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PRINCIPAL EIGENVALUES FOR PERIODIC PARABOLIC STEKLOV PROBLEMS WITH L^∞ WEIGHT FUNCTION

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Abstract. In this paper we give sufficient conditions for the existence of a positive principal eigenvalue for a periodic parabolic Steklov problem with a measurable and essentially bounded weight function. For this principal eigenvalue its uniqueness, simplicity and monotone dependence on the weight are stated. A related maximum principle with weight is also given

1. Introduction

Let Ω be a $C^{2+\theta}$ and bounded domain in $\mathbb{R}^N$ with $N \geq 2$ and $\theta \in (0,1)$, let $T > 0$ and let $\{a_{ij}\}_{1\leq i,j\leq N}$, $\{b_j\}_{1\leq,j\leq N}$ be two families of real functions defined on $\overline{\Omega} \times \mathbb{R}$ and $\Omega \times \mathbb{R}$ respectively, satisfying for $1 \leq i,j \leq N$ that $a_{ij} = a_{ij}(x,t)$ and $b_j = b_j(x,t)$ are T periodic in t, $a_{ij} = a_{ji}$, $\frac{\partial a_{ij}}{\partial x_i}{}_{|[0,T]} \in C\left(\overline{\Omega} \times \mathbb{R}\right)$ and $b_j \in L^\infty(\Omega \times \mathbb{R})$. Let $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative and T periodic function belonging to $L^s(\Omega \times \mathbb{R})$ for some $s > 1 + \frac{N}{2}$. Assume in addition that for some $\gamma \in \left(\frac{1}{2}, 1\right)$ and for all i,j

$$a_{ij} \in C^\gamma\left(\mathbb{R}, C\left(\overline{\Omega}\right)\right), \qquad b_j \in C^\gamma\left(\mathbb{R}, L^\infty(\Omega)\right) \tag{1}$$

and that

$$a_0 \in C^\gamma\left(\mathbb{R}, L^s(\Omega)\right) \tag{2}$$

where $a_{ij}(t)(x) := a_{ij}(x,t)$, $b_j(t)(x) := b_j(x,t)$ and $a_0(t)(x) := a_0(x,t)$. Let $b = (b_1, ..., b_N)$ and let A be the $N \times N$ matrix whose i,j entry is a_{ij}. Assume also that A is uniformly elliptic on $\overline{\Omega} \times [0,T]$, i.e., that there exists a positive constant α such that

$$\sum_{i,j} a_{ij}(x,t)\,\xi_i\xi_j \geq \alpha\,|\xi|^2 \tag{3}$$

for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$, $\xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N$. Let L be the periodic parabolic operator defined by

$$Lu := u_t - div\,(A\nabla u) + \langle b, \nabla u\rangle + a_0 u \tag{4}$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^N$. Finally, let b_0 be a nonnegative and T periodic function in $L^\infty(\partial\Omega \times \mathbb{R})$ and let ν be the unit exterior normal to $\partial\Omega$. Under the above hypothesis and notations (that we assume from now on)

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we consider, for a T periodic function (that may changes sign) $m \in L^{\infty}(\partial\Omega \times \mathbb{R})$, the periodic parabolic Steklov principal eigenvalue problem with weight function m

$$\begin{aligned} Lu &= 0 \text{ in } \Omega \times \mathbb{R} \\ \langle A\nabla u, \nu\rangle + b_0 u &= \lambda m u \text{ on } \partial\Omega \times \mathbb{R}, \\ u(x,t) &\ T \text{ periodic in } t \\ u &> 0 \text{ in } \Omega \times \mathbb{R}, \end{aligned} \tag{5}$$

the solutions understood in the sense of the definition 2.1 below. In order to describe our results let us introduce, for $m \in L^{\infty}(\partial\Omega \times \mathbb{R})$, the quantities

$$P(m) := \int_0^T ess \sup_{x\in\partial\Omega} m(x,t)\, dt, \qquad N(m) := \int_0^T ess \inf_{x\in\partial\Omega} m(x,t)\, dt \tag{6}$$

In this paper we prove (cf. Theorem 6.1) that if either $a_0 > 0$ and $b_0 \geq 0$ or $a_0 = 0$ and $b_0 > 0$ and if $P(m) > 0$ (respectively $N(m) < 0$) then there exists a positive (resp. negative) principal eigenvalue for (5), that is, a λ whose associated eigenfunction u satisfies (5). Under an additional assumption on m a similar existence result is also given for the case $a_0 = 0$, $b_0 = 0$.

Our approach, adapted from [4] and [8], reads as follows: If we change $\lambda m u$ in (5) by $\lambda m u + \mu u$, we have the following one parameter eigenvalue problem: given $\lambda \in \mathbb{R}$ find $\mu \in \mathbb{R}$ such that this modified (5) has a solution. We prove in section 4 that this problem has a unique solution $\mu = \mu_m(\lambda) \in \mathbb{R}$ which satisfies that $\lambda \to \mu_m(\lambda)$ is real analytic and concave. We also obtain an expression for $\mu_m'(0)$ which allows us to decide the sign of $\mu_m'(0)$. In section 5 we prove that $P(m) > 0$ (respectively $N(m) < 0$) implie $\lim_{\lambda\to\infty} \mu_m(\lambda) = -\infty$ (resp. $\lim_{\lambda\to-\infty} \mu_m(\lambda) = -\infty$). From these facts, and since the zeroes of the function μ_m are exactly the principal eigenvalues for (5), our results will follow.

Sections 2 and 3 have a preliminar character. In section 2 we collect some general facts about initial value parabolic problems and in section 3 we study existence and uniqueness of periodic solutions for parabolic problems and we prove some compactness and positivity properties of the corresponding solutions operators related.

2. Preliminaries

Let us start introducing the notations to be used along the paper. For a topological vector space E we put E^* for its topological dual and $\langle , \rangle_{E^*,E}$ for the corresponding evaluation bilinear map $\langle \Lambda, e\rangle_{E^*,E} = \Lambda(e)$. If E_1, E_2 are normed spaces and if $S : E_1 \to E_2$ is a bounded linear map we denote by $\|S\|_{E_1,E_2}$ (or simply by $\|S\|$ if no confusion arises) its corresponding operator norm. If E is a real Banach, $-\infty \leq t_0 < t_1 \leq \infty$ and $1 \leq p < \infty$ we put $L^p(t_0, t_1; E)$ for the space of the measurable functions (in the Bochner sense) $f : (t_0, t_1) \to E$ such that $\|f\|_{L^p(t_0,t_1;E)} := \left(\int_{t_0}^{t_1} \|f(t)\|_E^p\, dt\right)^{\frac{1}{p}} < \infty$. We define also $L^{\infty}(t_0, t_1; E)$ and, for $1 \leq p \leq \infty$, the space $L^p_{loc}(t_0, t_1; E)$, similarly (with the obvious changes) to the corresponding usual Lebesgue's spaces. For $1 \leq p \leq \infty$ we put $L^p_T(\mathbb{R}, E)$ for the space of the T periodic functions $f : \mathbb{R} \to E$ satisfying that $f_{|(0,T)} \in L^p(0,T;E)$. We write also $C_T(\overline{\Omega} \times \mathbb{R})$ (respectively $C_T(\partial\Omega \times \mathbb{R})$) for the space of the T periodic functions belonging to $C(\overline{\Omega} \times \mathbb{R})$ (resp. to $C_T(\partial\Omega \times \mathbb{R})$). The spaces $L^p(t_0,t_1;E)$, $L^p_T(\mathbb{R},E)$, $C_T(\overline{\Omega} \times \mathbb{R})$ and $C_T(\partial\Omega \times \mathbb{R})$, equipped with their respective norms $\|\|_{L^p(t_0,t_1;E)}$, $\|\|_{L^p(0,T;E)}$, $\|\|_{C(\overline{\Omega})\times[0,T]}$ and $\|\|_{C(\partial\Omega)\times[0,T]}$ are Banach spaces. For $t_0 < t_1$ we will identify (writing $f(x,t) = f(t)(x)$) the spaces

$$L^2(\Omega \times (t_0,t_1)) = L^2\left(t_0,t_1;L^2(\Omega)\right),$$
$$L^2_T(\Omega \times \mathbb{R}) = L^2\left(0,T;L^2(\Omega)\right)$$

and also the corresponding spaces of functions defined on $\partial\Omega \times (t_0,t_1)$

Let X, V be the real Hilbert spaces $X = L^2(\Omega)$, $V = H^1(\Omega)$ equipped with their usual norms. For $t_0 < t_1$ let $D = C_c^\infty(t_0,t_1;V)$ be the space of the indefinitely differentiable Frechet functions from (t_0,t_1) into V, D equipped with the topology of the uniform convergence on each compact subset of (t_0,t_1) of the function and all its derivatives. Let D' be its dual space. For $u \in L^1_{loc}(t_0,t_1;V)$, let u' be its distributional derivative defined by $\langle u',\varphi\rangle_{D',D} = -\int_{t_0}^{t_1} \langle u(t),\varphi_t(t)\rangle_X \, dt$ for all $\varphi \in D$ where $\langle,\rangle_X$ denotes the inner product in X. We will say that $u' \in L^2(t_0,t_1;V^*)$ if there exists a function (denoted by $t \to u'(t)$) belonging to $L^2(t_0,t_1;V^*)$ such that $\langle u',\varphi\rangle_{D',D} = \int_{t_0}^{t_1} \langle u'(t),\varphi(t)\rangle_{V^*,V} \, dt$ for all $\varphi \in D$.

For $t \in \mathbb{R}$, let $a_{L,b_0}(t,.,.) : V \times V \to \mathbb{R}$ be the bilinear form defined by

$$a_{L,b_0}(t,g,h) = \tag{7}$$
$$\int_\Omega \left[\langle A(.,t)\nabla g, \nabla h\rangle + \langle b(.,t),\nabla g\rangle h + a_0(.,t)gh\right] + \int_{\partial\Omega} b_0(.,t)gh$$

(the values on $\partial\Omega$ of g and h understood in the trace sense) and let $\mathcal{A}_{L,b_0}(t) : V \to V^*$ be the bounded linear operator defined by

$$\mathcal{A}_{L,b_0}(t)g = a_{L,b_0}(t,g,.) \tag{8}$$

For $t_0 < t_1$, $f \in L^2(\Omega \times (t_0,t_1))$, $\Phi \in L^2(\partial\Omega \times (t_0,t_1))$ and $t \in (t_0,t_1)$, let $\Lambda_{f,\Phi}(t) \in V^*$ be defined by

$$\langle \Lambda_{f,\Phi}(t), h\rangle_{V^*,V} = \int_\Omega f(.,t)h + \int_{\partial\Omega} \Phi(.,t)h, \qquad h \in V. \tag{9}$$

So $\Lambda_{f,\Phi} \in L^2(t_0,t_1;V^*)$ and

$$\|\Lambda_{f,\Phi}\|_{L^2(t_0,t_1;V^*)} \le c\left(\|f\|_{L^2(\Omega\times(t_0,t_1))} + \|\Phi\|_{L^2(,\partial\Omega\times(t_0,t_1))}\right) \tag{10}$$

for some positive constant depending only on t_0, t_1, Ω and N. We set also

$$W_{t_0,t_1} := \left\{u \in L^2(t_0,t_1;V) : u' \in L^2(t_0,t_1;V^*)\right\} \tag{11}$$

and $\|u\|_{W_{t_0,t_1}} := \|u\|_{L^2(t_0,t_1;V)} + \|u'\|_{L^2(t_0,t_1;V^*)}$. So W_{t_0,t_1}, equipped with the norm $\|.\|_{W_{t_0,t_1}}$, is a Banach space. With these notations we can formulate the following definition

Definition 2.1. For $-t_0 < t_1$, $f \in L^2(\Omega \times (t_0, t_1))$ and $\Phi \in L^2(\partial\Omega \times (t_0, t_1))$ we say that $u : \Omega \times (t_0, t_1) \to \mathbb{R}$ is a solution of the problem

$$\begin{aligned} Lu &= f \text{ in } \Omega \times (t_0, t_1) \\ \langle A\nabla u, \nu \rangle + b_0 u &= \Phi \text{ on } \partial\Omega \times (t_0, t_1) \end{aligned} \tag{12}$$

if $u \in W_{t_0,t_1}$ and $u'(t) + \mathcal{A}_{L,b_0}(t)\, u(t) = \Lambda_{f,\Phi}(t)$ a.e. $t \in (t_0, t_1)$.

¿From now on, a solution of a boundary problem like (12) (except if otherwise is explicitely stated) will mean a solution in the above sense.

Remark 2.2. For k, l, $t \in \mathbb{R}$ with $k > 0$, standard computations on the quadratic form $g \to a_{L+k,l}(t, g, g)$ give, for all $g \in V$,

$$a_{L+k,l}(t, g, g) \geq \left(k - \frac{\||b|\|^2_{L^\infty(\Omega \times \mathbb{R})}}{4\alpha} \right) \|g\|^2_X + l \int_{\partial\Omega} g^2$$

and also

$$a_{L+k,l}(t, g, g) \geq \left(\alpha - \frac{\||b|\|^2_{L^\infty(\Omega \times \mathbb{R})}}{4k} \right) \|\nabla g\|^2_X + l \int_{\partial\Omega} g^2$$

where α is the ellipticity constant of A. So, for $k > k_0 := \frac{\||b|\|^2_{L^\infty(\Omega \times \mathbb{R})}}{4\alpha}$ and $l \geq 0$, there exists a positive constant β depending only on α and $\||b|\|_{L^\infty(\Omega \times \mathbb{R})}$ such that

$$a_{L+k,l}(t, g, g) \geq \beta \|g\|^2_V \tag{13}$$

for all $t \in \mathbb{R}$ and $g \in V$. Moreover, for such k and l, the assumptions on the coefficients of L imply that there exists a positive constant c such that

$$a_{L+k,l}(t, g, h) \leq c \|g\|_V \|h\|_V \tag{14}$$

and that

$$|a_{L+k,l}(t, g, h) - a_{L+k,l}(s, g, h)| \leq c\, |t - s|^\gamma \|g\|_V \|h\|_V \tag{15}$$

for all $s, t \in \mathbb{R}$ and $g, h \in V$.■

For k_0 as in Remark 2.2, $k \geq k_0$, $-\infty < \tau < t < \infty$ and $u_0 \in X$ consider the problem

$$\begin{aligned} u &\in W_{\tau,t}, \\ u'(s) + \mathcal{A}_{L+k,l}(s)\, u(s) &= 0 \text{ a.e. } s \in (\tau, t) \\ u(\tau) &= u_0. \end{aligned} \tag{16}$$

Note that $W_{\tau,t} \subset C([\tau, t], X)$ (cf. ([12], Lemma 5.5.1) and so the initial condition $u(\tau) = u_0$ makes sense. Taking into account the facts in Remark 2.2, ([12], Theorem 5.5.1) applies to see that (16) has a unique solution u. Let $U_{L+k,l}(t, \tau) : X \to X$ be the linear operator defined by $U_{L+k,l}(t, \tau)\, u_0 = u(t)$.

Let us recall the following properties (cf. [12], Theorem 5.4.1) of the evolution operators $U_{L+k,l}(t, \tau)$

Remark 2.3. i) Given $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$ there exists a positive constant c such that, for $t_0 < \tau < t \leq t_1$,

$$\|U_{L+k,l}(t, \tau)\|_{X,V} \leq c\,(t - \tau)^{-\frac{1}{2}}. \tag{17}$$

ii) Since $V \subset X \simeq X^* \subset V^*$ (the isomorphism $X \simeq X^*$ given by duality) we can consider $X \subset V^*$. In this setting, it holds that for t_0, t_1 as above there exists a positive constant c' such that

$$\left\| U_{L+k,l}(t,\tau) u_0 \right\|_X \leq c' (t-\tau)^{-\frac{1}{2}} \left\| u_0 \right\|_{V^*} \tag{18}$$

for $t_0 < \tau < t \leq t_1$ and $u_0 \in X$. Since V (and then also X) is dense in V^*, it follows that $U_{L+k,l}(t,\tau) : X \to V$ has a unique bounded extension to an operator (still denoted $U_{L+k,l}(t,\tau)$) from V^* into X which satisfies, for c' as in (18),

$$\left\| U_{L+k,l}(t,\tau) \right\|_{V^*,X} \leq c' (t-\tau)^{-\frac{1}{2}}. \tag{19}$$

Finally, we recall also that for $\tau \leq s \leq t$ it holds that

$$U_{L+k,l}(t,\tau) = U_{L+k,l}(t,s)\, U_{L+k,l}(s,\tau). \tag{20}$$

For $-\infty < t_0 < t_1 < \infty$, $\Lambda \in L^2(t_0, t_1; V^*)$ and $u_0 \in X$ consider the problem

$$\begin{gathered} v_k \in W_{t_0.t_1}, \\ v_k'(t) + \mathcal{A}_{L+k,l}(t)\, v_k(t) = \Lambda(t) \text{ a.e. } t \in (t_0, t_1) \\ v_k(t_0) = u_0. \end{gathered} \tag{21}$$

Taking into account (13), (14) and (15), ([12], Theorem 5.5.1) applies to see that (21) has a unique solution v_k given by

$$v_k(t) = U_{L+k,l}(t,t_0)\, u_0 + \int_{t_0}^{t} U_{L+k,l}(t,\tau)\, \Lambda(\tau)\, d\tau. \blacksquare \tag{22}$$

Remark 2.4. Observe that $u \in W_{t_0,t_1}$ is a solution of the problem

$$\begin{gathered} u(t) + \mathcal{A}_{L,l}(t)\, u(t) = \Lambda(t) \text{ a.e. } t \in (t_0, t_1) \\ u(t_0) = u_0 \end{gathered} \tag{23}$$

if and only if $v_k(t) := e^{-k(t-t_0)} u(t)$ solves

$$\begin{gathered} v_k'(t) + \mathcal{A}_{L+k,l}(t)\, v_k(t) = \Lambda_k \text{ a.e. } t \in (t_0, t_1) \\ v_k(t_0) = u_0 \end{gathered} \tag{24}$$

with Λ_k defined by $\Lambda_k(t) := e^{-k(t-t_0)} \Lambda(t)$. Thus (23) has a unique solution u given by

$$u(t) = U_{L,l}(t,t_0)\, u_0 + \int_{t_0}^{t} U_{L,l}(t,\tau)\, \Lambda(\tau)\, d\tau \tag{25}$$

with $U_{L,l}(t,\tau)$ defined by

$$U_{L,l}(t,\tau) := e^{k(t-\tau)} U_{L+k,l}(t,\tau). \tag{26}$$

Moreover, for $t \in [t_0, t_1]$ we have (cf. [12], Lemma 5.5.2)

$$\begin{aligned} &\frac{1}{2} \left\| u(t) \right\|_X^2 + \int_{t_0}^{t} a_{L,l}(\tau, u(\tau), u(\tau))\, d\tau \\ &= \frac{1}{2} \left\| u_0 \right\|_X^2 + \int_{t_0}^{t} \left\langle \Lambda(\tau), u(\tau) \right\rangle_{V^*,V} d\tau. \end{aligned} \tag{27}$$

¿From (27), standard computations show that there exists a positive constant c independent of Λ and u_0 such that

$$\|u\|_{W_{t_0,t_1}} \leq c\left(\|\Lambda\|_{L^2(t_0,t_1,V^*)} + \|u_0\|_{L^2(\Omega)}\right). \blacksquare \tag{28}$$

Remark 2.5. The estimates (17), (18), (19) and (20) still hold (with another constants) for the operators $U_{L,l}(t,\tau)$ given by (26) and $u(t) := U_{L.l}(t,\tau)\,u_0$ satisfies

$$\begin{aligned} Lu &= \text{ in } \Omega \times (t_0,t_1), \\ \langle A\nabla u, \nu\rangle + lu &= 0 \text{ on } \partial\Omega \times (t_0,t_1) \\ u(t_0) &= u_0 \end{aligned} \tag{29}$$

for $u_0 \in L^2(\Omega)$. $\blacksquare$

Remark 2.6. For $l \geq 0$, $-\infty < t_0 < t_1 < \infty$, $f \in L^2(\Omega \times (t_0,t_1))$, $\Phi \in L^2(\partial\Omega \times (t_0,t_1))$ and $u_0 \in L^2(\Omega)$ the problem

$$\begin{aligned} Lu &= f \text{ in } \Omega \times (t_0,t_1), \\ \langle A\nabla u, \nu\rangle + lu &= \Phi \text{ on } \partial\Omega \times (t_0,t_1), \\ u(.,t_0) &= u_0 \end{aligned} \tag{30}$$

has a unique solution which satisfies in addition that

$$\|u\|_{W_{t_0,t_1}} \leq c\left(\|f\|_{L^2(\Omega\times(t_0,t_1))} + \|\Phi\|_{L^2(\partial\Omega\times(t_0,t_1))} + \|u_0\|_{L^2(\Omega)}\right). \tag{31}$$

for some positive constant c independent of f, Φ and u_0. Indeed, the solutions of (30) are those of (23) taking there $\Lambda = \Lambda_{f,\Phi}$, and Remark 2.4 applies. $\blacksquare$

Remark 2.7. It is easy to check that the constant c in (28) and so also in Remark 2.5 and Remark 2.6 can be chosen depending only on Ω, N, γ, α and on an upper bound of $\Sigma_{i,j}\|a_{ij}\|_{L^\infty(\Omega\times(t_0,t_1))} + \Sigma_j\|b_j\|_{L^\infty(\Omega\times(t_0,t_1))} + \|a_0\|_{L^s(\Omega\times(t_0,t_1))}$. $\blacksquare$

Lemma 2.8. *Let t_0, t_1, f, Φ and u_0 be as in Lemma 2.4 and let $\left\{L^{(n)}\right\}$ be a sequence of operators of the form*

$$L^{(n)}u == u_t - div\left(A^{(n)}\nabla u\right) + \left\langle b^{(n)}, \nabla u\right\rangle + a_0^{(n)}u$$

with $A^{(n)} = \left(a_{ij}^{(n)}\right)$, $b^{(n)} = \left(b_1^{(n)}, ..., b_N^{(n)}\right)$ and $a_0^{(n)}$ satisfying for each n the conditions stated for L at the introduction with the same γ, α and s given there for L. Assume also that for each i and j, $\left\{a_{ij}^{(n)}\right\}$ and $\left\{b_j^{(n)}\right\}$ converge uniformly on $\overline{\Omega} \times (t_0,t_1)$ to a_{ij} and b_j respectively and that $\left\{a_0^{(n)}\right\}$ converges to a_0 in $L^s(\Omega \times (t_0,t_1))$. Let $\left\{f^{(n)}\right\}$ and $\left\{\Phi^{(n)}\right\}$ be sequences in $L^2(\Omega \times (t_0,t_1))$ and in $L^2(\partial\Omega \times (t_0,t_1))$ respectively and assume that they converge to f and Φ in their respective spaces. Let $\left\{u_0^{(n)}\right\}$ be a sequence in $L^2(\Omega)$ that converges to u_0 in

$L^2(\Omega)$ and let $l \geq 0$. Thus the solution $u^{(n)} \in W_{t_0,t_1}$ of the problem

$$L^{(n)}u^{(n)} = f^{(n)} \text{ in } \Omega \times (t_0, t_1),$$
$$\left\langle A\nabla u^{(n)}, \nu \right\rangle + lu^{(n)} = \Phi^{(n)} \text{ on } \partial\Omega \times (t_0, t_1),$$
$$u^{(n)}(., t_0) = u_0^{(n)}.$$

converges in the W_{t_0,t_1} norm to the solution u of (30).

Proof. For k_0 as in Remark 2.2, $k \geq k_0$, $l \geq 0$ and $n \in \mathbb{N}$, let $v_k^{(n)} \in W_{t_0,t_1}$ be the solution of

$$\left(v_k^{(n)}\right)'(t) + \mathcal{A}_{L^{(n)}+k,l}(t)\, v_k^{(n)}(t) = \Lambda_{f_k^{(n)},\Phi_k^{(n)}}(t) \text{ a.e.} t \in (t_0, t_1),$$
$$v_k^{(n)}(t_0) = u_0^{(n)}$$

and let v_k be the solution of (24). We have

$$\left(v_k^{(n)} - v_k\right)'(t) + \mathcal{A}_{L+k,l}(t)\left(v_k^{(n)} - v_k\right)(t) = \widetilde{\Lambda}^{(n)}(t) \text{ a.e.} t \in (t_0, t_1), \tag{32}$$
$$\left(v_k^{(n)} - v_k\right)(t_0) = u_0^{(n)} - u_0$$

where

$$\widetilde{\Lambda}^{(n)}(t) \tag{33}$$
$$:= \Lambda_{f_k^{(n)},\Phi_k^{(n)}}(t) - \Lambda_{f_k,\Phi_k}(t) + \left(\mathcal{A}_{L+k,l}(t) - \mathcal{A}_{L^{(n)}+k,l}(t)\right) v_k^{(n)}(t).$$

Our assumptions imply that $\lim_{n\to\infty} \left\| \Lambda_{f_k^{(n)},\Phi_k^{(n)}} - \Lambda_{f_k,\Phi_k} \right\|_{L^2(t_0,t_1;V^*)} = 0$ and that $\lim_{n\to\infty} \left\| \mathcal{A}_{L+k,l}(t) - \mathcal{A}_{L^{(n)}+k,l}(t) \right\|_{V.V^*} = 0$ uniformly on $t \in [t_0, t_1]$. From Remarks 2.6 and 2.7 we have that $\left\{ \left\| v_k^{(n)} \right\|_{L^2(t_0,t_1;V)} \right\}$ is a bounded sequence. Then from (33) $\lim_{n\to\infty} \left\| \widetilde{\Lambda}^{(n)} \right\|_{L^2(t_0,t_1;V^*)} = 0$. Thus from Remark 2.6 applied to (32) we obtain $\lim_{n\to\infty} \left\| v_k^{(n)} - v_k \right\|_{W_{t_0,t_1}} = 0$. Since $u^{(n)}(t) = e^{k(t-t_0)} v_k^{(n)}$ and $u(t) = e^{k(t-t_0)} v_k$ the lemma follows.■

Lemma 2.9. *Assume that $f \in L^2(\Omega \times (t_0, t_1))$, $\Phi \in L^2(\partial\Omega \times (t_0, t_1))$ and $u_0 \in L^2(\Omega)$ are nonnegative. Then the solution u of (30) is nonnegative.*

Proof. We pick sequences $\{L_n\}$, $\left\{f^{(n)}\right\}$, $\left\{\Phi^{(n)}\right\}$ and $\left\{u_0^{(n)}\right\}$ as in Lemma 2.8 satisfying in addition that $f^{(n)} \geq 0$, $\Phi^{(n)} \geq 0$, $u_0^{(n)} \geq 0$ and such that $a_{ij}^{(n)}$, $b_j^{(n)}$, $a_0^{(n)}$ and $f^{(n)}$ belong to $C^\infty\left(\overline{\Omega} \times [t_0, t_1]\right)$, $\Phi^{(n)}$ belongs to $C^\infty(\partial\Omega \times [t_0, t_1])$ and $u_0^{(n)} \in C_c^\infty(\Omega)$. Let $\left\{v_k^{(n)}\right\}$ be as in the proof of Lemma 2.8. Thus $v_k^{(n)} \in C^{2+\sigma, 1+\frac{\sigma}{2}}(\Omega \times (t_0, t_1))$ (cf. e.g., Theorem 5.3 in [9], p. 320)). The classical maximum principle gives $v_k^{(n)} \geq 0$ and since by Lemma 2.8 $\lim_{n\to\infty} v_k^{(n)} = v_k$ in $L^2(\Omega \times (t_0, t_1))$ we get $v_k \geq 0$. Since the solution u of (30) is given by $u(t) = e^{kt} v_k(t)$ the lemma follows.■

Remark 2.10. Let us recall some well known facts concerning Sobolev spaces (see e.g. [9], Lemma 3.3, p 80 Lemma 3.4, p. 82)

i): For $-\infty < t_0 < t_1 < \infty$ and $u \in W_q^{2,1}(\Omega \times (t_0,t_1))$ with $1 \leq q < \infty$ we have $u_{|\partial\Omega\times(t_0,t_1)} \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (t_0,t_1))$ and the restriction map (in the trace sense) $u \to u_{|\partial\Omega\times(t_0,t_1)}$ is continuous from $W_q^{2,1}(\Omega \times (t_0,t_1))$ into $W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times (t_0,t_1))$.

ii) For $u \in W_q^{2,1}(\Omega \times (t_0,t_1))$ with $1 \leq q < \infty$ it holds that $u(.,t) \in W^{2-\frac{2}{q},q}(\Omega)$ for $t \in [t_0,t_1]$ and for such t there exists a positive constant c independent of u such that $\|u(.,t)\|_{W^{2-\frac{1}{q},q}(\Omega)} \leq c\|u\|_{W_q^{2,1}(\Omega\times(t_0,t_1))}$.

iii) For $q > N+2$ the following facts hold:

$W_q^{2,1}(\Omega \times (t_0,t_1)) \subset C^{1+\sigma,\frac{1+\sigma}{2}}\left(\overline{\Omega} \times [t_0,t_1]\right)$ for some $\sigma \in (0,1)$, with continuous inclusion.

$W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (t_0,t_1)) \subset C^{1+\sigma,\frac{1+\sigma}{2}}(\partial\Omega \times [t_0,t_1])$ for some $\sigma \in (0,1)$ and with continuous inclusion.

iv) For $1 \leq r \leq \infty$ let r^* be defined by $(r^*)^{-1} = r^{-1} - (N+1)^{-1}$ if $r < N+1$ and $r^* = \infty$ if $r \geq N+1$. Thus $W_r^{2,1}(\Omega \times (t_0,t_1)) \subset L^{r^*}(\Omega \times (t_0,t_1))$ if $r^* < \infty$ and $W_r^{2,1}(\Omega \times (t_0,t_1)) \subset L^q(\Omega \times (t_0,t_1))$ for all $q \in [1,\infty)$ if $r^* = \infty$, in both cases with continuous inclusion.■

Remark 2.11. For $q > N+2$ it holds that $W^{2-\frac{2}{q},q}(\Omega) \subset C^{1+\sigma}\left(\overline{\Omega}\right)$ continuously for some $\sigma \in (0,1)$. In this case, for $\tau \in \mathbb{R}$, let $W_{B_l(\tau)}^{2-\frac{2}{q},q}(\Omega)$ be the space of the functions $h \in W^{2-\frac{2}{q},q}(\Omega)$ that satisfy (in the pointwise sense) $B_l(\tau)h = 0$ where

$$B_l(\tau)h := \langle A(.,\tau)\nabla h, \nu\rangle + lh. \tag{34}$$

Let us recall that for such q and for $-\infty < t_0 < t_1 < \infty$, $f \in L^q(\Omega \times (t_0,t_1))$, $\Phi \in W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega \times (t_0,t_1))$ and $u_0 \in W_{B_l(t_0)}^{2-\frac{2}{q},q}(\Omega)$ there exists a unique $u \in W_q^{2,1}(\Omega \times (t_0,t_1))$ satisfying almost everywhere

$$\begin{aligned} Lu &= f \text{ in } \Omega \times (t_0,t_1), \\ \langle A\nabla u, \nu\rangle + lu &= \Phi \text{ on } \partial\Omega \times (t_0,t_1), \\ u(t_0) &= u_0. \end{aligned}$$

(for a proof, see [9], Theorem 9.1, p. 341, concerning the Dirichlet problem and its extension, to our boundary conditions, indicated there (at the end of chapter 4, paragraph 9, p. 351). Moreover, there exists a positive constant c independent of f, Φ and u_0 such that

$$\begin{aligned} &\|u\|_{W_q^{2,1}(\Omega\times(t_0,t_1))} \\ &\leq c\left(\|f\|_{L^q(\Omega\times(t_0,t_1))} + \|\Phi\|_{W_q^{2-\frac{1}{q},1-\frac{1}{2q}}(\partial\Omega\times(t_0,t_1))} + \|u_0\|_{W^{2-\frac{2}{q},q}(\Omega)}\right). \blacksquare \end{aligned}$$

Lemma 2.12. *i) For $\tau < t$, $U_{L,l}(t,\tau) : L^2(\Omega) \to L^2(\Omega)$ is a compact and positive operator.*

ii) Let t_0, $t_1 \in \mathbb{R}$ with $t_0 < t_1$. For $1 \leq q < \infty$, $t_0 < \tau \leq t_1$ and $u_0 \in L^2(\Omega)$ the restriction of $U_{L,l}(.,t_0)\, u_0$ to $\Omega \times (\tau, t_1)$ belongs to $W_q^{2,1}(\Omega \times (\tau, t_1))$ and there exists a positive constant c such that $\|U_{L,l}(.,t_0)\, u_0\|_{W_q^{2,1}(\Omega\times(\tau,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$ for all $u_0 \in L^2(\Omega)$.

iii) $U_{L,l}(t,\tau)\left(L^2(\Omega)\right) \subset W^{2-\frac{2}{q},q}(\Omega)$ for $\tau < t$ and $1 \leq q < \infty$ and $U_{L,l}(t,t_0)$ is a bounded operator from $L^2(\Omega)$ into $W^{2-\frac{2}{q},q}(\Omega)$.

iv) For $\tau < t$ it hold that $U_{L,l}(t,\tau)\left(L^2(\Omega)\right) \subset C^1\left(\overline{\Omega}\right)$ and $U_{L,l}(t,\tau)$ is a bounded operator from $L^2(\Omega)$ into $C^1\left(\overline{\Omega}\right)$. Moreover, if $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, and $u_0 \neq 0$ then $\min_{\overline{\Omega}} U_{L,l}(t,\tau)\, u_0 > 0$.

v) For $N+2 < q < \infty$ and $\tau < t$, $U_{L,l}(t,\tau)_{|W_{B_l(\tau)}^{2-\frac{2}{q},q}(\Omega)} : W_{B_l(\tau)}^{2-\frac{2}{q},q}(\Omega) \to W_{B_l(\tau)}^{2-\frac{2}{q},q}(\Omega)$ is a compact and strongly positive operator .

Proof. By Lemma 2.9 $U_{L,l}(t,\tau) : L^2(\Omega) \to L^2(\Omega)$ is a positive operator. It is also compact because $U_{L,l}(t,\tau) : L^2(\Omega) \to H^1(\Omega)$ is continuous (cf. Remark 2.5) and $H^1(\Omega)$ has compact inclusion in $L^2(\Omega)$. Thus (i) holds.

To see (ii) we pick a strictly increasing sequence of positive numbers $\{\eta_j\}_{j\in\mathbb{N}}$ such that $t_0 < t_0 + \eta_j < \tau$ for all $j \in \mathbb{N}$ and we pick also a sequence of functions $\{\varphi_j\}_{j\in\mathbb{N}}$ in $C^\infty(\mathbb{R})$ satisfying $\varphi_j(s) = 0$ for $s \leq t_0 + \eta_j$, $\varphi_j(s) = 1$ for $s \geq t_0 + \eta_{j+1}$. Let $u(t) := U_{L+k,l}(t,t_0)\, u_0$ and let $\{v_j\}_{j\in\mathbb{N}}$ and $\{w_j\}_{j\in\mathbb{N}}$ be the sequences of functions on $\Omega \times (t_{0,}, t_1)$ inductively defined by $v_1 := u\varphi_1$, $v_{j+1} := \varphi_{j+1} v_j$ and by $w_1 := \varphi_1' u$, $w_{j+1} =: \varphi_{j+1}' v_j + \varphi_{j+1} w_j$ respectively. Then, for all j,

$$\begin{aligned} Lv_j &= w_j \text{ in } \Omega \times (t_0 + \eta_j, t_1)\,, \\ \langle A\nabla v_j, \nu\rangle + lv_j &= 0 \text{ on } \partial\Omega \times (t_0 + \eta_j, t_1)\,, \\ v_j(t_0 + \eta_j) &= 0 \end{aligned} \tag{35}$$

Let $\{q_j\}_{j\in\mathbb{N}}$ be defined by $q_1 = 2$ and by $q_{j+1} = q_j^*$ (with q_j^* as in (iv) of Remark 2.10) and let $j_0 = \min\left\{j : q_j^* = \infty\right\}$. For the rest of the proof c will denote a positive constant independent of u_0 non necessarily the same at each occurrence (even in a same chain of inequalities). We claim that for $j \leq j_0$

$$v_j \in W_{q_j}^{2,1}(\Omega \times (t_0 + \eta_{j+1}, t_1)) \text{ and } w_j \in W_{q_j}^{2,1}(\Omega \times (t_0 + \eta_{j+1}, t_1)) \tag{36}$$

with their respective norms bounded by $c\,\|u_0\|_{L^2(\Omega)}$.

If (36) holds, for $1 \leq q < \infty$ Remark 2.10 (iv) gives $\|w_{j_0}\|_{L^q\left(\Omega\times\left(t_0+\eta_{j_0+1},t_1\right)\right)} \leq c\,\|u_0\|_{L^2(\Omega)}$. Taking into account that $u = v_{j_0}$ on $\Omega \times (\tau, t_1)$, Remark 2.11 gives

$$\begin{aligned} \|u\|_{W_q^{2,1}(\Omega\times(\tau,t_1))} = \|v_{j_0}\|_{W_q^{2,1}(\Omega\times(\tau,t_1))} &\leq \|v_{j_0}\|_{W_q^{2,1}\left(\Omega\times\left(t_0+\eta_{j_0+1},t_1\right)\right)} \\ &\leq c\,\|w_{j_0}\|_{L^q\left(\Omega\times\left(t_0+\eta_{j_0+1},t_1\right)\right)} \leq c\,\|u_0\|_{L^2(\Omega)} \end{aligned}$$

and so (ii) holds.

To prove the claim we proceed inductively. Since u satisfies 29, Remark 2.6 gives $\|u\|_{L^2(\Omega\times(t_0+\eta_1,t_1))} \leq \|u\|_{L^2(\Omega\times(t_0,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$ and so $\|w_1\|_{L^2(\Omega\times(t_0+\eta_1,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$. Then, by Remark 2.11, $\|v_1\|_{W_2^{2,1}(\Omega\times(t_0+\eta_1,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$ and so $\|v_1\|_{W_2^{2,1}(\Omega\times(t_0+\eta_2,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$. Since $u = v_1$ on $\Omega \times (t_0 + \eta_2, t_1)$ and $w_1 =$

$u\varphi_1$ we get also that $\|w_1\|_{W_2^{2,1}(\Omega\times(t_0+\eta_2,t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$. Thus (36) holds for $j=1$. Suppose that it holds for some $j<j_0$. Then

$$\begin{aligned}\|v_j\|_{W^{2,1}_{q_{j+1}}(\Omega\times(t_0+\eta_{j+1},t_1))} &\leq c\,\|w_{j+1}\|_{L^{q_j}(\Omega\times(t_0+\eta_{j+1},t_1))}\\ &= c\left\|\varphi'_{j+1}v_j+\varphi_{j+1}w_j\right\|_{L^{q_j}(\Omega\times(t_0+\eta_{j+1},t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}\end{aligned}$$

and so (since $u=v_{j+1}$ on $\Omega\times(t_0+\eta_{j+3},t_1)$)

$$\begin{aligned}\|u\|_{W^{2,1}_{q_{j+1}}(\Omega\times(t_0+\eta_{j+2},t_1))} &= \|v_{j+1}\|_{W^{2,1}_{q_{j+1}}(\Omega\times(t_0+\eta_{j+2},t_1))}\\ &\leq \|v_{j+1}\|_{W^{2,1}_{q_{j+1}}(\Omega\times(t_0+\eta_{j+1},t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}\end{aligned} \tag{37}$$

Since $w_{j+1}=u\sum_{1\leq k\leq j+1}\varphi'_k\prod_{\substack{1\leq r\leq j+1\\ r\neq j+1}}\varphi_r$ it follows that $\|w_{j+1}\|_{W^{2,1}_{q_{j+1}}(\Omega\times(t_0+\eta_{j+2},t_1))} \leq c\,\|u_0\|_{L^2(\Omega)}$ and so, from (35), a similar estimate holds for v_{j+1}. This complete the proof of the claim.

The imbedding theorems for Sobolev spaces and (ii) imply (iii). The first part of (iv) is again obtained applying (ii) with $q>N+2$. To see the second part of (iv), we observe that if $u_0>0$ and $u:=U_{L,l}(t,\tau)\,u_0$ then $u\neq 0$ and, by Lemma 2.9, $u\geq 0$. Let φ_1 and v_1 be as in the proof of (ii), Since $v_1=\varphi_1 u\in W_q^{2,1}(\Omega\times(t_0,t_1))\subset C^{1+\sigma,\frac{1+\sigma}{2}}(\Omega\times[t_0,t_1])$, the boundary condition for v_1 holds in the pointwise sense. Now, the Hopf parabolic maximum principle applied to

$$\begin{gathered}Lv_1=\varphi' u \text{ in } \Omega\times(t_0+\eta_1,t_1),\\ \langle A\nabla v_1,\nu\rangle+lv_1=0 \text{ on } \partial\Omega\times(t_0+\eta_1,t_1)\end{gathered}$$

jointly with the fact that $v_1=u$ on $\Omega\times(\tau,t_1)$ gives (iv).

To see (v), let $s\in(0,\tau)$, $q>N+2$ and let $\widetilde{q}>q$. Since $W^{2-2/q,q}_{B_l(\tau)}(\Omega)\subset L^2(\Omega)$ (with $B_l(\tau)$ given by (34)), from (ii) we can consider the bounded operator $S: W^{2-2/q,q}_{B_l(\tau)}(\Omega)\to W^{2,1}_{\widetilde{q}}(\Omega\times(\tau,t))$ defined by $Su_0=(U_{L,l}(.,s)\,u_0)_{|\Omega\times(\tau,T)}$. Since the operator $u\to u(t)$ is continuous from $W^{2,1}_{\widetilde{q}}(\Omega\times(\tau,t))$ into $W^{2-2/\widetilde{q},\widetilde{q}}(\Omega)$ and the inclusion map $i: W^{2-2/\widetilde{q},\widetilde{q}}(\Omega)\to W^{2-2/q,q}(\Omega)$ is compact, we obtain the compactness assertion of (v). Finally, the strong positivity in (v) follows from (iv).■

Lemma 2.13. *i) If* $\Lambda\in H^1(\Omega)^*$ *and* $\Lambda\geq 0$ *then* $U_{L,l}(t,\tau)\,\Lambda\geq 0$ *for* $\tau<t$.

ii) If $f\in L^2(\Omega\times(t_0,t_1))$ *and* $\Phi\in L^2(\partial\Omega\times(t_0,t_1))$ *are nonnegative functions and if either* $f\neq 0$ *or* $\Phi\neq 0$ *then*

$$\int_{t_0}^{t_1} U_{L,l}(t_1,\tau)\,\Lambda_{f,\Phi}(\tau)\,d\tau>0$$

Proof. Let $P_{L^2(\Omega)}$, $P_{H^1(\Omega)}$, $P_{H^1(\Omega)^*}$ be the positive cones in $L^2(\Omega)$, $H^1(\Omega)$ and $H^1(\Omega)^*$ respectively and let $\overline{P}_{H^1(\Omega)}$ be the closure of $P_{H^1(\Omega)}$ in $H^1(\Omega)^*$. Observe that if $\Lambda\in P_{H^1(\Omega)^*}\cup\{0\}$ then $\Lambda\in\overline{P}_{H^1(\Omega)}$. Indeed, if not, the Hann Banach Theorem gives $\eta\in H^1(\Omega)^{**}$ such that $\eta_{|\overline{P}_{H^1(\Omega)}}=0$ and $\eta(\Lambda)=1$. For $g\in H^1(\Omega)$ let $\lambda_g\in H^1(\Omega)^*$ be defined by $\lambda_g(f)=\int_\Omega fg$. Thus $\lambda_g\in P_{H^1(\Omega)^*}$ for all $g\in P_{H^1(\Omega)}$. Since $H^1(\Omega)$ is reflexive there exists $\varphi\in H^1(\Omega)$ such that

$\eta(\lambda) = \lambda(\varphi)$ for all $\lambda \in H^1(\Omega)^*$. In particular we have $0 = \eta(\lambda_g) = \int_\Omega fg$ for all $g \in P_{H^1(\Omega)}$. This implies that $\varphi = 0$ and so $\eta = 0$ which contradicts $\eta(\Lambda) = 1$. Thus $\Lambda \in \overline{P}_{H^1(\Omega)}$.

Let $\Lambda \in P_{H^1(\Omega)^*}$, so $\Lambda \in \overline{P}_{H^1(\Omega)}$ and then there exists a sequence $\{u_{0,j}\}_{j\in N}$ of nonnegative functions in $H^1(\Omega)$ that converges to Λ in $H^1(\Omega)^*$. Since $U_{L,l}(t,\tau) : H^1(\Omega)^* \to L^2(\Omega)$ is continuous and, by Lemma 2.12 (i), it is a positive operator on $L^2(\Omega)$, we have $U_{L,l}(t,\tau)\Lambda = \lim_{j\to\infty} U_{L,l}(t,\tau) u_{0,j} \geq 0$ and so (i) holds.

To see (ii), observe that $\Lambda_{f,\Phi} \geq 0$ and so (i) gives

$$U_{L,l}(t,\tau)\Lambda_{f,\Phi}(\tau) \geq 0 \text{ a.e. } \tau \in (t_0,t_1). \tag{38}$$

Moreover,

$$u(t) := \int_{t_0}^{t} U_{L,l}(t,\tau)\Lambda_{f,\Phi}(\tau)\, d\tau \tag{39}$$

is the solution of the problem

$$\begin{gathered} Lu = f \text{ in } \Omega \times (t_0,t_1), \\ \langle A\nabla u, \nu\rangle + lu = \Phi \text{ on } \partial\Omega \times (t_0,t_1), \\ u(0) = 0. \end{gathered}$$

Then, by (i), $u \geq 0$ in $\Omega \times (t_0,t_1)$ and since $u \neq 0$ (because either $f \neq 0$ or $\Phi \neq 0$) we conclude that for some $\overline{t} \in (t_0,t_1)$ the set

$$J_{\overline{t}} = \left\{\tau \in \left(0,\overline{t}\right) : U_{L,l}\left(\overline{t},\tau\right)\Lambda_{f,\Phi}(\tau) \in P_{L^2(\Omega)}\right\}$$

has positive measure. Then, since $U_{L,l}(T,\tau) = U_{L,l}\left(T,\overline{t}\right)U_{L,l}\left(\overline{t},\tau\right)$, Lemma 2.12 (iv) gives $U_{L,l}(T,\tau)\Lambda_{f,\Phi}(\tau) > 0$ for all $\tau \in J_{\overline{t}}$. Now (ii) follows from (38) and (39).■

Remark 2.14. Let us recall the following version of the Krein Rutman Theorem for Banach lattices and one of its corollaries (for a proof, see e.g., [5], Theorem 12.3 and Corollary 12.4)

i) Let E be a Banach lattice with cone positive P and let $S : E \to E$ be a bounded, compact, positive and irreducible linear operator. Then S has a positive spectral radius $\rho(S)$ which is an algebraically simple eigenvalue of S and S^*. The associated eigenspaces are spanned by a quasi interior eigenvector and a strictly positive eigenfunctional respectively. Moreover, $\rho(S)$ is the only eigenvalue of T having a positive eigenvector.

ii) For E and S as above and for a positive $v \in E$ the equation $ru - Su = v$ has a unique positive solution if $r > \rho(S)$, no positive solution if $r < \rho(S)$ and no solution at all if $r = \rho(S)$. In particular this implies that if $Sv \geq \rho(S)v$ for some positive v then $Sv = \rho(Sv)$.

We recall also that a point $a \in E$ is a quasi interior point if and only if $a \in P$ and the order interval $[0,a]$ is total (i.e. the linear span of $[0,a]$ is dense in E) and that for a measure space Z equipped with a positive measure $d\sigma$ on Z and $1 \leq p < \infty$ the quasi interior points in $L^p(Z,d\sigma)$ are the functions that are strictly positive almost everywhere. Moreover, for such p, a bounded and positive linear operator $S : L^p(Z,d\sigma) \to L^p(Z,d\sigma)$ satisfying that $S(f)(x) > 0$ *a.e.* $x \in Z$ for all $f > 0$ is an irreducible operator (cf [13], Proposition 3, p. 409).■

Lemma 2.15. *For $l > 0$ and $\tau < t$, $U_{L,l}(t,\tau) : L^2(\Omega) \to L^2(\Omega)$ is a positive irreducible operator and its spectral radius ρ satisfies $0 < \rho < 1$.*

Proof. By (i) and (iv) of Lemma 2.12, $U_{L,l}(t,\tau)$ is a positive, irreducible and compact operator. Thus, by the Krein Rutman Theorem, ρ is positive and that is the unique eigenvalue with positive eigenfunctions associated. Moreover, by Lemma 2.10 (iii), these eigenfunctions belong to $W^{2-\frac{2}{q},q}(\Omega)$ for $1 \leq q < \infty$. Take $q > N+2$. By Lemma 2.12 (v), $U_{L,l}(t,\tau) : W^{2-\frac{2}{q},q}_{B_l(\tau)}(\Omega) \to W^{2-\frac{2}{q},q}_{B_l(\tau)}(\Omega)$ is a compact and strongly positive operator which, by the Krein Rutman Theorem, has a positive spectral radius ρ_q. Since the eigenfunctions of $U_{L,l}(t,\tau)$ belong to $W^{2-\frac{2}{q},q}_{B_l(\tau)}(\Omega)$ we have $\rho = \rho_q$. Thus, to prove the lemma, it is enough to see that $\rho_q < 1$.

We proceed by contradiction. Suppose $\rho_q \geq 1$, let φ be a positive eigenfunction with eigenvalue ρ_q and let $w = U_{L,l}(.,\tau)(\varphi)$. Since $U_{L,l}(t,\tau)(\varphi) = \rho\varphi \geq \varphi$, . By Lemma 2.12 (ii), $w \in W^{2,1}_q(\Omega \times (\tau,t))$ and since $w(t) \geq w(\tau)$ the maximum principle gives that either w is a constant or $\max_{\overline{\Omega}\times[\delta,T]} w(x,t)$ is achieved at some point $(x^*,t^*) \in \partial\Omega \times (\tau,t)$. If w is a constant, since $l > 0$ the boundary condition (which is satisfied in the pointwise sense because $q > N+2$) implies $w = 0$ which is impossible and if the maximum is achieved at some point $(x^*,t^*) \in \partial\Omega \times (\tau,t)$ we would have $\langle A\nabla w, \nu\rangle (x^*,t^*) > 0$ in contradiction with the boundary condition.■

3. Periodic solutions

Let W be the Banach space

$$W := \left\{u \in L^2_T\left(\mathbb{R}, H^1(\Omega)\right) : u' \in L^2_T\left(\mathbb{R}, H^1(\Omega)^*\right)\right\} \tag{40}$$

with norm $\|u\|_W = \|u\|_{L^2_T(\mathbb{R},H^1(\Omega))} + \|u'\|_{L^2_T(\mathbb{R},H^1(\Omega)^*)}$.

Lemma 3.1. *For $l > 0$, $f \in L^2_T(\Omega \times \mathbb{R})$ and $\Phi \in L^2_T(\partial\Omega \times \mathbb{R})$ the problem*

$$\begin{gathered} Lu = f \text{ in } \Omega \times \mathbb{R} \\ \langle A\nabla u, \nu\rangle + lu = \Phi \text{ on } \partial\Omega \times \mathbb{R}, \\ u(x,t) \;\; T \text{ periodic in } t \end{gathered} \tag{41}$$

has a unique solution $u \in W$.

Proof. Let $\delta > 0$. For $u_0 \in L^2(\Omega)$ the solution of

$$\begin{gathered} Lu = f \text{ in } \Omega \times (0, T+\delta) \\ \langle A\nabla u, \nu\rangle + lu = \Phi \text{ on } \partial\Omega \times (0, T+\delta), \\ u(0) = u_0 \end{gathered} \tag{42}$$

is given by

$$u(t) = U_{L,l}(t,t_0)\, u_0 + \int_{t_0}^{t} U_{L,l}(t,\tau)\, \Lambda_{f,\Phi}(\tau)\, d\tau. \tag{43}$$

By Lemma 2.15, $I - U_{L,l}(T,0) : L^2(\Omega) \to L^2(\Omega)$ has a bounded inverse. From (25), $u(0) = u(T)$ if and only if

$$u_0 = \left(I - U_{L,l}(T,0)\right)^{-1} \int_0^T U_{L,l}(T,\tau)\, \Lambda_{f,\Phi}(\tau)\, d\tau \tag{44}$$

then there exists a unique solution u of $Lu = f$ in $\Omega \times (0, T + \delta)$, $\langle A\nabla u, \nu\rangle + lu = \Phi$ on $\partial\Omega \times (0, T + \delta)$ and $u(0) = u(T)$. For such a u and for $t \in [0, T + \delta]$, let $v(t) = u(t + T)$. Thus $Lv = f$ in $\Omega \times (0, \delta)$, $\langle A\nabla v, \nu\rangle + lv = \Phi$ on $\partial\Omega \times (0, \delta)$ and $v(0) = u(0)$. Then $v(t) = u(t)$ (i.e., $u(t + T) = u(t)$) for $[0, T + \delta]$. Thus u can be extended to a solution of (41) which is unique by (44).■

Let $tr : H^1(\Omega) \to L^2(\partial\Omega)$ be the trace operator on $H^1(\Omega)$ and for $v \in W$ let $Tr(v) \in L^2_T(\partial\Omega \times \mathbb{R})$ be the trace operator defined by $Tr(v)(t) = tr(v(t))$.

For $l > 0$ we define the linear operators

$$\begin{aligned} S_1^l &: L^2_T(\Omega \times \mathbb{R}) \times L^2_T(\partial\Omega \times \mathbb{R}) \to W, \\ S_2^l &: L^2_T(\Omega \times \mathbb{R}) \times L^2_T(\partial\Omega \times \mathbb{R}) \to L^2_T(\partial\Omega \times \mathbb{R}), \\ S^l &: L^2_T(\partial\Omega \times \mathbb{R}) \to L^2_T(\partial\Omega \times \mathbb{R}) \end{aligned}$$

by

$S_1^l(f, \Phi) = u$ where u is the solution of (41) given by Lemma 3.1,

$S_2^l(f, \Phi) = Tr\left(S_1^l(f, \Phi)\right)$,

$S^l(\Phi) = S_2^l(0, \Phi)$

respectively.

Remark 3.2. Let B, B_0 and B_1 be Banach spaces, B_0 and B_1 reflexive. let $i : B_0 \to B$ be a compact and linear map and $j : B \to B_1$ an injective bounded linear operator. For T finite and $1 < p_i < \infty$, $i = 0, 1$

$$W := \left\{ v \in L^{p_0}(0, T; B_0) : \frac{d}{dt}(j \circ i \circ v) \in L^{p_1}(0, T; B_1) \right\}$$

is a Banach space under the norm $\|v\|_{L^{p_0}(0,T;B_0)} + \left\|\frac{d}{dt}(j \circ i \circ v)\right\|_{L^{p_1}(0,T;B_1)}$. A variant of an Aubin-Lions ´s theorem (for a proof see [10], p. 57 or Lemma 3 in [6]) asserts that if $V \subset W$ is bounded then the set $\{i \circ v : v \in V\}$ is precompact in $L^{p_0}(0, T; B)$.

We will apply this result to $B = L^2(\partial\Omega)$, $B_0 = H^1(\Omega)$ and $B_1 = H^1(\Omega)^*$. The map i is the trace map, $j : L^2(\partial\Omega) \to H^1(\Omega)^*$ is defined by

$$\langle j(g), h\rangle_{H^1(\Omega)^*, H^1(\Omega)} = \int_{\partial\Omega} tr(h)\, g, \qquad g \in L^2(\partial\Omega)$$

and $p_0 = p_1 = 2$. Hence W above is a special case of W in (11) for $(t_0, t_1) = (0, T)$ which is naturally isometric to the space W of (40).■

Lemma 3.3. *i) For $l > 0$, S_1^l and S_2^l are bounded linear operators and S_2^l is also compact*

ii) If $f \in L^2_T(\Omega \times \mathbb{R})$ and $\Phi \in L^2_T(\partial\Omega \times \mathbb{R})$ are nonnegative and if either $f \neq 0$ or $\Phi \neq 0$ then $ess\inf_{\Omega\times\mathbb{R}} S_1^l(f, \Phi) > 0$ and $ess\inf_{\partial\Omega\times\mathbb{R}} S_2^l(f, \Phi) > 0$. Moreover, if $\Phi > 0$ then $ess\inf_{\partial\Omega\times\mathbb{R}} S^l(\Phi) > 0$.

iii) S^l is a bounded, positive, irreducible and compact operator on $L^2_T(\partial\Omega \times \mathbb{R})$.

Proof. For $f \in L^2_T(\Omega \times \mathbb{R})$ and $\Phi \in L^2_T(\partial\Omega \times \mathbb{R})$ the T periodic solution of (42) is given by (43) with u_0 given by (44). Remark 2.6 gives

$$\|u\|_W \leq c\left(\|f\|_{L^2_T(\Omega\times\mathbb{R})} + \|\Phi\|_{L^2_T(\partial\Omega\times\mathbb{R})} + \|u_0\|_{L^2(\Omega)}\right).$$

So, to see that S_1^l is a bounded operator, it is enough to obtain see that

$$\|u_0\|_{L^2(\Omega)} \leq c \left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right) \tag{45}$$

(for the rest of the proof c will denote a positive constant independent of f and Φ, non necessarily the same at each occurrence, even in a same chain of inequalities). Let $v(t) := \int_0^t U_{L+k,l}(t,\tau)\Lambda_{f,\Phi}(\tau)$. Thus v solves $(L+k)v = f$ in $\Omega\times(0,T)$, $\langle A\nabla v,\nu\rangle + lv = \Phi$ on $\partial\Omega\times(0,T)$ and $v(0)=0$. Since

$$\|\Lambda_{f,\Phi}\|_{L^2(0,T,H^1(\Omega)^*)} \leq c\left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right)$$

(27) (applied to this problem and used with $t_0 = 0$ and $t = T$) gives

$$\begin{aligned}
\frac{1}{2}\|v(T)\|_{L^2(\Omega)}^2 &\leq \int_0^T \langle \Lambda_{f,\Phi}(\tau), v(s)\rangle_{H^1(\Omega)^*,H^1(\Omega)}\, ds \\
&\leq c\left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right)\|v\|_{L^2(0,T,H^1(\Omega))} \\
&\leq c\left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right)^2,
\end{aligned}$$

the last inequality by Remark 2.6. So

$$\|v(T)\|_{L^2(\Omega)} \leq c\left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right).$$

Now,

$$\begin{aligned}
&\left\| \int_0^T U_{L,l}(T,\tau)\Lambda_{f,\Phi}(\tau)\, d\tau \right\|_{L^2(\Omega)} \\
&= \left\| \int_0^T e^{k(T-\tau)} U_{L+k,l}(T,\tau)\Lambda_{f,\Phi}(\tau)\, d \right\|_{L^2(\Omega)} \leq e^{kT}\|v(T)\|_{L^2(\Omega)}
\end{aligned}$$

and so

$$\left\| \int_0^T U_{L,l}(T,\tau)\Lambda_{f,\Phi}(\tau)\, d\tau \right\|_{L^2(\Omega)} \leq c\left(\|f\|_{L_T^2(\Omega\times\mathbb{R})} + \|\Phi\|_{L_T^2(\partial\Omega\times\mathbb{R})} \right). \tag{46}$$

By Lemma 2.5, $I - U_{L,l}(t,\tau) : L^2(\Omega) \to L^2(\Omega)$ has a bounded inverse, and so (44) and (46) give (45). Then S_1^l is bounded and this implies the boundedness, first of S_2^l, and then of S^l.

To see that S_2^l and S^l are compact, we consider a bounded sequence $\{(f_n,\Phi_n)\} \subset L_T^2(\mathbb{R}; L^2(\Omega)) \times L_T^2(\mathbb{R}; L^2(\partial\Omega))$. Then, from Remark 3.2 $\left\{S_2^l(f_n,\Phi_n)\right\}$ is bounded in W, so $\left\{Tr\left(S_1^l(f_n,\Phi_n)\right)\right\}$ has a convergent subsequence in $L_T^2(\mathbb{R}; L^2(\partial\Omega))$. From $S^l(\Phi) = S_2^l(0,\Phi)$ we have that S^l is also compact.

Suppose now that either $f > 0$ or $\Phi > 0$ and let u_0 be given by (44). For $\delta > 0$. Lemma 2.13 (iv) gives $ess\inf U_{L,l}(t,0)u_0 > 0$ for $\delta \leq t \leq T+\delta$ and, by Lemma 2.12 (ii), we have $U_{L,l}(.,0)u_0 \in C\left(\overline{\Omega}\times(\delta,T+\delta)\right)$. Then $U_{L,l}(.,0)u_0$ has a positive minimum M on $\overline{\Omega}\times[\delta,T+\delta]$. Now,

$$S_1^l(f,\Phi)(t) = U_{L,l}(t,0)u_0 + \int_0^t U_{L,l}(t,\tau)\Lambda_{f,\Phi}(\tau)\, d\tau \geq U_{L,l}(t,0)u_0 \geq M$$

for $t \in [\delta, T+\delta]$ and so, by periodicity, $S_1^l(f,\Phi) \geq M$. Since $S_2^l(f,\Phi) = Tr(S_1^l(f,\Phi))$ and $S^l(\Phi) = Tr\left(S_1^l(0,\Phi)\right)$ we get that $S_2^l(\Phi) \geq M$ and also that $S^l(\Phi) \geq M$. Then (ii) holds and S^l is irreducible.■

Lemma 3.4. $\lim_{l\to\infty} \left\| S^l \right\| = 0$.

Proof. For $l > 0$ consider $\Phi \in L_T^2(\partial\Omega \times \mathbb{R})$ and let $u = S_2^l(0,\Phi)$. Let $u_1 = S_1^l(0,\Phi^+)$, $u_2 = S_1^l(0,\Phi^-)$ with $\Phi^+ = \max(\Phi, 0)$, $\Phi^- = \max(-\Phi, 0)$. Thus $u_1 \geq 0$, $u_2 \geq 0$ and $u = u_1 - u_2$.

Along the proof c will denote a positive constant independent of f and Φ (non necessarily the same even in a same chain of inequalities). Since $Lu_1 = 0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u_1, \nu\rangle + lu_1 = \Phi^+ \leq |\Phi|$ and u_1 is T periodic, Remark 2.6 gives $0 \leq u_1 \leq S_1^l(0,|\Phi|)$. So

$$\begin{aligned}\|u_1\|_{L_T^2(\Omega\times\mathbb{R})} \leq \|u_1\|_{L_T^2(\mathbb{R},H^1(\Omega))} &= c\left\|S_1^l(0,|\Phi|)\right\|_{L_T^2(\mathbb{R},H^1(\Omega))} \\ &\leq c\,\|\Phi\|_{L_T^2(\mathbb{R},L^2(\partial\Omega))}.\end{aligned}$$

and a similar estimate hold for u_2, and then also for u. Now, u solves $Lu = 0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u, \nu\rangle + lu = \Phi$ on $\partial\Omega\times\mathbb{R}$ and u is T periodic. Then, from (27) used with $t_0 = 0$ and $t = T$ we get

$$\begin{aligned} l\left\|S^l(\Phi)\right\|^2_{L^2(\partial\Omega\times(0,T))} &= \int_{\partial\Omega\times(0,T)} lu^2 \qquad (47)\\ &= \int_{\partial\Omega\times(0,T)} u\Phi - \int_{\Omega\times(0,T)} \left[\langle A\nabla u,\nabla u\rangle + \langle b,\nabla u\rangle u + a_0u^2\right]\end{aligned}$$

Now

$$-\int_{\Omega\times(0,T)} \left[\langle A\nabla u,\nabla u\rangle + \langle b,\nabla u\rangle u + a_0u^2\right] \qquad (48)$$

$$= -\int_{\Omega\times(0,T)} \left\langle A\left(\nabla u + \frac{1}{2}A^{-1}b\right), \nabla u + \frac{1}{2}A^{-1}b\right\rangle + \int_{\Omega\times(0,T)} \left[\left\langle \frac{1}{4}A^{-1}b, b\right\rangle - a_0\right]u^2$$

$$\leq \left\|\left\langle \frac{1}{4}A^{-1}b, b\right\rangle\right\|_{L^\infty(\Omega\times(0,T))} \int_{\Omega\times(0,T)} u^2 \leq c\,\|\Phi\|^2_{L_T^2(\mathbb{R}\times\partial\Omega)}.$$

the last inequality by Remark 2.6. Lemma 3.3 (iii) and Remark 2.6 give also

$$\int_{\partial\Omega\times(0,T)} u\Phi \leq \|u\|_{L^2(\partial\Omega\times(0,T),)} \|\Phi\|_{L^2(\partial\Omega\times(0,T),)} \leq c\,\|\Phi\|^2_{L^2(\partial\Omega\times(0,T))}.$$

Thus $l\left\|S^l(\Phi)\right\|^2_{L^2(\partial\Omega\times(0,T))} \leq c\,\|\Phi\|^2_{L^2(\partial\Omega\times(0,T),)}$ and the lemma holds.■

We will use the multiplication operator M_ζ given by

$$M_\zeta(\Phi) = \zeta\Phi, \quad \zeta \in L_T^\infty(\partial\Omega\times\mathbb{R}),\ \Phi \in L_T^2(\partial\Omega\times\mathbb{R}). \qquad (49)$$

For $\zeta \in L_T^\infty(\partial\Omega\times\mathbb{R})$ and $\Phi \in L_T^2(\partial\Omega\times\mathbb{R})$ let us observe that $u \in W$ satisfies

$$\begin{aligned} Lu = 0 \text{ in } \Omega\times\mathbb{R}, \qquad (50)\\ \langle A\nabla u,\nu\rangle + lu = \zeta Tr(u) + \Phi \text{ on } \partial\Omega\times\mathbb{R}\end{aligned}$$

(in the sense of the definition 2.1) if and only if for each $R \in \mathbb{R}$ it satisfies $Lu = 0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u,\nu\rangle + (l+R)u = (\zeta+R)Tr(u) + \Phi$ on $\partial\Omega\times\mathbb{R}$, i.e., we can "add" Ru to both sides in the boundary condition of (50).

Lemma 3.5. *i) For each $R>0$ there exists $l_0=l_0(R)$ such that for $l\geq l_0$ and $\zeta\in L_T^{\infty}(\partial\Omega\times\mathbb{R})$ such that $\|\zeta\|_{L_T^{\infty}(\partial\Omega\times\mathbb{R})}\leq R$ the problem (50) has a unique solution $u\in W$ for all $\Phi\in L_T^2(\partial\Omega\times\mathbb{R})$. Moreover, it satisfies $ess\inf_{\Omega\times\mathbb{R}}u>0$ if $\Phi>0$.*

ii) For such R, l and ζ, the solution operator $\Phi\to u$ is a bounded linear operator from $L_T^2(\partial\Omega\times\mathbb{R})$ into W whose norm is uniformly bounded on ζ for $\|\zeta\|_{L_T^{\infty}(\partial\Omega\times\mathbb{R})}\leq R$.

Proof. Let $\zeta\in L_T^{\infty}(\partial\Omega\times\mathbb{R})$ such that $\|\zeta\|_{L_T^{\infty}(\partial\Omega\times\mathbb{R})}\leq R$. By Lemma 3.4 there exists $l_0=l_0(R)>0$ such that $\left\|S^{l+R}\right\|\leq\frac{1}{4R}$ for $l\geq l_0$. For $l\geq l_0(R)$ we have $\left\|S^{l+R}M_{\zeta+R}\right\|\leq\frac{1}{2}$ and so $I-S^{l+R}M_{\zeta+R}$ has a bounded inverse. If $u\in W$ solves (50), it solves $Lu=0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u,\nu\rangle+(l+R)u=(\zeta+R)Tr(u)+\Phi$ on $\partial\Omega\times\mathbb{R}$ and so

$$Tr(u)=S^{l+R}\left(M_{\zeta+R}(Tr(u)+\Phi)\right),i.e.,Tr(u)=\left(I-S^{l+R}M_{\zeta+R}\right)^{-1}S^{l+R}(\Phi).$$

Then

$$u=S_1^{l+R}\left(0,M_{\zeta+R}\left(\left(I-S^{l+R}M_{\zeta+R}\right)^{-1}S^{l+R}(\Phi)\right)+\Phi\right). \tag{51}$$

Thus the solution of (50), if exists, is unique and given by (51).

To prove existence, consider the function u defined by (51). It solves

$$Lu=0 \text{ in } \Omega\times\mathbb{R}, \tag{52}$$
$$\langle A\nabla u,\nu\rangle+(l+R)u=(\zeta+R)\left(I-S^{l+R}M_{\zeta+R}\right)^{-1}S^{l+R}(\Phi)+\Phi \text{ on } \partial\Omega\times\mathbb{R}$$
$$u(x,t)\ \ T \text{ periodic in } T$$

and so

$$\begin{aligned}Tr(u)&=S^{l+R}M_{\zeta+R}\left(I-S^{l+R}M_{\zeta+R}\right)^{-1}S^{l+R}(\Phi)+S^{l+R}(\Phi)\\&=\left(I-S^{l+R}M_{\zeta+R}\right)^{-1}S^{l+R}(\Phi).\end{aligned} \tag{53}$$

Then (52) can be rewritten as

$$Lu=0 \text{ in } \Omega\times\mathbb{R},$$
$$\langle A\nabla u,\nu\rangle+(l+R)u=(\zeta+R)Tr(u)+\Phi \text{ on } \partial\Omega\times\mathbb{R}$$
$$u(x,t)\ \ T \text{ periodic in } T$$

and so u solves (50).

Suppose now $\Phi>0$. By (ii) and (iii) of Lemma 3.3, S_1^{l+R} and S^{l+R} are positive operators and also $ess\inf_{\Omega\times\mathbb{R}}S_1^{l+R}(\Phi)>0$. Thus (51) gives $ess\inf_{\Omega\times\mathbb{R}}u>0$ and so (i) holds. Finally, from (51) and since S^{l+R} and S_1^{l+R} are bounded and $\left\|S^{l+R}M_{\zeta+R}\right\|\leq\frac{1}{2}$ and $\|M_{\zeta+R}\|\leq 2R$, we obtain (ii).■

We will need to introduce two news operators. For $R>0$, $l\geq l_0((R))$, $\|\zeta\|_{L_T^{\infty}(\partial\Omega\times\mathbb{R})}\leq R$ let

$$\begin{aligned}&S_1^{l,\zeta}:L_T^2(\partial\Omega\times\mathbb{R})\to W,\\&S^{l,\zeta}:L_T^2(\partial\Omega\times\mathbb{R})\to L_T^2(\partial\Omega\times\mathbb{R})\end{aligned} \tag{54}$$

be defined by $S_1^{l,\zeta}(\Phi) = u$ where u is the solution of (50) given by Lemma 3.5 and by $S^{l,\zeta}(\Phi) = Tr\left(S_1^{l,\zeta}(\Phi)\right)$ respectively.

Corollary 3.6. *For R, l and ζ as in Lemma 3.5, $S^{l,\zeta}$ is a bounded, compact, positive and irreducible operator.*

Proof. By (53) we have

$$S^{l,\zeta}(\Phi) = Tr\left(S_1^{l,\zeta}(\Phi)\right) = S^l\left(I - S^{l+R}M_{\zeta+R}\right)^{-1} S^{l+R}(\Phi) + S^l(\Phi\ )$$

and the corollary follows from Lemma 3.3 (iv)■.

4. A one parameter eigenvalue problem

Lemma 4.1. *i) For $m \in L_T^\infty(\partial\Omega \times \mathbb{R})$ and $\lambda \in \mathbb{R}$ there exists a unique $\mu = \mu_m(\lambda) \in \mathbb{R}$ such that the problem*

$$\begin{gathered} Lu = 0 \text{ in } \Omega \times \mathbb{R}, \\ \langle A\nabla u, \nu\rangle + b_0 u = \lambda m u + \mu u \text{ on } \partial\Omega \times \mathbb{R}, \\ u(x,t)\ T \text{ periodic in } t \end{gathered} \tag{55}$$

has a positive solution. Moreover, for l positive and large enough let $\rho\left(S^{l,\lambda m-b_0}\right)$ be the spectral radius of $S^{l,\lambda m-b_0}$. It holds that $\mu_m(\lambda) = \left(\rho\left(S^{l,\lambda m-b_0}\right)\right)^{-1} - l$ (where $\rho\left(S^{l,\lambda m-b_0}\right)$ is the spectral radius of $S^{l,\lambda m-b_0}$).

ii) The solution space for this problem is one dimensional and for l positive and large enough $(l+\mu_m(\lambda))^{-1}1$ is an algebraically simple eigenvalue of $S^{l,\lambda m-b_0}$.

iii) Each positive solution u of (55) satisfies $ess\inf_{\Omega\times\mathbb{R}} u > 0$.

Proof. Let $R > \|\lambda m - b_0\|_{L^\infty(\partial\Omega\times\mathbb{R})}$, let $l_0 = l_0(R)$ be as in Lemma 3.5 and for $l \geq l_0$, let ρ be the spectral radius of $S^{l,\lambda m-b_0}$. From Lemma 3.6 $S^{l,\lambda m-b_0}$ is a compact, positive and irreducible operator on $L_T^2(\partial\Omega\times\mathbb{R})$. Then, by the Krein Rutman theorem, ρ is a positive eigenvalue of $S^{l,\lambda m-b_0}$ with a positive eigenfunction w associated. Let $u = S_1^{l,\lambda m-b_0}(w)$. Thus u is a T periodic solution of $Lu = 0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u,\nu\rangle + lu = (\lambda m - b_0)u + w$ on $\partial\Omega\times\mathbb{R}$. It is also positive because, by Lemma 3.5, $S_1^{l,\lambda m-b_0}$ is a positive operator. Since $Tr(u) = Tr\left(S_1^{l,+\lambda m-b_0}(w)\right) = S^{l,+\lambda m-b_0}(w) = \rho w$ it follows that u solves (55) for $\mu = \frac{1}{\rho} - l$.

On the other hand, if v is a positive solution of (55) then $Lv = 0$ in $\Omega\times\mathbb{R}$ and $\langle A\nabla u,\nu\rangle + (b_0+l)u = \lambda m u + (\mu+l)u$ on $\partial\Omega\times\mathbb{R}$. So, for $l \geq l_0(R)$ $S^{l,\lambda m-b_0}(Tr(u)) = \frac{1}{\mu+l}Tr(u)$. From Corollary 3.6 and the Krein Rutman theorem it follows that $\frac{1}{\mu+l} = \rho$ and so $\mu = \frac{1}{\rho} - l$. Thus (55) has a positive solution if and only if $\mu = \frac{1}{\rho} - l$. In particular, this gives that μ does not depend on the choice of R and l. If v is another positive solution of (55), for R and as above, and since $Tr(v) > 0$ and $Tr(v)$ is an eigenfunction of $S^{l,\lambda m-b_0}$ with eigenvalue ρ, the Krein Rutman theorem gives $Tr(v) = \eta Tr(u)$ for some $\eta\in\mathbb{R}\backslash\{0\}$. Thus

$$\begin{aligned} v &= S_1^{l,-b_0}(\lambda m Tr(v) + (\mu+l)Tr(v)) \\ &= \eta S_1^{l,-b_0}(\lambda m Tr(u) + (\mu+l)Tr(u)) = \eta u, \end{aligned}$$

then the solution space for (55) is one dimensional. Again by the Krein Rutmnan theorem, $(l+\mu_m(\lambda))^{-1}$ is an algebraically simple eigenvalue of $S^{l+R,\lambda m b_0}$.

Finally, each positive solution u of (55) satisfies

$$u = S_1^{l,-b_0}\left((\lambda m Tr(u) + (\mu + l) Tr(u))\right),$$

and so Lemma 3.5 (iii) gives $ess\inf_{\Omega\times\mathbb{R}} u > 0$.■

The aim of the rest of this section is to given some properties of the function $\mu_m(\lambda)$, $\lambda \in \mathbb{R}$ defined, for $m \in L_T^\infty(\partial\Omega\times\mathbb{R})$, by Lemma 4.1. Each zero of μ_m provides a principal eigenvalue with weight m and the corresponding solutions u in (55) are the respective positive eigenfunctions. We will prove that the map $m \to \mu_m(\lambda)$ is strictly decreasing in m (Lemma 4.6) and continuous for the $a.e.$ convergence in m (Lemma 4.7) hence continuous in $L_T^\infty(\partial\Omega\times\mathbb{R})$. $\mu_m(\lambda)$ is concave and analytic in λ (cf. Corollary 4.9 and Remark 4.11).

Remark 4.2. For $q > N+2$ let $W_{q,T}^{2,1}(\Omega\times\mathbb{R})$ be the space of the T periodic functions on $\Omega\times R$ whose restriction to $(0,T)$ belongs to $W_q^{2,1}(\Omega\times(0,T))$ and for $\gamma\in(0,1)$ let $C_T^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega\times\mathbb{R})$ be the space of the T periodic functions on $\partial\Omega\times R$ belonging to $C^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega\times\mathbb{R})$.

We recall that if

$a_{ij}\in C^{\gamma,\gamma/2}\left(\overline{\Omega}\times\mathbb{R}\right)$, $b_j\in C^1\left(\overline{\Omega}\times\mathbb{R}\right)$ for $1\le i,j\le N$; $a_0\in C^{\gamma,\gamma/2}\left(\overline{\Omega}\times\mathbb{R}\right)$,

$m,\, b_0 \in C_T^{1+\gamma\frac{1+\gamma}{2}}(\partial\Omega\times\mathbb{R})$

for such a γ, then (cf. Remark 3.1 in [8]) the solutions u of (55) belong to $W_{q,T}^{2,1}(\Omega\times\mathbb{R})$ and so $\lambda m u + \mu_m(\lambda) u \in C_T^{1+\eta\frac{1+\eta}{2}}(\partial\Omega\times\mathbb{R})$ for some $\eta\in(0,1)$. Thus Theorem 2.5 in [8] gives $u\in C^{2,1}\left(\overline{\Omega}\times\mathbb{R}\right)$.■

In order to make explicit the dependence on m, L and b_0, we will write sometimes μ_{m,L,b_0} or $\mu_{,m,L}$for the function μ_m.

Lemma 4.3. *Let $m\in L_T^\infty(\Omega\times\mathbb{R})$ and suppose that $v\in W$ satisfies*

$$\begin{aligned} Lv &= f \text{ in } \Omega\times\mathbb{R}, \\ \langle A\nabla v,\nu\rangle + b_0 v &= \Phi + \lambda m v + \mu v \text{ on } \partial\Omega\times\mathbb{R}, \\ v &> 0 \text{ on } \Omega\times\mathbb{R} \end{aligned} \tag{56}$$

for some $\lambda,\mu\in\mathbb{R}$, $f\in L_T^2(\Omega\times\mathbb{R})$ and $\Phi\in L_T^2(\partial\Omega\times\mathbb{R})$. If $f\ge 0$ and $\Phi\ge 0$ then $\mu_m(\lambda)\ge\mu$. If in addition either $f>0$ or $\Phi>0$ then $\mu_m(\lambda)>\mu$.

Proof. If $f=0$ and $\Phi=0$ then, by Lemma 4.1, $\mu=\mu_m(\lambda)$. Assume that either $f>0$ or $\Phi>0$. Since $\mu_{m,L,b_0}(\lambda)=\mu_{m+\sigma,L,b_0+\sigma\lambda}(\lambda)$ for all $\lambda,\sigma\in\mathbb{R}$, it suffices to prove the lemma in the case $m\ge 0$. For $R>0$ let $l_0(R)$ be as in Lemma 3.5 and let $l\ge l_0(\|b_0\|_\infty)+l_0(\|\lambda m-b_0\|_\infty)$. Let $w=S_1^{l,-b_0}(f,0)$, and let $z=S_1^{l,-b_0}(0,(\lambda m+\mu+l)Tr(v)+\Phi)$. Thus $w\ge 0$, $z\ge 0$ and, since $v=w+z$, $v\ge z$. So also $Tr(v)\ge Tr(z)$. Now,

$$\begin{aligned} Lz &= 0 \text{ in } \Omega\times\mathbb{R}, \\ \langle A\nabla z,\nu\rangle + b_0 z &= \Phi + (\lambda m+\mu+l)\,Tr(v) \\ &= \lambda m Tr(z) + \Phi + \lambda m Tr(v-z) + (\mu+l)\,Tr(v) \text{ on } \partial\Omega\times\mathbb{R}, \end{aligned}$$

then

$$z = S_1^{l,\lambda m - b_0}\left(\Phi + \lambda m Tr\left(v - z\right) + \left(\mu + l\right) Tr\left(v\right)\right) \geq S^{l,\lambda m - b_0}\left(\left(\mu + l\right) Tr\left(z\right)\right). \tag{57}$$

If $\Phi > 0$ since $m \geq 0$ we have $\Phi + \lambda m Tr\left(v - z\right) + \left(\mu + l\right) Tr\left(v\right) > 0$. If $f > 0$ then (by Lemma 4.3) $ess\inf_{\Omega\times\mathbb{R}} w > 0$ and so $Tr\left(w\right) > 0$. Then $Tr\left(v - z\right) > 0$ and thus, from (57), $ess\inf_{\Omega\times\mathbb{R}} z > 0$. Then $Tr\left(z\right) > 0$. Also, from (57),

$$Tr\left(z\right) \geq S_1^{l,\lambda m - b_0}\left(\left(\mu + l\right) Tr\left(v\right)\right) = \left(\mu + l\right) S^{l,\lambda m - b_0}\left(Tr\left(z\right)\right).$$

Let $\rho\left(S^{l,\lambda m - b_0}\right)$ be the spectral radius of $S^{l,\lambda m - b_0}$. Remark 2.14 (ii) gives $\frac{1}{\mu+l} \geq \rho\left(S^{l,\lambda m - b_0}\right) = \frac{1}{\mu_m(\lambda)+l}$ and so $\mu_m\left(\lambda\right) \geq \mu$.■

Lemma 4.4. *Suppose $v \in W$ satisfies*

$$\begin{aligned} Lv &= f \text{ in } \Omega \times \mathbb{R}, \\ \langle A\nabla v, \nu\rangle + b_0 v &= \Phi + \lambda m v + \mu v \text{ on } \partial\Omega \times \mathbb{R}, \\ ess\inf_{\Omega\times\mathbb{R}} v &> 0 \end{aligned} \tag{58}$$

for some $\lambda, \mu \in \mathbb{R}$, $f \in L_T^2\left(\Omega \times \mathbb{R}\right)$ and $\Phi \in L_T^2\left(\partial\Omega \times \mathbb{R}\right)$. If $f \leq 0$ and $\Phi \leq 0$ then $\mu_m\left(\lambda\right) \leq \mu$. If in addition either $f < 0$ or $\Phi < 0$ then $\mu_m\left(\lambda\right) < \mu$.

Proof. Consider first the case when $\lambda \geq 0$ and $m \geq 0$. For $R > 0$ let $l_0\left(R\right)$ be as in Lemma 3.5 and let $l \geq l_0\left(\left\|\lambda m - b_0\right\|_\infty\right)$. Let w be the T periodic solution of $Lw = f$ in $\Omega\times\mathbb{R}$, $\langle A\nabla w, \nu\rangle + \left(b_0 + l\right) w = 0$ on $\partial\Omega\times\mathbb{R}$ and let z be the T periodic solution of $Lz = 0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla z, \nu\rangle + \left(b_0 + l\right) z = \Phi + \lambda m v + \left(\mu + l\right) v$ on $\partial\Omega\times\mathbb{R}$. Thus $v = z + w$ and, by Lemma 3.3 (iv), $w \leq 0$. Then $0 < ess\inf_{\Omega\times\mathbb{R}} v \leq v \leq z$ and so also $0 < Tr\left(v\right) \leq Tr\left(z\right)$. Let

$$\widetilde{\Phi} := \left(\lambda m + l + \mu\left(\lambda\right)\right)\left(Tr\left(v\right) - Tr\left(z\right)\right) + \left(\mu - \mu\left(\lambda\right)\right) Tr\left(v\right) + \Phi.$$

Since z is T periodic and

$$\begin{aligned} Lz &= 0 \text{ in } \Omega \times \mathbb{R}, \\ \langle A\nabla z, \nu\rangle + \left(b_0 + l\right) z &= \lambda m z + \left(\mu\left(\lambda\right) + l\right) z + \widetilde{\Phi} \text{ on } \partial\Omega \times \mathbb{R} \end{aligned}$$

we have $Tr\left(z\right) = S^{l,\lambda m - b_0}\left(\left(\mu\left(\lambda\right) + l\right) Tr\left(z\right) + \widetilde{\Phi}\right)$. Thus

$$\frac{1}{\mu\left(\lambda\right) + l} Tr\left(z\right) = S^{l,\lambda m - b_0}\left(Tr\left(z\right) + \frac{1}{\mu\left(\lambda\right) + l}\widetilde{\Phi}\right) \tag{59}$$

If $\mu\left(\lambda\right) > \mu$ then $\widetilde{\Phi} \leq 0$ and so $S^{l,\lambda m - b_0}\left(Tr\left(z\right)\right) \geq \rho\left(S^{l,\lambda m - b_0}\right) Tr\left(z\right)$ where $\rho\left(S^{l,\lambda m - b_0}\right)$ is the spectral radius of $S^{l,\lambda m - b_0}$. Thus, Remark 2.14 (ii) gives $\frac{1}{\mu(\lambda)+l} \times Tr\left(z\right) = S^{l,\lambda m - b_0}\left(Tr\left(z\right)\right)$ and so $S^{l,\lambda m - b_0}\left(\widetilde{\Phi}\right) = 0$. Then, by Lemma 3.3 (iii), $\widetilde{\Phi} = 0$. This implies $\mu = \mu\left(\lambda\right)$ in contradiction with the assumption $\mu\left(\lambda\right) > \mu$. Thus $\mu\left(\lambda\right) \leq \mu$.

Assume now that either $f < 0$ or $\Phi < 0$ and that $\mu\left(\lambda\right) < \mu$. If $f < 0$ then $\sup w < 0$ and so $0 < v < z$ and $0 < Tr\left(v\right) < Tr\left(z\right)$ This implies $\widetilde{\Phi} < 0$ and if $\Phi < 0$ the same conclusion is obtained. So, in both cases, (59) gives now $S^{l,\lambda m - b_0}\left(Tr\left(z\right)\right) > \rho\left(S^{l,\lambda m - b_0}\right) Tr\left(z\right)$ in contradiction with Remark 2.14, (ii).

Since for $\sigma \in \mathbb{R}$ we have $\mu_{L,m,b_0}(\lambda) = \mu_{L,m+\sigma,b_0+\sigma\lambda}(\lambda)$, the case $\lambda \geq 0$ and m arbitrary follows from the previous one and, finally, the case $\lambda < 0$ follows from the case $\lambda > 0$ by considering the identity $\mu_m(\lambda) = \mu_{-m}(-\lambda)$.■

Let L_0 be the operator defined by $L_0 u = \frac{\partial u}{\partial t} - div(A\nabla u) + \langle b, \nabla u\rangle$. We have

Corollary 4.5. *i) Suppose $a_0 > 0$. Then $\mu_{m,L,b_0}(\lambda) > \mu_{m,L_0,b_0}(\lambda)$ for all $\lambda \in \mathbb{R}$.*

ii) Suppose $b_0 > 0$. Then $\mu_{m,L,b_0}(\lambda) > \mu_{m,L,0}(\lambda)$ for all $\lambda \in \mathbb{R}$.

Proof. let u be the solution of (55). Thus

$$L_0 u = -a_0 u \text{ in } \Omega \times \mathbb{R}, \tag{60}$$
$$\langle A\nabla u, \nu\rangle + b_0 u = \lambda m u + \mu_{b_0,m,L}(\lambda) u \text{ on } \partial\Omega \times (0,T).$$

If $a_0 > 0$, since $ess\inf u > 0$ we have $-a_0 u < 0$, then Lemma 4.4 gives (i). If $b_0 > 0$ then $-b_0 Tr(u) < 0$. Since

$$Lu = 0 \text{ in } \Omega \times \mathbb{R},$$
$$\langle A\nabla u, \nu\rangle = -b_0 u + \lambda m u + \mu_{m,L,b_0}(\lambda) u \text{ on } \partial\Omega \times (0,T),$$

(ii) follows again from Lemma 4.4.■

Lemma 4.6. *For $m_1, m_2 \in L_T^\infty(\partial\Omega \times \mathbb{R})$, $m_1 \leq m_2$ with $m_1 \neq m_2$ imply $\mu_{m_1}(\lambda) > \mu_{m_2}(\lambda)$ for all $\lambda > 0$ and $\mu_{m_1}(\lambda) < \mu_{m_2}(\lambda)$ for all $\lambda < 0$.*

Proof. Suppose $\lambda > 0$ and $\mu_{m_1}(\lambda) \leq \mu_{m_2}(\lambda)$. Let u_1 be a positive and T periodic solution of

$$Lu_1 = 0 \text{ in } \Omega \times \mathbb{R},$$
$$\langle A\nabla u_1, \nu\rangle + b_0 u_1 = \lambda m_1 u_1 + \mu_{m_1}(\lambda) u_1$$

Since $\lambda m_1 u_1 + \mu_{m_1}(\lambda) u_1 < \lambda m_2 u_1 + \mu_{m_2}(\lambda) u_1$ on $\partial\Omega \times (0,T)$ and $ess\inf_{\Omega\times\mathbb{R}} u_1 > 0$, Lemma 4.4 applies to give $\mu_{m_2}(\lambda) < \mu_{m_2}(\lambda)$ which contradicts our assumption $\mu_{m_1}(\lambda) \leq \mu_{m_2}(\lambda)$. The case $\lambda < 0$ follows from the case $\lambda > 0$ using that $\mu_m(\lambda) = \mu_{-m}(-\lambda)$.■

Lemma 4.7. *Let $\{m_n\}$ be a bounded sequence in $L_T^\infty(\partial\Omega \times \mathbb{R})$ which converges a.e. to m in $\partial\Omega \times \mathbb{R}$. Then $\lim_{n\to\infty} \mu_{m_n}(\lambda) = \mu_m(\lambda)$ for each $\lambda \in \mathbb{R}$.*

Proof. To prove the lemma it suffices to show that for each $\{m_n\}$ as in the statement of the lemma there exists a subsequence $\{m_{n_k}\}$ such that $\lim_{k\to\infty} \mu_{m_k}(\lambda) = \mu_m(\lambda)$.

Let M be a positive number such that $|m_n| \leq M$ for all n and let $\lambda \in \mathbb{R}$. Thus, by Corollary 4.5,

$$\mu_M(\lambda) \leq \mu_{m_n}(\lambda) \leq \mu_{-M}(\lambda). \tag{61}$$

Let u_n be the positive T periodic solution of

$$Lu_n = 0 \text{ in } \Omega \times \mathbb{R}, \tag{62}$$
$$\langle A\nabla u_n, \nu\rangle + b_0 u_n = \lambda m_n u_n + \mu_{m_n}(\lambda) u_n$$

normalized by $\|Tr(u_n)\|_{L_T^2(\partial\Omega\times\mathbb{R})} = 1$. We observe that $\{\lambda m_n u_n + \mu_{m_n}(\lambda) u_n\}$ is a bounded sequence in $L_T^2(\partial\Omega \times \mathbb{R})$ and so, by Lemma 3.3 (i), $\{u_n\}$ is bounded in W. Thus $\{u_n\}$ is bounded in $L_T^2(\mathbb{R}, H^1(\Omega))$ and $\{(j \circ i \circ u_n)'\}$ is bounded in $L_T^2(\mathbb{R}, H^1(\Omega)^*)$ where $i : H^1(\Omega) \to L^2(\partial\Omega) \times L^2(\Omega)$ and $j : L^2(\partial\Omega) \times L^2(\Omega) \to H^1(\Omega)^*$ are the linear maps defined in Remark 3.2 Then there exists a subsequence

$\{u_{n_k}\}$ that converges in $L^2_T(\partial\Omega\times\mathbb{R})$ to some u. From (61), after pass to a furthermore subsequence, we can assume also that $\lim_{k\to\infty}\mu_{m_{n_k}}(\lambda)=\mu$ for some $\mu\in\mathbb{R}$. Thus $\left\{\lambda m_{n_k}u_{n_k}+\mu_{m_{n_k}}(\lambda)\,u_{n_k}\right\}$ converges in $L^2_T(\partial\Omega\times\mathbb{R})$ to $\lambda mu+\mu u$. Since $u_n=S_2^{l,-b_0}(\lambda m_n u_n+\mu_{m_n}(\lambda)\,u_n)$ and $S_2^{l,-b_0}$ is continuous we obtain that $\{u_{n_k}\}$ converges in W to $S_2^{l,-b_0}(\lambda mu+\mu u)$. It follows that $u=S_2^{l,-b_0}(\lambda mu+\mu u)$ i.e., that u is a T periodic solution of $Lu=0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u,\nu\rangle+b_0u=\lambda mu+\mu$ in $\partial\Omega\times\mathbb{R}$. Since $u_{n_k}>0$ and $\{Tr\,(u_{n_k})\}$ converges in $L^2_T(\partial\Omega\times\mathbb{R})$ to u and since $\|Tr\,(u_{n_k})\|_{L^2_T(\partial\Omega\times\mathbb{R})}=1$ we get $u>0$. Then $\mu=\mu_m(\lambda)$.■

Corollary 4.8. *For each $\lambda\in\mathbb{R}$ the map $m\to\mu_m(\lambda)$ is continuous from $L^\infty_T(\partial\Omega\times\mathbb{R})\to\mathbb{R}$.*

Corollary 4.9. *μ_m is a concave function.*

Proof. Choose a sequence $\{m_n\}$ in $C^\infty_T(\partial\Omega\times\mathbb{R})$ that converges *a.e.* to m in $\partial\Omega\times\mathbb{R}$ and such that $\|m_j\|_\infty\le 1+\|m\|_\infty$ for all n. By ([8], lemma 3.3), each μ_{m_n} is concave and the corollary follows from Lemma 3.8.■

Let $B\left(L^2_T(\partial\Omega\times\mathbb{R})\right)$ denote the space of the bounded linear operators on $L^2_T(\partial\Omega\times\mathbb{R})$ and for $\rho>0$, $\zeta\in L^\infty_T(\partial\Omega\times\mathbb{R})$, let $B_\rho(\zeta)$ be the open ball in $L^\infty_T(\partial\Omega\times\mathbb{R})$ with center ζ and radius ρ.

Lemma 4.10. *Let $R>0$ and let $l_0=l_0(R)$ be as in Lemma 3.5. For $l\ge l_0$ the map $\zeta\to S^{l,-b_0+\zeta}$ is real analytic from $B_R(\zeta)$ into $B\left(L^2_T(\partial\Omega\times\mathbb{R})\right)$.*

Proof. Let $l\ge l_0$, $\zeta_0\in B_R(0)$ and $\Phi\in L^2_T(\partial\Omega\times\mathbb{R})$. For $\zeta\in B_{R-\|\zeta_0\|}(\zeta_0)$, the solution $u_\zeta=S^{l,\zeta}(\Phi)$ of (50) is T periodic and solves $Lu_\zeta=0$ in $\Omega\times\mathbb{R}$, $\langle A\nabla u_\zeta,\nu\rangle+(b_0+l)\,u_\zeta=\Phi+\zeta_0Tr\,(u_\zeta)+(\zeta-\zeta_0)\,Tr\,(u_\zeta)$ on $\partial\Omega\times\mathbb{R}$, Then $Tr\,(u_\zeta)=S^{l,\zeta_0-b_0}\Phi+S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}Tr\,(u_\zeta)$, i.e., we have

$$S^{l,\zeta-b_0}=S^{l,\zeta_0-b_0}+S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}S^{l,\zeta-b_0} \tag{63}$$

Also, $\left\|S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}\right\|\le\|\zeta-\zeta_0\|\left\|S^{l,\zeta_0-b_0}\right\|<1$ and then, from (63), $\left\|S^{l,\zeta-b_0}\right\|\le 2\left\|S^{l,\zeta_0-b_0}\right\|$. An iteration of (63) gives, for $n\in\mathbb{N}$,

$$S^{l,\zeta-b_0}=S^{l,\zeta_0-b_0}\sum_{j=1}^{n}\left(S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}\right)^j+S^{l,\zeta_0-b_0}\left(M_{\zeta-\zeta_0}S^{l,\zeta_0-b_0}\right)^{n+1}$$

Since $\left\|S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}\right\|<1$ we have $\lim_{n\to\infty}\left\|S^{l,\zeta_0-b_0}\left(M_{\zeta-\zeta_0}S^{l,\zeta_0-b_0}\right)^{n+1}\right\|=0$. Thus

$$S^{l,\zeta-b_0}=S^{l,\zeta_0-b_0}\sum_{j=1}^{\infty}\left(S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}\right)^j=S^{l,\zeta_0-b_0}\left(I-S^{l,\zeta_0-b_0}M_{\zeta-\zeta_0}\right)^{-1}.$$

Since $\zeta\to M_{\zeta-\zeta_0}$ is real analytic the lemma follows.■

Remark 4.11. Corollary 4.9 implies that μ_m is continuous. So, taking into account Corollary 3.3 and Lemma 4.10, ([3] lemma 1.3) applies to obtain that $\mu_m(\lambda)$ is real analytic in λ. Moreover, a positive solution u_λ for (55) can be chosen such that $\lambda\to u_{\lambda|\partial\Omega\times R}$ is a real analytic map from $\mathbb{R}$ into $L^2_T(\partial\Omega\times\mathbb{R})$.

Observe also that if $a_0=0$ and $b_0=0$ then $\mu_m(0)=0$ and that, in this case, the eigenfunctions associated for (55) are the constant functions. Finally, for the case when either $a_0 > 0$ or $b_0 \neq 0$, applying Lemma 4.3 with $v = 1$, $\lambda = 0$ and $\mu = 0$ we obtain that $\mu_m(0) > 0$.■

Remark 4.12. Assume that $a_0 = 0$, $b_0 = 0$ and for l large enough, consider the spectral radius ρ_l of the operator $S^{l,\lambda m - b_0} : L^2_T(\partial\Omega \times \mathbb{R}) \to L^2_T(\partial\Omega \times \mathbb{R})$. Since $\Phi = 1$ is a positive eigenfunction associated to the eigenvalue $\frac{1}{l}$, the Krein Rutman Theorem asserts that $\rho_l = \frac{1}{l}$ and that there exists a positive eigenvector $\Psi \in L^2_T(\partial\Omega \times \mathbb{R})$ for the adjoint operator $\left(S^{l,\lambda m - b_0}\right)^*$ satisfying $\left(S^{l,\lambda m - b_0}\right)^* \Psi = \Psi$. Moreover, such a Ψ is unique up a multiplicative constant.■

Lemma 4.13. *Suppose that $a_0 = 0$, $b_0 = 0$ and let $S^{l,\lambda m - b_0}$ and Ψ be as in remark 3.7. Then $\mu_m'(0) = -\frac{\langle \Psi, m\rangle}{\langle \Psi, 1\rangle}$.*

Proof. For $\lambda \in \mathbb{R}$, let u_λ be a solution of (55) such that $\lambda \to u_\lambda$ is real analytic and $u_\lambda = 1$ for $\lambda = 0$. Since

$$\begin{cases} Lu_\lambda = 0 \text{ on } \Omega \times \mathbb{R} \\ \langle A\nabla u_\lambda, \nu\rangle + (b_0 + l)\, u_\lambda = (\lambda m + \mu_m(\lambda) + l)\, u_\lambda \text{ on } \partial\Omega \times \mathbb{R} \\ u_\lambda(x,t) \;\; T \text{ periodic in } t \end{cases}$$

we get $Tr(u_\lambda) = \lambda S^{l,\lambda m - b_0}(mTr(u_\lambda)) + (\mu_m(\lambda) + l)\, S^{l,\lambda m - b_0}(Tr(u_\lambda))$ and so

$$\lambda \langle \Psi, mTr(u_\lambda)\rangle + \mu_m(\lambda)\langle \Psi, Tr(u_\lambda)\rangle = 0.$$

Taking the derivative with respect to λ at $\lambda = 0$ and using that $\mu_m(0) = 0$ and that $u_\lambda = 1$ for $\lambda = 0$, the lemma follows.■

5. The behavior of μ_m at $\pm\infty$

We fix $m \in L^\infty_T(\partial\Omega \times \mathbb{R})$, $\partial\Omega$ seen as compact Riemannian C^2 manifold of dimension $N-1$. For $\rho > 0$ fixed in $\mathbb{R}$, we will find a closed curve $\Gamma \in C_T(\mathbb{R}; \partial\Omega)$ of class C^2 and $\delta = \delta(\rho)$ such that the tube

$$B_{\Gamma,\delta} = \left\{(x,t) \in \partial\Omega \times [0,T] : x \in \exp_{\Gamma(t)} D_{\delta,\Gamma(t)}\right\} \tag{64}$$

satisfies

$$\frac{1}{\omega_{N-1}\delta^{N-1}} \int_{B_{\Gamma,\delta}} m\, d\sigma dt \geq \int_a^b \sup_{x\in\partial\Omega} m(x,t)\, dt - 2\rho. \tag{65}$$

To do let us introduce some additional notations to explain $\exp_{\Gamma(t)}\left(D_{\delta,\Gamma(t)}\right)$. For $x \in \partial\Omega$ let $T_x(\partial\Omega)$ denote the tangent space to $\partial\Omega$ at x as a subspace of $\mathbb{R}^N$ with the usual inner product of $\mathbb{R}^N$. This Riemannian structure gives an exponential map $\exp_x : T_x(\partial\Omega) \to \partial\Omega$ and an area element $d\sigma(x)$. For each $X \in T_x(\partial\Omega)$, $\exp_x X = \eta(1)$ where $\eta(t)$ is the geodesic satisfying $\eta(0) = x$, $\eta'(0) = X$. We have also the geodesic distance $d_{\partial\Omega}$ on $\partial\Omega$ and geodesic balls $B_r(x)$, $x \in \partial\Omega$, $r > 0$. We denote d the distance on $\partial\Omega \times (0,T)$ given by

$$d((x,t),(y,s)) = \max\left(d_{\partial\Omega}(x,y), |t-s|\right) \tag{66}$$

and, for $(x,t) \in \partial\Omega \times (0,T)$ and $r > 0$ we put $B_r(x,t)$ for the corresponding open ball with center (x,t) and radius r. So we have that $B_r(x,t) = B_r(x) \times (t-r, t+r)$ is a cylinder. Concerning the measures $d\sigma$ on $\partial\Omega$ and $d\sigma dt$ on $\partial\Omega \times (0,T)$ we denote indistinctly $|E|$ the measure of a Borel subset of $\partial\Omega$ or of $\partial\Omega \times (0,T)$.

For $x \in \partial\Omega$ let $\{X_{1,x}, ..., X_{N-1,x}\}$ be an orthonormal basis of $T_x(\partial\Omega)$ and let $\varphi_x : \{z \in \mathbb{R}^{N-1} : |z| < r\} \to \partial\Omega$ be the map defined by $\varphi_x(z_1, ...z_{N-1}) = \exp_x\left(\sum_{j=1}^{N-1} z_j X_{j,x}\right)$. From well known properties of the exponential map there exists $\varepsilon > 0$ such that $\varphi_x : \{z \in \mathbb{R}^{N-1} : |z| < r\} \to B_r(x)$ is a diffeomorphism for $0 < r < \varepsilon$, $x \in \partial\Omega$. For such r and $x \in \partial\Omega$ let $y \to (z_1(y), ..., z_{N-1}(y))$ be the coordinate system defined by φ_x on $B_r(x)$, let $\left\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_{N-1}}\right\}$ be the corresponding coordinate frame, let $g_{ij}(y) := \left\langle \frac{\partial}{\partial z_i}_{|y}, \frac{\partial}{\partial z_j}_{|y} \right\rangle$, $1 \leq i, j \leq N-1$, $y \in B_r(x)$ and let $(g_{ij}(y))$ be the $(N-1) \times (N-1)$ matrix whose i, j entry is $g_{ij.}(y)$. Finally, we put ω_{N-1} for the area of the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

Lemma 5.1. *i) For $x \in \partial\Omega$ it holds that $\lim_{r\to 0} \frac{|B_r(x)|}{\omega_{N-1} r^{N-1}} = 1$ uniformly in $x \in \partial\Omega$.*

ii) $d\sigma$ is doubling, that is $|B_{2r}(x)| \leq c\,|B_r(x)|$ for some $c > 0$ independent of $x \in \partial\Omega$ and $r > 0$.

iii) Let $E \subset \partial\Omega \times R$ be a Borel set. Then $\lim_{|B|\to 0,\ (x,t)\in B} \frac{|E\cap B|}{|B|} = 1$ a.e. $(x, t) \in E$ (the limit taken on balls B in $\partial\Omega \times \mathbb{R}$)

Proof. To obtain (i) we consider an orthonormal basis $\{X_{1.x}, ...X_{N-1.x}\}$ of $T_x(\partial\Omega)$ and $z \in \mathbb{R}^{N-1}$. For ε small enough and $0 < r < \varepsilon$ we have

$$\frac{|B_r(x)|}{\omega_{N-1} r^{N-1}} - 1 = \frac{1}{\omega_{N-1} r^{N-1}} \int_{|z|<r} (f(x, z) - 1)\, dz_1...dz_{N-1}$$

where $f(x, z) := \det^{\frac{1}{2}}\left(g_{ij}\left(\exp_x\left(\sum_{j=1}^{N-1} z_j X_{j,x}\right)\right)\right)$. Since $(x, z) \to f(x, z) - 1$ is uniformly continuous on $\partial\Omega \times D_1$ and $f(x, 0) = 1$, $x \in \partial\Omega$ we obtain (i) by taking limits.

As $\partial\Omega$ has finite diameter for $d_{\partial\Omega}$ we have (ii).

Finally, $d\sigma dt$ is also doubling in $\partial\Omega \times \mathbb{R}$ and so (iii) holds (cf. e.g. [11]).■

Lemma 5.2. *For each $\rho > 0$ there exists $\delta > 0$, a partition $\{t_0, ...t_n\}$ of $[0, T]$ and points $x_1, ..., x_n$* in $\partial\Omega$ *with $x_n = x_1$ such that $\{B_\delta(x_1) \times (t_{i-1}, t_i)\}_{1\leq i\leq n}$ is a family of disjoint sets and*

$$\frac{1}{\omega_{N-1}\delta^{N-1}} \int_{\cup_{i=1}^n B_\delta(x_i)\times(t_{i-1},t_i)} m(x, t)\, d\sigma(x)\, dt \geq \int_0^T ess \sup_{x\in\partial\Omega} m(x, t)\, dt - \rho$$

Proof. Without lost of generality we can assume that $\|m\|_\infty \leq 1$. For $t \in [0, T]$ let $\widetilde{m}(t) = ess \sup_{x\in\partial\Omega} m(x, t)$ and for $\eta > 0$ let

$$E(\eta) = \{(x, t) \in \partial\Omega \times \mathbb{R} : m(x, t) > \widetilde{m}(t) - \eta\}. \tag{67}$$

and let $E(\eta)^d$ be the set of the density points (in the sense of Lemma 5.1, (iii)) in $E(\eta)$. We fix $\alpha \in \left(0, \frac{1}{2}\right)$. For $k \in \mathbb{N}$, let $E(\eta)^{(k)}$ be the set of the points $(x, t) \in E(\eta)^d$ such that

$$\frac{|B_\rho(y, s) \cap E(\eta)|}{|B_\rho(y, s)|} > 1 - \alpha$$

for all open ball $B_{\rho}(y,s) \subset \partial\Omega \times \mathbb{R}$ containing (x,t) and with radius $\rho < \frac{1}{k}$. Observe that $E(\eta)^{(k)} \subset E(\eta)^{(s)}$ for $k < s$ and that (from Lemma 3.16 (iii) $E(\eta) = \cup_{k\in\mathbb{N}} E(\eta)^{(k)}$. Thus $\lim_{k\to\infty} \left|\pi\left(E(\eta)^{(k)}\right)\right| = |\pi(E(\eta))| = T$ where $\pi(x,t) := t$.

Given $\varepsilon > 0$ we fix $k \in \mathbb{N}$ such that $\left|\pi\left(E(\eta)^{(k)}\right)\right| \geq T - \varepsilon$. For $n \in \mathbb{N}$ let $l = \frac{T}{2n}$ and let $\{t_0, ...t_n\}$ be the partition of $[0,T]$ given by $t_i = 2il$.
Let $I = \left\{i \in \{1,2,...n\} : (\partial\Omega \times (t_{i-1}, t_i)) \cap E(\eta)^{(k)} \neq \emptyset\right\}$ and let $I^c = \{1,2,...n\} \setminus I$. Denote $\delta = \frac{T}{4n}$. For $i \in I \setminus \{n\}$ let $(x_i, t_i^*) \in (\partial\Omega \times (t_{i-1}, t_i)) \cap E(\eta)^{(k)}$ and let $Q_i = B_{\delta}(x_i) \times (t_{i-1}, t_i)$ and, for $j \in I^c \setminus \{n\}$ let $x_j \in \partial\Omega$ and let $Q_j = B_{\delta}(x_j) \times (t_{j-1}, t_j)$. We also set $x_n = x_1$ and $Q_n = B_{\delta}(x_n) \times (t_{n-1}, t_n)$. Since $\left|\pi\left(E(\eta)^{(k)}\right)\right| \geq T - \varepsilon$ we have $\sum_{i\in I^c} (t_i - t_{i-1}) \leq \varepsilon$.
Consider the case $i \in I$. We have $\int_{Q_i} m(x,t)\, d\sigma(x)\, dt = \int_{Q_i \cap E(\eta)} m(x,t)\, d\sigma(x)\, dt + \int_{Q_i \cap E(\eta)^c} m(x,t)\, d\sigma(x)\, dt$. Also,

$$\begin{aligned}
&\int_{Q_i\cap E(\eta)} m(x,t)\, d\sigma(x)\, dt \geq \int_{Q_i\cap E(\eta)} \widetilde{m}(t)\, d\sigma(x)\, dt - \eta\, |Q_i \cap E(\eta)| \\
&\geq \int_{t_{i-1}}^{t_i} \widetilde{m}(t) \left(\left|(Q_i \cap E(\eta))_t\right| - \left|(Q_i)_t\right|\right) dt + \int_{t_{i-1}}^{t_i} \widetilde{m}(t) \left|(Q_i)_t\right| dt - \eta\, |Q_i| \\
&\geq |Q_i \cap E(\eta)| - |Q_i| + |B_{\delta}(x_i)| \int_{t_{i-1}}^{t_i} \widetilde{m}(t)\, dt - 2l\eta\, |B_{\delta}(x_i)|.
\end{aligned}$$

Since $(x_i, t_i^*) \in E(\eta)^{(k)}$ and $(x_i, t_i^*) \in B_{\delta}(x_i) \times \left(\frac{t_i + t_{i-1}}{2} - l, \frac{t_i + t_{i-1}}{2} + l\right)$ we get $|Q_i \cap E(\eta)| \geq (1-\alpha)\, |Q_i|$). So, the above inequalities give

$$\int_{Q_i\cap E(\eta)} m(x,t)\, d\sigma(x)\, dt \geq \left(-2l(\alpha+\eta) + \int_{t_{i-1}}^{t_i} \widetilde{m}(t)\, dt\right) |B_{\delta}(x_i)|.$$

Moreover, $\int_{Q_i \cap E(\eta)^c} m(x,t)\, d\sigma(x)\, dt \leq \left|(Q_i \cap E(\eta))^c\right| = |Q_i| - |Q_i \cap E(\eta)| \leq 2l\alpha\, |B_{\delta}(x_i)|$. Thus

$$\int_{Q_i} m(x,t)\, d\sigma(x)\, dt \geq \left(-2l(2\alpha+\eta) + \int_{t_{i-1}}^{t_i} \widetilde{m}\right) |B_{\delta}(x_i)|. \tag{68}$$

Also, for $j \in I^c$,

$$\int_{Q_j} m(x,t)\, d\sigma(x)\, dt \geq -|Q_j| = -2l\, |B_{\delta}(x_j)| \tag{69}$$

For $i \in I$ let $\varepsilon_i(\delta) = \frac{|B_\delta(x_i)|}{\omega_{N-1}\delta^{N-1}} - 1$. From (68) and (69) we have

$$\begin{aligned}
&\int_{\cup_{i=1}^n Q_i} m(x,t)\, d\sigma(x)\, dt \\
&= \sum_{i\in I\setminus\{n\}} \int_{Q_i} m(x,t)\, d\sigma(x)\, dt + \sum_{i\in I^c\setminus\{n\}} \int_{Q_i} m(x,t)\, d\sigma(x)\, dt + \int_{Q_n} m(x,t)\, d\sigma(x)\, dt \\
&\geq \sum_{i\in I\setminus\{n\}} \left(\int_{t_{i-1}}^{t_i} \widetilde{m}(t)\, dt - 2l(2\alpha+\eta) \right) |B_\delta(x_i)| - \sum_{i\in I^c} 2\alpha l\, |B_\delta(x_i)| - \frac{T}{n} |B_\delta(x_n)| \\
&= \omega_{N-1}\delta^{N-1} \left(\int_0^T \widetilde{m}(t)\, dt - \sum_{i\in I^c\setminus\{n\}} \int_{t_{i-1}}^{t_i} \widetilde{m}(t)\, dt - 2l\#(I)(2\alpha+\eta) - 2l\#(I^c)\,\alpha \right) \\
&\qquad -\omega_{N-1}\delta^{N-1}\frac{T}{n} \\
&+\omega_{N-1}\delta^{N-1} \left(\sum_{i\in I\setminus\{n\}} \varepsilon_i(\delta) \left(-2l(2\alpha+\eta) + \int_{t_{i-1}}^{t_i} \widetilde{m}(t)\, dt \right) - \sum_{i\in I^c\setminus\{n\}} 2\alpha l \varepsilon_i(\delta) \right) \\
&\qquad -\omega_{N-1}\delta^{N-1}\frac{T}{n}\varepsilon_n(\delta).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\cup_{i=1}^n Q_i} m(x,t)\, d\sigma(x)\, dt &\geq \omega_{N-1}\delta^{N-1} \int_0^T \widetilde{m}(dt) \\
&- \omega_{N-1}\delta^{N-1} \left(\varepsilon + \varepsilon\alpha + T(2\alpha+\eta) - \frac{T}{n} \right) \\
&- \omega_{N-1}\delta^{N-1} \max_{1\leq i\leq n} |\varepsilon_i(\delta)| \left(2\alpha + \eta + T + \alpha\varepsilon + \frac{T}{n} \right).
\end{aligned}$$

where $\#(I)$ and $\#(I^c)$ denote the cardinals of I and I^c respectively. Since $\delta = \frac{T}{4n}$ and Lemma 3.11 gives that $\lim_{n\to\infty} \max_{1\leq i\leq n} \left|\varepsilon_i\left(\frac{T}{4n}\right)\right| = 0$, taking n large enough and α, η and ε small enough the lemma follows.■

For a T periodic curve $\Gamma \in C^2(\mathbb{R}, \partial\Omega)$ and $\delta > 0$, let $B_{\Gamma,\delta}$ defined by (64). We have

Lemma 5.3. *Assume that $\partial\Omega$ is connected. Then for each $\rho > 0$ there exist $\Gamma \in C_T^2(\mathbb{R}, \partial\Omega)$ and $\delta > 0$ such that*

$$\frac{1}{\omega_{N-1}\delta^{N-1}} \int_{B_{\Gamma,\delta}} m(x,t)\, d(x)\, \sigma dt \geq \int_0^T ess \sup_{x\in\partial\Omega} m(x,t)\, dt - 2\rho$$

Proof. Let $\rho > 0$ and let $x_1, ..., x_n,\ t_0, ..., t_n$ and δ be as in Lemma 5.2. For $\theta < \frac{T}{2n}$ and $i = 1, ..., n-1$, let $\gamma_i : [t_i - \theta, t_i + \theta] \to \partial\Omega$ be a C^2 map satisfying $\gamma_i(t_i - \theta) = x_{i-1}$, $\gamma_i(t_i + \theta) = x_i$ and $\gamma_i^{(j)}(t) = 0$ for $j = 1, 2$ and $t = t_i \pm \theta$. Let $\Gamma \in C_T^2(\mathbb{R}, \partial\Omega)$ be defined by $\Gamma(t) = x_1$ for $t \in [t_0, t_1 - \theta]$, $\Gamma(t) = x_n$ for

$t\in[t_n+\theta,t_n]$ and by

$$\Gamma(t)=x_{i-1}\text{ for }t\in(t_{i-1}+\theta,t_i-\theta),$$
$$\Gamma(t)=\gamma_i(t)\text{ for }t\in(t_i-\theta,t_i+\theta),$$
$$\Gamma(t)=x_i\text{ for }t\in(t_i+\theta,t_{i+1}-\theta).$$

for $i=1,...,n-1$. For θ small enough Γ satisfies the conditions of the lemma.■

Corollary 5.4. *Assume that $\partial\Omega$ is connected and let $P(m)$ be defined by (6). If $P(m)>0$ then for δ positive and small enough there exists $\Gamma\in C_T^2(\mathbb{R},\partial\Omega)$ such that $\int_{B_{\Gamma,\delta}}m>0$.*

Remark 5.5. Let $\Gamma\in C_T^2\left(\mathbb{R},\mathbb{R}^N\right)$ as in Lemma 5.3. Since the map $t\to\nu(\Gamma(t))$ belongs to $C^{1+\theta}\left(\mathbb{R},\mathbb{R}^N\right)$ there exists a $C^{1+\theta}$ and T periodic map $t\to A(t)$ from $\mathbb{R}$ into $SO(N)$ such that $A(t)\nu(\Gamma(0))=\nu(\Gamma(t))$ for $t\in\mathbb{R}$. Let $\{X_{1,0},...,X_{N-1,0}\}$ be an orthonormal basis of $T_{\Gamma(0)}(\partial\Omega)$ and let $X_j(t)=A(t)X_{j,0}$, for $j=1,2,...N-1$, $t\in\mathbb{R}$. Thus each X_j is a T periodic map, $X_j\in C^{1+\gamma}\left(\mathbb{R},\mathbb{R}^N\right)$ and for each t, $\{X_1(t),...,X_{N-1}(t)\}$ is an orthonormal basis of $T_{\Gamma(t)}(\partial\Omega)$. For $z\in\mathbb{R}^N$ and $t\in\mathbb{R}$ we set

$$x(z,t) \tag{70}$$
$$:=\exp_{\Gamma(t)}\left(\sum_{1\le j\le N-1}z_jX_j(t)\right)-z_{N+1}\nu\left(\exp_{\Gamma(t)}\left(\sum_{1\le j\le N-1}z_jX_j(t)\right)\right),$$

and

$$\Lambda(z,t):=(x(z,t),t). \tag{71}$$

For $\delta>0$ let $D_\delta=\left\{z\in\mathbb{R}^{N-1}:|z|<\delta\right\}$ and $Q_\delta:=D_\delta\times(0,\delta)\times\mathbb{R}$. Thus, for δ positive and small enough Λ is a diffeomorphism from Q_δ onto an open neighborhood $W_\delta\subset\mathbb{R}^N\times\mathbb{R}$ of the set $\{(T(t),t):t\in\mathbb{R}\}$ satisfying

$\Lambda(Q_\delta)=W_\delta\cap(\Omega\times\mathbb{R})$,
$\Lambda(Q_\delta)=W_\delta\cap(\partial\Omega\times\mathbb{R})$,
$\Lambda(D_\delta\times\{0\}\times\{t\})=B_\delta(\Gamma(t))\times\{t\}$,
$\Lambda(0,t)=(\Gamma(t),t)$,
$\Lambda(.,t)$ is T periodic in t.

Moreover, $\Lambda:Q_\delta\to W_\delta$ and its inverse $\Theta:W_\delta\to Q_\delta$ are of class $C^{2,1}$ on their respective domains. For $\delta,\Lambda,\Theta,W_\delta$ as above, with $\Theta(x,t)=(\Theta_1(x,t),...,\Theta_{N+1}(x,t))$ we have $\Theta_{N+1}(x,t)=t$ and also (cf. (3.13) and (3.14) in [8])

$$\nabla\Theta_N=-g\nu\qquad\text{on }W_\delta\cap(\partial\Omega\times\mathbb{R})$$

for some $g\in C^1(W_\delta\cap(\partial\Omega\times\mathbb{R}))$ satisfying $g(x,t)\neq0$ for $(x,t)\in W_\delta\cap(\partial\Omega\times\mathbb{R})$ and $g(\Gamma(t),t)=1$ for $t\in\mathbb{R}$. Moreover, if $\Lambda'(\Gamma(t),t)$ denotes the Jacobian matrix of Λ at $(\Gamma(t),t)$, from the definition of Λ and taking into account that the differential of $\exp_x$ at the origin is the identity on $T_x(\partial\Omega)$, we have that $\det\Lambda'(\Gamma(t),t)=1$ for $t\in\mathbb{R}$.

Lemma 5.6. *Assume that $\partial\Omega$ is connected and that $P(m)>0$. Then $\lim_{\lambda\to\infty}\mu_m(\lambda)=-\infty$.*

Proof. Let $\{m_n\}$ be a sequence in $C_T^\infty(\partial\Omega\times\mathbb{R})$ that converges to m a.e in $\partial\Omega\times\mathbb{R}$ and satisfying $\|m_n\|_\infty\le1+\|m\|_\infty$ for $n\in\mathbb{N}$, let $\left\{L^{(n)}\right\}$ be a sequence of operators as in Lemma 2.8 and let $A^{(n)}$ be the $N \times N$ matrix whose i, j entry $a_{ij}^{(n)}$, let $\{b_0{}^{(n)}\}$ be a sequence in $W_{q,T}^{2-\frac{1}{q},1-\frac{1}{2q}}$ for some $q > N+2$ and such $\lim_{n\to\infty} b_0{}^{(n)} = b_0$ $a.e.$ in $\partial\Omega \times \mathbb{R}$.

For δ positive and small enough let Γ be as in Corollary 5.4 and let Q_δ, W_δ, Λ and Θ be as in Remark 5.5.

For $(s,t) \in Q_\delta$ let

$$\widetilde{a}_{ij}^{(n)}(s,t) = \sum_{1\le l.r\le N} a_{lr}(\Lambda(s,t)) \frac{\partial \Theta_i}{\partial x_l}(\Lambda(s,t)) \frac{\partial \Theta_j}{\partial x_r}(\Lambda(s,t)),$$

let $\widetilde{b}^{(n)}(s,t) = \left(\widetilde{b}_1^{(n)}(s,t), ..., \widetilde{b}_N^{(n)}(s,t)\right)$ with

$$\begin{aligned}\widetilde{b}_j^{(n)}(s,t) := & \frac{\partial \Theta_j}{\partial t}(\Lambda(s,t)) + \sum_{1\le r\le N} b_r(\Lambda(s,t)) \frac{\partial \Theta_j}{\partial x_r}(\Lambda(s,t)) \\ & - \sum_{1\le i,l,r\le N} \frac{\partial \widetilde{a}_{ir}}{\partial s_l}(s,t) \frac{\partial \Theta_i}{\partial x_r}(\Lambda(s,t)) \frac{\partial \Theta_j}{\partial x_r}(\Lambda(s,t)) \\ & - \sum_{1\le i,r\le N} \widetilde{a}_{ij}(s,t) \frac{\partial^2 \Theta_j}{\partial x_i \partial x_r}(\Lambda(s,t))\end{aligned}$$

and let $\widetilde{A}^{(n)}(s,t)$ be the $N \times N$ symmetric and positive matrix whose (i,j) entry is $\widetilde{a}_{ij}^{(n)}(s,t)$, let $\widetilde{a}_0^{(n)}$ be defined on Q_δ by $\widetilde{a}_0 = a_0 \circ \Lambda$, let $\widetilde{m}_n$, $\widetilde{b}_0$ be defined on $D_\delta \times \{0\} \times [0,T]$ by $\widetilde{m}_n = m_n \circ \Lambda$ and $\widetilde{b}_0 = b_0 \circ \Lambda$. For $\lambda > 0$ let $u_{n,\lambda}$ be a positive and T periodic solution of

$$\begin{gathered} L^{(n)} u_{n,\lambda} = 0 \text{ in } \Omega \times \mathbb{R}, \\ \left\langle A^{(n)} \nabla u_{n,\lambda}, \nu \right\rangle + b_0^{(u)} u_{n,\lambda} = \lambda m_n u_{n,\lambda} + \mu_{m_n, L^{(n)}}(\lambda) u_{n,\lambda} \text{ on } \partial\Omega \times \mathbb{R} \end{gathered}$$

normalized by $\|u_{n,\lambda}\|_W = 1$. Let $\widetilde{u}_{n,\lambda} \in C^{2,1}(Q_\delta)$ be defined by $\widetilde{u}_{n,\lambda} = u_{n,\lambda} \circ \Lambda$. Then, a computation shows that

$$\begin{gathered} \widetilde{L}^{(n)} \widetilde{u}_{n,\lambda} = 0 \text{ in } Q_\delta \times (0,\delta) \times \mathbb{R}, \\ \left\langle \widetilde{A}^{(n)} \nabla \widetilde{u}_{n,\lambda}, e_N \right\rangle + \widetilde{b}_0^{(u)} \widetilde{u}_{n,\lambda} = \lambda \widetilde{m}_n \widetilde{u}_{n,\lambda} + \mu_{m_n, L^{(n)}}(\lambda) \widetilde{u}_{n,\lambda} \text{ on } Q_\delta \times \{0\} \times \mathbb{R} \end{gathered}$$

Let $\beta \in (0,\delta)$ (to be chosen latter), let $h \in C^\infty(\mathbb{R})$ such that $0 \le h \le 1$, $h(\zeta) = 1$ for $\zeta < \delta - \beta$, $h(\zeta) = 0$ for $\zeta \ge \delta$ and let $G \in C^\infty\left(\mathbb{R}^{N+!}\right)$ be defined by $G(z,s,t) = h(|(z,s)|)$ for $(z,s,t) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}$. Finally, we set $\widetilde{g} = g \circ \Lambda$ and, for a definite positive matrix $P \in M_N(\mathbb{R})$ and $w \in \mathbb{R}^N$ we put $\|w\|_P := \langle Pw, w \rangle$. With these notations we have, as in the proof of Lemma 3.11 in [8],

$$\mu_{m_n,L^{(n)},b_0^{(n)}}(\lambda) \int\limits_{D_\delta\times(0,T)} \left(G^2\widetilde{g}\right)(\xi,0,t)\,d\xi dt \tag{72}$$
$$\leq -\lambda \int\limits_{D_\delta\times(0,T)} \left(G^2\widetilde{g}\widetilde{m}_n\right)(\xi,0,t)\,d\xi dt$$
$$+\int_{\{s\in\mathbb{R}^N:|s|<\delta:\}\times(0,T)} \left[\left\|\left(\nabla G+\frac{G}{2}\widetilde{A}^{(n)}\widetilde{b}^{(n)}\right)\right\|^2_{\widetilde{A}^{(n)}(s,t)} + \widetilde{a}_0^{(n)}(s,t)\,G^2\right](s,t)\,dsdt.$$

Also

$$\int_{D_\delta\times(0,T)} \widetilde{m}(z,0,t)\sqrt{\det\left(g_{ij}\left(\exp_{\Gamma(t)}\left(\sum_{j=1}^{N-1} z_jX_j(t)\right)\right)\right)}dzdt = \int_{B_{\Gamma,\delta}} m > 0.$$

Thus, since $\sqrt{\det\left(g_{ij}\left(\Gamma(t)\right)\right)} = 1$ and $z \to \sqrt{\det\left(g_{ij}\left(\exp_{\Gamma(t)}\left(\sum_{j=1}^{N-1} z_jX_j(t)\right)\right)\right)}$ is continuous, we get $\int_{D_\delta\times(0,T)} \widetilde{m}(z,0,t)\,dzdt > 0$ for δ positive and small enough. Then (for a smaller δ if necessary) and some positive constant c we have

$$\int_{D_\delta\times(0,T)} \widetilde{m}_n(z,0,t)\,dzdt > c,$$

for n large enough. Since $\widetilde{g}$ is continuous on $D_\delta\times\{0\}\times\mathbb{R}$ and $\widetilde{g}(0,t) = 1$ we can assume also (diminishing δ and c if necessary) that, for n large enough,

$$\int_{D_\delta\times(0,T)} \left(\widetilde{m}_n\widetilde{g}\right)(z,0,t)\,dzdt > c \text{ and } \int_{D_\delta\times(0,T)} \widetilde{g}(z,0,t)\,dzdt > c$$

¿From these inequalities it is clear that we can pick β small enough in the definition of G such that for n large enough

$$\int_{D_\delta\times(0,T)} \left(G^2m_n^*g^*\right)(\sigma,0,t)\,d\sigma dt > c/2, \tag{73}$$
$$\int_{D_\delta\times(0,T)} \left(G^2g^*\right)(\sigma,0,t)\,d\sigma dt > c/2. \tag{74}$$

We have also

$$\lim_{n\to\infty}\int_{B_{\Gamma,\eta}} \left\|\left(\nabla G+\frac{G}{2}\widetilde{A}^{(n)}\widetilde{b}^{(n)}\right)(s,t)\right\|^2_{A^{(n)*}(s,t)} dsdt$$
$$= \int_{B_{\Gamma,\eta}} \left\|\left(\nabla G+\frac{G}{2}Ab^*\right)(s,t)\right\|^2_{A^*(s,t)} dsdt$$

so, from (73), we get positive constants c_1 and c_2 independent of n and λ such that $\mu_{m_n,L^{(n)},b_0^{(n)}}(\lambda) \leq -c_1 - c_2\lambda$ for all n large enough. Also, since

$$L^{(n)}1 \geq 0 \text{ in } \Omega \times \mathbb{R},$$
$$\left\langle A^{(n)}\nabla 1, \nu \right\rangle + b_0^{(u)}1 \geq \lambda m_n 1 - (1 + \|m\|_\infty)\lambda - (1 + \|b_0\|_\infty) \text{ on } \partial\Omega \times \mathbb{R},$$

Lemma 4.3 gives $\mu_{m_n,L^{(n)}}(\lambda) \geq -(1 + \|m\|_\infty)\lambda - (1 + \|b_0\|_\infty)$. Thus $\{\mu_{m_n,L^{(n)}}(\lambda)\}$ is bounded, and so, after pass to a subsequence we can assume that $\{\mu_{m_n,L^{(n)}}(\lambda)\}$ converges to some $\mu \leq -c_1 - c_2\lambda$. Since $\{\lambda m_n Tr(u_{n,\lambda}) + \mu_{m_n,L^{(n)}}(\lambda) Tr(u_{n,\lambda})\}$ is bounded in $L_T^2(\partial\Omega \times \mathbb{R})$, by Lemma 3.3 and after pass to a furthermore subsequence, we can assume that $\{u_{n,\lambda}\}$ converges in W to some $u_\lambda \geq 0$. By Lemma 2.8 u satisfies $Lu = 0$ in $\Omega \times \mathbb{R}$, $\langle A\nabla u, \nu\rangle + b_0 u = \lambda m u + \mu u$ on $\partial\Omega \times \mathbb{R}$. Thus $\mu_{m,L,b_0}(\lambda) = \mu$ and so $\mu_{m,L,b_0}(\lambda) \leq -c_1 - c_2\lambda$.■

6. Principal eigenvalues for periodic parabolic Steklov problems

Let $P(m)$ and $N(m)$ be defined by (6). We have

Theorem 6.1. *Suppose one of the following assertions i), ii), iii), holds.*

i) $P(m) > 0$ *(respectively* $N(m) < 0$*) and either* $a_0 > 0$ *or* $b_0 > 0$

ii) $a_0 = 0$, $b_0 = 0, P(m) > 0$ *(respectively* $N(m) < 0$*)*, $\langle \Psi, m\rangle < 0$ *(resp.* $\langle \Psi, m\rangle > 0$*) with* Ψ *defined as in remark 3.7.*

Then there exists a unique positive (resp. negative) principal eigenvalue for (55) and the associated eigenspace is one dimensional.

proof. Suppose $a_0 = 0$, $b_0 = 0$, $P(m) > 0$ and $\langle \Psi, m\rangle < 0$. Since $\mu_m(0) = 0$ and, by Lemma 3.14, $\mu_m'(0) > 0$ the existence of a positive principal eigenvalue $\lambda = \lambda_1(m)$ for (55) follows from Lemma 5.6. Since μ_m does not vanish identically, the concavity of μ_m gives the uniqueness of the positive principal eigenvalue.

Moreover, if u, v are solutions in W for (55), then, from Lemma 4.1, $u = cv$ on $\partial\Omega \times R$ for some constant c. Since, for $l \in R$, $L(u - cv) = 0$ on $\Omega \times R$, $B_{b_0+l}(u - cv) = \lambda m(u - cv) + \mu_m(\lambda)(u - cv)$ and $u - cv = 0$ on $\partial\Omega \times R$. Thus, taking l large enough, Lemma 2.9 gives $u = cv$ on $\Omega \times R$.

If either $a_0 > 0$ or $b_0 > 0$ then (by Remark 3.12) $\mu_m(0) > 0$ and so the existence follows from Lemma 5.6. The other assertions of the theorem follow as in the case $a_0 = 0$. Since $\mu_m(-\lambda) = \mu_{-m}(\lambda)$ and $N(m) = -P(-m)$, the assertions concerning negative principal eigenvalues reduce to the above.■

Theorem 6.2. *Let* $\lambda \in \mathbb{R}$ *such that* $\mu_m(\lambda) > 0$. *Then for all* $\Phi \in L_T^2(\partial\Omega \times \mathbb{R})$ *the problem*

$$Lu = 0 \text{ in } \Omega \times \mathbb{R}, \tag{75}$$
$$B_{b_0}u = \lambda m u + \Phi \text{ on } \partial\Omega \times \mathbb{R}$$
$$u(x,t) \ T \text{ periodic in } t$$

has a unique solution. Moreover $\Phi > 0$ *implies that* $ess\inf_{\Omega\times\mathbb{R}} u > 0$.

proof. Since $\mu_m(\lambda) > 0$ for l large enough we have $\rho\left(S^{l,\lambda m - b_0}\right) < \frac{1}{l}$ and so , $\left(\frac{1}{l}I - S^{l,\lambda m - b_0}\right)^{-1}$ is a well defined and positive operator. If u is a solution of (75) then $u = S_{\lambda m + l}^{l,-b_0}\Phi$ so the solution, if exists, is unique. To see that it exists, consider

$$w := \frac{1}{l} S^{l,\lambda m - b_0} \left(\frac{1}{l} I - S^{l,\lambda m - b_0} \right)^{-1} \Phi.$$

and observe that $u = S_1^{l,-b_0} \left((\lambda m + l) w + \Phi \right)$ solves (75). Finally, if $\Phi > 0$, then $w > 0$ on $\partial\Omega \times \mathbb{R}$ and since

$$u = S_1^{l+R,-b_0} \left((\lambda m Tr (u) + (\mu + l + R) Tr (u)) \right),$$

Lemma 2.18 (iii) gives $ess \inf_{\Omega \times \mathbb{R}} u > 0$.■

Let $\lambda_1 (m)$ (respectively $\lambda_{-1} (m)$) be the positive (resp. negative) principal eigenvalue for the weight m with the convention that $\lambda_1 (m) = +\infty$ (respectively $\lambda_{-1} (m) = -\infty$) if there not exists such a principal eigenvalue. From the properties of μ_m, Theorem 6.2 gives the following

Corollary 6.3. *Assume that either* $a_0 > 0$ *or* $b_0 > 0$*. Then the interval* $(\lambda_{-1} (m), \lambda_1 (m))$ *does not contains eigenvalues for problem (55). If* $a_0 = 0$ *and* $b_0 = 0$*, the same is true for the intervals* $(\lambda_{-1} (m), 0)$ *and* $(0, \lambda_1 (m))$.

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