SYMMETRY AND SYMMETRY BREAKING FOR MINIMIZERS IN THE TRACE INEQUALITY

ENRIQUE J. LAMI DOZO AND OLAF TORNÉ

Abstract. We consider symmetry properties of minimizers in the variational characterization of the best constant in the trace inequality $C\|u\|^p_{L^q(\partial B_\rho)} \leq \|u\|^p_{W^{1,p}(B_\rho)}$ in the ball B_ρ of radius ρ. When p is fixed minimizers in this problem can be radial or nonradial depending on the parameters q and ρ. We prove that there is a global radial function $u_0 > 0$, with u_0 independent of q, such that any radial minimizer is a multiple of the restriction of u_0 to B_ρ. Next we prove that if either q or ρ is sufficiently large then the minimizers are nonradial. In the case when $p = 2$ we consider a generalization of the minimization problem and improve some of the above symmetry results. We also present some numerical results describing the exact values of q and ρ for which radial symmetry breaking occurs.

1. Introduction

Let B_ρ denote the ball of radius ρ centered at the origin in $\mathbb{R}^N$ with $N \geq 2$. Let $1 < p < \infty$ be fixed and denote by p^* the critical trace exponent given by $p^* = p(N-1)/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$. Let $1 < q < p^*$. The trace inequality states that there exists a constant C which depends on q and ρ such that

$$C\left(\int_{\partial B_\rho} |u|^q \, d\sigma\right)^{p/q} \leq \int_{B_\rho} |\nabla u|^p + |u|^p \, dx \qquad \forall u \in W^{1,p}(B_\rho).$$

The best constant is given by

$$S_q(\rho) = \inf_{u \in W^{1,p}(B_\rho)} \frac{\int_{B_\rho} |\nabla u|^p + |u|^p \, dx}{\left(\int_{\partial B_\rho} |u|^q \, d\sigma\right)^{p/q}}. \tag{1}$$

If $1 < q < p^*$ it is standard to show that this infimum is reached by a function u which has definite sign and that any nonzero multiple of u is again a minimizer. If $q = p$ then u is the first eigenfunction in a Steklov type problem (see equation (4) below). If $q \neq p$ we can assume that u is positive and normalized in such a way that

$$S_q(\rho)\left(\int_{\partial B_\rho} u^q \, d\sigma\right)^{\frac{p}{q}-1} = 1. \tag{2}$$

The function u is then a solution of the boundary value problem

$$\begin{cases} \Delta_p u = u^{p-1} & \text{in } B_\rho, \\ u > 0 & \text{in } B_\rho, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = u^{q-1} & \text{on } \partial B_\rho. \end{cases} \tag{3}$$

Given the inherent symmetries of this minimization problem it is natural to ask if u is radial. Two previous articles study this question in the case $p = 2$. The following results are known:

(1) In [8] M. del Pino and C. Flores consider the best trace constant in an expanding smooth domain when $2 < q < 2^*$. They show that when the parameter governing the expansion is sufficiently large the minimizing functions concentrate near a single point on the boundary where the mean curvature is maximum. In the case of a ball this result implies that the minimizing functions are nonradial when ρ is sufficiently large.
(2) In [6] J. Fernandez Bonder, E. Lami Dozo and J. Rossi proved the following results: Let $N \geq 3$. There exists $R > 0$ such that for any $\rho < R$ and for any $1 < q \leq 2^*$ the minimizer for $S_q(\rho)$ is radial. If $N = 2$ then for any $1 < q < \infty$ there exists $R(q)$ such that for any $\rho < R(q)$ the minimizer for $S_q(\rho)$ is radial. Now let $\rho > 0$ be fixed. The authors show that if there is a radial minimizer for $S_{q_0}(\rho)$ then for any $q \leq q_0$ there exists a radial minimizer for $S_q(\rho)$. In particular, since there always exists a radial minimizer when $q = 2$ it follows that $S_q(\rho)$ has a radial minimizer for any $q \leq 2$ and any $\rho > 0$.

We have considered these questions in the more general setting $1 < p < +\infty$. We extend many of the results known for the case $p = 2$ and find new methods of proof for some of them. We also prove various results which are new even in the case $p = 2$. We now state our main results.

Let $\rho > 0$ be fixed and consider the following Steklov type eigenvalue problem,

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } B_\rho, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial B_\rho. \end{cases} \tag{4}$$

It is well known that the first eigenvalue $\lambda_1(\rho) = S_p(\rho)$ is simple (see [7]). Let u_0 be an eigenfunction associated to $\lambda_1(\rho)$. Since u_0 is unique up to a constant factor and (4) is invariant under rotation it is clear that u_0 is radial.

Theorem 1. *Let $\rho > 0$ and $1 < p < +\infty$ be fixed.*
(1) *If there exists a radial minimizer for $S_q(\rho)$ then it is a multiple of u_0.*
(2) *Assume there exists a radial minimizer for $S_{q_0}(\rho)$. If $1 < q < q_0$ then any minimizer for $S_q(\rho)$ is a multiple of u_0.*
(3) *Let $1 < q < p$. Then the solution of the boundary value problem (3) is unique and it is a multiple of u_0. In particular any minimizer for $S_q(\rho)$ is a multiple of u_0.*

The second and third statements of Theorem 1 are partly known in the case $p = 2$. Indeed it is shown in [6] that if $q \leq q_0$ or $q \leq 2$ then there exists a radial minimizer for $S_q(\rho)$. In fact the second statement in Theorem 1 asserts that in this case any minimizer is radial and is given up to a constant factor by u_0. Moreover the third statement in Theorem 1 asserts that when $q < p$ we have uniqueness in the associated boundary value problem as well as in the minimization problem.

Under certain conditions radial symmetry is lost if either q or ρ is sufficiently large. Define the function $\rho \mapsto Q(\rho)$ by

$$Q(\rho) = \frac{1}{\lambda_1(\rho)^{p/(p-1)}}\left(1 - (N-1)\frac{\lambda_1(\rho)}{\rho}\right) + 1. \tag{5}$$

Theorem 2. *Let $1 < p < +\infty$ be fixed.*
(1) *Let $\rho > 0$. If $q > Q(\rho)$ then there is no radial minimizer for $S_q(\rho)$.*
(2) *Let $p < q < p^*$. There exists $R(q)$ such that for any $\rho > R(q)$ there is no radial minimizer for $S_q(\rho)$.*

The first statement in Theorem 2 appears to be new even in the case $p = 2$. Moreover, although there are several studies in the literature concerning symmetry breaking in variational problems it is unusual to find precise quantitative results in higher dimension such as the first statement in Theorem 2. The second statement in Theorem 2 appears to be new in the case $p \neq 2$ but follows from [8] in the case $p = 2$. The authors of [8] consider the more general setting of an expanding smooth bounded domain. Our proof of Theorem 2 is different and only works in the case of a ball. However it is more simple and direct than [8]. We use a method of "desymmetrization" whereby one starts with a hypothetical radial minimizer and uses it to construct a nonradial function which has a smaller Rayleigh quotient.

It will be shown that if $N \geq 3$ and if ρ is sufficiently small then $Q(\rho) > p^*$. When $q > p^*$ the minimization problem (1) is not a priori well defined so the first statement of Theorem 2 does not apply. This is in perfect agreement with the result of [6], in the case when $p = 2$ and $N \geq 3$, which states that there exists $R > 0$ such that for any $\rho < R$ and for any $2 < q \leq 2^*$ the minimizers are radial.

When the minimizer in (1) is nonradial it is still possible to get some symmetry by using the technique of spherical symmetrization, also known as foliated Schwarz symmetrization (see [4] for a description of this symmetrization method). We describe the shape of nonradial minimizers in section 5.

Using theorems 1 and 2 we may define a function $\tilde{Q}(\rho)$ such that

$$\begin{aligned} &q \leq \tilde{Q}(\rho) \Rightarrow \text{ Any minimizer for } S_q(\rho) \text{ is a multiple of } u_0 \\ &q > \tilde{Q}(\rho) \Rightarrow \text{ There is no radial minimizer for } S_q(\rho) \end{aligned} \tag{6}$$

It is clear that $p \leq \tilde{Q}(\rho) \leq Q(\rho)$. Furthermore we will prove the following statements:

1. If $N = 2$ and $p = 2$ then $\lim_{\rho\to 0} \tilde{Q}(\rho) = \lim_{\rho\to 0} Q(\rho) = +\infty$
2. For any N and p we have $\lim_{\rho\to+\infty} \tilde{Q}(\rho) = \lim_{\rho\to+\infty} Q(\rho) = p$

In light of this it seems natural to ask if $\tilde{Q}(\rho) = Q(\rho)$ for all $\rho > 0$. In other words, we ask if the converse of the first statement in Theorem 2 is true. We do not know of a proof (or counter proof) of this statement so we have tested the equality $\tilde{Q}(\rho) = Q(\rho)$ numerically in a special case. More precisely we present numerical data in section 6 which suggests that $\tilde{Q}(\rho) = Q(\rho)$ at least when $p = 2$ and $N = 2$. We have not found any values of the parameters for which $\tilde{Q}(\rho) \neq Q(\rho)$. This numerical study also yields a means of visualizing the graphs of minimizers when $N = 2$.

In the case $p = 2$ we can improve Theorem 1. To do so, we next consider the problem

$$\begin{cases} \Delta u = u & \text{in } B_\rho, \\ u > 0 & \text{in } B_\rho, \\ \frac{\partial u}{\partial \nu} = \lambda u + u^{q-1} & \text{on } \partial B_\rho, \end{cases} \tag{7}$$

where B_ρ is as above, $1 < q < 2^*$ and $-\infty < \lambda < \lambda_1(\rho)$ where $\lambda_1(\rho)$ is the first Steklov eigenvalue (see equation (10) below). A solution u is called a least energy solution if it is a minimizer for

$$S_q^\lambda(\rho) = \inf_{v\in H^1(B_\rho)} \frac{\int_{B_\rho} |\nabla v|^2 + v^2\, dx - \lambda \int_{\partial B_\rho} v^2\, d\sigma}{\left(\int_{\partial B_\rho} |v|^q\, d\sigma\right)^{2/q}}. \tag{8}$$

Conversely if a function u minimizes (8) then an appropriate multiple of u is a solution of (7). The following result can be proved using standard variational methods.

Theorem 3. *Let $\rho > 0$. Let $q \neq 2$ and $1 < q < 2^*$. Then (7) has a least energy solution if and only if $-\infty < \lambda < \lambda_1(\rho)$.*

Given the symmetries of the minimization problem (8) it is again natural to ask if the minimizers are radial functions. A similar problem has been studied elsewhere in the literature: Consider a positive solution of the equation

$$-\Delta u = u^{q-1} - \lambda u \quad \text{in } B_\rho \tag{9}$$

with homogeneous Dirichlet ($u = 0$) or Neumann ($\frac{\partial u}{\partial \nu} = 0$) boundary conditions where q is subcritical and $\lambda > 0$. In the case of the Dirichlet boundary condition any solution is radial by the Gidas-Ni-Nirenberg theorem. When the Neumann condition is imposed least energy solutions can be defined in a similar way to (8). The authors of [5] prove that there are no nonconstant radial least energy solutions and they prove some axial symmetry properties for least energy solutions, which do not depend on any of the parameters ρ, q or λ.

In contrast to the situation for (9) we find that the symmetry properties of minimizers in (8) depend on each of the parameters ρ, q and λ. First let us denote by $\lambda_1(\rho)$ the first eigenvalue in the Steklov type problem

$$\begin{cases} \Delta u = u & \text{in } B_\rho, \\ \frac{\partial u}{\partial \nu} = \mu u & \text{on } \partial B_\rho. \end{cases} \tag{10}$$

Let u_0 be an eigenfunction associated to $\lambda_1(\rho)$. This function is positive, unique up to a constant factor and it is radial. As in the case $\lambda = 0$ we show that if $q < 2$ then the minimizer for (8) is unique up to a constant factor and is given by u_0 up to a normalization. It remains to consider the case $q > 2$. Let $-\infty < \lambda < 1$ be fixed. By the proof of Proposition 6 below, the function $\rho \mapsto \lambda_1(\rho)$ is strictly increasing, $\lambda_1(0) = 0$ and $\lim_{\rho\to+\infty} \lambda_1(\rho) = 1$. It follows that we can define

$$\rho_0(\lambda) = \inf\{\rho > 0;\, \lambda < \lambda_1(\rho)\}. \tag{11}$$

Note that when $\lambda \leq 0$ we have $\rho_0 = 0$.

Theorem 4. *Consider $\rho > 0$, $2 < q < 2^*$ and $-\infty < \lambda < 1$. Then there exist positive numbers $\delta_1(q,\lambda)$, $\delta_2(q,\rho)$ and $\delta_3(\lambda,\rho)$ such that if one or more of the following conditions is true*

1. $\rho_0(\lambda) < \rho < \rho_0(\lambda) + \delta_1$,
2. $\lambda_1(\rho) - \delta_2 < \lambda < \lambda_1(\rho)$,
3. $2 < q < 2 + \delta_3$,

then any minimizer for $S_q^\lambda(\rho)$ is a multiple of u_0.

Notice that the third statement of Theorem 4 improves the third statement of Theorem 1 in the case $p = 2$. Moreover it will be shown that the numbers δ_1, δ_2 and δ_3 can not be chosen independently of the parameters ρ, q and λ. Concerning the loss of radial symmetry we have the following result.

Theorem 5. *Let $\rho > 0$, $2 < q < 2^*$ and $-\infty < \lambda < \lambda_1(\rho)$ be fixed. If*

$$1 - (N-1)\frac{\lambda_1(\rho)}{\rho} - (q-1)\lambda_1^2(\rho) + (q-2)\lambda_1(\rho)\lambda < 0. \tag{12}$$

then there is no radial minimizer for $S_q^\lambda(\rho)$.

Note that (12) can be used to write quantitative estimates for the values of q and λ where radial symmetry breaking occurs by isolating either of these parameters in the inequality (cf. equation (5)). In fact we will deduce from (12) that minimizers for $S_q^\lambda(\rho)$ may be nonradial if either ρ or q is sufficiently large or if λ is sufficiently small.

In section 5 we apply spherical symmetrization to describe the shape of nonradial minimizers. Lastly, in section 6 we give numerical examples which suggest that the converse of Theorem 5 is also true: If the inequality $\geq$ holds in (12) then any minimizer is radial.

2. Radial minimizers

From now on let $1 < p < \infty$ and $\rho > 0$ be fixed. Theorem 1 follows from propositions 2, 3 and 4 below. First we show that the eigenfunction associated to the first eigenvalue $\lambda_1(\rho) = S_p(\rho)$ in the Steklov type problem (4) is independent of ρ in the following sense.

Proposition 1. *There exists a positive radial function u_0 such that*

$$\Delta_p u_0 = u_0^{p-1} \quad \text{in } \mathbb{R}^N.$$

This u_0 unique up to a constant factor and for any $\rho > 0$ the restriction of u_0 to B_ρ is the first eigenfunction of (4).

Proof. We construct u_0 such that, say, $u_0(0) = 1$. For $\alpha > 0$ let B_α be the ball of radius α centered at the origin. Let u_α denote a solution of the Dirichlet problem $\Delta_p u_\alpha = u_\alpha^{p-1}$ in B_α and $u_\alpha \equiv 1$ on ∂B_α. This function u_α is unique by regularity theory and the comparison principle (see [10] and [3]). For any $\alpha > 0$ we define the restriction of u_0 to B_α by $u_0 = \frac{u_\alpha}{u_\alpha(0)}$. Using the comparison principle as above one can check that u_0 is well defined and has the desired properties. □

Another useful property is given in the following

Proposition 2. *Let v be a radial solution of (3). Then v is a multiple of u_0. In particular any radial minimizer of (1) is a multiple of u_0.*

Proof. Fix $a > 0$ such that $au_0 \equiv v$ on ∂B_ρ. The solution of the Dirichlet problem for the equation $\Delta_p w = w^{p-1}$ is unique by the results of [10] and [3]. It follows that $au_0 \equiv v$ in B_ρ. □

The following proposition contains the second statement of Theorem 1.

Proposition 3. *Let $1 < p < \infty$ and let $\rho > 0$ be fixed. Let $1 < q_0 < p^*$ and assume there exists a radial minimizer for $S_{q_0}(\rho)$. If $1 < q < q_0$, then any minimizer for $S_q(\rho)$ is a multiple of u_0.*

Proof. Let v be a minimizer for $S_q(\rho)$. If v is constant on the boundary then v is a multiple of u_0 by the same argument as in the proof of Proposition 2. Assume now that v is not constant on the boundary. To simplify notations we write $B = B_\rho$. It follows from the strict Holder inequality that

$$\left(\int_{\partial B} v^q \, d\sigma\right)^{p/q} < |\partial B|^{\frac{p}{q} - \frac{p}{q_0}} \left(\int_{\partial B} v^{q_0} \, d\sigma\right)^{p/q_0}.$$

Now, by Proposition 2, u_0 is a minimizer for $S_{q_0}(\rho)$. Using the previous inequality we get the following

$$\begin{aligned}
\frac{\|v\|_{W^{1,p}}^p}{\left(\int_{\partial B} v^{q_0} \, d\sigma\right)^{p/q_0}} &< |\partial B|^{\frac{p}{q} - \frac{p}{q_0}} \frac{\|v\|_{W^{1,p}}^p}{\left(\int_{\partial B} v^q \, d\sigma\right)^{p/q}} \\
&= \frac{\left(\int_{\partial B} u_0^q \, d\sigma\right)^{p/q}}{\left(\int_{\partial B} u_0^{q_0} \, d\sigma\right)^{p/q_0}} \frac{\|v\|_{W^{1,p}}^p}{\left(\int_{\partial B} v^q \, d\sigma\right)^{p/q}} \\
&\leq \frac{\|u_0\|_{W^{1,p}}^p}{\left(\int_{\partial B} u_0^{q_0} \, d\sigma\right)^{p/q_0}} \\
&= S_{q_0}(\rho)
\end{aligned}$$

where the last inequality follows from the fact that v is a minimizer for $S_q(\rho)$. This contradicts the definition (1) of $S_{q_0}(\rho)$. Thus if $1 < q < q_0$, then any minimizer v for $S_q(\rho)$ must be radial. By Proposition 2 it follows that v is a multiple of u_0. □

Recall that any minimizer for $S_p(\rho)=\lambda_1(\rho)$ is a multiple of u_0. Thus the above Proposition implies that the minimizer for $S_q(\rho)$ is given by u_0 whenever $q\le p$. In fact a stronger property holds since we have uniqueness not only of the minimizer but also in the associated boundary value problem.

Proposition 4. *Let $1<q<p$. The solution of* (3) *is unique and it is a multiple of u_0.*

Proof. Assume that there exist two solutions u and v of (3). By the regularity results of [10] and the maximum principle of [9] it follows that $u,v>0$ in $\bar{B}_\rho$. Using first Picone's identity (see [1]) and then the weak formulation of (3) we have

$$\begin{aligned}
0 &\le \int_{B_\rho}|\nabla u|^p\,dx-\int_{B_\rho}|\nabla v|^{p-2}\nabla v\nabla\left(\tfrac{u^p}{v^{p-1}}\right)dx\\
&= -\int_{B_\rho}u^p\,dx+\int_{\partial B_\rho}u^q\,d\sigma+\int_{B_\rho}v^{p-1}\tfrac{u^p}{v^{p-1}}\,dx-\int_{\partial B_\rho}v^{q-1}\tfrac{u^p}{v^{p-1}}\,d\sigma\\
&= \int_{\partial B_\rho}u^q\,d\sigma-\int_{\partial B_\rho}v^{q-p}u^p\,d\sigma\\
&= \int_{\partial B_\rho}u^p\left(u^{q-p}-v^{q-p}\right)d\sigma.
\end{aligned}$$

Clearly we can swap u and v in the above equation. Combining the inequality thus obtained with the above inequality we get

$$0\le\int_{\partial B_\rho}\left(u^p-v^p\right)\left(u^{q-p}-v^{q-p}\right)d\sigma.$$

Since $q<p$ the above integrand is nonpositive so that in fact $u\equiv v$ on ∂B_ρ. By uniqueness of the solution to the Dirichlet problem we get $u\equiv v$ in B_ρ as in the proof of Proposition 2. Since it is unique the solution u must be radial and Proposition 2 implies u is a multiple of u_0. □

3. Loss of radial symmetry

Let $1<p<\infty$ be fixed in what follows. Recall that the function $\rho\mapsto Q(\rho)$ is defined by (5) and that $\lambda_1(\rho)=S_p(\rho)$ is the first eigenvalue in the Steklov type problem (4). Theorem 2 will follow from Proposition 5 and corollary 1 below.

Proposition 5. *Let $\rho>0$. If $q>Q(\rho)$ there is no radial minimizer for $S_q(\rho)$.*

Proof. Let $\rho>0$ be fixed and consider u_0 the radial solution of $\Delta_p u_0=u_0^{p-1}$ in $\mathbb{R}^N$ such that $u_0\equiv 1$ on ∂B_ρ (see Proposition 1). By Proposition 2 it is enough to check that u_0 is not a minimizer for $S_q(\rho)$ when $q>Q(\rho)$. We write u instead of u_0 to simplify notations. For any $t\in\mathbb{R}$ and $x\in\mathbb{R}^N$ denote $x^t=(x_1-t,x_2,\dots,x_N)$. Consider the function

$$\Phi(t)=\frac{\int_{B_\rho}|\nabla u(x^t)|^p+u(x^t)^p\,dx}{\left(\int_{\partial B_\rho}u(x^t)^q\,d\sigma\right)^{p/q}}.$$

Then

$$\begin{aligned}
\Phi'(t) &= p\left(\int_{\partial B_\rho}u(x^t)^q\,d\sigma\right)^{-\frac{p}{q}-1}\cdot\\
&\quad\left[-\left(\int_{B_\rho}\tfrac{1}{2}|\nabla u(x^t)|^{p-2}\tfrac{\partial|\nabla u(x^t)|^2}{\partial x_1}+\tfrac{1}{p}\tfrac{\partial u(x^t)^p}{\partial x_1}\,dx\right)\left(\int_{\partial B_\rho}u(x^t)^q\,d\sigma\right)\right.\\
&\quad\left.+\left(\int_{B_\rho}|\nabla u(x^t)|^p+u(x^t)^p\,dx\right)\left(\tfrac{1}{q}\int_{\partial B_\rho}\tfrac{\partial u(x^t)^q}{\partial x_1}\,d\sigma\right)\right].
\end{aligned}$$

The first and second integrands in square brackets are odd functions when $t = 0$, so $\Phi'(0) = 0$. Using the fact that $u \equiv 1$ on ∂B_ρ and the divergence Theorem we get

$$\begin{aligned}\Phi''(0) &= p\,(|\partial B_\rho|)^{-\frac{p}{q}-1}\Big(|\partial B_\rho|\int_{B_\rho}\tfrac{\partial}{\partial x_1}\Big(\tfrac{1}{2}|\nabla u|^{p-2}\tfrac{\partial|\nabla u|^2}{\partial x_1}+\tfrac{1}{p}\tfrac{\partial u^p}{\partial x_1}\Big)\,dx\\ &\quad -\tfrac{1}{q}\|u\|_{1,p}^p\int_{\partial B_\rho}\tfrac{\partial^2 u^q}{\partial x_1^2}\,d\sigma\Big)\\ &= C\Big(|\partial B_\rho|\int_{\partial B_\rho}\Big(\tfrac{1}{2}|\nabla u|^{p-2}\tfrac{\partial|\nabla u|^2}{\partial x_1}+\tfrac{1}{p}\tfrac{\partial u^p}{\partial x_1}\Big)\,\nu_1\,d\sigma\\ &\quad -\|u\|_{1,p}^p\int_{\partial B_\rho}(q-1)\Big(\tfrac{\partial u}{\partial x_1}\Big)^2+\tfrac{\partial^2 u}{\partial x_1^2}\,d\sigma\Big)\end{aligned}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the outer normal vector and C is a positive constant. Now since u is radial we can write

$$\begin{aligned}\Phi''(0) &= C\Big(\tfrac{|\partial B_\rho|}{N}\int_{\partial B_\rho}\tfrac{1}{2}|\nabla u|^{p-2}\tfrac{\partial|\nabla u|^2}{\partial\nu}+\tfrac{1}{p}\tfrac{\partial u^p}{\partial\nu}\,d\sigma\\ &\quad -\tfrac{\|u\|_{1,p}^p}{N}\int_{\partial B_\rho}(q-1)|\nabla u|^2+\Delta u\,d\sigma\Big).\end{aligned}$$

By definition $u = u(r)$ satisfies

$$\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}u^{p-1} \quad \forall r > 0. \tag{13}$$

Proposition 1 states that for any $r > 0$ the function u is an eigenfunction associated to $\lambda_1(r)$ in $B(r)$. The boundary condition satisfied by eigenfunctions implies that

$$u'(r)^{p-1} = \lambda_1(r)u(r)^{p-1} \quad \forall r > 0. \tag{14}$$

Using (13) and (14) a straightforward calculation shows that

$$\tfrac{1}{2}|\nabla u|^{p-2}\tfrac{\partial|\nabla u|^2}{\partial\nu}+\tfrac{1}{p}\tfrac{\partial u^p}{\partial\nu} = \tfrac{\lambda_1^{1/(p-1)}}{p-1}\Big(1-(N-1)\tfrac{\lambda_1}{\rho}\Big)+\lambda_1^{1/(p-1)}$$

and that

$$(q-1)|\nabla u|^2+\Delta u = (q-1)\lambda_1^{2/(p-1)}+\frac{1}{(p-1)\lambda_1^{\frac{p-2}{p-1}}}\Big(1-(N-1)\tfrac{\lambda_1}{\rho}\Big)+(N-1)\frac{\lambda_1^{1/(p-1)}}{\rho}$$

on ∂B_ρ. We also have that

$$\|u\|_{1,p}^p = \lambda_1|\partial B_\rho|.$$

Collecting equations we get $\Phi'(0) = 0$ and

$$\Phi''(0) = C\left(1-(N-1)\frac{\lambda_1(\rho)}{\rho}-(q-1)\lambda_1(\rho)^{p/(p-1)}\right),$$

where C is a positive constant. If $q > Q(\rho)$ then $\Phi''(0) < 0$ and $t = 0$ is a local maximum for Φ. Thus $u = u_0$ can not be a minimizer. □

In order to get symmetry breaking in large balls we must study the asymptotic behavior of $Q(\rho)$ as $\rho \to +\infty$. This will follow from the following lemma.

Lemma 1. *Denote by $\lambda_1(\rho) = S_p(\rho)$ the first eigenvalue in the Steklov type problem (4). Then the function $\rho \mapsto \lambda_1(\rho)$ is a solution of the following differential equation*

$$\lambda' = 1-(p-1)\lambda^{p/(p-1)}-(N-1)\frac{\lambda}{\rho} \quad \forall\rho > 0 \tag{15}$$

satisfying the initial condition $\lambda_1(0) = 0$.

Proof. Let u_0 denote as before the positive radial solution of $\Delta_p u_0 = u_0^{p-1}$ in $\mathbb{R}^N$ normalized in such a way that, say, $u_0(0) = 1$. For any $\rho > 0$ the first eigenfunction of the Steklov problem (4) is given by the restriction of u_0 to B_ρ (see Proposition 1). From the boundary condition satisfied by an eigenfunction we get

$$\lambda_1(\rho) = \frac{u_0'(\rho)^{p-1}}{u_0(\rho)^{p-1}} \qquad \forall \rho > 0. \tag{16}$$

Deriving (16) with respect to ρ and using the equation (13) satisfied by u_0 we get the desired equation for λ_1. Now, choosing $u \equiv 1$ as testing function in (1) we get

$$\lambda_1(\rho) \equiv S_p(\rho) \leq \frac{|B_\rho|}{|\partial B_\rho|} = \frac{\rho}{N} \tag{17}$$

so that $\lambda_1(0) = 0$. □

Proposition 6. *The function $Q(\rho)$ has the following asymptotic behavior*

$$\lim_{\rho\to 0} Q(\rho) = +\infty \qquad \text{and} \qquad \lim_{\rho\to+\infty} Q(\rho) = p.$$

Proof. As mentioned above any minimizer for $S_p(\rho)$ is radial. Consequently Proposition 5 implies that $p \leq Q(\rho)$ for any $\rho > 0$. On the other hand using (5) and (15) we see that $\lambda_1'(\rho) = (Q(\rho) - p)\, \lambda_1(\rho)^{p/(p-1)}$ so that $\lambda_1'(\rho) > 0$ for all $\rho > 0$. It follows from (15) that the function $\lambda_1(\rho)$ is bounded by some positive constant. Consequently $\lambda_1'(\rho) \to 0$ as $\rho \to +\infty$. It follows that $\lim_{\rho\to\infty} \lambda_1(\rho)^{\frac{p}{p-1}} = \frac{1}{p-1}$. Hence $\lim_{\rho\to\infty} Q(\rho) = p$.

Now (17) implies that $\frac{\lambda_1(\rho)}{\rho}$ is bounded by $1/N$ and that $\lambda_1(\rho) \to 0$ as $\rho \to 0$, so $\lim_{\rho\to 0} Q(\rho) = +\infty$. □

A remarkable consequence of propositions 5 and 6 is that radial symmetry is lost in large balls.

Corollary 1. *Let $q > p$. If ρ is sufficiently large there is no radial minimizer for $S_q(\rho)$.*

Proof. By Proposition 6 we have $q > Q(\rho)$ for all ρ sufficiently large. The result then follows from Proposition 5. □

Recall $\tilde{Q}(\rho)$ is defined by (6). It is clear that $Q(\rho)$ is an upper bound for $\tilde{Q}(\rho)$, so that $p \leq \tilde{Q}(\rho) \leq Q(\rho)$ for all $\rho > 0$. We may also show that $\tilde{Q}$ and Q have the same asymptotic behavior. Indeed from Proposition 6 we get

$$\lim_{\rho\to+\infty} Q(\rho) = \lim_{\rho\to+\infty} \tilde{Q}(\rho) = p.$$

When $p = 2$ and $N = 2$ the results of [6] imply that $\lim_{\rho\to 0} \tilde{Q}(\rho) = +\infty$ so

$$\lim_{\rho\to 0} Q(\rho) = \lim_{\rho\to 0} \tilde{Q}(\rho),$$

at least in this special case. In light of this it seems natural to ask if $\tilde{Q} = Q$. We do not know of a proof (or counter proof) of this statement but we have checked numerically that it is true at least in the case $N = 2$ and $p = 2$ (see section 6).

4. A problem involving a nonlinear boundary condition

In this section we consider the minimization problem (8) and prove theorems 4 and 5. We first state a symmetry result which is analogous to Theorem 1. Since the proof is very similar to the proof of Theorem 1 we will not include it here.

Theorem 6. *Let $\rho > 0$ be fixed.*

(1) *If there exists a radial minimizer for (8) then it is a multiple of u_0.*

(2) *Assume there exists a radial minimizer for* $S_{q_0}^{\lambda_0}(\rho)$. *If* $1 < q \le q_0$ *and if* $\lambda_0 \le \lambda < \lambda_1(\rho)$ *then any minimizer for* $S_q^\lambda(\rho)$ *is a multiple of* u_0.
(3) *Let* $q < 2$ *and let* $-\infty < \lambda < \lambda_1(\rho)$. *Then the solution of the boundary value problem* (7) *is unique and it is a multiple of* u_0. *In particular any minimizer for* $S_q^\lambda(\rho)$ *is a multiple of* u_0.

Let us now prove Theorem 4.

Proof of Theorem 4. Statement (1). Recall that ρ_0 is defined such that $\lambda_1(\rho_0) = \lambda$ when $0 < \lambda < 1$ and $\rho_0 = 0$ when $\lambda \le 0$. Let v_ρ be a sequence of positive minimizers for $S_q^\lambda(\rho)$ with $\rho \to \rho_0$. We have that $S_q^\lambda(\rho) \to S_q^\lambda(\rho_0) = 0$ as $\rho \to \rho_0$. We may change scale by setting $u_\rho(x) = v_\rho(\rho x)$ and denoting the unit ball by B. The u_ρ satisfy

$$(18) \quad \int_B \nabla u_\rho \nabla\varphi + \rho^2 u_\rho\varphi\, dx - \lambda\rho\int_{\partial B} u_\rho\varphi\, d\sigma - \rho^{\frac{N-2}{q}(2^*-q)} S_q^\lambda(\rho)\int_{\partial B} u_\rho^{q-1}\varphi\, d\sigma = 0$$

for any $\varphi \in H^1(B)$ where u_ρ is normalized in such a way that $\int_{\partial B} u_\rho^q\, d\sigma = 1$. If $\rho_0 > 0$ let $\tilde u(x) = u_0(\rho x)$ where u_0 is the first Steklov eigenfunction in B_ρ normalized so that $\tilde u \equiv \frac{1}{|\partial B|^{1/q}}$ on ∂B, whereas if $\rho_0 = 0$ let $\tilde u \equiv \frac{1}{|\partial B|^{1/q}}$ in B. It can be shown using standard arguments of functional analysis that $u_\rho \to \tilde u$ in $H^1(B)$ as $\rho \to \rho_0$.

Consider now the function

$$F : H^1(B)\times\mathbb{R}\times\mathbb{R} \to H^1(B)'\times\mathbb{R} : (u,t,\rho) \mapsto (F_1(u,t,\rho), F_2(u,t,\rho))$$

$$< F_1(u,t,\rho),\varphi >= \int_B \nabla u\nabla\varphi + \rho^2 u\varphi\, dx - \lambda\rho\int_{\partial B} u\varphi\, d\sigma - t\int_{\partial B}|u|^{q-2}u\varphi\, d\sigma$$

$$F_2(u,t,\rho) = \int_{\partial B}|u|^q\, d\sigma - 1$$

We have $F(\tilde u, 0, \rho_0) = 0$. Let $(v,s) \in H^1(B)\times\mathbb{R}$. The derivative of F with respect to (u,t) at the point $(\tilde u, 0, \rho_0)$ and in the direction (v,s) is given by

$$\begin{aligned}\langle \tfrac{\partial F_1}{\partial(u,t)}\big|_{(\tilde u,0,\rho_0)}(v,s),\varphi\rangle &= \int_B \nabla v\nabla\varphi + \rho_0^2 v\varphi\, dx - \lambda\rho_0\int_{\partial B} v\varphi\, d\sigma \\ &\quad -s|\partial B|^{-\frac{q-1}{q}}\int_{\partial B}\varphi\, d\sigma \qquad \forall\varphi\in H^1(B) \\ \tfrac{\partial F_2}{\partial(u,t)}\big|_{(\tilde u,0,\rho_0)}(v,s) &= q|\partial B|^{-\frac{q-1}{q}}\int_{\partial B} v\, d\sigma\end{aligned}$$

Let $(\phi,\alpha) \in H^1(B)'\times\mathbb{R}$ and consider the minimization problem

$$\inf_{v\in X} \tfrac{1}{2}\int_B |\nabla v|^2 + \rho_0^2 v^2\, dx - \tfrac{\lambda_1(\rho_0)\rho_0}{2}\int_{\partial B} v^2\, d\sigma - \langle\phi, v\rangle,$$

$$X = \left\{v\in H^1(B);\ \int_{\partial B} v\, d\sigma = 0\right\}.$$

It can be shown that any minimizing sequence is bounded and that the infimum is achieved by some function $v_0 \in X$ satisfying

$$\int_B \nabla v_0\nabla\varphi + \rho_0^2 v_0\varphi\, dx - \lambda_1(\rho_0)\rho_0\int_{\partial B} v_0\varphi\, d\sigma - \langle\phi,\varphi\rangle = \eta\int_{\partial B}\varphi\, d\sigma,$$

for all $\varphi \in H^1(B)$, where η is a Lagrange multiplier. Setting $v = v_0 + \frac{\alpha}{q}\tilde u$ and $s = \eta|\partial B|^{\frac{q-1}{q}}$ we have $\frac{\partial F}{\partial(u,t)}\big|_{(\tilde u,0,\rho_0)}(v,s) = (\phi,\alpha)$ so that the differential is surjective. One may check that it is also injective. By the implicit function Theorem there is a neighborhood V of $(\tilde u, 0, \rho_0)$ in $H^1(B)\times\mathbb{R}\times\mathbb{R}$ such that for any ρ sufficiently near to ρ_0 there is a unique point $(u,t,\rho)\in V$ such that $F(u,t,\rho) = 0$.

Now consider again the sequence u_ρ above. We have $\left(u_\rho, S_q^\lambda(\rho), \rho\right) \to (\tilde u, 0, \rho_0)$ in $H^1(B)\times\mathbb{R}\times\mathbb{R}$ as $\rho\to\rho_0$, and $F\left(u_\rho, S_q^\lambda(\rho),\rho\right) = 0$. By the uniqueness property in the implicit function Theorem and the fact that F is invariant by rotation we have that u_ρ is radial for ρ close to ρ_0. Going back to B_ρ by a change of scale we see that any minimizer v_ρ for $S_q^\lambda(\rho)$ is radial when ρ is sufficiently near to ρ_0. By Theorem 6 the function v_ρ is then a multiple of u_0.

Statement (2). Let $\rho > 0$ and $2 < q < 2^*$ be fixed. Let u_i be a sequence of positive minimizers for $S_q^{\lambda_i}(\rho)$ with $\lambda_i \to \lambda_1(\rho)$ as $i \to \infty$. We have that $S_q^{\lambda_i}(\rho) \to S_q^{\lambda_1(\rho)}(\rho) = 0$ as $i \to \infty$. Moreover the u_i satisfy the following equation

$$\int_{B_\rho} \nabla u_i \nabla \varphi + u_i \varphi \, dx - \lambda_i \int_{\partial B_\rho} u_i \varphi \, d\sigma - S_q^{\lambda_i}(\rho) \int_{\partial B_\rho} u_i^{q-1} \varphi \, d\sigma = 0 \tag{19}$$

for any $\varphi \in H^1(B_\rho)$ where u_i is normalized in such a way that $\int_{\partial B_\rho} u_i^q \, d\sigma = 1$. It follows from standard arguments of functional analysis that u_i converges in $H^1(B_\rho)$ to u_0, an eigenfunction associated to $\lambda_1(\rho)$ and normalized so that $u_0 \equiv \frac{1}{|\partial B_\rho|^{1/q}}$ on ∂B_ρ.

Similarly to above, define the function

$$F : H^1(B_\rho) \times \mathbb{R} \times \mathbb{R} \to H^1(B_\rho)' \times \mathbb{R} : (u,t,\lambda) \mapsto (F_1(u,t,\lambda), F_2(u,t,\lambda))$$

$$< F_1(u,t,\lambda), \varphi >= \int_{B_\rho} \nabla u \nabla \varphi + u\varphi \, dx - \lambda \int_{\partial B_\rho} u\varphi \, d\sigma - t \int_{\partial B_\rho} |u|^{q-2} u \varphi \, d\sigma$$

$$F_2(u,t,\lambda) = \int_{\partial B_\rho} |u|^q \, d\sigma - 1$$

We have $F(u_0, 0, \lambda_1) = 0$. Let $(v,s) \in H^1(B) \times \mathbb{R}$ and consider the derivative of F with respect to (u,t) at the point $(u_0, 0, \lambda_1)$ and in the direction (v,s):

$$\begin{aligned} \langle \tfrac{\partial F_1}{\partial(u,t)} \big|_{(u_0,0,\lambda_1)} (v,s), \varphi \rangle &= \int_{B_\rho} \nabla v \nabla \varphi + v\varphi \, dx - \lambda_1(\rho) \int_{\partial B_\rho} v\varphi \, d\sigma \\ &\quad - s|\partial B|^{-\frac{q-1}{q}} \int_{\partial B_\rho} \varphi \, d\sigma \qquad \forall \varphi \in H^1(B_\rho) \\ \tfrac{\partial F_2}{\partial(u,t)} \big|_{(u_0,0,\lambda_1)} (v,s) &= q|\partial B_\rho|^{-\frac{q-1}{q}} \int_{\partial B_\rho} v \, d\sigma \end{aligned}$$

Let $(\phi, \alpha) \in H^1(B)' \times \mathbb{R}$. The infimum

$$\inf_{v \in X} \tfrac{1}{2} \int_{B_\rho} |\nabla v|^2 + v^2 \, dx - \tfrac{\lambda_1(\rho)}{2} \int_{\partial B_\rho} v^2 \, d\sigma - \langle \phi, v \rangle,$$

$$X = \left\{ v \in H^1(B_\rho);\ \int_{\partial B_\rho} v \, d\sigma = 0 \right\}.$$

is achieved by a function $v_0 \in X$ such that

$$\int_{B_\rho} \nabla v_0 \nabla \varphi + v_0 \varphi \, dx - \lambda_1(\rho) \int_{\partial B_\rho} v_0 \varphi \, d\sigma - \langle \phi, \varphi \rangle = \eta \int_{\partial B_\rho} \varphi \, d\sigma,$$

for all $\varphi \in H^1(B_\rho)$, where η is a Lagrange multiplier. Setting $v = v_0 + \frac{\alpha}{q} u_0$ and $s = \eta |\partial B|^{\frac{q-1}{q}}$ we see that $\partial F / \partial(u,t)|_{(u_0,0,\lambda_1)}(v,s) = (\phi, \alpha)$ so that the differential is surjective. One may check that it is also injective. Arguing as above, with use of the implicit function theorem, we reach the desired conclusion.

Statement (3). Let $\rho > 0$ and $-\infty < \lambda < \lambda_1(\rho)$. Let u_i be a sequence of positive minimizers for $S_{q_i}^\lambda(\rho)$ with $q_i \to 2$, normalized so that $\int_{\partial B_\rho} u_i^{q_i} \, d\sigma = 1$. It can be shown that $S_q^\lambda(\rho) \to S_2^\lambda(\rho) = \lambda_1(\rho) - \lambda$ and that $u_i \to u_0$ in $H^1(B_\rho)$ where u_0 is an eigenfunction associated to $\lambda_1(\rho)$ and normalized in such a way that $u_0 \equiv \frac{1}{|\partial B_\rho|^{1/2}}$ on ∂B_ρ.

Consider the function:

$F : H^1(B_\rho) \times \mathbb{R} \times \mathbb{R} \to H^1(B_\rho)' \times \mathbb{R} : (u,t,q) \mapsto (F_1(u,t,q), F_2(u,t,q))$

$< F_1(u,t,q), \varphi >= \int_{B_\rho} \nabla u \nabla \varphi + u\varphi \, dx - \lambda \int_{\partial B_\rho} u\varphi \, d\sigma - t \int_{\partial B_\rho} |u|^{q-2} u\varphi \, d\sigma$

$F_2(u,t,q) = \int_{\partial B_\rho} |u|^q \, d\sigma - 1$

We have $F(u_0, \lambda_1 - \lambda, 2) = 0$. Using the implicit function Theorem as above one shows that there is a neighborhood V of $(u_0, \lambda_1(\rho) - \lambda, 2)$ in $H^1(B_\rho) \times \mathbb{R} \times \mathbb{R}$ such that for any q sufficiently near to 2 there is a unique point $(u,t,q) \in V$ such that $F(u,t,q) = 0$. Arguing as above we get that u_i is radial for q_i near 2. □

Proof of Theorem 5. Let $\rho > 0$ be fixed and consider u_0 the radial solution of $\Delta u_0 = u_0$ in $\mathbb{R}^N$ such that $u_0 \equiv 1$ on ∂B_ρ (see Proposition 1). Note that the restriction of u_0 to the ball B_ρ is just the first eigenfunction in problem (10). By Theorem 6 it is enough to check that u_0 is not a minimizer for $S_q^\lambda(\rho)$ when inequality (12) holds. For any $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ denote $x^t = (x_1 - t, x_2, \dots, x_N)$. Define the function

$$\Phi(t) = \frac{\int_{B_\rho} |\nabla u_0(x^t)|^2 + u_0(x^t)^2 \, dx - \lambda \int_{\partial B_\rho} u_0(x^t)^2 \, d\sigma}{\left(\int_{\partial B_\rho} u_0(x^t)^q \, d\sigma\right)^{2/q}}.$$

As in the proof of Proposition 5 we show that

$$\Phi'(0) = 0$$

$$\Phi''(0) = C\left(1 - (N-1)\frac{\lambda_1(\rho)}{\rho} - (q-1)\lambda_1^2(\rho) + (q-2)\lambda_1(\rho)\lambda\right)$$

where C is a positive constant. If $\Phi''(0) < 0$ then $t = 0$ is a local maximum for Φ and u_0 cannot be a minimizer. □

Define the functions

$$Q(\rho,\lambda) = 1 + \frac{1}{\lambda_1(\rho) - \lambda}\left(\frac{1}{\lambda_1(\rho)} - \frac{N-1}{\rho} - \lambda\right) \tag{20}$$

and

$$\Lambda(\rho,q) = \frac{-1}{(q-2)\lambda_1(\rho)}\left(1 - (N-1)\frac{\lambda_1(\rho)}{\rho} - (q-1)\lambda_1^2(\rho)\right). \tag{21}$$

If either $q > Q(\rho,\lambda)$ or $\lambda < \Lambda(\rho,q)$ then inequality (12) holds and the minimizer for $S_q^\lambda(\rho)$ is nonradial. Now let $\rho > 0$ and let $-\infty < \lambda < \lambda_1(\rho)$. By Theorem 4 there is a $\delta_3 > 0$ such that if $2 < q < 2 + \delta_3$ then the minimizer for $S_q^\lambda(\rho)$ is given by a multiple of u_0. This δ_3 is not bounded below by a positive constant independent of ρ and λ. Indeed let $q > 2$ be fixed. We have that $\lim_{\rho\to\infty} Q(\rho,\lambda) = \lim_{\lambda\to-\infty} Q(\rho,\lambda) = 2$ so that the minimizer for $S_q^\lambda(\rho)$ is nonradial if ρ is sufficiently large or if λ is sufficiently near to $-\infty$. Consequently $\inf_{\rho,\lambda}\delta_3 = 0$. In a similar manner it can be shown that the number δ_1 (respectively δ_2) appearing in Theorem 4 can not be chosen independently of λ and q (respectively q and ρ).

5. Symmetry properties of nonradial minimizers

The technique of spherical symmetrization, also known as foliated Schwartz symmetrization, is well adapted to the minimization problem (1). For a description of this technique see for instance [4]. Let u be a minimizer for (1) and let $\tilde{u} \in W^{1,p}(B_\rho)$ denote the foliated Schwartz symmetrization of u with respect to the north pole. It is well known that for any ball B_ρ we have

$$\|\tilde{u}\|_{W^{1,p}(B_\rho)} \le \|u\|_{W^{1,p}(B_\rho)} \quad \text{and} \quad \|\tilde{u}\|_{L^q(\partial B_\rho)} = \|u\|_{L^q(\partial B_\rho)} \tag{22}$$

so that $\tilde{u}$ is also a minimizer for $S_q(\rho)$. The function $\tilde{u}$ depends only on two variables: The radial variable and φ the geodesic distance from the north pole on the unit sphere. Also the restriction of $\tilde{u}$ to any sphere centered at the origin and contained in B_ρ is an increasing function of φ. This fact, together with the maximum principle of [9], implies that $\tilde{u}$ achieves it's maximum at a single point which is situated on the boundary of B_ρ.

Now let u be a minimizer for problem (8) and let $\tilde{u}$ be the foliated Schwartz symmetrization of u with respect to the north pole. The relations (22) again hold with $p=2$ so that $\tilde{u}$ is also a minimizer. In fact Denzler has shown in [2] that, when $p=2$, either the inequality in (22) is strict or u and $\tilde{u}$ coincide on every sphere up to a rotation. This implies that any minimizer for (8) is foliated Schwartz symmetric.

6. Numerical computations

Recall the function $\tilde{Q}(\rho)$ is defined by

$$q \leq \tilde{Q}(\rho) \Rightarrow \text{ Any minimizer for } S_q(\rho) \text{ is a multiple of } u_0$$

$$q > \tilde{Q}(\rho) \Rightarrow \text{ There is no radial minimizer for } S_q(\rho)$$

Based on the remarks at the end of section 3 we may guess that $\tilde{Q} = Q$ where Q is given by (5). We do not know of a proof (or counter proof) of this statement so we have checked it numerically in the special case when $p=2$ and $N=2$. We found that $Q(\rho) = \tilde{Q}(\rho)$ for a large range of values of ρ. In this section we explain our methods then quote some precise numerical results.

Denote by $S_q^{rad}(\rho)$ the infimum (1) restricted to radial functions. By the definition of $\tilde{Q}(\rho)$ we have $S_q^{rad}(\rho) = S_q(\rho)$ if and only if $q \leq \tilde{Q}(\rho)$. Since $\tilde{Q}(\rho) \leq Q(\rho)$ it follows that $Q(\rho) = \tilde{Q}(\rho)$ if and only if $S_{Q(\rho)}^{rad}(\rho) = S_{Q(\rho)}(\rho)$. Thus it suffices to compute approximations of $S_{Q(\rho)}^{rad}(\rho)$ and $S_{Q(\rho)}(\rho)$ and compare these numbers.

In practice it is straightforward to obtain an approximation of $S_q^{rad}(\rho)$. By Palais' principle any minimizer v for $S_q^{rad}(\rho)$ is a solution of (3) and is thus a multiple of u_0 by Proposition 2. Thus

$$S_q^{rad}(\rho) = \frac{\|u_0\|_{1,p}^p}{\|u_0\|_{L^q(\partial B_\rho)}^p} = S_p(\rho)\frac{\|u_0\|_{L^p(\partial B_\rho)}^p}{\|u_0\|_{L^q(\partial B_\rho)}^p} = S_p(\rho)|\partial B_\rho|^{1-\frac{p}{q}}.$$

When $p=2$ we can get $\lambda_1(\rho) = S_p(\rho)$ directly using expression (23) below. When $p \neq 2$ we can get $\lambda_1(\rho)$ by solving the Cauchy problem in Lemma 1.

Computing $S_q(\rho)$ is far more tricky and we are only able to consider this problem when $p=2$. Consider the Steklov type problem

$$\begin{cases} \Delta u = u & \text{in } B_\rho, \\ \frac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial B_\rho. \end{cases}$$

A complete set of eigenfunctions and the associated eigenvalues are given by

$$\begin{aligned} u_{kj}(x) &= |x|^{1-\frac{N}{2}} I_{k+\frac{N}{2}-1}(|x|) Y_{kj}\left(\tfrac{x}{|x|}\right) \\ \lambda_k &= \tfrac{1-N/2}{\rho} + \tfrac{I'_{k+N/2-1}(\rho)}{I_{k+N/2-1}(\rho)} \end{aligned} \tag{23}$$

where I_ν is the modified Bessel function of the first kind and of order ν and the Y_{kj} are the spherical harmonics of order k indexed by j. The functions u_{kj} form a basis of $H^1(B_\rho)$. Now denote by V_n the subspace of $H^1(B_\rho)$ spanned by the first

n eigenfunctions in (23). Let

$$S_n = \inf_{v \in V_n} \frac{\int_{B_\rho} |\nabla v|^2 + v^2 \, dx}{\left(\int_{\partial B_\rho} |v|^q \, d\sigma \right)^{2/q}}. \tag{24}$$

Using that $V = \bigcup_{n=1}^{\infty} V_n$ is dense in $H^1(B_\rho)$ we get $S_n \searrow S_q(\rho)$ as $n \to +\infty$. Therefore an approximation to $S_q(\rho)$ is obtained by computing S_n given by (24) or equivalently by

$$S_n = \inf \left\{ \int_{B_\rho} |\nabla v|^2 + v^2 \, dx; \, v \in V_n, \, \int_{\partial B_\rho} |v|^q \, d\sigma = 1 \right\}. \tag{25}$$

Moreover if S_n is achieved by $u_n \in V_n$ then there exists a subsequence u_{n_i} such that $u_{n_i} \to u$ where u is a minimizer for $S_q(\rho)$. Notice that (24) and (25) are nonlinear optimization problems in $\mathbb{R}^n$. We considered the case $N = 2$ and used routines from the Nag library to solve this minimization problem. For safety we used various routines and both formulations (24) and (25) to get our numerical data. We now give a sample of our results. The graph of

$$Q(\rho) = \frac{1}{\lambda_1(\rho)^2} \left(1 - \frac{\lambda_1(\rho)}{\rho} \right) + 1$$

is plotted in figure 1. For $2 < q < \infty$ let $\rho^* = Q^{-1}(q)$. We tested a large range

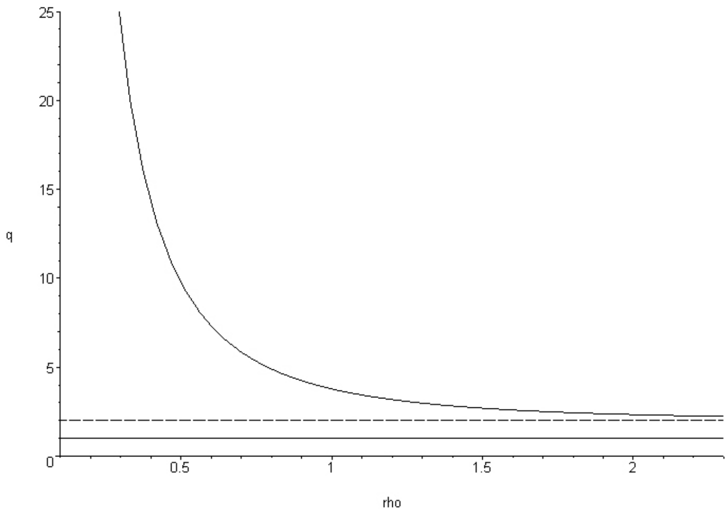

FIGURE 1. $Q(\rho)$ with $p = 2$ and $N = 2$

of values of q and found that $S_q^{rad}(\rho^*) = S_q(\rho^*)$. This confirms that $Q = \tilde{Q}$ when $p = 2$ and $N = 2$. We provide a small sample of our numerical data in table 1 where we have computed $S_q^{rad}(\rho)$ and $S_q(\rho)$ with $\rho = 0.95\rho^*$, $\rho = \rho^*$ and $\rho = 1.05\rho^*$.

We use a minimizer u_n for S_n as an approximation of a minimizer u for $S_q(\rho)$. The function u_0 is plotted in figure 2. An approximate minimizer for $S_{3.0}(1.3)$ is plotted in figure 3.

Now let us adapt the preceding analysis to the minimization problem (8). By Theorem 5 there is no radial minimizer for $S_q^\lambda(\rho)$ if

$$1 - (N-1)\frac{\lambda_1(\rho)}{\rho} - (q-1)\lambda_1^2(\rho) + (q-2)\lambda_1(\rho)\lambda < 0. \tag{26}$$

$q = 2.7$
$\rho^* = 1.49611$

ρ	$S_q^{rad}(\rho)$	$S_q(\rho)$
1.42130	1.01623	1.01623
1.49611	1.06399	1.06399
1.57091	1.11013	1.10566

$q = 9.0$
$\rho^* = 0.52553$

ρ	$S_q^{rad}(\rho)$	$S_q(\rho)$
0.49925	0.58920	0.58920
0.52553	0.64340	0.64340
0.55180	0.69935	0.69475

TABLE 1. $S_q(\rho)$ and $S_q^{rad}(\rho)$ with $N = 2$ and $p = 2$

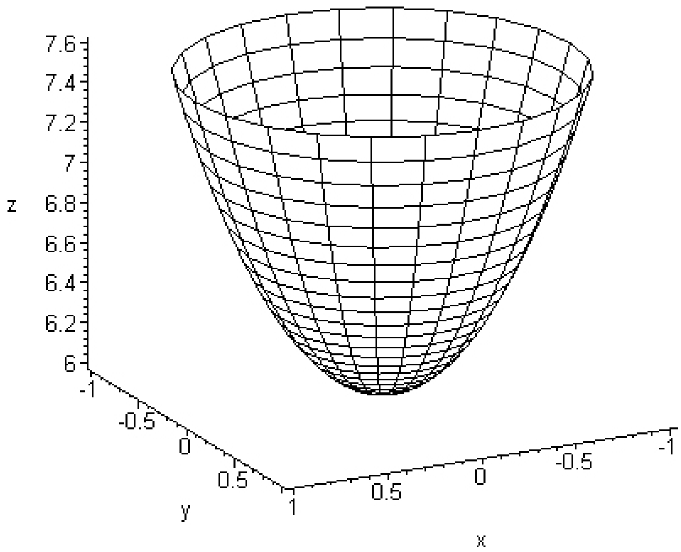

FIGURE 2. The function u_0 with $N = 2$ and $p = 2$.

We wish to show numerically that any minimizer for $S_q^\lambda(\rho)$ is a multiple of u_0 if the inequality $\geq$ holds in (26) or, equivalently, if $q < Q(\rho, \lambda)$ where $Q(\rho, \lambda)$ is defined by (20). Following the procedure described above we let $S_q^{\lambda, rad}$ denote the infimum (8) restricted to radial functions. By Theorem 6 it suffices to check that $S_{Q(\rho,\lambda)}^\lambda(\rho) = S_{Q(\rho,\lambda)}^{\lambda, rad}(\rho)$ for all admissible ρ and λ. As above we have that

$$S_q^{\lambda, rad} = (\lambda_1(\rho) - \lambda) |\partial B_\rho|^{1-\frac{2}{q}}$$

and an approximation of $S_q^\lambda(\rho)$ is given by

$$S_n^\lambda = \inf \left\{ \int_{B_\rho} |\nabla v|^2 + v^2 \, dx - \lambda \int_{\partial B_\rho} v^2 \, d\sigma; \; v \in V_n, \; \int_{\partial B_\rho} |v|^q \, d\sigma = 1 \right\}. \tag{27}$$

We consider only the case $N = 2$. In table 2 we give a sample of numerical results comparing $S_{Q(\rho,\lambda)}^\lambda(\rho)$ and $S_{Q(\rho,\lambda)}^{\lambda, rad}(\rho)$. To prepare this table we fixed $2 < q < +\infty$ and chose $-\infty < \lambda < 1$. Then we computed ρ^* such that $Q(\rho^*, \lambda) = q$. We see that $S_q^\lambda(\rho^*) = S_q^{\lambda, rad}(\rho^*)$. This sample, in addition to the many other similar tables that we computed, confirm that if the inequality $\geq$ holds in (26) then any minimizer for $S_q^\lambda(\rho)$ is a multiple of u_0, at least when $N = 2$.

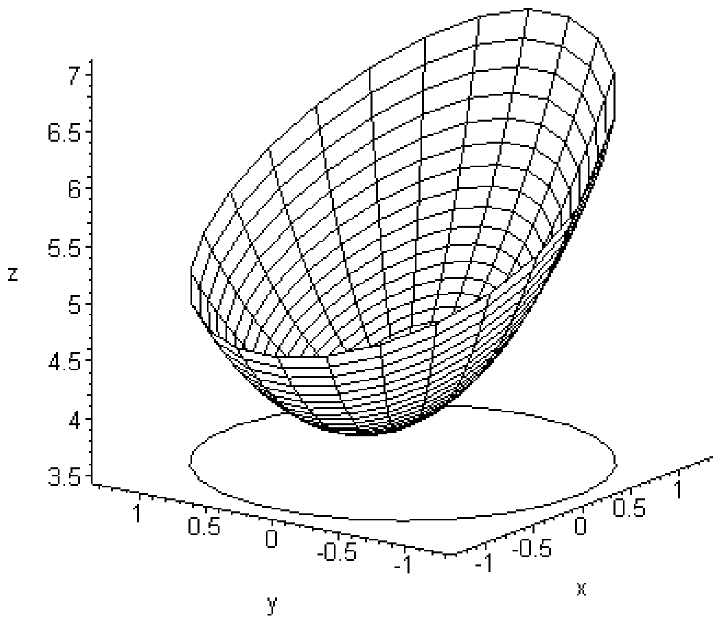

FIGURE 3. A minimizer for $S_{3.0}$ (1.3) with $N = 2$ and $p = 2$.

$q = 3.0$
$\lambda = 0.5$
$\rho^* = 2.09525$

ρ	$S_q^{\lambda,\,rad}(\rho)$	$S_q^{\lambda}(\rho)$
1.99048	0.45543	0.45543
2.09525	0.50249	0.50249
2.20001	0.54726	0.54231

$q = 7.5$
$\lambda = -1.5$
$\rho^* = 0.11630$

ρ	$S_q^{\lambda,\,rad}(\rho)$	$S_q^{\lambda}(\rho)$
0.11048	1.18990	1.18990
0.11630	1.23786	1.23786
0.12211	1.28529	1.28271

TABLE 2. $S_q^{\lambda}(\rho)$ and $S_q^{\lambda,\,rad}(\rho)$ with $N = 2$

ACKNOWLEDGMENTS

The authors would like to thank Yvan Notay for advising them in the initial stages of the numerical part of this work. The authors also extend their thanks to Georges Destree for advising them on programming issues and carefully explaining how to use the facilities at the ULB-VUB Computing Center.

REFERENCES

[1] W. Allegretto and Y. X. Huang, A Picone's identity for the p-Laplacian and applications. *Nonlinear Analysis T.M.A.*, 1998.

[2] J. Denzler, Windows of given area with minimal heat diffusion. *Transactions AMS* 351:569-580, 1999.

[3] L. Damascelli, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. *Ann. Inst. Henri Poincaré*, 15(4):493-516, 1998.

[4] B. Kawohl, *Rearrangement and Convexity of Level Sets in PDE.* Number 1150 in Lecture Notes in Math. Springer-Verlag, 1985.

[5] J.-L. Chern and C.-S. Lin, The symmetry of least-energy solutions for semilinear elliptic equations. *J. Differential Equations* 187:240-268, 2003.

[6] J. Fernández Bonder, E. Lami Dozo and J. D. Rossi, Symmetry properties for the extremals of the Sobolev trace embedding. *To appear in Ann. Inst. Henri Poincaré.*

[7] S. Martinez and J. D. Rossi, Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition. *Abstract and Applied Analysis* 7(5):287-293, 2002.

[8] M. del Pino and C. Flores, Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains. *Commun. in P.D.E.*, 26(11):2189-2210, 2001.

[9] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations. *Applied Math. Opt.*, 12:191-202, 1984.

[10] G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Analysis T.M.A.*, 12(11):1203-1219, 1988.

Enrique J. Lami Dozo, Université Libre de Bruxelles, Campus de la Plaine, ULB CP214, Boulevard du Triomphe,1050 Brussels, Belgium and IAM-CONICET and Universidad de Buenos Aires

E-mail address: lamidozo@ulb.ac.be

Olaf Torné, Université Libre de Bruxelles, Campus de la Plaine, ULB CP214, Boulevard du Triomphe, 1050 Brussels, Belgium

E-mail address: otorne@ulb.ac.be