$ is a unary operation on L that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$.

A *bounded lattice* is a lattice having a greatest element and a least element. For the sake of simplicity we use the same symbols either for the numbers 1,0 or for the greatest element and least element of a lattice.

An *orthomodular lattice* is an algebra $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ that satisfies the following:

1. $\langle L, \wedge, \vee, \neg, 0, 1 \rangle$ is a bounded lattice with involution,
2. $x \wedge \neg x = 0$,
3. $x \vee (\neg x \wedge (x \vee y)) = x \vee y$.

In the tradition of the quantum logical research, physical properties of the quantum system are organized in the orthomodular lattice of closed subspaces $\mathcal{L}(\mathcal{H}) =< \mathcal{P}(\mathcal{H}), \vee, \wedge, ^{\perp}, \mathbf{0}, \mathbf{1} >$ of a Hilbert space $\mathcal{H}$. Such lattices are called *Hilbert lattices*. This first event structure was introduced in the thirties by Birkhoff and von Neumann [2].

Boolean algebras are orthomodular lattices satisfying the *distributive law* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We denote by **2** the Boolean algebra of two elements. Let L be an orthomodular lattice. Two elements a, b in L are *orthogonal* (noted $a \perp b$) iff $a \leq \neg b$. An element $c \in L$ is said to be a *complement* of a iff $a \wedge c = 0$ and $a \vee c = 1$. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. An element z of L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L.

Proposition 1 *Let L be an orthomodular lattice. Then we have:*

1. $Z(L)$ *is a Boolean sublattice of* L [13, Theorem 4.15].
2. $z \in Z(L)$ *iff for each* $a \in L$, $a = (a \wedge z) \vee (a \wedge \neg z)$ [13, Lemma 29.9].

Let $\mathcal{A}$ a be variety whose algebras have an orthomodular reduct. Then, an important characterization of the equations in $\mathcal{A}$ is given by:

$$\models_{\mathcal{A}} t = s \quad iff \quad \models_{\mathcal{A}} (t \wedge s) \vee (\neg t \wedge \neg s) = 1 \tag{1}$$

Thus, we can safely assume that all $\mathcal{A}$-equations are of the form $t = 1$, where $t \in Term_{\mathcal{A}}$.

3 Two-Valued States on Orthomodular Lattices

In general two-valued states represent probability measures $s : \mathcal{O} \to \{0, 1\}$ where $\mathcal{O}$ is a set equipped with an orthostructure usually called *event structure*. From a physical point of view, two-valued measures are distinguished among the set of all classes of states because of their relation to the hidden variable approach to quantum mechanics. For a detailed discussion about the notion of two-valued state in orthostructures and hidden variables we remit to [9].

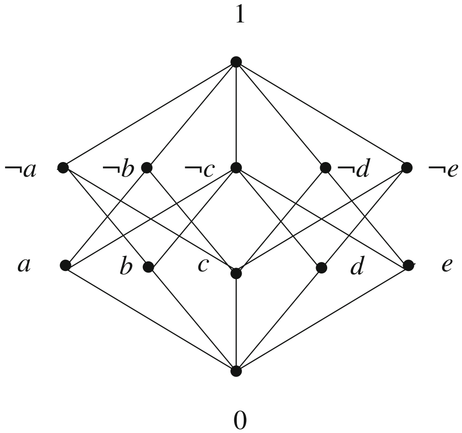
Definition 1 Let L be an orthomodular lattice. A *two-valued state* on L is a function $\sigma : L \to \{0, 1\}$ such that:

1. $\sigma(1) = 1$,
2. if $x \perp y$ then $\sigma(x \vee y) = \sigma(x) + \sigma(y)$.

This notion of two-valued state is introduced in [22] for orthoposets. In order to develop notions of hidden variables on orthostructures more general that Hilbert lattices we also note that a similar definition of two-valued states was introduced in [19]. In the mentioned work the second condition of Definition 1 is restricted by considering the orthogonal pairs containing a central element.

It is well known that Hilbert lattices do not admit two-valued states. However there exists a class of orthomodular lattices in which these states can be defined. The following example shows this fact.

Example 1 Let us consider the orthomodular lattice $MO2 \times \mathbf{2}$ whose Hasse diagram has the following form:



If we define the function $\sigma : MO2 \times \mathbf{2} \to \{0, 1\}$ such that:

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in \{1, \neg a, \neg b, c, \neg d, \neg e\} \\ 0, & \text{if } x \in \{0, a, b, \neg c, d, e\} \end{cases}$$

we can see that σ is a two-valued state.

We denote by $\mathcal{E}_2$ the class of pairs (L, σ) such that L is an orthomodular lattice and σ is a two-valued states on L (E_2-lattices for short). It is straightforward to prove the following proposition

Proposition 2 *Let* $(L, \sigma) \in \mathcal{E}_2$. *Then:*

1. $\sigma(1) = 1$ *and* $\sigma(0) = 0$,
2. $\sigma(\neg x) = 1 - \sigma(x)$,
3. *if* $x \leq y$ *then* $\sigma(x) \leq \sigma(y)$,
4. $\sigma(x \wedge y) \leq \min\{\sigma(x), \sigma(y)\}$,
5. $\sigma(x \vee y) = \sigma(x) + \sigma((x \vee y) \wedge \neg x)$.

If we consider the set $\{0, 1\}$ endowed with the natural Boolean structure, the above properties allow us to see two-valued states as functions to a Boolean algebra preserving order and orthocomplementation. This suggests the possibility of thinking the two-valued state as an unary operation added to the orthomodular structure. In this way we introduce the following definition:

Definition 2 [5] An *orthomodular lattice with an internal two-valued state* or IE_2-lattice for short, is an algebra $\langle L, \wedge, \vee, \neg, s, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that $\langle L, \wedge, \vee, \neg, 0, 1\rangle$ is an orthomodular lattice and s satisfies the following equations:

s1. $s(1) = 1$,
s2. $s(\neg x) = \neg s(x)$,
s3. $s(x \vee s(y)) = s(x) \vee s(y)$,
s4. $y = (y \wedge s(x)) \vee (y \wedge \neg s(x))$,
s5. $s(x \wedge y) \leq s(x) \wedge s(y)$,
s6. $s(x \vee (y \wedge \neg x)) = s(x) \vee s(y \wedge \neg x)$.

We shall refer to s as a *internal two-valued state*. We also define the subset of L given by $s(L) = \{s(x) : x \in L\}$. Clearly the class of IE_2-lattices is a variety that we call $\mathcal{IE}_2$. Since $\mathcal{IE}_2$ admits an orthomodular reduct, by (1), all the equations in $\mathcal{IE}_2$ can be referred to 1. The following proposition provides some basic properties about IE_2-lattices.

Proposition 3 *Let L be a IE_2-lattice. Then we have:*

1 $\langle s(L), \vee, \wedge, \neg, 0, 1\rangle$ *is a Boolean sublattice of* $Z(L)$*,*
2 *If* $x \leq y$ *then* $s(x) \leq s(y)$*,*
3 $s(x) \vee s(y) \leq s(x \vee y)$*,*
4 $s(s(x)) = s(x)$*,*
5 $x \in s(L)$ *iff* $s(x) = x$*,*
6 $s(x \wedge s(y)) = s(x) \wedge s(y)$*,*
7 *if* $x \perp y$ *then* $s(x \vee y) = s(x) \vee s(y)$*,*
8 L *is directly indecomposable iff* $s(L) = \{0, 1\}$*.*

Proof 1...7) See [5, Proposition 3.5]. 8) See [5, Proposition 5.6] □

Proposition 4 [5, Proposition 5.6] *Let L be a IE_2-lattice. Then L is directly indecomposable iff $s(L) = \{0, 1\}$.*

We denote by $\mathcal{D}(\mathcal{IE}_2)$ the class of directly indecomposable algebras of the variety $\mathcal{IE}_2$. The following theorem provide the relation between the class of orthomodular lattices that admit two-valued states and the IE_2-lattices.

Theorem 1 [8, Theorem 2.5]

1. *For each* $(L, \sigma) \in \mathcal{E}_2$, $s_\sigma(x) = \begin{cases} 1^L, & \text{if } \sigma(x) = 1 \\ 0^L, & \text{if } \sigma(x) = 0 \end{cases}$ *defines an internal two-valued state on L such that* $(L, s_\sigma) \in \mathcal{D}(\mathcal{IE}_2)$*.*
2. *For each* $(L, s) \in \mathcal{D}(\mathcal{IE}_2)$, $\sigma_s(x) = \begin{cases} 1, & \text{if } s(x) = 1^L \\ 0, & \text{if } s(x) = 0^L \end{cases}$ *is a two-valued state on L.*

Theorem 1 states that every orthomodular lattice with a two-valued state is univocally identifiable to a directly indecomposable algebra of the variety $\mathcal{IE}_2$ and viceversa. Moreover since $\mathcal{D}(\mathcal{IE}_2)$ contains the subdirectly irreducible algebras of $\mathcal{IE}_2$, it is immediate that

$$\models_{\mathcal{D}(\mathcal{IE}_2)} t = 1 \quad iff \quad \models_{\mathcal{IE}_2} t = 1 \tag{2}$$

Equation (2) shows that the the variety $\mathcal{IE}_2$ captures in detail the notion of two-valued state. Indeed, the equational theory of $\mathcal{IE}_2$ is ruled by the directly indecomposable IE_2-lattices which, by Theorem 1, represent the orthomodular lattices with a two-valued state.

4 Hilbert-Style Calculus for $\mathcal{IE}_2$

In [8] a Hilbert-style calculus with completeness respect to $\mathcal{IE}_2$ was introduced. In this section we briefly describe this calculus by introducing some basic ideas that will allow us to define in the next section a Kripke style semantic in which it can be interpreted.

Each subset $T \subseteq Term_{\mathcal{IE}_2}$ is referred as a *theory*. If u is a valuation, $u(T) = 1$ means that for each $\gamma \in T$, $u(\gamma) = 1$. Let $\alpha \in Term_{\mathcal{IE}_2}$ and T be a theory. We use $T \models_{\mathcal{A}} \alpha$

when for each valuation u over the algebras of $\mathcal{IE}_2$, if $u(T) = 1$ then $u(\alpha) = 1$. If for each possible valuation u, $u(\alpha) = 1$ then α is said to be a *tautology* and it is denoted $\models_{\mathcal{IE}_2} \alpha$.

Definition 3 Let us consider the syntactic abbreviation in $Term_{\mathcal{IE}_2}$ given by

$$\alpha R \beta \quad \text{for} \quad (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta).$$

Then, the calculus $\langle Term_{\mathcal{IE}_2}, \vdash \rangle$ is defined by the following axioms:

A0. 1, $1R(\alpha \vee \neg\alpha)$ and $\neg 1 R 0$,
A1. $\alpha R \alpha$,
A2. $\neg(\alpha R \beta) \vee (\neg(\beta R \gamma) \vee (\alpha R \gamma))$,
A3. $\neg(\alpha R \beta) \vee (\neg\alpha R \neg\beta)$,
A4. $\neg(\alpha R \beta) \vee ((\alpha \wedge \gamma) R (\beta \wedge \gamma))$,
A5. $(\alpha \wedge \beta) R (\beta \wedge \alpha)$,
A6. $(\alpha \wedge (\beta \wedge \gamma)) R ((\alpha \wedge \beta) \wedge \gamma)$,
A7. $(\alpha \wedge (\alpha \vee \beta)) R \alpha$,
A8. $(\neg\alpha \wedge \alpha) R ((\neg\alpha \wedge \alpha) \wedge \beta)$,
A9. $\alpha R \neg\neg\alpha$,
A10. $\neg(\alpha \vee \beta) R (\neg\alpha \wedge \neg\beta)$,
A11. $(\alpha \vee (\neg\alpha \wedge (\alpha \vee \beta)) R (\alpha \vee \beta)$,
A12. $(\alpha R \beta) R (\beta R \alpha)$,
A13. $\neg(\alpha R \beta) \vee (\neg\alpha \vee \beta)$,
A14. $s(1) R 1$,
A15. $s(\neg\alpha) R \neg s(\alpha)$,
A16. $s(\alpha \vee s(\beta)) R (s(\alpha) \vee s(\beta))$,
A17. $((\alpha \wedge s(\beta)) \vee (\alpha \wedge \neg s(\beta)) R \alpha$,
A18. $(s(\alpha) \wedge s(\beta)) R (s(\alpha \wedge \beta)) \vee (s(\alpha) \wedge s(\beta)))$,
A19. $s(\alpha \vee (\beta \wedge \neg\alpha)) R (s(\alpha) \vee s(\beta \wedge \neg\alpha))$,

and the following inference rules:

$$\frac{\alpha, \neg\alpha \vee \beta}{\beta} \qquad disjunctivesyllogism(DS)$$

$$\frac{\alpha}{s(\alpha)} \qquad (S)$$

Let T be a theory. A *proof* from T is a sequence $\alpha_1, ..., \alpha_n$ in $Term_{\mathcal{IE}_2}$ such that each member is either an axiom or a member of T or follows from some preceding member of the sequence using *DS* or *S*. $T \vdash \alpha$ means that α is provable in T, that is, α is the last element of a proof from T. If $T = \emptyset$, we use the notation $\vdash \alpha$ and in this case we will say that α is a theorem of $\langle Term_{\mathcal{IE}_2}, \vdash \rangle$. T is *inconsistent* if and only if $T \vdash \alpha$ for each $\alpha \in Term_{\mathcal{IE}_2}$; otherwise it is *consistent*.

Proposition 5 [8, Theorem 3.3] *Axioms of the* $\langle Term_{\mathcal{IE}_2}, \vdash \rangle$ *are tautologies and inference rules preserves valuations equal to 1.*

The following theorem establishes the strong completeness for the calculus $\langle Term_{\mathcal{IE}_2}, \vdash \rangle$ with respect to the variety $\mathcal{IE}_2$ and, by (2), with respect to the sub class $\mathcal{D}(\mathcal{IE}_2)$ of direct indecomposable algebras.

Theorem 2 [8, Theorem 3.5] *Let $\alpha \in Term_{\mathcal{IE}_2}$ and T be a theory. Then we have that:*

$$T \vdash \alpha \quad iff \quad T \models_{\mathcal{IE}_2} \alpha \quad iff \quad T \models_{\mathcal{D}(\mathcal{IE}_2)} \alpha$$

5 Kripke-Style Semantics for $\langle Term_{\mathcal{IE}_2}, \vdash\rangle$

In this section we develop Kripke models for the calculus $\langle Term_{\mathcal{IE}_2}, \vdash\rangle$ based on Baer*-semigroups.

Definition 4 A *Baer *-semigroup* [1, 6, 11], also called Foulis semigroup, is an algebra $\langle G, \cdot, ^*, ', 0\rangle$ of type $\langle 2, 1, 1, 0\rangle$ such that, upon defining $1 = 0'$, the following conditions are satisfied:

1. $\langle G, \cdot\rangle$ is a semigroup,
2. $0 \cdot x = x \cdot 0 = 0$,
3. $1 \cdot x = x \cdot 1 = x$,
4. $(x \cdot y)^* = y^* \cdot x^*$,
5. $x^{**} = x$,
6. $x \cdot x' = 0$,
7. $x' \cdot x' = x' = (x')^*$,
8. $x' \cdot y \cdot (x \cdot y)' = y \cdot (x \cdot y)'$.

Let G be a Baer *-semigroup. An element $e \in S$ is a *projector* iff $e = e^* = e \cdot e$. The set of all projectors of G is denoted by $P(G)$. A projector $e \in P(G)$ is said to be closed iff $e'' = e$. We denote by $P_c(G)$ the set of all closed projectors. Moreover we can prove that $P_c(G) = \{x' : x \in S\}$. We can establish a partial order $\langle P(G), \leq\rangle$ given by $e \leq f$ iff $e \cdot f = e$. If $x \in G$ then we define the set $x \cdot G$ as

$$x \cdot G = \{x \cdot g : g \in G\}.$$

Proposition 6 [14, Proposition 3.11] *Let G be a Baer*- semigroup and $e_1, e_2 \in P(G)$. If $e_1 \cdot G = e_2 \cdot G$ then $e_1 = e_2$.*

Let G be a Baer*- semigroup. Let us consider the following operations for any $e_1, e_2 \in P_c(G)$:

$$e_1 \wedge e_2 = e_1 \cdot (e_2' \cdot e_1)' \quad and \quad e_1 \vee e_2 = (e_1' \wedge e_2')'.$$

Then we can see that $\langle P_c(G), \wedge, \vee, ', 0, 1\rangle$ is an orthomodular lattice with respect to the order $\langle P(G), \leq\rangle$ [13, Theorem 37.8].

Definition 5 A IE_B^*-semigroup [7] is an algebra $\langle G, \cdot, ^*, ', s, 0\rangle$ of type $\langle 2, 1, 1, 1, 0\rangle$ such that $\langle G, \cdot, ^*, ', 0\rangle$ is a Baer *-semigroup and s satisfies the following equations:

1. $s(1) = 1$,
2. $s(x') = s(x)'$,
3. $s(x)'' = s(x)$,
4. $s(x' \vee s(y')) = s(x') \vee s(y')$,
5. $y' = (y' \wedge s(x)) \vee (y' \wedge s(x)')$,
6. $s(x' \wedge y') \leq s(x') \wedge s(y')$.

Proposition 7 *Let G be an IE_B^*-semigroup. Then,*

1. $s(x) \in Z(P_c(G))$,
2. $s(x'') = s(x)$,
3. $\langle P_c(G), \vee, \wedge, ', s/_{P_c(G)}, 0, 1\rangle$ *is an IE_B-lattice and* $\langle s(G), \vee, \wedge, ', 0, 1\rangle$ *is a Boolean subalgebra of* $Z(P_c(G))$.

Proof 1) By Axiom 3, for each $x \in G$, $s(x) \in P_c(G)$. By Proposition 1–2 and Axiom 5, $s(x) \in Z(P_c(G))$. 2) Immediate from Axiom 2 and Axiom 3. 3) Straightforward calculation. □

Proposition 8 *Let L be a IE_B-lattice, then there exists a IE_B^*-semigroup $G(L)$ such that L is $\mathcal{IE}_B$-isomorphic to $P_c(G(L))$.*

Proof In [1, Proposition 2] it is proved that there exists a Baer*-semigroup $G(L)$ and an orthonmodular isomorphism $f : L \to P_c(G(L))$. Note that f naturally defines an IE_B-lattice structure on $P_c(G(L))$. In this way f is a IE_B-isomorphism. Since for each $x \in G(L)$, $x'' \in P_c(G(L))$ then, we can define the operation $\hat{s}$ on $G(L)$ such that $\hat{s}(x) = s(x'')$. We have to prove that $\langle G(L), \cdot, ^*, ', \hat{s}, 0\rangle$ is a IE_B^*-semigroup. Indeed,

Ax1) Is immediate. Ax2) $\hat{s}(x') = s(x''') = s(x'')' = \hat{s}(x)'$. Ax3) $\hat{s}(x)'' = s(x'')'' = s(x'')$ because $s(x) \in P_c(G(L))$ and $'$ is an orthocomplementation on $P_c(G(L))$. Thus $\hat{s}(x)'' = \hat{s}(x)$. The rest of the axioms hold because $\hat{s}/_{P_c(G(L))} = s$ and $\langle P_c(G), \vee, \wedge, ', s, 0, 1\rangle$ is an IE_B-lattice. Hence $G(L)$ is a IE_B^*-semigroup as is required. □

Definition 6 A *two valued state frame* is a pair $\langle G, u\rangle$ such that G is a IE_B^*-semigroup and u is a valuation $u : Term_{\mathcal{IE}_2} \to P_c(G)$

We denote by $\mathcal{F}_2^*$ the class of all two valued state frames.

Definition 7 Let $\langle G, u\rangle$ be a two valued state frame. Then we define inductively the *forcing relation* $\models^x_{\langle G,u\rangle} \subseteq G \times Term_{\mathcal{IE}_2}$ as follows:

1. $\models^x_{\langle G,u\rangle} p$ iff $x \in u(p) \cdot G$, for each variable $p \in Term_{\mathcal{IE}_2}$,
2. $\models^x_{\langle G,u\rangle} \alpha \wedge \beta$ iff $\models^x_{\langle G,u\rangle} \alpha$ and $\models^x_{\langle G,u\rangle} \beta$,
3. $\models^x_{\langle G,u\rangle} \neg\alpha$ iff $\forall g \in G, \models^g_{\langle G,u\rangle} \alpha \Longrightarrow g^* \cdot x = 0$,
4. $\models^x_{\langle G,u\rangle} s(\alpha)$ iff $x = s(\alpha) \cdot x$.

The relation $\models^x_{\langle G,u\rangle} \alpha$ is read as α *is true at the point x in the two valued state frame* $\langle G, u\rangle$ and, by $\models_{\langle G,u\rangle} \alpha$, we understand that for each $x \in G$, $\models^x_{\langle G,u\rangle} \alpha$. If T is a theory, $\models_{\langle G,u\rangle} T$ means that, for each $\beta \in T$ we have that $\models_{\langle G,u\rangle} \beta$. The notion of *consequence* in the Kripke-style sense, denoted by $T \models_{\mathcal{F}_2^*} \alpha$, is introduced as follows:

$$T \models_{\mathcal{F}_2^*} \alpha \quad iff \quad \forall \langle G, u\rangle \in \mathcal{F}_2^*, \quad \models_{\langle G,u\rangle} T \Longrightarrow \models_{\langle G,u\rangle} \alpha.$$

Let $\alpha \in Term_{\mathcal{IE}_2}$, T be a theory and $\langle G, u\rangle$ be a two valued state frame. Then we consider the following sets:

$$|\alpha|_{\langle G,u\rangle} = \{x \in G : \models^x_{\langle G,u\rangle} \alpha\},$$

$$|T|_{\langle G,u\rangle} = \bigcap_{\alpha \in T} |\alpha|_{\langle G,u\rangle}.$$

Proposition 9 *Let* $\alpha \in Term_{\mathcal{IE}_2}$, T *be a theory and* $\langle G, u\rangle$ *be a two valued state frame. Then:*

1. $|\alpha|_{\langle G,u\rangle} = u(\alpha) \cdot G$,
2. $\models_{\langle G,u\rangle} T$ *iff* $|T|_{\langle G,u\rangle} = G$.

Proof 1) If α is a variable the proposition results trivial. Suppose that $\models^x_{\langle G,u\rangle} s(\alpha)$. Then $x = s(\alpha) \cdot x$ proving that $x \in s(\alpha) \cdot G$. For the converse, let us suppose that $x \in s(\alpha) \cdot G$ i.e., $x = s(\alpha) \cdot g$. By Proposition 7-1, $s(\alpha)$ is a projector and then, it is idempotent. Thus $s(\alpha) \cdot x = s(\alpha) \cdot s(\alpha) \cdot g = s(\alpha) \cdot g = x$. It proves that $\models^x_{\langle G,u\rangle} s(\alpha)$. Hence $|s(\alpha)|_{\langle G,u\rangle} = u(s(\alpha)) \cdot G$. If α has the form $\beta \wedge \gamma$ or $\neg\beta$ it follows by induction on the complexity of terms and we refer to [14, Lemma 3.16] for a detailed proof.

2) By the above item we have that , $\models_{\langle G,u\rangle} T$ iff $\forall\alpha \in T$, $|\alpha|_{\langle G,u\rangle} = G$ iff $G = \bigcap_{\alpha\in T} |\alpha|_{\langle G,u\rangle} = |T|_{\langle G,u\rangle}$ □

Theorem 3 [Kripke style completeness] $\alpha \in Term_{\mathcal{IE}_2}$ *and* T *be a theory. Then,*

$$T \models_{\mathcal{F}_2^*} \alpha \Longleftrightarrow T \vdash \alpha.$$

Proof Let us assume that $T \models_{\mathcal{F}_2^*} \alpha$. Let $L \in \mathcal{IE}_2$ and a valuation $u : Term_{\mathcal{IE}_2} \to L$ such that $u(T) = 1$. By Theorem 8, there exists a IE_B^*-semigroup $G(L)$ such that L is $\mathcal{IE}_2$-isomorphic to $P_c(G(L))$. Thus, we can assume that the valuation u has the form $u : Term_{\mathcal{IE}_2} \to P_c(G(L))$. Let us consider the two valued state frame $\langle G(L), u\rangle$. Since for each $\beta \in T$, $u(\beta) = 1$, by Proposition 9, $G(L) = u(\beta) \cdot G(L) = |\beta|_{\langle G(L),u\rangle}$ and consequently $|T|_{\langle G(L),u\rangle} = 1 \cdot G(L) = G(L)$. Then, by hypothesis, $\models_{\langle G(L),u\rangle} \alpha$ and $u(\alpha) \cdot G(L) = G(L)$. Hence, by Proposition 6, $u(\alpha) = 1$ proving that $T \models_{\mathcal{IE}_2} \alpha$. Then, by Proposition 2, $T \vdash \alpha$.

For the converse let us assume that $T \vdash \alpha$ and then $T \models_{\mathcal{IE}_2} \alpha$. Suppose that $\models_{\langle G,u\rangle} T$. By Proposition 9-2, for each $\beta \in T$, $u(\beta) \cdot G = 1 \cdot G = G$. Then, by Proposition 6, $u(\beta) = 1$ for each $\beta \in T$. Thus $u(T) = 1$ and, by hypothesis, $u(\alpha) = 1$. Consequently $G = u(\alpha) \cdot G$ and, by Proposition 9, $\models_{\langle G,u\rangle} \alpha$. It proves that $T \models_{\mathcal{F}_2^*} \alpha$. □

6 Conclusions

In this work a Kripke style semantics for a logical calculus related to orthomodular lattices admitting two-valued states is studied. Although it is well known that Hilbert lattices do not admit two-valued states, there exists a class of orthomodular lattices in which these states can be defined and its study could be useful to better understand the constraints imposed by Hilbert lattices. This fact has motivated our logical approach to the mentioned class. A completeness theorem respect to this semantic is established.

A generalization of the present study —i.e., the consideration of Kripke models based on *-semigroups for families of two-valued states defined on different orthostructures— will be analyzed in a future work.

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