Spectral Shorted Matrices

Jorge Antezana Gustavo Corach * Demetrio Stojanoff †

Jorge Antezana and Demetrio Stojanoff
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET
e-mail: antezana@mate.unlp.edu.ar and demetrio@mate.unlp.edu.ar

Gustavo Corach
Depto. de Matemática, FI-UBA, Buenos Aires, Argentina and IAM-CONICET
e-mail: gcorach@fi.uba.ar

Abstract

Given a $n \times n$ positive semidefinite matrix A and a subspace $\mathcal{S}$ of $\mathbb{C}^n$, $\Sigma(\mathcal{S}, A)$ denotes the shorted matrix of A to $\mathcal{S}$. We consider the notion of *spectral shorted* matrix

$$\rho(\mathcal{S}, A) = \lim_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m}.$$

We completely characterize this martix in terms of $\mathcal{S}$ and the spectrum and the eigenspaces of A. We show the relation of this notion with the spectral order of matrices and the Kolmogorov's complexity of A to a vector $\xi \in \mathbb{C}^n$.

Keywords: shorted matrix, spectral order, positive matrices.

AMS Subject Classifications: Primary 47A30, 47B15.

*Partially supported by CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT X050) and ANPCYT (PICT03-09521)

†Partially supported CONICET (PIP 4463/96), Universidad de La PLata (UNLP 11 X350) and ANPCYT (PICT03-09521)

1 Introduction

Consider a fixed $n \times n$ (Hermitian semidefinite) positive matrix A and a subspace $\mathcal{S}$ of $\mathbb{C}^n$. In this paper we define and study the properties of a positive matrix $\rho(\mathcal{S}, A)$ associated to the pair $(A, \mathcal{S})$ which is related to the shorted matrix $\Sigma(\mathcal{S}, A)$ of Anderson [1] by means of a spectral radius-type formula.

Denote by $M_n(\mathbb{C})^+$ the set of all positive semidefinite matrices. Given a matrix C denote by $R(C)$ the subspace spanned by the columns of C (i.e. the range of C).

The shorting $\Sigma(\mathcal{S}, A)$ can be defined as follows. Suppose, for simplicity, that $\mathcal{S}$ is the subspace spanned by the first s canonical vectors and consider the partitioned matrix $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ where $a \in M_s(\mathbb{C})^+$, $b \in M_{(n-s)\times s}(\mathbb{C})$ and $c \in M_{n-s}(\mathbb{C})^+$. Then, $\Sigma(\mathcal{S}, A) = \begin{pmatrix} a - bc^\dagger b^* & 0 \\ 0 & 0 \end{pmatrix}$ is the biggest element D of $M_n(\mathbb{C})^+$ such that $D \leq A$ (i.e. $A - D$ is a positive matrix) and $R(D) \subseteq \mathcal{S}$ (where $c^\dagger$ is the Moore-Penrose pseudoinverse of c). This result and many others were proved by W. N. Anderson in [1] and applied to electrical circuit theory. Observe that $\Sigma(\mathcal{S}, A)$ can also be seen as an $s \times s$-matrix (or, which is the same, as a linear transformation on $\mathcal{S}$) Observe also that there is no canonical notation for $\Sigma(\mathcal{S}, A)$. W. N. Anderson [1] denotes $\mathcal{S}(A)$, T. Ando [3] denotes $A/_{\mathcal{S}}$ and E. L. Pekarev [17] uses $A_{\mathcal{S}}$.

Later on, W. N. Anderson and G. E. Trapp [2] extended the concept to a Hilbert space context; indeed, it was M. G. Krein [11] in 1946 who first defined and used this construction in his study of extensions of selfadjoint operators, see also J. L. Smul'jan [19]. Many generalizations and applications came later. The reader is referred to the papers by T. Ando [3], R. Cottle [7], D. Carlson [6], S. K. Mitra [15], C. A. Butler and T. D. Morley [5], E. L. Pekarev [17] and C. K. Li and R. Mathias [13], [14] to have a complete panorama on these matters.

For a positive number t, consider the power matrix A^t and its shorted matrix $\Sigma(\mathcal{S}, A^t)$. It turns out that the map $t \to \Sigma(\mathcal{S}, A^t)^{1/t}$ is decreasing for $t \geq 1$. Its limit

$$\rho(\mathcal{S}, A) = \lim_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m},$$

which we call the *spectral shorted matrix* of A to $\mathcal{S}$, is the main subject of the present paper. The limit should be understood respect to any matrix norm, for instance, the operator norm induced by the Euclidean norm of $\mathbb{C}^n$.

Suppose that $\mathcal{S} = \{\xi \in \mathbb{C}^n : \xi_1 = \ldots = \xi_{n-1} = 0\}$. Denote by $P = P_{\mathcal{S}}$, the orthogonal projection onto $\mathcal{S}$. Then, for every non negative definite matrix A, we can identify $\Sigma(\mathcal{S}, A)$ and $\rho(\mathcal{S}, A)$ with non negative numbers, because $\dim \mathcal{S} = 1$. With this convention, if A is invertible, then

$$\Sigma(\mathcal{S}, A) = \frac{\det A}{\det A_{nn}},$$

where $A_{nn} = (1-P)A(1-P)$ acts on $\mathcal{S}^\perp$ (i.e. it is identified with the $(n-1)\times(n-1)$ principal submatrix of A obtained by deleting the last column and the last row of A). Indeed, it follows from the well known formula $\det A = \det A_{nn} \det \Sigma(\mathcal{S}, A)$, which is in the origin of the study of Schur complements (see [10], [7], [3], [6]). Therefore

$$\Sigma(\mathcal{S}, A^t)^{1/t} = \frac{\det A}{[\det(A^t)_{nn}]^{1/t}} \ ,$$

so that

$$\rho(\mathcal{S}, A) = \frac{\det A}{\lim_{t\to\infty}[\det(A^t)_{nn}]^{1/t}} \ .$$

This relation can be used in the following way: if $\mu_1(B) \ge \cdots \ge \mu_n(B)$ are the eigenvalues of the selfadjoint $n \times n$ matrix B, then, by interlacing, $\mu_i(A)^t \ge \mu_i((A^t)_{nn}) \ge \mu_{i+1}(A)^t$, for $i = 1, \ldots, n-1$. Therefore, for every $t \in [1, \infty)$,

$$[\det(A^t)_{nn}]^{1/t} \le \frac{\det A}{\mu_n(A)} \ , \quad \text{so that} \quad \rho(\mathcal{S}, A) \ge \mu_n(A).$$

Conversely, in this paper we completely characterize the matrix $\rho(\mathcal{S}, A)$ in terms of the subspace $\mathcal{S}$ and the eigenspaces of A. Then the $\lim\limits_{t\to\infty}[\det(A^t)_{nn}]^{1/t}$, and the corresponding limit for every one dimensional subspace $\mathcal{S}$, can be described as in formulae (6), (7) and (12). For instance, from these formulae we can deduce that $\lim\limits_{t\to\infty}[\det(A^t)_{nn}]^{1/t} = \dfrac{\det A}{\mu_n(A)}$ if and only if $\ker(A - \mu_n(A)I) \not\subseteq \mathcal{S}^\perp$.

In [9], J. I. Fujii and M. Fujii consider the Kolmogorov's complexity

$$K\left(A, \xi\right) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi, \xi\rangle)}{n}$$

for an invertible positive matrix A and a unit vector ξ and show several properties of K. In section 6 we show that, if $\mathcal{S}$ is the subspace generated by ξ, then

$$K\left(A, \xi\right) = \log \rho\left(\mathcal{S}, A^{-1/2}\right)^{-2} = \log \rho\left(\mathcal{S}, A^{-1}\right)^{-1},$$

where we are identifying the rank one spectral shorted matrices with the positive number which characterizes it. With this identification, several results of [9] can be deduced from the properties of the spectral shorted operator, see Remark 6.2. Moreover, it shows that $\rho\left(\mathcal{S}, A\right)$ can be seen as a higher dimensional version of K.

Section 2 contains preliminaries and a brief account of the main properties of the shorting operation. In section 3 the properties of ρ are compared to those of Σ. On one side, several properties of both operations are analogous. For instance, we prove that for every positive number t it holds that

$$\rho\left(\mathcal{S}, A^t\right) = \rho\left(\mathcal{S}, A\right)^t. \tag{1}$$

A key property of the spectral shorted operator, similar to a property satisfied by the usual shorted operator is the following (see Corollary 3.9): given $A \in M_n(\mathbb{C})^+$ and two subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathbb{C}^n$, it holds

$$\rho\left(\mathcal{S} \cap \mathcal{T}, A\right) = \rho\left(\mathcal{T}, \rho\left(\mathcal{S}, A\right)\right).$$

On the other side, to get the monotonicity ($0 \le A \le B$ implies $\Sigma\left(\mathcal{S}, A\right) \le \Sigma\left(\mathcal{S}, B\right)$) for ρ we are forced to change the order relation, because in general it is not true that

$\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B)$ (see Example 7.2). Recall the definition of the spectral order $\preccurlyeq$ in $M_n(\mathbb{C})^+$: given $A, B \in M_n(\mathbb{C})^+$, we write $A \preccurlyeq B$ if $A^m \leq B^m$ for all $m \geq 1$. This order provides the following link with Krein's definition of the shorted operator: $\rho(\mathcal{S}, A)$ is the biggest (in both orders $\leq$ and $\preccurlyeq$) element D of $M_n(\mathbb{C})^+$ such that $D \preccurlyeq A$ and $R(D) \subseteq \mathcal{S}$ (see Proposition 5.5).

The spectral order was studied by M. P. Olson in [16], where the following characterization is proved: given $A, B \in M_n(\mathbb{C})^+$, then $A \preccurlyeq B \iff f(A) \leq f(B)$ for every monotone non-decreasing map $f : [0, +\infty) \to \mathbb{R}$. In section 5 the properties of the spectral shorted operator are used to prove a new characterization of the spectral order. For $A, B \in M_n(\mathbb{C})^+$, the following statements are equivalent:

1. $A \preccurlyeq B$
2. For every subspace $\mathcal{S}$, it holds $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B)$.
3. For every one dimensional subspace $\mathcal{S}$, it holds $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B)$.
4. If $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$ and $\lambda > \mu$, then $\ker(A - \lambda) \subseteq (\ker(B - \mu))^\perp$.
5. There is a positive integer $k \leq n$ and a sequence of positive matrices $\{D_i\}_{0 \leq i \leq k}$ such that, $D_0 = A$, $D_k = B$, $D_i \leq D_{i+1}$ and $D_i D_{i+1} = D_{i+1} D_i$ $(i = 0, \cdots, k-1)$.

Using this result, formula (1) can be generalized as follows: for every non-decreasing function f defined on $[0, +\infty)$ it holds

$$f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A)) \tag{2}$$

if both $\rho(\mathcal{S}, A)$ and $\rho(\mathcal{S}, f(A))$ are considered as acting on $\mathcal{S}$. Moreover, a complete characterization of the spectrum of $\rho(\mathcal{S}, A)$ (which is contained in the spectrum of A) and the eigenspaces of $\rho(\mathcal{S}, A)$ are given in terms of $\mathcal{S}$ and the eigenspaces of A. For example:

1. $\min \sigma(\rho(\mathcal{S}, A)) = \min\{\lambda \in \sigma(A) : \ker(A - \lambda I) \not\subseteq \mathcal{S}^\perp\}$, where $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$. In particular, if A is invertible, then $\rho(\mathcal{S}, A) : \mathcal{S} \to \mathcal{S}$ is invertible too.
2. $\|\rho(\mathcal{S}, A)\| = \max \sigma(\rho(\mathcal{S}, A)) = \min\{\lambda \in \sigma(A) : \oplus_{\mu > \lambda} \ker(A - \mu I) \cap \mathcal{S} = \{0\}\ \}$. In particular, $\|\rho(\mathcal{S}, A)\| = \|A\| \iff \ker(A - \|A\| I) \cap \mathcal{S} \neq \{0\}$.
3. For $\lambda \in \mathbb{R}$,
$$\bigoplus_{\mu \geq \lambda} \ker(\rho(\mathcal{S}, A) - \mu I) = \mathcal{S} \cap \bigoplus_{\mu \geq \lambda} \ker(A - \mu I)$$

In section 6. we study the particular case of one-dimensional subspaces and show that several results by J. I. Fujii and M. Fujii [9] on what they call Kolmogorov's complexity, become corollaries of our results. We should mention, however, that Fujii and Fujii have proven a one dimensional version of Theorem 4.3. The last section contains several examples.

Several results of this paper remain valid, with almost the same proofs, for operators on a separable Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$; in particular, the spectral shorted operator $\rho(\mathcal{S}, A)$ can be defined in this setting. However, a complete characterization of $\rho(\mathcal{S}, A)$ in this case is still an open problem. These matters will be discussed elsewhere.

We wish to acknowledge Professor T. Ando for several useful comments about the properties of the spectral order.

2 Preliminaries

For a matrix $A \in M_n(\mathbb{C})$, we denote by $R(A)$ the range of A, $\ker A$ the kernel of A, $\sigma(A)$ the spectrum (i.e. the set of eigenvalues) of A, A^* the adjoint matrix of A, $\rho(A)$ the spectral radius of A, $\|A\|$ the spectral norm (i.e. the operator norm induced by the Euclidean norm of $\mathbb{C}^n$) of A and $A^\dagger$ the Moore-Penrose pseudoinverse of A. If $A = A^*$, we denote by $\lambda_{min}(A) = \min \sigma(A) = \min_{\|\xi\|=1} \langle A\xi, \xi\rangle$.

Given a subspace $\mathcal{S}$ of $\mathbb{C}^n$, we denote by $P_\mathcal{S}$ the orthogonal (i.e. selfadjoint) projection onto $\mathcal{S}$. If $B \in M_n(\mathbb{C})$ satisfies $P_\mathcal{S} B P_\mathcal{S} = B$, we consider the compression of B to $\mathcal{S}$, (i.e. the restriction of B to $\mathcal{S}$ as a linear transformation form $\mathcal{S}$ to $\mathcal{S}$), and we say that we think B as *acting* on $\mathcal{S}$. Several times this is done in order to consider $\sigma(B)$ just in terms of the action of B on $\mathcal{S}$. For example, if $B \geq \lambda P_\mathcal{S}$ for some $\lambda > 0$, then we can deduce that $0 \notin \sigma(B)$, if we think B as acting on $\mathcal{S}$.

Along this note we use the fact that every subspace $\mathcal{S}$ of $\mathbb{C}^n$ induces a representation of elements of $M_n(\mathbb{C})$ by 2×2 block matrices, that is, we shall identify each $A \in M_n(\mathbb{C})$ with a 2×2-matrix, let us say $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$. Observe that $\begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$ is the matrix which represents A^*.

Shorted operator

W. N. Anderson [1] showed that if $A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$ is a $n \times n$ positive matrix and B is a square $k \times k$ submatrix, then the matrix

$$\Sigma(\mathcal{S}, A) = \begin{pmatrix} B - CD^\dagger C^* & 0 \\ 0 & 0 \end{pmatrix},$$

where $D^\dagger$ is the Moore-Penrose pseudoinverse of D and $\mathcal{S}$ the subspace of $\mathbb{C}^n$ generated by the first k canonical vectors, has the following interpretation in electrical network theory: if A is the impedance matrix of a resistive n-port network, then $\Sigma(\mathcal{S}, A)$ is the impedance matrix of the network obtained by shorting the last $n - k$ ports. In his paper, Anderson proved that

$$\Sigma(\mathcal{S}, A) = \max\{X \in M_n(\mathbb{C})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\}. \tag{3}$$

Although the existence of this maximum has already been observed by M.G. Krein [11] in an infinite demensional context, this result has been widely used only after it was rediscovered by Anderson and Trapp [1], [2]. In this note, we use equation (3) as the definition of shorted matrices.

Definition 2.1. Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. Then, the *shorted matrix* of A to $\mathcal{S}$ is defined by

$$\Sigma(\mathcal{S}, A) = \max\{X \in M_n(\mathbb{C})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\}$$

where the maximum is taken for the natural order relation in $M_n(\mathbb{C})^+$ (see [2]).

In the next theorem we state some results on shorted operators proved by Anderson and Trapp [2], M.G. Krein [11] and E. L. Pekarev [17] which are relevant in this paper.

Theorem 2.2. *Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathbb{C}^n$ and let $A, B \in M_n(\mathbb{C})^+$. Then*

1. *For every $c \in \mathbb{R}^+$ we have that $\Sigma(\mathcal{S}, c\,A) = c\,\Sigma(\mathcal{S}, A)$.*

2. *If $\mathcal{S} \subseteq \mathcal{T}$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{T}, A)$.*

3. $\Sigma(\mathcal{S} \cap \mathcal{T}, A) = \Sigma(\mathcal{S}, \Sigma(\mathcal{T}, A))$.

4. *If $A \leq B$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{S}, B)$.*

5. $\Sigma(\mathcal{S}, A^2) \leq \Sigma(\mathcal{S}, A)^2$

6. $\Sigma(\mathcal{S}, A) = \inf\{QAQ^* \;:\; Q^2 = Q,\; R(Q) = \mathcal{S}\}$ ■

There is also a result about the continuity of the shorting operation (see [2], Corollary 2).

Theorem 2.3. *Let A_n ($n \in \mathbb{N}$) be a sequence of positive matrices such that $A_n \underset{n\to\infty}{\searrow} A$. Then, for every subspace $\mathcal{S}$ it holds*

$$\Sigma(\mathcal{S}, A_n) \underset{n\to\infty}{\searrow} \Sigma(\mathcal{S}, A).$$

■

3 Definition of $\rho(\mathcal{S}, A)$ and basic properties

Proposition 3.1. *Let $A \in M_n(\mathbb{C})^+$ and let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$. Then, for every $t \geq 1$, it holds*

$$\Sigma(\mathcal{S}, A^t)^{1/t} \leq \Sigma(\mathcal{S}, A).$$

Moreover, if $1 \leq s \leq t$ then $\Sigma(\mathcal{S}, A^s)^{1/s} \geq \Sigma(\mathcal{S}, A^t)^{1/t}$.

Proof. Note that $\Sigma(\mathcal{S}, A^t) \leq A^t$. Since $0 \leq 1/t \leq 1$, by Löwner's theorem [12], it follows that $\Sigma(\mathcal{S}, A^t)^{1/t} \leq A$. On the other hand $R(\Sigma(\mathcal{S}, A^t)^{1/t}) \subseteq \mathcal{S}$. So the statement follows from the definition of shorted matrix. If $t \geq s \geq 1$, let us denote $u = t/s \geq 1$ and $B = A^s$. Note that $B^u = A^t$. Then

$$\Sigma\left(\mathcal{S}, A^t\right)^{s/t} = \Sigma(\mathcal{S}, B^u)^{1/u} \leq \Sigma(\mathcal{S}, B) = \Sigma(\mathcal{S}, A^s).$$

Therefore, because $1/s \leq 1$, we get $\Sigma(\mathcal{S}, A^t)^{1/t} \leq \Sigma(\mathcal{S}, A^s)^{1/s}$.

■

Corollary 3.2. *Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S} \subseteq \mathbb{C}^n$. Then, for every $0 \leq r \leq 1$, it holds that $\Sigma(\mathcal{S}, A)^r \leq \Sigma(\mathcal{S}, A^r)$.*

Proof. Apply Proposition 3.1 to A^r with $t = 1/r$ ■

Consider the map $[1, \infty) \to M_n(\mathbb{C})^+$ given by $t \mapsto \Sigma(\mathcal{S}, A^t)^{1/t}$. By Proposition 3.1, this map is nonincreasing. This fact motivates the following definition:

Definition 3.3. Given $A \in M_n(\mathbb{C})^+$, the *spectral shorted* matrix of A by $\mathcal{S}$ is

$$\rho(\mathcal{S}, A) = \inf_{t \geq 1} \Sigma(\mathcal{S}, A^t)^{1/t} = \lim_{t \to +\infty} \Sigma(\mathcal{S}, A^t)^{1/t}.$$

In the next proposition we sum up some simple properties of spectral shorted matrices.

Proposition 3.4. *Let $A \in M_n(\mathbb{C})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathbb{C}^n$. Then:*

a. $R(\rho(\mathcal{S}, A)) \subseteq R(A) \cap \mathcal{S}$.

b. $\rho(\mathcal{S}, cA) = c\,\rho(\mathcal{S}, A)$ *for every* $c \in [0, +\infty)$.

c. *If* $\mathcal{S} \subseteq \mathcal{T}$*, then,* $\rho(\mathcal{S}, A) \leq \rho(\mathcal{T}, A)$.

d. $\Sigma(\mathcal{S}, \rho(\mathcal{S}, A)) = \rho(\mathcal{S}, A)$ *and* $\rho(\mathcal{S}, \Sigma(\mathcal{S}, A)) = \Sigma(\mathcal{S}, A)$.

e. $\rho(\mathcal{S}, \rho(\mathcal{S}, A)) = \rho(\mathcal{S}, A)$.

f. $\rho(\mathcal{S} \cap \mathcal{T}, A) \leq \rho(\mathcal{T}, \Sigma(\mathcal{S}, A))$.

Proof.

a, b and c. These properties follow from the definition of $\rho(\mathcal{S}, A)$ and Proposition 2.2.

d. Since $R\left(\Sigma\left(\mathcal{S}, A^t\right)^{1/t}\right) \subseteq \mathcal{S}$ for each $t \geq 1$, it holds $R(\rho(\mathcal{S}, A)) \subseteq \mathcal{S}$, so $\Sigma(\mathcal{S}, \rho(\mathcal{S}, A)) = \rho(\mathcal{S}, A)$.

e. It is a consequence of the previous equality.

f. It can be deduced from inequalities

$$\Sigma\left(\mathcal{S} \cap \mathcal{T}, A^{2^m}\right) \leq \Sigma\left(\mathcal{T}, \Sigma\left(\mathcal{S}, A^{2^m}\right)\right) \leq \Sigma\left(\mathcal{T}, \Sigma(\mathcal{S}, A)^{2^m}\right) \qquad \forall m \in \mathbb{N}.$$

■

Examples 3.5.

1. If A is the projection with range $\mathcal{T}$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = P_{\mathcal{S} \cap \mathcal{T}}$ for every $t \in [1, \infty)$.

2. If A commutes with the orthogonal projection $P = P_{\mathcal{S}}$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = PA$ for every $t \in [1, \infty)$. ▲

The next result exhibites one of the main advantages of the spectral shorting over the classical shorting.

Theorem 3.6. *Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. Then, for every $t \in (0, \infty)$ it holds*

$$\rho(\mathcal{S}, A)^t = \rho\left(\mathcal{S}, A^t\right).$$

In particular, $\rho(\mathcal{S}, A)^t \leq A^t$ for every $t \in (0, \infty)$.

Proof. Given $t \in (0, \infty)$, since $st \to \infty$ as $s \to \infty$ and the map $x \to x^{1/t}$ is continuous, we have that

$$\rho\left(\mathcal{S}, A^t\right)^{1/t} = \left(\lim_{s\to\infty} \Sigma\left(\mathcal{S}, (A^t)^s\right)^{1/s}\right)^{1/t} = \lim_{s\to\infty} \Sigma\left(\mathcal{S}, A^{st}\right)^{1/st} = \rho(\mathcal{S}, A).$$

■

Before going on, let us recall the definition of spectral order (see [16]).

Definition 3.7. Let $A, B \in M_n(\mathbb{C})^+$. We write $A \preccurlyeq B$ if for every $m \in \mathbb{N}$ it holds that $A^m \leq B^m$. The relation $\preccurlyeq$ defined on $M_n(\mathbb{C})^+$ is a partial order and it is called *spectral order*.

The next result replaces the monotony property (4 of Theorem 2.2) of the classical shorting operation with respect to the usual order $\leq$.

Proposition 3.8. *Given $A, B \in M_n(\mathbb{C})^+$ such that $A \preccurlyeq B$. Then, for every subspace $\mathcal{S}$ of $\mathbb{C}^n$, it holds*

$$\rho(\mathcal{S}, A) \preccurlyeq \rho(\mathcal{S}, B).$$

Proof. Let $\mathcal{S}$ be subspace. Given $m > 1$, since $A^m \leq B^m$, by Theorem 2.2 (4) it holds $\Sigma(\mathcal{S}, A^m) \leq \Sigma(\mathcal{S}, B^m)$. Moreover, as the function $f(x) = x^{1/m}$ is operator monotone (see [4]), we get

$$(\Sigma(\mathcal{S}, A^m))^{1/m} \leq (\Sigma(\mathcal{S}, B^m))^{1/m}$$

and taking limit we obtain

$$\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B).$$

On the other hand, note that $A \preccurlyeq B$ implies that $A^k \preccurlyeq B^k$ for every $k \geq 1$. Thus, by what we have already proved, it holds

$$\rho\left(\mathcal{S}, A^k\right) \leq \rho\left(\mathcal{S}, B^k\right) \qquad (\forall\, k \geq 1).$$

Using Theorem 3.6, these inequalities can be rewritten as

$$\rho(\mathcal{S}, A)^k \leq \rho(\mathcal{S}, B)^k \qquad (\forall\, k \geq 1)$$

which is equivalent to $\rho(\mathcal{S}, A) \preccurlyeq \rho(\mathcal{S}, B)$. ■

In section 5 there is a deeper study about the relationship between the operator $\rho(\mathcal{S}, A)$ and the spectral order.

Theorem 3.9. *Let $A \in M_n(\mathbb{C})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be subspaces. Then*

$$\rho(\mathcal{S} \cap \mathcal{T}, A) = \rho(\mathcal{T}, \rho(\mathcal{S}, A))$$

Proof. Given $t \geq 1$, we get

$$\begin{aligned}\Sigma\left(\mathcal{T}, \rho\left(\mathcal{S}, A\right)^{t}\right)^{1/t} &= \Sigma\left(\mathcal{T}, \rho\left(\mathcal{S}, A^{t}\right)\right)^{1/t} = \Sigma\left(\mathcal{S}\cap\mathcal{T}, \rho\left(\mathcal{S}, A^{t}\right)\right)^{1/t} \\ &\geq \Sigma\left(\mathcal{S}\cap\mathcal{T}, \rho\left(\mathcal{S}\cap\mathcal{T}, A^{t}\right)\right)^{1/t} = \rho\left(\mathcal{S}\cap\mathcal{T}, A^{t}\right)^{1/t} \\ &= \rho\left(\mathcal{S}\cap\mathcal{T}, A\right)\end{aligned}$$

and taking limit we obtain the following inequality

$$\rho\left(\mathcal{T}, \rho\left(\mathcal{S}, A\right)\right) \geq \rho\left(\mathcal{S}\cap\mathcal{T}, A\right).$$

On the other hand, by Proposition 3.6, for every $t \geq 1$, $\rho\left(\mathcal{S}, A\right)^{t} = \rho\left(\mathcal{S}, A^{t}\right) \leq A^{t}$; then

$$\Sigma\left(\mathcal{T}, \rho\left(\mathcal{S}, A\right)^{t}\right)^{1/t} = \Sigma\left(\mathcal{S}\cap\mathcal{T}, \rho\left(\mathcal{S}, A\right)^{t}\right)^{1/t} \leq \left(\Sigma\left(\mathcal{S}\cap\mathcal{T}, A^{t}\right)\right)^{1/t}$$

and taking limit again we get $\rho\left(\mathcal{T}, \rho\left(\mathcal{S}, A\right)\right) \leq \rho\left(\mathcal{S}\cap\mathcal{T}, A\right)$. ■

Proposition 3.10. *Let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$ and let $\{A_m\}$ be a sequence in $M_n(\mathbb{C})^+$ such that $A_m \xrightarrow[m\to\infty]{} A$ and $A_{m+1} \preccurlyeq A_m$ for every $m \in \mathbb{N}$. Then*

$$\rho\left(\mathcal{S}, A_m\right) \underset{n\to\infty}{\searrow} \rho\left(\mathcal{S}, A\right)$$

Proof. Since $\rho\left(\mathcal{S}, A_{m+1}\right) \leq \rho\left(\mathcal{S}, A_m\right)$ (by Corollary 3.8), there is a positive operator L such that
$\rho\left(\mathcal{S}, A_m\right) \xrightarrow[m\to\infty]{} L$. Clearly $\rho\left(\mathcal{S}, A\right) \leq L$.

On the other hand, for every $m, k \geq 1$

$$L \leq \rho\left(\mathcal{S}, A_m\right) \leq \Sigma\left(\mathcal{S}, A_m^k\right)^{1/k} \tag{4}$$

Now fix $k \geq 1$. As $A_n^k \underset{m\to\infty}{\searrow} A^k$, by proposition 2.3 it follows

$$\Sigma\left(\mathcal{S}, A_m^k\right)^{1/k} \xrightarrow[m\to\infty]{} \Sigma\left(\mathcal{S}, A^k\right)^{1/k} \tag{5}$$

hence, joining (4) and (5) we obtain $L \leq \Sigma\left(\mathcal{S}, A^k\right)^{1/k}$, which implies $L \leq \rho\left(\mathcal{S}, A\right)$ ■

Remark 3.11. In section 7 we show an example for which the statements of Propositions 3.8 and 3.10 fail if the spectral order is replaced by the usual one.

4 Spectrum of $\rho\left(\mathcal{S}, A\right)$

In this section $\mathcal{S}$ is a subspace of $\mathbb{C}^n$ and $P = P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$.

Proposition 4.1. *For $A \in M_n(\mathbb{C})^+$ let $\mu = \min \sigma\left(A\right)$. Then*

$$\mu P \leq \rho(\mathcal{S}, A).$$

In particular, if A is invertible then $\rho(\mathcal{S}, A) : \mathcal{S} \to \mathcal{S}$ is invertible.

Proof. Note that $\mu^m = \min \sigma(A^m)$ for all $m \in \mathbb{N}$. Then $\mu^m P \le \mu^m I \le A^m$ for all $m \in \mathbb{N}$, so that $\mu P \le \Sigma(\mathcal{S}, A^m)^{1/m}$ and the result follows ■

Proposition 4.2. *Let $A \in M_n(\mathbb{C})^+$. Then, if $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$, it holds*

$$\min \sigma(\rho(\mathcal{S}, A)) = \max\{\lambda \ge 0 : A^m - \lambda^m P \ge 0, \ \forall\, m \in \mathbb{N}\}. \tag{6}$$

Proof. Recall that P is the identity on $\mathcal{S}$, which is the space where $\rho(\mathcal{S}, A)$ and $\Sigma(\mathcal{S}, A^m)^{1/m}$ act. Then, for $\lambda \ge 0$,

$$\begin{aligned} \lambda P \le \rho(\mathcal{S}, A) &\Leftrightarrow \lambda\ P \le \Sigma(\mathcal{S}, A^m)^{1/m} && \forall\, m \in \mathbb{N} \\ &\Leftrightarrow \lambda^m P \le \Sigma(\mathcal{S}, A^m) && \forall\, m \in \mathbb{N} \\ &\Leftrightarrow \lambda^m\ P \le A^m && \forall\, m \in \mathbb{N} \end{aligned}$$

and the result is proved. ■

Theorem 4.3. *Let $A \in M_n(\mathbb{C})^+$. Then*

$$\min \sigma(\rho(\mathcal{S}, A)) = \min\{\lambda \in \sigma(A) : \ker(A - \lambda I) \not\subseteq \mathcal{S}^\perp\}, \tag{7}$$

if $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$.

Proof. Let $\mu = \min\{\lambda \in \sigma(A) : \ker(A - \lambda I) \not\subseteq \mathcal{S}^\perp\}$. Fix $m \in \mathbb{N}$. It is clear that $\mu^m = \min\{\lambda \in \sigma(A^m) : \ker(A^m - \lambda I) \not\subseteq \mathcal{S}^\perp\}$. Then

$$\bigoplus_{\lambda < \mu^m} \ker(A^m - \lambda I) \subseteq \mathcal{S}^\perp \quad \Rightarrow \quad \mathcal{S} \subseteq \bigoplus_{\lambda \ge \mu^m} \ker(A^m - \lambda I),$$

so that $\mu^m P \le A^m$ for all $m \in \mathbb{N}$. Therefore $\mu P \le \rho(\mathcal{S}, A)$ and $\min \sigma(\rho(\mathcal{S}, A)) \ge \mu$.

On the other hand, if $\mathcal{L} = \ker(A - \mu I)$, let ρ be a unit vector in $\mathcal{L}$ such that $\langle P\rho, \rho\rangle \ne 0$, and let $\lambda \ge 0$ such that $A^m - \lambda^m P \ge 0$, for every $m \in \mathbb{N}$. Then

$$0 \le \langle (A^m - \lambda^m P)\rho,\ \rho\rangle = \mu^m - \lambda^m \langle P\rho,\ \rho\rangle\,.$$

This implies that $\lambda^m \langle P\rho, \rho\rangle \le \mu^m$, for every $m \in \mathbb{N}$. Since $\langle P\rho, \rho\rangle > 0$, it must be $\mu \ge \lambda$. Then, by the above Proposition, we get $\min \sigma(\rho(\mathcal{S}, A)) \le \mu$. ■

Corollary 4.4. *Let $A \in M_2(\mathbb{C})$ and suppose that $\dim \mathcal{S} = 1$. If $AP \ne PA$, then*

$$\rho(\mathcal{S}, A) = \min \sigma(A)\ \ P$$

Proof. If $\min \sigma(A) = \mu$ and A is not diagonal, then $\ker(A - \mu I) \not\subseteq \mathcal{S}^\perp$. ■

Proposition 4.5. *If $A \in M_n(\mathbb{C})^+$, then $\sigma(\rho(\mathcal{S}, A)) \subseteq \sigma(A)$.*

Proof. Given $\lambda \in \sigma(\rho(\mathcal{S}, A))$, let $\mathcal{T} = \bigoplus_{\mu \geq \lambda} \ker(\rho(\mathcal{S}, A) - \mu)$. As $\mathcal{T}$ reduce $\rho(\mathcal{S}, A)$ we have that

$$\rho(\mathcal{T}, \rho(\mathcal{S}, A)) = P_{\mathcal{T}}\, \rho(\mathcal{S}, A)$$

On the other hand, according to Proposition 3.9

$$\rho(\mathcal{T}, \rho(\mathcal{S}, A)) = \rho(\mathcal{T} \cap \mathcal{S}, A)$$

Now, the minimum eigenvalue of $\rho(\mathcal{T} \cap \mathcal{S}, A)$ belongs to $\sigma(A)$, as we have shown in Theorem 4.3. But, by construction, $\lambda = \min \sigma(\rho(\mathcal{T} \cap \mathcal{S}, A))$. Thus $\lambda \in \sigma(A)$. ■

Remark 4.6. Given a matrix A, the condition number of A is defined by means of

$$\operatorname{cond}(A) = \|A\|\, \|A^\dagger\|,$$

where $A^\dagger$ denotes the Moore-Penrose pseudoinverse of A. In particular, when $A \in M_n(\mathbb{C})^+$, then $\operatorname{cond}(A) = \lambda_{\max}(A)\, \lambda$, where λ is the inverse of the smallest eigenvalue of A different from zero. Taking this into account, by the previous Proposition we obtain

$$\operatorname{cond}(A) \geq \operatorname{cond}(\rho(\mathcal{S}, A)).$$

▲

At the end of the next section we shall give a more detailed description of $\sigma(\rho(\mathcal{S}, A))$.

5 Spectral order and the spectral shorted matrix

In this section we profundize the study of the relationship between the spectral order (recall Definition 3.7) and the properties of the spectral shorting operation. We begin with the following examples, whose verifications are easy to see.

Examples 5.1. *Given $A, B \in M_n(\mathbb{C})^+$ such that $A \leq B$, it holds*

1. *If $AB = BA$ then $A \preccurlyeq B$.*

2. *If $\lambda_{\max}(A) \leq \lambda_{\min}(B)$ then $A \preccurlyeq B$.*

3. *In $M_2(\mathbb{C})^+$, $A \preccurlyeq B$ if and only if either $\lambda_{\max}(A) \leq \lambda_{\min}(B)$ or $AB = BA$. Indeed, it is an easy consequence of Corollary 4.4.*

4. *If there is a matrix C such that $A \leq C \leq B$, $AC = CA$, and $BC = CB$, then $A \preccurlyeq B$.*

▲

One of the main results of the paper is the following theorem, which provides some useful characterizations of the spectral order. Observe that the equivalence $a \iff b$ is related to a similar result of J. I. Fujii and M. Fujii [9].

Theorem 5.2. *Let $A, B \in M_n(\mathbb{C})^+$. Then, the following statements are equivalent:*

a. $A \preccurlyeq B$

b. *For every one dimensional subspace $\mathcal{S}$, it holds $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B)$.*

c. *If $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$ and $\lambda > \mu$, then $\ker(A - \lambda) \subseteq (\ker(B - \mu))^\perp$.*

d. *There is a positive integer $k \leq n$ and an sequence of positive matrices $\{D_i\}_{0 \leq i \leq k}$ such that, $D_0 = A$, $D_k = B$, $D_i \leq D_{i+1}$ and $D_i D_{i+1} = D_{i+1} D_i$ ($i = 0, \cdots, k-1$).*

Proof.

a $\Rightarrow$ b Use Proposition 3.8

b $\Rightarrow$ c Let $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$ such that $\lambda > \mu$, and suppose that there exists $\xi \in \ker(A - \lambda) \setminus (\ker(B - \mu))^\perp$. Let $\mathcal{S}$ be the subspace generated by ξ. Then $\ker(B - \mu I) \nsubseteq \mathcal{S}^\perp$ and, by the Theorem 4.3,

$$\rho(\mathcal{S}, A) = \lambda > \mu \geq \rho(\mathcal{S}, B)$$

which contradicts b.

c $\Rightarrow$ d Let us proceed by induction over the dimension of the space $\mathbb{C}^n$. If $n = 1$, it is clearly true.

Now, let $n > 1$ and suppose that (c.$\Rightarrow$d.) for $n-1$. Let define

$$N = \{\lambda \in \sigma(A) : \quad \lambda > \lambda_{\min}(B)\}.$$

If $N = \varnothing$, then, $A \leq \lambda_{\min}(B) I \leq B$. On the other hand, if $N \neq \varnothing$, let P be the projection onto the subspace $\bigoplus_{\lambda \in N} \ker(A - \lambda)$ and D_1 the operator defined by

$$D_1 = \lambda_{\min}(B)(I - P) + PA.$$

Since $PA = AP$, it is clear that $AD_1 = D_1 A$ and $A \leq D_1$. On the other hand, the pair (D_1, B) also satisfy (c). D_1 and B have a common eigenvector ξ, which corresponds to $\lambda_{\min}(B)$ (because $\ker(B - \lambda_{\min}(B)) \subseteqq R(I - P)$). Let $\mathcal{L}$ be the subspace generated by ξ. Then D_1 and B can be represented

$$D_1 = \begin{pmatrix} \lambda_{\min}(B) & 0 \\ 0 & \widehat{D_1} \end{pmatrix} \begin{matrix} \mathcal{L} \\ \mathcal{L}^\perp \end{matrix} \qquad \text{and} \qquad B = \begin{pmatrix} \lambda_{\min}(B) & 0 \\ 0 & \widehat{B} \end{pmatrix} \begin{matrix} \mathcal{L} \\ \mathcal{L}^\perp \end{matrix}.$$

As $(\widehat{D_1}, \widehat{B})$ satisfy (c), applying the inductive hypothesis we find an increasing sequence $\{\widehat{D_j}\}_{j=2,\cdots,k}$ ($k \leq n$), such that $\widehat{D_k} = \widehat{B}$ and $\widehat{D_j}\widehat{D_{j+1}} = \widehat{D_{j+1}}\widehat{D_j}$ ($j = 1, \cdots, k-1$). Finally, the sequence that we are looking for is

$$D_0 = A$$

$$D_j = \begin{pmatrix} \lambda_{\min}(B) & 0 \\ 0 & \widehat{D_j} \end{pmatrix} \qquad (j = 1, \cdots, k).$$

d $\Rightarrow$ a Since $D_iD_{i+1} = D_{i+1}D_i$ $(i = 0, \cdots, k-1)$, it holds that $A \preccurlyeq D_1 \preccurlyeq \cdots \preccurlyeq D_k \preccurlyeq B$. ■

Remark 5.3. *Another proof of the equivalence between (a) and (c) can be found in [16]. In the following Corollary we give a short proof, using Theorem 5.2, of Olson's characterization of spectral order in the finite dimensional case.*

Corollary 5.4. *Let $A, B \in M_n(\mathbb{C})^+$, $\mathcal{S}$ a subspace of $\mathbb{C}^n$, and f a non-decreasing function. If $A \preccurlyeq B$ then $f(A) \preccurlyeq f(B)$.*

Proof. According to Theorem 5.2, there exist an increasing sequence $\{D_i\}_{i=1,\cdots,k}$ such that $D_0 = A$, $D_k = B$, $D_i \leq D_{i+1}$ and $D_iD_{i+1} = D_{i+1}D_i$ $(i = 0, \cdots, k)$. Therefore the sequence $\{f(D_i)\}_{i=1,\cdots,k}$ is non-decreasing. On the other hand, $f(D_0) = f(A)$, $f(D_k) = f(B)$ and $f(D_i)f(D_{i+1}) = f(D_{i+1})f(D_i)$. Thus, again by Theorem 5.2, $f(A) \preccurlyeq f(B)$. ■

Proposition 5.5. *Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. If*

$$\mathcal{M}_\rho(\mathcal{S}, A) = \ \{D \in M_n(\mathbb{C})^+ : \ \ D \preccurlyeq A, \ \ R(D) \subseteq \mathcal{S}\}$$

then

$$\rho\,(\mathcal{S}, A) = \max \mathcal{M}_\rho(\mathcal{S}, A),$$

where the "maximum" is taken for any of the orders $\leq$ and $\preccurlyeq$.

Proof. Firstly, note that $\rho\,(\mathcal{S}, A) \in \mathcal{M}_\rho(\mathcal{S}, A)$. In fact, $\rho\,(\mathcal{S}, A)^m \leq A^m$ for every $m \in \mathbb{N}$ by Proposition 3.6, and clearly $R(\rho\,(\mathcal{S}, A)) \subseteq \mathcal{S}$ by definition.

Next, suppose that $D \in \mathcal{M}_\rho(\mathcal{S}, A)$. As $D^m \leq A^m$, it holds that

$$\Sigma\,(\mathcal{S}, D^m)^{1/m} \leq \Sigma\,(\mathcal{S}, A^m)^{1/m}$$

and, since $\Sigma\,(\mathcal{S}, D^m)^{1/m} = D$ for every $m \in \mathbb{N}$, taking limit we have

$$D \leq \rho\,(\mathcal{S}, A)\,.$$

Note also that, if $D \in \mathcal{M}_\rho(\mathcal{S}, A)$, then for every $k \in \mathbb{N}$, $D^k \preccurlyeq A^k$ and, with the same proof as before one gets that

$$D^k \leq \rho\left(\mathcal{S}, A^k\right) = \rho\,(\mathcal{S}, A)^k\,.$$

Hence $D \preccurlyeq \rho\,(\mathcal{S}, A)$. ■

Corollary 5.6. *Let $A \in M_n(\mathbb{C})^+$, and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. Then $R(\rho\,(\mathcal{S}, A)) = R(A) \cap \mathcal{S}$.*

Proof. Since $0 \leq \rho\,(\mathcal{S}, A) \preccurlyeq A$, then $\rho\,(\mathcal{S}, A)^2 \leq A^2$ and, by Douglas' majorization theorem [8], $R(\rho\,(\mathcal{S}, A)) \subseteq R(A) \cap \mathcal{S}$. On the other hand, let P be the orthogonal projection onto $R(A)$. Then, there is a constant $\lambda > 0$ such that $P \leq \lambda A$. Since $AP = PA$, we have that $P \preccurlyeq \lambda A$, and by Proposition 3.8, $\rho\,(\mathcal{S}, P)^2 \leq \lambda^2 \rho\,(\mathcal{S}, A)^2$. But $\rho\,(\mathcal{S}, P)$ is the projection on $R(A) \cap \mathcal{S}$, so that, again by Douglas' theorem, $R(A) \cap \mathcal{S} \subseteq R(\rho\,(\mathcal{S}, A))$. ■

Proposition 5.7. *Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. Then, for every non-decreasing function $f : [0, +\infty) \to [0, +\infty)$, it holds that*

$$f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A)) \tag{8}$$

where $\rho(\mathcal{S}, A)$ and $\rho(\mathcal{S}, f(A))$ are considered as acting on $\mathcal{S}$.

Proof. Let A be the 2×2-matrix $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, according to the decomposition induced by $\mathcal{S}$. Since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \succcurlyeq \begin{pmatrix} \rho(\mathcal{S}, A) & 0 \\ 0 & 0 \end{pmatrix},$$

using Corollary 5.4 we get

$$f\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\right) \succcurlyeq \begin{pmatrix} f(\rho(\mathcal{S}, A)) & 0 \\ 0 & f(0) \end{pmatrix} \succcurlyeq \begin{pmatrix} f(\rho(\mathcal{S}, A)) & 0 \\ 0 & 0 \end{pmatrix}.$$

So, by Proposition 5.5

$$\begin{pmatrix} \rho(\mathcal{S}, f(A)) & 0 \\ 0 & 0 \end{pmatrix} \succcurlyeq \begin{pmatrix} f(\rho(\mathcal{S}, A)) & 0 \\ 0 & 0 \end{pmatrix}$$

and we have that $\rho(\mathcal{S}, f(A)) \succcurlyeq f(\rho(\mathcal{S}, A))$.

In order to prove the other inequality we first suppose that f strictly increasing. In this case there exist a positive, non-decreasing function g on $[0, +\infty)$ such that

$$g|_{[f(0), +\infty)} = f^{-1}.$$

Since $\sigma(f(A)) \subseteq [f(0), +\infty)$, we can use the part already proved and obtain

$$g(\rho(\mathcal{S}, f(A))) \preccurlyeq \rho(\mathcal{S}, g \circ f(A))) = \rho(\mathcal{S}, A).$$

But, applying f to both sides and taking into account Corollary 5.4 we get

$$\rho(\mathcal{S}, f(A)) \preccurlyeq f(\rho(\mathcal{S}, A))$$

Now, consider a general non-decreasing function f defined on $[0, +\infty)$. Let $\{g_m\}$ the sequence of function defined by $g_m(x) = f(x) + \dfrac{x}{m}$. Since g_m is strictly increasing and $g_m \underset{m\to\infty}{\searrow} f$, using what we have already done and Proposition 3.10 we get

$$f(\rho(\mathcal{S}, A)) = \lim_{m\to\infty} g_m(\rho(\mathcal{S}, A)) = \lim_{m\to\infty} \rho(\mathcal{S}, g_m(A)) = \rho(\mathcal{S}, f(A)).$$

■

Proposition 5.8. *Let $A \in M_n(\mathbb{C})^+$ and $\mathcal{S}$ a subspace of $\mathbb{C}^n$. Then,*

$$\begin{aligned} \bigoplus_{\mu\geq\lambda} \ker(\rho(\mathcal{S}, A) - \mu) &= \bigoplus_{\mu\geq\lambda} \ker(A - \mu) \cap \mathcal{S}, \quad \textit{and} \\ \bigoplus_{\mu>\lambda} \ker(\rho(\mathcal{S}, A) - \mu) &= \bigoplus_{\mu>\lambda} \ker(A - \mu) \cap \mathcal{S}. \end{aligned} \tag{9}$$

Proof. Let us consider the function $f = \aleph_{[\lambda,+\infty)}$. By Proposition 5.7 we know that

$$f(\rho(\mathcal{S},A)) = \rho(\mathcal{S}, f(A)).$$

Therefore, by comparing the ranges of these matrices we obtain

$$\bigoplus_{\mu\geq\lambda} \ker(\rho(\mathcal{S},A)-\mu) = R(f(\rho(\mathcal{S},A))) = R(\rho(\mathcal{S},f(A))) = \bigoplus_{\mu\geq\lambda} \ker(A-\mu)\cap\mathcal{S}.$$

The other equality can be proved in a similar way by using the function $f = \aleph_{(\lambda,+\infty)}$ ■

5.9. Now, after proving Proposition 5.8, we have all the technical tools in order to find the spectrum and the eigenspaces of $\rho(\mathcal{S},A)$ in terms of the spectral decomposition of A and the subspace $\mathcal{S}$.

Let $A \in M_n(\mathbb{C})^+$, let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$ and suppose that $\sigma(A) = \{\lambda_1, \cdots, \lambda_m\}$ $(\lambda_1 < \cdots < \lambda_m)$. Since by Proposition 4.5 $\sigma(\rho(\mathcal{S},A)) \subseteq \sigma(A)$, we have that $\sigma(\rho(\mathcal{S},A)) = \{\lambda_{i_1}, \ldots, \lambda_{i_p}\}$. The smallest eigenvalue of $\rho(\mathcal{S},A)$ was characterized by Proposition 4.3 in the following way

$$\lambda_{i_1} = \min\{\lambda \in \sigma(A) : \ker(A-\lambda I) \not\subseteq \mathcal{S}^\perp\}.$$

The other ones can be identified in this way

$$\lambda_{i_2} = \min\Big\{\lambda \in \sigma(A) :\ \lambda > \lambda_{i_1} \text{ and } \bigoplus_{\mu\geq\lambda} \ker(\rho(\mathcal{S},A)-\mu) \neq \bigoplus_{\mu>\lambda} \ker(\rho(\mathcal{S},A)-\mu)\Big\},$$

$$\vdots$$

$$\lambda_{i_{k+1}} = \min\Big\{\lambda \in \sigma(A) :\ \lambda > \lambda_{i_k} \text{ and } \bigoplus_{\mu\geq\lambda} \ker(\rho(\mathcal{S},A)-\mu) \neq \bigoplus_{\mu>\lambda} \ker(\rho(\mathcal{S},A)-\mu)\Big\},$$

and finally

$$\lambda_{i_p} = \min\Big\{\lambda \in \sigma(A) : \bigoplus_{\mu>\lambda} \ker(\rho(\mathcal{S},A)-\mu) = \{0\}\Big\}.$$

These formulae can be rewritten using Proposition 5.8 in the following way

$$\lambda_{i_2} = \min\Big\{\lambda \in \sigma(A) :\ \lambda > \lambda_{i_1} \text{ and } \bigoplus_{\mu\geq\lambda} \ker(A-\mu)\cap\mathcal{S} \neq \bigoplus_{\mu>\lambda} \ker(A-\mu)\cap\mathcal{S}\Big\},$$

$$\vdots$$

$$\lambda_{i_{k+1}} = \min\Big\{\lambda \in \sigma(A) :\ \lambda > \lambda_{i_k} \text{ and } \bigoplus_{\mu\geq\lambda} \ker(A-\mu)\cap\mathcal{S} \neq \bigoplus_{\mu>\lambda} \ker(A-\mu)\cap\mathcal{S}\Big\},$$

$$\vdots$$

$$\lambda_{i_p} = \min\Big\{\lambda \in \sigma(A) : \bigoplus_{\mu>\lambda} \ker(A-\mu)\cap\mathcal{S} = \{0\}\Big\}.$$

On the other hand, having characterized the eigenvalues of $\rho(\mathcal{S}, A)$ and using Proposition 5.8, the spaces of eigenvectors of $\rho(\mathcal{S}, A)$ can be writing in the following way

$$\ker(\rho(\mathcal{S}, A) - \lambda_{i_p}) = \bigoplus_{\mu \geq \lambda_{i_p}} \ker(A - \mu) \cap \mathcal{S}, \text{ and}$$

$$\ker(\rho(\mathcal{S}, A) - \lambda_{i_k}) = \Big(\bigoplus_{\mu \geq \lambda_{i_k}} \ker(A - \mu) \cap \mathcal{S} \Big) \cap \Big(\bigoplus_{\mu \geq \lambda_{i_{k+1}}} \ker(A - \mu) \cap \mathcal{S} \Big)^{\perp} k = 1, \ldots, p-1.$$

We summarized the previous discussion in the next Theorem:

Theorem 5.10. *Let $A \in M_n(\mathbb{C})^+$ and let $\mathcal{S}$ be a subspace of $\mathbb{C}^n$. Suppose that $\sigma(A) = \{\lambda_1, \cdots, \lambda_m\}$ ($\lambda_1 < \cdots < \lambda_m$) and let $i_1, \cdots, i_p$ be the subindexes defined by*

i. $\lambda_{i_1} = \min\{\lambda \in \sigma(A) : \ker(A - \lambda I) \not\subseteq \mathcal{S}^{\perp}\}$

ii. *For $k = 2, \ldots, p-1$ we define λ_{i_k} as the smallest eigenvalue of A such that $\lambda_{i_k} > \lambda_{i_{k-1}}$ and*

$$\bigoplus_{\mu \geq \lambda_{i_k}} \ker(A - \mu) \cap \mathcal{S} \supsetneqq \bigoplus_{\mu > \lambda_{i_k}} \ker(A - \mu) \cap \mathcal{S} \neq 0.$$

iii. $\lambda_{i_p} = \min \Big\{ \lambda \in \sigma(A) : \bigoplus_{\mu > \lambda} \ker(A - \mu I) \cap \mathcal{S} = \{0\} \Big\}.$

Then,

a. $\sigma(\rho(\mathcal{S}, A)) = \{\lambda_{i_1}, \ldots, \lambda_{i_p}\}$

b. $\|\rho(\mathcal{S}, A)\| = \lambda_{i_p} = \min \Big\{ \lambda \in \sigma(A) : \bigoplus_{\mu > \lambda} \ker(A - \mu I) \cap \mathcal{S} = \{0\} \Big\}\}$

c. *If P_p is the (orthogonal) projection onto the subspace*

$$\bigoplus_{\mu \geq \lambda_{i_p}} \ker(A - \mu) \cap \mathcal{S},$$

and P_k ($k = 1, \ldots, p-1$) is the (orthogonal) projection onto the subspace

$$\Big(\bigoplus_{\mu \geq \lambda_{i_k}} \ker(A - \mu) \cap \mathcal{S} \Big) \cap \Big(\bigoplus_{\mu \geq \lambda_{i_{k+1}}} \ker(A - \mu) \cap \mathcal{S} \Big)^{\perp},$$

it holds that

$$\rho(\mathcal{S}, A) = \sum_{k=1}^{p} \lambda_{i_k} P_k \tag{10}$$

6 The case $\dim \mathcal{S} = 1$

Suppose that $\dim \mathcal{S} = 1$ and let $P = P_{\mathcal{S}}$. For every $A \geq 0$ there exist $\lambda \geq 0$ such that $\rho(\mathcal{S}, A) = \lambda P$. In this section we shall study the one dimensional case, and, for simplicity of the notations, we shall identify $\rho(\mathcal{S}, A)$ with this number λ, instead of λP.

Recall that, using Theorem 4.3, it holds

$$\rho(\mathcal{S}, A) = \lambda_{min}(\rho(\mathcal{S}, A)) = \min\{\lambda \in \sigma(A) : \ker(A - \lambda I) \nsubseteq \mathcal{S}^{\perp}\}. \tag{11}$$

Proposition 6.1. *Let $A \in M_n(\mathbb{C})^+$ and let $\mathcal{S}$ be the subspace of $\mathbb{C}^n$ generated by the unit vector ξ. If A is invertible, then*

$$\rho(\mathcal{S}, A) = \lim_{m\to\infty} \|A^{-m}\xi\|^{-1/m} = \inf_{m\in\mathbb{N}} \|A^{-m}\xi\|^{-1/m} \tag{12}$$

If A is not invertible, then

1. $\rho(\mathcal{S}, A) = 0$ *if* $\ker A \nsubseteq \mathcal{S}^{\perp}$,
2. $\rho(\mathcal{S}, A) = \lim_{m\to\infty} \|B^m\xi\|^{-1/m} = \inf_{m\in\mathbb{N}} \|B^m\xi\|^{-1/m}$ *if* $\ker A \subseteq \mathcal{S}^{\perp}$ *and* $B = A^{\dagger}$.

Proof. The general case easily reduces to the invertible case by Theorem 4.3, by taking the restriction of A to $R(A)$. Note that $\ker A \subseteq \mathcal{S}^{\perp}$ implies that $\mathcal{S} \subseteq R(A)$.

Suppose that A is invertible and write $\xi = \sum_{k=1}^{n} a_i\xi_i$, where $\{\xi_i\}$ is a orthonormal basis of eigenvectors of A such that $A\xi_i = \lambda_i\xi$ and $\lambda_i \leq \lambda_{i+1}$, $1 \leq i \leq n-1$. Let j the first index such that $a_j \neq 0$. By Theorem 4.3, it holds $\rho(\mathcal{S}, A) = \lambda_j$. Therefore

$$\frac{A^{-m}\xi}{\lambda_j^{-m}} = \sum_{i\geq j} a_i \frac{\lambda_j^m}{\lambda_i^m}\xi_i \xrightarrow[m\to\infty]{} \sum_{\lambda_i=\lambda_j} a_i\xi_i.$$

and $\lim_{m\to\infty} \frac{\|A^{-m}\xi\|^{-1/m}}{\lambda_j} = 1$, since $\| \sum\limits_{\lambda_i=\lambda_j} a_i\xi_i\|^{-1/m} \xrightarrow[m\to\infty]{} 1$.

Finally, let us show that the sequence $\{\|A^{-m}\xi\|^{-1/m}\}$ is decreasing. Given $k \geq h$, as $\sum\limits_{i\geq j} a_i^2 = \|\xi\| = 1$, by Jensen's inequality, we have

$$\|A^{-k}\xi\|^{2h/k} = \left(\sum_{i\geq j} \frac{1}{\lambda_i^{2k}}\, a_i^2\right)^{h/k} \geq \sum_{i\geq j} \left(\frac{1}{\lambda_i^{2k}}\right)^{h/k} a_i^2 = \sum_{i\geq j} \frac{1}{\lambda_i^{2h}}\, a_i^2 = \|A^{-h}\xi\|^2$$

and applying the function $f(x) = x^{-1/2h}$ to both sides of the inequality we get $\|A^{-k}\xi\|^{-1/k} \leq \|A^{-h}\xi\|^{-1/h}$. ∎

Remark 6.2. Given an invertible matrix $A \in M_n(\mathbb{C})^+$ and ξ a unit vector, J. I. Fujii and M. Fujii [9] define the Kolmogorov's complexity:

$$K(A, \xi) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi, \xi\rangle)}{n} = \log \lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}.$$

Among several results, they prove

1. $\sigma(A) = \{\exp(K(A,\xi)) : \|\xi\| = 1\}$
2. $K(A,\xi) = \min\{\log\lambda : \lambda \in \sigma(A),\ \xi \in \bigoplus_{\mu\leq\lambda} \ker(A-\mu)\}$
3. $A \preccurlyeq B \Leftrightarrow K(A,\xi) \leq K(B,\xi)$ for every ξ

Let us show that their results can be deduced from the knowledge of the spectral shorted matrix $\rho(\mathcal{S}, A^{-1})$. Using Propositions 6.1 and 5.7, if $\mathcal{S}$ is the subspace generated by ξ, it is easy to see that

$$K(A,\xi) = \log \rho\left(\mathcal{S}, A^{-1/2}\right)^{-2} = \log \rho\left(\mathcal{S}, A^{-1}\right)^{-1}.$$

With this identification, the above mentioned results of [9] can be deduced from Proposition 4.5, formula (11) and Theorem 5.2, respectively.

7 Some examples

Let us show first an example of a pair $(A, \mathcal{S})$ such that $\rho(\mathcal{S}, A)$ is explicitely computed.

Example 7.1. Consider the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 10 & -2 \\ 2 & -2 & 6 \end{pmatrix},$$

and the subspace $\mathcal{S}$ generated by the vectors $(1,0,0)$ and $(0,1,0)$. The eigenvalues of A are 4, 6 and 12, and their eigenvectors are $(-1,0,1)$, $(1,1,1)$ and $(1,-2,1)$ respectively.

Let us begin calculating the eigenvalues of $\rho(\mathcal{S}, A)$. According to Theorem 4.3 the smallest eigenvalue of $\rho(\mathcal{S}, A)$ is the minimum element of the spectrum of A such that

$$\ker(A - \lambda I) \not\subseteq \mathcal{S}^{\perp}.$$

As it can be checked easily, this value is 4. Now, as it was explained before Theorem 5.10 the second eigenvalue of $\rho(\mathcal{S}, A)$ will be the smallest eigenvalue μ of A such that

$$\mathcal{S} \cap \bigoplus_{\lambda\geq\mu} \ker(A-\lambda) \subsetneq \mathcal{S} \cap \bigoplus_{\lambda\geq 4} \ker(A-\lambda) = \mathcal{S}.$$

This number is 6. So, by a dimension argument, the spectrum of $\rho(\mathcal{S}, A)$ is $\{4, 6\}$.

We shall use part (d) of Theorem 5.10 to calculate the eigenvectors associated to each eigenvalue. An eigenvector for the eigenvalue 6 is any non zero vector in

$$\mathcal{S} \cap \ \mathrm{Span}\{(1,1,1),(1,-2,1)\},$$

for instance, $(0,1,0)$. On the other hand, an eigenvector for the eigenvalue 4 can be found by looking for a vector in $\mathcal{S}$ orthogonal to $(0,1,0)$, for instance $(1,0,0)$. In this way we get

$$\rho(\mathcal{S}, A) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

According to Propositions 5.5, it follows that $\rho(\mathcal{S}, A) \preccurlyeq A$. Therefore, by Theorem 5.2 there must be intermediate matrices D_1 and D_2, such that

a. $\rho(\mathcal{S}, A) \leq D_1 \leq D_2 \leq A$ and

b. $\rho(\mathcal{S}, A)\, D_1 = D_1\, \rho(\mathcal{S}, A)$, $D_1 D_2 = D_2 D_1$ and $D_2 A = A D_2$.

Following the algorithm suggested by the induction used to prove $(c \Rightarrow d)$ in Proposition 5.2 we get

$$D_1 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \text{and} \qquad D_2 = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 6 & 0 \\ 1 & 0 & 5 \end{pmatrix}.$$

▲

Now we are going to exhibit some examples which show that some hypothesis can not be relaxed. For example, let us begin with Proposition 3.8 where we have proved that given a subspace $\mathcal{S}$ of $\mathbb{C}^n$ and A, B in $M_n(\mathbb{C})^+$ such that $A \preccurlyeq B$, then $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}, B)$. This Proposition may fail if we put $A \leq B$ instead of $A \preccurlyeq B$ as the following example shows:

Example 7.2. Let us consider the following matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

and the one dimensional subspace $\mathcal{S}$ generated by the vector $(1, 0)$. Clearly, $A \leq B$; on the other hand, $\rho(\mathcal{S}, A) = P_{\mathcal{S}}$ and $\rho(\mathcal{S}, B) = \dfrac{3 - \sqrt{5}}{2}\, P_{\mathcal{S}} < P_{\mathcal{S}}$ by Corollary 4.4. ▲

In the statement of Proposition 3.10, the hypothesis of being non-increasing respect to the spectral order seems very strong. Nevertheless, the result may fail if the sequence is only non-increasing respect to the usual order, as the following example shows:

Example 7.3. Consider the following sequence of matrices:

$$A_m = \begin{pmatrix} 1 + 1/m & 1/m \\ 1/m & 1/m \end{pmatrix} \in M_2(\mathbb{C}) \ , \quad m \in \mathbb{N}.$$

It is clear that, for every $m \in \mathbb{N}$, $0 \leq A_{m+1} \leq A_m$, and $\lambda_{min}(A_m) \leq \langle A_m e_2, e_2 \rangle = 1/m$. On the other hand, $A_m \xrightarrow[m \to \infty]{} P$, the orthogonal projector onto the subspace generated by e_1.

Let $\mathcal{S} = R(P)$. Then, by Corollary 4.4, $\rho(\mathcal{S}, A_m) = \lambda_{min}(A_m) P \leq \dfrac{1}{m} P$, so that $\rho(\mathcal{S}, A_m) \xrightarrow[m \to \infty]{} 0$, and $\rho(\mathcal{S}, P) = P$. ▲

References

[1] W. N. Anderson, Shorted operators, SIAM J. Appl. Math. 20 (1971), 520-525.

[2] W. N. Anderson and G. E. Trapp, Shorted operators II, SIAM J. Appl. Math. 28 (1975), 60-71.

[3] T. Ando, Generalized Schur complements, Linear Algebra Appl. 27 (1979), 173-186.

[4] R. Bhatia, Matrix Analysis, Berlin-Heildelberg-New York , Springer 1997.

[5] C. A. Butler and T. D. Morley, A note on the shorted operator, SIAM J. Matrix Anal. Appl. 9 (1988), 147-155.

[6] D. Carlson, What are Schur complements, anyway?, Linear Algebra Appl. 74 (1986), 257-275.

[7] R. W. Cottle, Manifestations of the Schur complement, Linear Algebra Appl. 8 (1974), 189-211.

[8] R. G. Douglas, On majorization, factorization and range inclusion of operators in a Hilbert space, Proc. Amer. Math. Soc. 17 (1966), 413-416.

[9] Jun Ichi Fujii and Masatoshi Fujii, Kolmogorov's complexity for positive definite matrices. Special issue dedicated to Professor T. Ando. Linear Algebra Appl. 341 (2002), 171-180.

[10] E. Haynsworth, Determination of the inertia of a partitioned Hermitian matrix, Linear Algebra Appl. 1 (1968), 73-81.

[11] M. G. Krein, The theory of self-adjoint extensions of semibounded Hermitian operators and its applications, Mat. Sb. (N. S.) 20 (62) (1947), 431-495

[12] K. Löwner, Über monotone Matrixfunktionen. Math. Zeit. 38 (1934), 177-216.

[13] Chi-Kwong Li and Roy Mathias, Extremal characterizations of the Schur complement and resulting inequalities. SIAM Rev. 42 (2000), 233-246.

[14] Chi-Kwong Li and Roy Mathias, Some interlacing theorems on the Schur complement. Linear and Multilinear Algebra 44 (1998), no. 4, 373-382.

[15] S. K. Mitra and M. L. Puri, Shorted matrices–An extended concept and some applications, Linear Alg. Appl. 42 (1982), 57-79.

[16] M. P. Olson, The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice, Proc. Amer. Math. Soc., 28 (1971) 537-544.

[17] E. L. Pekarev, Shorts of operators and some extremal problems, Acta Sci. Math. (Szeged) 56 (1992), 147-163.

[18] E. L. Pekarev and J. L. Smul'jan, Parallel addition and parallel substraction of operators, Math. USSR Izvestija 10 (1976), 351-370.

[19] J. L. Smul'jan, A Hellinger operator integral. (Russian) Mat. Sb. (N.S.) 49 (91) 1959 381-430. English transl. AMS Transl. 22 (1962), 289-337.