C*-MODULAR VECTOR STATES*

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Abstract

Let $\mathcal{B}$ be a C*-algebra and X a Hilbert C* $\mathcal{B}$-module. If $p \in \mathcal{B}$ is a projection, let $\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}$ be the p-sphere of X. For φ a state of $\mathcal{B}$ with support p in $\mathcal{B}$ and $x \in \mathcal{S}_p(X)$, consider the *modular vector* state φ_x of $\mathcal{L}_\mathcal{B}(X)$ given by $\varphi_x(t) = \varphi(\langle x, t(x)\rangle)$. The spheres $\mathcal{S}_p(X)$ provide fibrations

$$\mathcal{S}_p(X) \to \mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\},\ x \mapsto \varphi_x,$$

and

$$\mathcal{S}_p(X) \times \{\text{states with support } p\} \to \Sigma_{p,X} = \{ \text{ modular vector states } \},$$

$$(x, \varphi) \mapsto \varphi_x.$$

These fibrations enable us to examine the homotopy type of the sets of modular vector states, and relate it to the homotopy type of unitary groups and spaces of projections. We regard modular vector states as generalizations of pure states to the context of Hilbert C*-modules, and the above fibrations as generalizations of the projective fibration of a Hilbert space.

Keywords: state space, C*–module.

1 Introduction

Unit vectors ξ in a Hilbert space H induce states ω_ξ, called pure states or vector states, in the algebra $\mathcal{B}(H)$ of bounded operators on H:

$$\omega_\xi(t) =< \xi, t\xi >,\ t \in \mathcal{B}(H).$$

There is a natural map

$$\mathcal{S}_1(H) \to \{ \text{ pure states } \},\ \xi \mapsto \omega_\xi,$$

where $\mathcal{S}_1(H)$ denotes the unit sphere of H. The fibres of this map are (homeomorphic to) the torus $\{\lambda \in \mathbb{C} : |\lambda| = 1\}$, i.e. this map is the fibration of the unit sphere over the projective space of H.

Hilbert C*-modules (see the definition below) are generalizations of Hilbert space. In this paper we construct analogues of pure states in the context of Hilbert C*-modules, and consider the

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projective fibrations which correspond to these states. We study the topological properties of this set of states, which are states of the C*-algebra $\mathcal{L}_{\mathcal{B}}(X)$ of adjointable operators of the module X.

Let $\mathcal{B}$ be a C*- algebra. A a right C*-module X over $\mathcal{B}$ [10] is a right $\mathcal{B}$-module, with a $\mathcal{B}$-valued inner product $\langle\ ,\ \rangle$, which is linear in the second variable, having the following properties $(x, y, z \in X,\ b \in \mathcal{B})$,

1. $\langle x, yb\rangle = \langle x, y\rangle b$.
2. $\langle x, y\rangle^* = \langle y, x\rangle$.
3. $\langle x, x\rangle \geq 0$.
4. $x \neq 0$ implies $\langle x, x\rangle \neq 0$.

Furthermore, X is assumed to be complete with the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. As is usual notation, $\mathcal{L}_{\mathcal{B}}(X)$ is the C* algebra of operators $t : X \to X$ which posess an adjoint t^* satisfying $\langle tx, y\rangle = \langle x, t^*y\rangle$. The spheres of X where considered in [1],[2]: if $p \in \mathcal{B}$ is a projection, let $\mathcal{S}_p(X)$ be the p-sphere of X,

$$\mathcal{S}_p(X) = \{x \in X : \langle x, x\rangle = p\}.$$

If $x \in X$, the map $\mathcal{L}_{\mathcal{B}}(X) \to \mathcal{B}$, $t \mapsto \langle x, tx\rangle$ is positive. Therefore, if φ is a positive functional of $\mathcal{B}$, then $t \mapsto \varphi(\langle x, tx\rangle)$ is a positive functional of $\mathcal{L}_{\mathcal{B}}(X)$. This functional is a state if we require that $x \in \mathcal{S}_p(X)$ and $\varphi(p) = 1$, i.e. the support projection of φ is at least p. Let us consider states of $\mathcal{B}$ with support *equal* to p, and denote this set of states by $\Sigma_p(\mathcal{B})$. For each $\varphi \in \Sigma_p(\mathcal{B})$ and each $x \in \mathcal{S}_p(X)$, we denote by φ_x the modular vector state of $\mathcal{L}_{\mathcal{B}}(X)$ with density (φ, x), given by

$$\varphi_x(t) = \varphi(\langle x, tx\rangle).$$

The sphere $\mathcal{S}_p(X)$ lies over a set of modular vector states: fix $\varphi \in \Sigma_p(\mathcal{B})$, and put

$$\mathcal{S}_p(X) \to \mathcal{O}_\varphi := \{\varphi_x : x \in \mathcal{S}_p(X)\},\ x \mapsto \varphi_x.$$

If we let both $x \in \mathcal{S}_p(X)$ and $\varphi \in \Sigma_p(\mathcal{B})$ vary, we obtain

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X} := \{\varphi_x : x \in \mathcal{S}_p(X), \varphi \in \Sigma_p(\mathcal{B})\},\ (x, \varphi) \mapsto \varphi_x.$$

We introduce in $\mathcal{O}_\varphi$ and $\Sigma_{p,X}$ the metrics d_φ and d (respectively), given by:

1. $d_\varphi(\Phi, \Psi) = \inf\{\|x - y\| : \varphi_x = \Phi, \varphi_y = \Psi\}$.
2. $d(\Phi, \Psi) = \|\Phi - \Psi\| + \|supp(\Phi) - supp(\Psi)\|$.

The (convex) set $\Sigma_p(\mathcal{B})$ is considered with the relative topology induced by the usual norm of the conjugate space of $\mathcal{B}$. These metrics d and d_φ do come up naturally if we look for continuity of the projective maps above.

If $x, y \in X$, let $\theta_{x,y} \in \mathcal{L}_{\mathcal{B}}(X)$ be the "rank one" operator given by $\theta_{x,y}(z) = x\langle y, z\rangle$. If $\langle x, x\rangle = p$ then the operator $\theta_{x,x} = e_x$ is a selfadjoint projection, and all projections arising in this manner, from vectors on $\mathcal{S}_p(X)$, are mutually (Murray-von Neumann) equivalent. It turns out that modular vector states are precisely the states of $\mathcal{L}_{\mathcal{B}}(X)$ with support of rank one, i.e. equal to one of these projections e_x.

The contents of the paper are as follows. In section 2 we establish basic facts concerning modular vector states. Section 3 is devoted to the set $\mathcal{O}_\varphi$ and the map $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$, $\sigma(x) = \varphi_x$. Under a suitable hypothesis, we prove that σ is a locally trivial fibre bundle, a fact which enables us to examine the homotopy type of $\mathcal{O}_\varphi$. For example, we obtain that if $p\mathcal{B}p$ is a finite von Neumann algebra and the $p\mathcal{B}p$-module Xp is selfdual [11], then $\mathcal{O}_\varphi$ is simply connected. In section 4 we study $\Sigma_{p,X}$ and the map $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X}$, $(x, \varphi) \mapsto \varphi_x$. This map is a principal fibre bundle. Again this fact is used to study the homotopy type of $\Sigma_{p,X}$. For instance, if $p\mathcal{B}p$ is a von Neumann algebra and Xp is selfdual, then $\Sigma_{p,X}$ is simply connected. Both in sections 3 and 4, the special case of Kasparov's module $X = H_{\mathcal{B}} = H \otimes \ell^2$ is considered. Here we use the fact that $\mathcal{S}_p(H_{\mathcal{B}})$ is contractible [2], to compute the homotopy groups of $\mathcal{O}_\varphi$ and $\Sigma_{p,X}$.

2 Basic facts

Let us establish some basic facts and notations about the vector states φ_x.

We shall be concerned with states of $\mathcal{B}$ that have their support in $\mathcal{B}$, a fact which holds automatically if $\mathcal{B}$ is a von Neumann algebra and φ is normal.

Each element $x \in \mathcal{S}_p(X)$ gives rise to a (non unital) $*$-isomorphism

$$i_x : p\mathcal{B}p \to \mathcal{L}_{\mathcal{B}}(X),\ i_x(a) = \theta_{xa,x}.$$

Fix $x_0 \in \mathcal{S}_p(X)$. Let us recall from [2] the following principal fibre bundle, which we call the projective bundle

$$\rho : \mathcal{S}_p(X) \to \mathcal{E}_{e_{x_0}} = \{\text{projections in } \mathcal{L}_{\mathcal{B}}(X) \text{ equivalent to } e_{x_0}\}$$

given by $\rho(x) = e_x$. Note that $\mathcal{E}_{e_{x_0}}$ depends only on p and not on the choice of x_0 (all projections on $\mathcal{E}_{e_{x_0}}$ are of the form e_x for some $x \in \mathcal{S}_p(X)$). The structure group of the projective bundle is the unitary group $U_{p\mathcal{B}p}$ of $p\mathcal{B}p$.

As is usual notation, if φ is a faithful state of $\mathcal{B}$, $\mathcal{B}^\varphi$ is the centralizer algebra of φ, i.e. $\mathcal{B}^\varphi = \{a \in \mathcal{B} : \varphi(ab) = \varphi(ba) \text{ for all } b \in \mathcal{B}\}$. If the support $supp(\varphi) = p < 1$, then denote by $\mathcal{B}_p^\varphi$ the centralizer of the restriction of φ to the reduced algebra $p\mathcal{B}p$.

Typically a, b, c will denote elements of $\mathcal{B}$, x, y, z elements of X and r, s, t elements of $\mathcal{L}_{\mathcal{B}}(X)$. $\mathcal{B}''$ will denote the von Neumann enveloping algebra of $\mathcal{B}$, and X' the selfdual completion of X, which is a C*-module over $\mathcal{B}''$ [11]. By fibre bundle we mean a locally trivial fibre bundle, and by fibration we mean a surjective map having the homotopy lifting property [13].

Lemma 2.1 *Let φ be a state of $\mathcal{B}$ with $supp(\varphi) = p \in \mathcal{B}$, and x an element in $\mathcal{S}_p(X)$. Then $supp(\varphi_x) = e_x$.*

Proof. Clearly $\varphi_x(e_x) = \varphi(\langle x, e_x(x)\rangle) = \varphi(\langle x, xp\rangle) = \varphi(p) = 1$. Put $r = supp(\varphi_x) \in \mathcal{L}_{\mathcal{B}''}(X')$. We have $r \leq e_x$, i.e. $e_x r = r e_x = r$. This implies that r is of the form $e_y = \theta_{y,y}$, namely, $y = r(x) \in X'$. Now $\langle r(x), r(x)\rangle = q$ is a projection in $\mathcal{B}''$, with $q \leq p$. Indeed,

$$\begin{aligned}\langle r(x), r(x)\rangle\langle r(x), r(x)\rangle &= \langle r(x), r(x)\langle r(x), r(x)\rangle\rangle \\ &= \langle r(x), \theta_{r(x),r(x)}(r(x))\rangle = \langle r(x), r(x)\rangle.\end{aligned}$$

And $\langle r(x), r(x)\rangle p = \langle r(x), r(xp)\rangle = \langle r(x), r(x)\rangle$, i.e. $q \leq p$. Now it is clear that $\varphi(q) = \varphi(\langle r(x), r(x)\rangle) = \varphi(\langle x, r(x)\rangle) = \varphi_x(r) = 1$, which implies that $q = p$, Therefore $\langle r(x) - x, r(x) - x\rangle = \langle r(x), r(x)\rangle + \langle x, x\rangle - \langle r(x), x\rangle - \langle x, r(x)\rangle = 0$, since all these products equal p (because $\langle r(x), x\rangle = \langle r^2(x), x\rangle = \langle r(x), r(x)\rangle$). Finally, $r(x) = x$ implies that $r = e_{r(x)} = e_x$. □

Lemma 2.2 *Let Φ be a state of $\mathcal{L}_{\mathcal{B}}(X)$ with $supp(\Phi) = e_x$ for some $x \in \mathcal{S}_p(X)$. Then $\Phi = \varphi_x$ for φ a state in $\mathcal{B}$ with $supp(\varphi) = p$. Namely $\varphi(a) = \Phi(i_x(a))$.*

Proof. Put $\varphi = \Phi \circ i_x$ as above. First note that if $t \in \mathcal{L}_{\mathcal{B}}(X)$, then $e_x t e_x = \theta_{x\langle x,t(x)\rangle,x}$. Then $\varphi_x(t) = \Phi(i_x(\langle x, t(x)\rangle)) = \Phi(\theta_{x\langle x,t(x)\rangle,x}) = \Phi(e_x t e_x) = \Phi(t)$. It remains to see that $supp(\varphi) = p$. Clearly $\varphi(p) = \Phi(\theta_{xp,x}) = \Phi(e_x) = 1$. Suppose that $q \leq p$ is a projection in $\mathcal{B}''$ with $\varphi''(q) = \Phi(\theta_{xq,x}) = 1$ (φ'' here denotes the normal extension of the former φ to $\mathcal{B}''$). Note that $\theta_{xq,x} = \theta_{xq,xq} = e_{xq}$ is in fact a projection (associated to $xq \in \mathcal{S}_q(X')$), and verifies $e_{xq} \leq e_x$. It follows that $\theta_{xq,xq} = \theta_{x,x}$. Then $xq = \theta_{xq,xq}(x) = \theta_{x,x}(x) = x$, and therefore $q = p$. □

Remark 2.3 If $\mathcal{B}$ is a von Neumann algebra, the inner product of X is weakly continuous, and the state Φ of the preceding result is normal, then $\varphi = \Phi \circ i_x$ is also normal.

Proposition 2.4 *Let $\psi, \varphi \in \Sigma_p(\mathcal{B})$, $x, y \in \mathcal{S}_p(X)$. Then*

a) $\varphi_x = \psi_x$ *if and only if* $\varphi = \psi$.

b) $\varphi_x = \psi_y$ *if and only if* $\psi = \varphi \circ Ad(u)$, *with* $y = xu$ *and* $u \in U_{p\mathcal{B}p}$.

c) $\varphi_x = \varphi_y$ *if and only if* $y = xv$, *for* v *a unitary element in* $\mathcal{B}_p^\varphi$.

Proof. Let us start with a): $\varphi(b) = \varphi_x(\theta_{xb,x}) = \psi_x(\theta_{xb,x}) = \psi(b)$.

To prove b), suppose that $\varphi_x = \psi_y$. Then they have the same support, i.e. $e_x = e_y$, which implies that there exists a unitary element $u \in U_{p\mathcal{B}p}$ such that $y = xu$ (see [2]). Then

$$\varphi_x(t) = \psi_y(t) = \psi(\langle xu, t(xu)\rangle) = \psi(u^*\langle x, t(x)\rangle u) = [\psi \circ Ad(u^*)]_x(t).$$

Using part a), this implies that $\varphi = \psi \circ Ad(u^*)$, or $\psi = \varphi \circ Ad(u)$.

To prove c), use b), and note that the unitary element $u \in U_{p\mathcal{B}p}$ satisfies $\varphi = \varphi \circ Ad(u)$, i.e. $u \in \mathcal{B}_p^\varphi$. □

3 The set $\mathcal{O}_\varphi$

In this section we consider the set $\mathcal{O}_\varphi = \{\varphi_x : x \in \mathcal{S}_p(X)\}$ for a fixed state φ of $\mathcal{B}$ with $supp(\varphi) = p$. Note that in the particular case when $X = \mathcal{B}$ is a finite von Neumann algebra and $p = 1$, then $\mathcal{O}_\varphi$ is the unitary orbit of φ. In the general case, there is a natural map

$$\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi, \ \sigma(x) = \varphi_x.$$

Let us consider the following metric in $\mathcal{O}_\varphi$:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x' - y'\| : x', y' \in \mathcal{S}_p(X), \varphi_{x'} = \varphi_x, \varphi_{y'} = \varphi_y\}.$$

It is clear that this metric induces the same topology as the quotient topology given by the map σ. Moreover, in view of 2.4, it can be computed as follows:

$$d_\varphi(\varphi_x, \varphi_y) = \inf\{\|x - yv\| : v \text{ unitary in } \mathcal{B}_p^\varphi\}.$$

First note that this is indeed a metric. For instance, if $d_\varphi(\varphi_x, \varphi_y) = 0$ then there exist unitaries v_n in $\mathcal{B}_p^\varphi$ such that $\|x - yv_n\| \to 0$, i.e. $yv_n \to x$ in $\mathcal{S}_p(X)$. In particular yv_n is a Cauchy sequence, and therefore v_n is a Cauchy sequence, converging to a unitary v in $\mathcal{B}_p^\varphi$. Then $x = yv$ and $\varphi_x = \varphi_y$. The other properties follow similarly.

With this metric $\mathcal{O}_\varphi$ is homeomorphic to the quotient $\mathcal{S}_p(X)/U_{\mathcal{B}_p^\varphi}$. The following result implies that the inclusion $\mathcal{O}_\varphi \subset \mathcal{B}^*$ (=conjugate space of $\mathcal{B}$) is continuous.

Lemma 3.1 *If $x, y \in \mathcal{S}_p(X)$, then $\|\varphi_x - \varphi_y\| \le 2\|x - y\|$. In particular*

$$\|\varphi_x - \varphi_y\| \le 2 d_\varphi(\varphi_x, \varphi_y)$$

where the norm $\| \ \|$ of the functionals denotes the usual norm of the conjugate space $\mathcal{B}^*$.

Proof. If $t \in \mathcal{L}_\mathcal{B}(X)$, then $|\varphi_x(t) - \varphi_y(t)| \le |\varphi(\langle x, t(x - y)\rangle)| + |\varphi(\langle x - y, ty\rangle)|$. By the Cauchy-Schwarz inequality $\|\langle x, t(x - y)\rangle\| \le \|t\| \, \|x - y\|$, and $\|\langle x - y, ty\rangle\| \le \|x - y\| \, \|t\|$. Then $\|\varphi_x(t) - \varphi_y(t)\| \le 2\|t\| \, \|x - y\|$, and the result follows. □

In order that the map $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$ be a locally trivial fibre bundle, we make the following assumption:

Hypothesis 3.2 *There exists a conditional expectation* $E_\varphi : p\mathcal{B}p \to \mathcal{B}_p^\varphi$*.*

This is the case if for example $\mathcal{B}$ is a von Neumann algebra and φ is normal. For the remaining of this section, we suppose that 3.2 holds.

Theorem 3.3 *The map* $\sigma : \mathcal{S}_p(X) \to \mathcal{O}_\varphi$*,* $\sigma(x) = \varphi_x$ *is a locally trivial fibre bundle. The fibre of this bundle is the unitary group* $U_{\mathcal{B}_p^\varphi}$ *of* $\mathcal{B}_p^\varphi$*.*

Proof. Since the spaces are homogeneous spaces, it suffices to show that there exist continuous local cross sections at every point x_0 of $\mathcal{S}_p(X)$. Suppose that $d_\varphi(\varphi_x, \varphi_{x_0}) < r < 1$. Then there exists a unitary operator v in $\mathcal{B}_p^\varphi$ such that $\|xv - x_0\| < 1$. Then

$$\|p - \langle xv, x_0\rangle\| = \|\langle x_0, x_0\rangle - \langle xv, x_0\rangle\| = \|\langle x_0 - xv, x_0\rangle\| \leq \|x_0 - xv\| < 1.$$

It follows that $\langle xv, x_0\rangle$ is invertible in $p\mathcal{B}p$. Therefore, we can find r such that also $E_\varphi(\langle xv, x_0\rangle) = v^* E_\varphi(\langle x, x_0\rangle)$ is invertible in $\mathcal{B}_p^\varphi$. Then $E_\varphi(\langle x, x_0\rangle)$ is invertible. Let us put

$$\eta_{x_0}(\varphi_x) = x\mu(E_\varphi(\langle x, x_0\rangle))$$

defined on $\{\varphi_x : d_\varphi(\varphi_x, \varphi_{x_0}) < r\}$, where μ denotes the unitary part in the polar decomposition (of invertible elements) in $\mathcal{B}_p^\varphi$: $c = \mu(c)(c^*c)^{1/2}$. First note that η_{x_0} is well defined. If x' is a vector in the fibre of φ_x, then $x' = xv$ for $v \in U_{\mathcal{B}_p^\varphi}$. Then $x'\mu(E_\varphi(\langle x', x_0\rangle)) = xv\mu(v^* E_\varphi(\langle x, x_0\rangle)) = x\mu(E_\varphi(\langle x, x_0\rangle))$, where the last equality holds because $\mu(ua) = u\mu(a)$ if u is unitary. Note that $\eta_{x_0}(\varphi_{x_0}) = x_0\mu(E_\varphi(\langle x_0, x_0\rangle)) = x_0$, and η_{x_0} is a cross section for σ, because $\mu(E_\varphi(\langle x, x_0\rangle))$ is a unitary in $\mathcal{B}_p^\varphi$. Finally, let us prove that η_{x_0} is continuous. Suppose that $d_\varphi(\varphi_{x_n}, \varphi_x) \to 0$, then there exist unitaries v_n in $\mathcal{B}_p^\varphi$ such that $x_n v_n \to x$. Then by the continuity of the operations, $x_n v_n \mu(E_\varphi(\langle x_n v_n, x_0\rangle)) = x_n\mu(E_\varphi(\langle x_n, x_0\rangle)) \to x\mu(E_\varphi(\langle x, x_0\rangle))$, i.e., $\eta_{x_0}(x_n) \to \eta_{x_0}(x)$. It is clear from 2.4 that the fibre is $U_{\mathcal{B}_p^\varphi}$. Namely, $\sigma^{-1}(\varphi_x) = \{xv : v \in U_{\mathcal{B}_p^\varphi}\}$. Note that $xv_n \to xv$ in $\sigma^{-1}(\varphi_x) \subset \mathcal{S}_p(X)$ if and only if $v_n \to v$ in $U_{\mathcal{B}_p^\varphi}$. □

We shall use the following result, which is a straightforward fact from the theory of fibrations.

Lemma 3.4 *Suppose that one has the following commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & X \\ & \searrow^{\pi_2} & \downarrow p \\ & & Y, \end{array}$$

where E, X, Y *are topological spaces,* π_1*,* π_2 *are fibrations and* p *is continuous and surjective. Then* p *is also a fibration.*

There is another natural bundle associated to $\mathcal{O}_\varphi$, which is the mapping

$$\mathcal{O}_\varphi \to \mathcal{E}_e, \ \varphi_x \mapsto e_x,$$

where e is any projection of the form e_{x_0} for some $x_0 \in \mathcal{S}_p(X)$. Since $e_x = supp(\varphi_x)$, we shall call this map *supp*. In general, taking support of positive functionals does not define a continuous map. However it is continuous in this context, i.e. restricted to the set $\mathcal{O}_\varphi$ with the metric d_φ. Indeed, as seen before, convergence of $\varphi_{x_n} \to \varphi_x$ in this metric implies the existence of unitaries v_n of $\mathcal{B}_p^\varphi \subset p\mathcal{B}p$ such that $x_n v_n \to x$ in $\mathcal{S}_p(X)$. This implies that $e_{x_n v_n} = e_{x_n} \to e_x$. Moreover, we have:

Theorem 3.5 *The map supp* $: \mathcal{O}_\varphi \to \mathcal{E}_e$ *is a fibration with fibre* $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$*. The following diagram of fibre bundles*

$$\begin{array}{ccc} \mathcal{S}_p(X) & \xrightarrow{\rho} & \mathcal{O}_\varphi \\ & \searrow_\sigma & \downarrow supp \\ & & \mathcal{E}_e \end{array}$$

is commutative.

Proof. It is apparent that the diagram commutes. Since ρ and σ are fibre bundles, it follows using 3.4 that *supp* is a fibration. Note that if $e_x = e_y$, then there exists $u \in U_{p\mathcal{B}p}$ such that $y = xu$. Then $\varphi_y = \varphi_{xu} = (\varphi \circ Ad(u^*))_x$, therefore $supp^{-1}(e_x) = \{(\varphi \circ Ad(u^*))_x : u \in U_{p\mathcal{B}p}\}$. In 2.4 it was shown that for φ and ψ states of $\mathcal{B}$ with support p, $\varphi_x = \psi_x$ implies $\varphi = \psi$. Then $\{\varphi \circ Ad(u) : u \in U_{p\mathcal{B}p}\}$ parametrizes the fibres of *supp*. Clearly this set is in one to one correspondence with $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. Moreover, $(\varphi \circ Ad(u_n))_x \to (\varphi \circ Ad(u))_x$ in $\mathcal{O}_\varphi$ if and only if $\inf_{v \in U_{\mathcal{B}_p^\varphi}} \|xu_n - xuv\| = \inf_{v \in U_{\mathcal{B}_p^\varphi}} \|u_n - uv\|$, i.e. the class of u_n converges to the class of u in $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$ (with the quotient topology induced by the norm of $\mathcal{B}$). □

The homotopy exact sequences of these bundles can be used to relate the homotopy groups of $\mathcal{O}_\varphi$, $\mathcal{S}_p(X)$, $\mathcal{E}_e$, $U_{p\mathcal{B}p}$, $U_{\mathcal{B}_p^\varphi}$ and $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$. Namely:

$$\ldots \pi_n(U_{\mathcal{B}_p^\varphi}, p) \to \pi_n(\mathcal{S}_p(X), x_0) \xrightarrow{\sigma_*} \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \to \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p) \to \ldots$$

where x_0 is a fixed element in $\mathcal{S}_p(X)$, and

$$\ldots \pi_n(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \xrightarrow{supp_*} \pi_n(\mathcal{E}, e_{x_0}) \to \pi_{n-1}(U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}, [p]) \to \ldots$$

with φ a fixed state in $\Sigma_p(\mathcal{B})$.

The first result uses the fact that σ is continuous and surjective.

Corollary 3.6 *If* $p\mathcal{B}p$ *is a finite von Neumann algebra, then* $\mathcal{O}_\varphi$ *is arcwise connected.*

Proof. If $p\mathcal{B}p$ is finite, it was shown in [2] that $\mathcal{S}_p(X)$ is connected. □

Corollary 3.7 *If* $p\mathcal{B}p$ *is a von Neumann algebra and the restriction of* φ *to* $p\mathcal{B}p$ *is normal, then*

$$\pi_1(\mathcal{O}_\varphi, \varphi_x) \cong \pi_1(\mathcal{E}_e, e_x).$$

If moreover Xp *is selfdual, then* $\pi_1(\mathcal{O}_\varphi, \varphi_x) = 0$.

Proof. The proof of the first assertion follows regarding the tail of the homotopy exact sequence of the bundle *supp*. Recall from [4] that the fibre $U_{p\mathcal{B}p}/U_{\mathcal{B}_p^\varphi}$ is simply connected. If Xp is selfdual, it was proven in [2] that the connected components of $\mathcal{E}_e$ are simply connected. □

Corollary 3.8 *If* $p\mathcal{B}p$ *is a von Neumann algebra,* φ *restricted to* $p\mathcal{B}p$ *is normal and* Xp *is selfdual, then, for a fixed* $x_0 \in \mathcal{S}_p(X)$*, the inclusion map*

$$i : U_{\mathcal{B}_p^\varphi} \hookrightarrow \mathcal{S}_p(X), \;\; v \mapsto x_0 v$$

induces an epimorphism

$$i_* : \pi_1(U_{\mathcal{B}_p^\varphi}, p) \to \pi_1(\mathcal{S}_p(X), x_0).$$

Proof. This fact is proved using the homotopy exact sequence of σ, and the fact that in this case $\pi_1(\mathcal{O}_\varphi, \varphi_{x_0}) = 0$. $\square$

In other words, this result says that regardless of the size of the selfdual module X, any closed continuous curve $x(t) \in \mathcal{S}_p(X)$ with $x(0) = x(1) = x_0$ is homotopic to a closed curve of the form $x_0(t) = x_0 v(t)$, with $v(t)$ a curve of unitaries in $\mathcal{B}_p^\varphi$, such that $v(0) = v(1) = p$.

Corollary 3.9 *Suppose that X is selfdual. If either*

a) $p\mathcal{B}p$ is a properly infinite von Neumann algebra,
or

b) $p\mathcal{B}p$ is a von Neumann algebra of type II_1 with $\mathcal{L}_\mathcal{B}(X)$ properly infinite,

then for $n \geq 1$

$$\pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \cong \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p).$$

Proof. In either situation, a) or b), we have that $\mathcal{S}_p(X)$ is contractible (see [2]). The proof follows by examining the homotopy exact sequence of the fibre bundle σ. $\square$

Situation b) occurs for example if $p\mathcal{B}p$ is a II_1 factor and Xp is not finitely generated over $p\mathcal{B}p$.

Finally let us state an analogous result for general C*-algebras $\mathcal{B}$ (under the hypothesis 3.2), for the module $X = H_\mathcal{B} = \mathcal{B} \otimes \ell^2$. Here we use the fact [2], that $\mathcal{S}_p(H_\mathcal{B})$ is contractible. The proof follows similarly as above.

Corollary 3.10 *If 3.2 holds, and $X = H_\mathcal{B}$, then for $n \geq 1$*

$$\pi_n(\mathcal{O}_\varphi, \varphi_{x_0}) \cong \pi_{n-1}(U_{\mathcal{B}_p^\varphi}, p).$$

These two results establish that in these cases, if $U_{\mathcal{B}_p^\varphi}$ is connected, then $\pi_1(\mathcal{O}_\varphi, \varphi_{x_0})$ is trivial. This is granted if $p\mathcal{B}p$ is a von Neumann algebra, and the restriction of φ to $p\mathcal{B}p$ is normal. However, note that $\pi_2(\mathcal{O}_\varphi, \varphi_{x_0})$ is not trivial. This is because $\mathcal{B}_p^\varphi$ is a finite von Neumann algebra and therefore $U_{\mathcal{B}_p^\varphi}$ has non trivial fundamental group (see [7], [12]).

4 Vector states in $\mathcal{L}_\mathcal{B}(X)$

In 2.2 it was shown that a state Φ of $\mathcal{L}_\mathcal{B}(X)$ with support $e = e_x$ for some $x \in \mathcal{S}_p(X)$, is of the form $\Phi = \varphi_x$ for some state φ in $\mathcal{B}$ with support p. Recall that we denote by $\Sigma_p(\mathcal{B})$ the set of states of $\mathcal{B}$ with support p, and by $\Sigma_{p,X}$ the set of states of $\mathcal{L}_\mathcal{B}(X)$ with support equivalent to e. In other words, $\Sigma_{p,X} = \cup_{\varphi \in \Sigma_p(\mathcal{B})} \mathcal{O}_\varphi$. There is a natural assignment

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X} \quad , \quad (x, \varphi) \mapsto \varphi_x.$$

Recall that $\varphi_x = \psi_y$, with $\varphi, \psi \in \Sigma_p(\mathcal{B})$, $x, y \in \mathcal{S}_p(X)$ if and only if $\psi = \varphi \circ Ad(u)$ and $y = xu$ with $u \in U_{p\mathcal{B}p}$ (see 2.4 part c)).

The unitary group $U_{p\mathcal{B}p}$ acts both on $\mathcal{S}_p(X)$ and $\Sigma_p(\mathcal{B})$. We may consider the diagonal action on $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$, defined by $u \cdot (x, \varphi) = (xu, \varphi \circ Ad(u))$. It follows that if we denote the quotient

$$\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) / \{(x, \varphi) \sim (xu, \varphi \circ Ad(u)), u \in U_{p\mathcal{B}p}\} := \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$$

(as is usual notation), then the assignment above induces a bijection

$$\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \simeq \Sigma_{p,X}.$$

If we endow $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$ with the quotient topology (where $\mathcal{S}_p(X)$ and $\Sigma_p(\mathcal{B})$ are considered with the norm topologies), a natural question is: what topology does this bijection induce in $\Sigma_{p,X}$? The following result states that convergence of a sequence in the quotient topology is equivalent in $\Sigma_{p,X}$ to convergence (in norm) of the states and their supports.

Proposition 4.1 *Consider in $\Sigma_{p,X}$ the metric d given by*

$$d(\Phi,\Psi) = \|\Phi - \Psi\| + \|supp(\Phi) - supp(\Psi)\|.$$

Then the metric space $(\Sigma_{p,X}, d)$ is homeomorphic to $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$, where the homeomorphism is given by the above bijection.

Proof. Denote by $[(x,\varphi)]$ the class of (x,φ) in $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Suppose that $[(x_n,\varphi_n)]$ converge to $[(x,\varphi)]$ in $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Then there exist unitaries u_n in $p\mathcal{B}p$ such that $x_n u_n$ converges to x and $\varphi_n \circ Ad(u_n)$ converges to φ, in the respective norms. By continuity of the inner product, it is clear then that $e_{x_n} = \theta_{x_n,x_n} = \theta_{x_n u_n, x_n u_n} \to e_x$ and $\varphi_{n\,x_n} = (\varphi_n \circ Ad(u_n))_{x_n u_n} \to \varphi_x$. Therefore the assignment $[(x,\varphi)] \mapsto \varphi_x$ is continuous. In the other direction, suppose that $d(\Phi_n, \Phi)$ tends to zero. There exist $\varphi_n, \varphi \in \Sigma_p(\mathcal{B})$ and $x_n, x \in \mathcal{S}_p(X)$ such that $\Phi_n = \varphi_{n\,x_n}$ and $\Phi = \varphi_x$. We have that $supp(\Phi_n) = e_{x_n} \to supp(\Phi) = e_x$. Since $e_{x_n} = \rho(x_n)$, $e_x = \rho(x)$, and $\rho : \mathcal{S}_p(X) \to \mathcal{E}_e$ is a fibre bundle with fibre $U_{p\mathcal{B}p}$, then there exist unitaries u_n in $p\mathcal{B}p$ such that $x_n u_n \to x$. We may replace x_n by $y_n = x_n u_n$ and φ_n by $\psi_n = \varphi_n \circ Ad(u_n)$, and still have $\Phi_n = \psi_{n\,y_n}$, with $y_n \to x$. We claim that $\psi_n \to \varphi$. Indeed, if $a \in \mathcal{B}$, by a typical argument

$$|\psi_n(a) - \varphi(a)| = |\Phi_n(\theta_{y_n a, y_n}) - \Phi(\theta_{xa,x})| \leq \|\Phi_n\|\|\theta_{y_n a, y_n} - \theta_{xa,x}\| + \|\Phi_n - \Phi\|\|\theta_{xa,x}\|.$$

The first summand is bounded (using the Cauchy-Schwarz inequality):

$$\|\theta_{y_n a, y_n} - \theta_{xa,x}\| \leq \|\theta_{y_n a, y_n - x}\| + \|\theta_{y_n a - xa, x}\| \leq \|y_n\|\|a\|\|y_n - x\| + \|y_n - x\|\|a\|\|x\|,$$

which equals $2\|a\|\|y_n - x\|$. The other summand equals $\|\Phi_n - \Phi\|\|a\|$. It follows that $[(y_n,\psi_n)] = [(x_n,\varphi_n)] \to [(x,\varphi)]$. □

By the inequality $\|\Phi - \Psi\| \leq d(\Phi,\Psi)$, it is clear that the inclusion $(\Sigma_{p,X}, d) \subset (\mathcal{L}_{\mathcal{B}}(X)^*, \|\ \|)$ is continuous. The following example shows that the topology given by the metric d in $\Sigma_{p,X}$ does not coincide with the norm topology of the conjugate space of $\mathcal{L}_{\mathcal{B}}(X)$. In other words, convergence of the vector states (which a priori have equivalent supports) does not imply convergence of the supports.

Example 4.2 Let $\mathcal{B} = D \subset B(\ell^2(\mathbb{N}))$ be the subalgebra of diagonal matrices (with respect to the canonical basis). Consider the conditional expectation $E : B(\ell^2(\mathbb{N})) \to D$ which consists on deleting all non diagonal entries. Let $a \in D$ be a trace class positive diagonal operator with trace one, and no zero entries in the diagonal. Put $\varphi(x) = Tr(ax)$, $x \in B(\ell^2(\mathbb{N}))$. Clearly, φ is faithful and $B(\ell^2(\mathbb{N}))^{\varphi} = D$. Let b be the unilateral shift in $\ell^2(\mathbb{N})$. Denote by q_n the $n \times n$ Jordan nilpotent, and w_n the unitary operator on $\ell^2(\mathbb{N})$ having the unitary matrix $q_n + {q_n^*}^{\,n-1}$ on the first $n \times n$ corner and the rest of the diagonal completed with 1.

Consider X the D-right module obtained as the completion of $B(\ell^2(\mathbb{N}))$ with the D-valued inner product given by E, i.e. $\langle x, y\rangle = E(x^* y)$, $x, y \in B(\ell^2(\mathbb{N}))$. Note that X is also a $B(\ell^2(\mathbb{N}))$-left module, and so the elements of $B(\ell^2(\mathbb{N}))$ act as adjointable operators in X. Consider the faithful state φ of D equal to the restriction of the former φ. The elements w_n and b lie in the unit sphere of X.

We claim that the projections e_{w_n} do not converge to e_b. Suppose that they do converge. Using that the map $x \mapsto e_x$ is a bundle, there would exist unitaries $v_n \in D$ such that $w_n v_n \to b$ in $\mathcal{S}_1(X)$. In [4] it was shown that the element b cannot be approximated by unitaries of $B(\ell^2(\mathbb{N}))$ in the norm topology of the module X.

On the other hand, the states φ_{w_n} converge to φ_b in the norm topology of the conjugate space of $\mathcal{L}_{\mathcal{B}}(X)$. Indeed

$$|\varphi_{w_n}(t) - \varphi_b(t)| = |Tr\,(a(\langle w_n, t(w_n)\rangle - \langle b, t(b)\rangle))|$$

$$\leq |Tr\,(a(\langle w_n, t(w_n) - t(b)\rangle))| + |Tr\,(a(\langle w_n - b, t(b)\rangle))|\,.$$

The first summand can be bounded by $\|t\|\ Tr(a(2 - E(w_n^* b) - E(b^* w_n)))$. Since $Tr(a) = 1$ and E is trace invariant, this term equals

$$\|t\|\ \ (2 - Tr(aw_n^* b + ab^* w_n)) = 2\|t\| \sum_{k\geq n} a_k,$$

where a_k are the diagonal entries of a. It is clear that this term tends to zero when $n \to \infty$. The other summand can be dealt in a similar way, establishing our claim.

Summarizing, the states φ_{w_n} converge in norm, but their supports e_{w_n} do not.

Next we shall see that the quotient map

$$\wp_1 : \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \quad , \quad \wp_1(x,\varphi) = [(x,\varphi)]$$

and the projection map

$$\wp_2 : \mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B}) \to \mathcal{S}_p(X)/U_{p\mathcal{B}p} \quad , \quad \wp_2([(x,\varphi)]) = [x]$$

are fibrations. Equivalently, if $\Sigma_{p,X}$ is considered with the topology induced by the metric d, the maps $(x,\varphi) \mapsto \varphi_x$ and $\varphi_x \mapsto e_x$ are fibrations. In what follows, for brevity, we shall use $\Sigma_{p,X}$ (considered with the metric d) instead of $\mathcal{S}_p(X) \times_{U_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$. Therefore $\wp_1(x,\varphi) = \varphi_x$.

Theorem 4.3 *The map $\wp_1 : \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) \to \Sigma_{p,X}$, $\wp_1(x,\varphi) = \varphi_x$ is a principal fibre bundle with fibre $U_{p\mathcal{B}p}$.*

Proof. It suffices to exhibit a local cross section around a generic base point φ_x. We claim that there is a neighborhood of φ_x such that elements ψ_y in this neighborhood satisfy that $\langle y, x\rangle$ is invertible. Indeed, if $d(\varphi_x, \psi_y) < r$, then $\|e_x - e_y\| < r$. If we choose r small enough so that e_y lies in the ball around e_x in which a local cross section of $\rho(x) = e_x$ is defined, then there exists a unitary u in $p\mathcal{B}p$ such that $\|x - yu\| < 1$. Note that

$$\|p - \langle yu, x\rangle\| = \|\langle x - yu, x\rangle\| \leq \|x - yu\| < 1.$$

Then $\langle yu, x\rangle = u^*\langle y, x\rangle$ is invertible in $p\mathcal{B}p$, and therefore also $\langle y, x\rangle$. In this neighborhood put

$$s(\psi_y) = (y\mu(\langle y, x\rangle), \psi \circ Ad(\mu(\langle y, x\rangle)),$$

where μ denotes the unitary part in the polar decomposition of invertible elements in $p\mathcal{B}p$ as before. We claim that s is a well defined continuous local cross section.

Suppose that $\psi_y = \psi'_{y'}$, then there exists a unitary u in $p\mathcal{B}p$ such that $y' = yu$ and $\psi = \psi' \circ Ad(u)$. Then $y'\mu(\langle y', x\rangle) = yu\mu(u^*\langle y, x\rangle) = y\mu(\langle y, x\rangle)$. Also, $\psi' \circ Ad(\mu(\langle y', x\rangle)) = \psi \circ Ad(u) \circ Ad(\mu(u^*\langle y, x\rangle)) = \psi \circ Ad(u) \circ Ad(u^*) \circ Ad(\mu(\langle y, x\rangle)) = \psi \circ Ad(\mu(\langle y, x\rangle))$.

It is apparent that s is a cross section. Let us see that s is continuous. Suppose that $\psi_{n_{y_n}} \to \varphi'_{x'}$ for $\varphi'_{x'}$ in the neighborhood of φ_x where s is defined. This implies that there exist unitaries u_n in $U_{p\mathcal{B}p}$ such that $y_n u_n \to x'$ and $\psi_n \circ Ad(u_n) \to \varphi'$ in the norm topologies. The continuity of the inner product implies that $y_n u_n \mu(\langle y_n u_n, x\rangle) = y_n \mu(\langle y_n, x\rangle) \to x'\mu(\langle x', x\rangle)$. Also $\psi_n \circ Ad(u_n) \circ Ad(\mu(\langle y_n u_n, x\rangle)) = \psi_n \circ Ad(\mu(\langle y_n, x\rangle)) \to \varphi' \circ Ad(\mu(\langle x', x\rangle))$. □

Next we consider the map $\wp_2 : \Sigma_{p,X} \to \mathcal{S}_p(X)/U_{p\mathcal{B}p}$. Recall that $\mathcal{S}_p(X)/U_{p\mathcal{B}p} \simeq \mathcal{E}_e$, where the homeomorphism is given by $[x] \mapsto e_x$. In other words, $\wp_2(\varphi_x)$ is the support of φ_x. The following result states that taking support of a state in $\Sigma_{p,X}$ (regarded with the d topology) is a fibration.

Theorem 4.4 *The map $\wp_2 : \Sigma_{p,X} \to \mathcal{S}_p(X)/U_{p\mathcal{B}p}$, given by $\wp_2(\varphi_x) = [x]$ is a fibration with fibre $\Sigma_p(\mathcal{B})$.*

Proof. Consider the diagram

$$\begin{array}{ccc} \mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}) & \xrightarrow{\wp_1} & \Sigma_{p,X} \\ & \searrow p & \downarrow \wp_2 \\ & & \mathcal{S}_p(X)/U_{p\mathcal{B}p}, \end{array}$$

where p is given by $p(x, \varphi) = [x]$. Clearly p is a fibre bundle, because it is the composition of the projective bundle $x \mapsto [x]$ with the projection $(x, \varphi) \mapsto x$. The map $\wp_1$ was shown to be a fibration. It follows from 3.4 that $\wp_2$ is a fibration. The fibre $\wp_2^{-1}([x])$ consists of all states φ_y with $[y] = [x]$. If $[x] = [y]$, there exists $u \in U_{p\mathcal{B}p}$ such that $\varphi_y = (\varphi \circ Ad(u^*))_x$. Then we may fix x (and not just $[x]$). Then $\varphi_x = \psi_x$ implies $\varphi = \psi$. It follows that the fibre over $[x]$ is the set $\{\varphi_x : \varphi \in \Sigma_p(\mathcal{B})\}$, which identifies with $\Sigma_p(\mathcal{B})$. □

We will use the fibrations $\wp_1$ and $\wp_2$ to obtain information about the homotopy type of these spaces.

As in the previous section, applying the homotopy exact sequences of these fibrations, we obtain

$$\ldots \pi_n\left(U_{p\mathcal{B}p}, p\right) \to \pi_n(\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B}), (x_0, \varphi)) \stackrel{(\wp_1)_*}{\to} \pi_n(\Sigma_{p,X}, \varphi_{x_0}) \to \pi_{n-1}(U_{p\mathcal{B}p}, p) \to \ldots$$

and

$$\ldots \pi_n(\Sigma_p(\mathcal{B}), \varphi) \to \pi_n(\Sigma_{p,X}, \varphi_{x_0}) \stackrel{(\wp_2)_*}{\to} \pi_n(\mathcal{E}, e_{x_0}) \to \pi_{n-1}(\Sigma_p(\mathcal{B}), \varphi) \ldots$$

First note that since $\Sigma_p(\mathcal{B})$ is convex, $\mathcal{S}_p(X) \times \Sigma_p(\mathcal{B})$ has the same homotopy type as $\mathcal{S}_p(X)$, and

$$\pi_*(\Sigma_{p,X}) = \pi_*(\mathcal{E}_{e_x}).$$

Note that the space $\mathcal{S}_p(H_\mathcal{B}) \times \Sigma_p(\mathcal{B})$ is contractible.

Corollary 4.5 *For $\varphi_0 \in \Sigma_p(\mathcal{B})$ and $x_0 \in \mathcal{S}_p(H_\mathcal{B})$ fixed,*

$$\pi_n\left(\Sigma_{p,H_\mathcal{B}}, \varphi_{0x_0}\right) \cong \pi_{n-1}(U_{p\mathcal{B}p}, p), \quad n \geq 1.$$

In particular, if $U_{p\mathcal{B}p}$ is connected, $\pi_1\left(\Sigma_{p,H_\mathcal{B}}, \varphi_{0x_0}\right) = 0$. If moreover $p\mathcal{B}p$ is a properly infinite von Neumann algebra, $\Sigma_{p,H_\mathcal{B}}$ has trivial homotopy groups of all orders.

Proof. The first fact follows from the contractibility of $\mathcal{S}_p(H_\mathcal{B}) \times \Sigma_p(\mathcal{B})$. If $p\mathcal{B}p$ is properly infinite, then $U_{p\mathcal{B}p}$ is contractible [6]. □

The fundamental group of the unitary group of a C*-algebra has been computed in many cases ([7], [12], [15]). For example, in the von Neumann algebra case, the fundamental group of the unitary group can be computed in terms of the type decomposition of the algebra.

Corollary 4.6 *If $p\mathcal{B}p$ is a finite von Neumann algebra, then $\Sigma_{p,X}$ is connected.*

Proof. It was noted that if $p\mathcal{B}p$ is finite, then $\mathcal{S}_p(X)$ is connected. □

Corollary 4.7 *If $p\mathcal{B}p$ is a properly infinite algebra, then for $n \geq 0$*

$$\pi_n\left(\Sigma_{p,X}, \varphi_{0x_0}\right) \cong \pi_n(\mathcal{S}_p(X), x_0).$$

If moreover Xp is selfdual, then

$$\pi_n\left(\Sigma_{p,X}, \varphi_{0x_0}\right) = 0$$

for all $n \geq 0$.

Proof. The proof follows considering the homotopy exact sequence of $\wp_1$. If $p\mathcal{B}p$ is properly infinite, its unitary group is contractible. If moreover Xp is selfdual, it was pointed out before that $\mathcal{S}_p(X)$ is contractible. □

We turn now our attention to the bundle $\wp_2$.

Corollary 4.8 *If $p\mathcal{B}p$ is a von Neumann algebra and Xp is selfdual, then $\pi_1(\Sigma_{p,X}, \varphi_x) = 0$*

Proof. It was shown in [2] that $\pi_1(\mathcal{E}_{e_x}) = 0$ □

Remark 4.9 There is another map related to this situation, namely the other projection $\wp_3$,

$$\wp_3 : \Sigma_{p,X} \to \Sigma_p(\mathcal{B})/U_{p\mathcal{B}p}, \ \wp_3(\varphi_x) = [\varphi].$$

This map is well defined and continuous. Going back to the notation $\mathcal{S}_p(X) \times_{u_{p\mathcal{B}p}} \Sigma_p(\mathcal{B})$, $\wp_3$ is the map $(x, \varphi) \mapsto \varphi$ at the quotient level,

$$[(x, \varphi)] \mapsto [\varphi].$$

However this map is not, in general, a fibration. Consider the case when $X = \mathcal{B}$ is a finite algebra, and $p = 1$. Here $\mathcal{L}_\mathcal{B}(\mathcal{B}) = \mathcal{B}$ and $\Sigma_{1,\mathcal{B}}$ consist of the states of $\mathcal{B}$ with support equivalent to 1 (note that $x \in \mathcal{S}_1(X)$ verifies $x^*x = 1$, i.e. $x \in U_\mathcal{B}$, and $e_x = 1$). That is, $\Sigma_{1,\mathcal{B}}$ is the set of faithful states of $\mathcal{B}$ ($= \Sigma_1(\mathcal{B})$ in our notation). Then $\wp_3$ is the quotient map

$$\Sigma_1(\mathcal{B}) \to \Sigma_1(\mathcal{B})/U_\mathcal{B}.$$

Take $\mathcal{B} = M_n(\mathbb{C})$ ($n < \infty$). Then the quotient map above is not a weak fibration. Indeed, both sets $\Sigma_1(M_n(\mathbb{C}))$ and $\Sigma_1(M_n(\mathbb{C}))/U_{M_n(\mathbb{C})}$ are convex metric spaces. The latter can be identified, using density matrices, to the n-tuples of eigenvalues $(\lambda_1, ..., \lambda_n)$ arranged in decreasing order and normalized such that $\sum \lambda_k = 1$, with the ℓ_1 distance. If this quotient map were a weak fibration, then the fibre would have trivial homotopy groups of all orders. This is not the case, because the fibre is the unitary group $U(n)$ of $M_n(\mathbb{C})$.

Remark 4.10 The set $\mathcal{O}_\varphi$ lies inside $\Sigma_{p,X}$, as the states φ_x with φ fixed. If we regard $\mathcal{O}_\varphi$ with the metric d_φ and $\Sigma_{p,X}$ with the metric d, it is clear that the inclusion is continuous. Indeed, it was noted that *supp* is continuous in $\mathcal{O}_\varphi$. Therefore if $d_\varphi(\varphi_{x_n}, \varphi_x) \to 0$, then $e_{x_n} \to e_x$, which implies that $d(\varphi_{x_n}, \varphi_x) \to 0$.

However, the identity mapping $(\mathcal{O}_\varphi, d_\varphi) \to (\mathcal{O}_\varphi, d)$ is not (in general) a homeomorphism. Take $X = \mathcal{B}$ and φ faithful. Then $\mathcal{O}_\varphi$ is the unitary orbit $\{\varphi \circ Ad(u) : u \in U_\mathcal{B}\} \sim U_\mathcal{B}/U_{\mathcal{B}^\varphi}$, and d_φ induces the same topology as the quotient topology ($U_\mathcal{B}$ with the norm topology). On the other hand, $\Sigma_{p,X}$ coincides in this case with $\Sigma_1(\mathcal{B})$, the set of faithful states of $\mathcal{B}$, and the metric d is just the usual norm of the conjugate space $\mathcal{B}^*$. In [3] it was shown that in general, the unitary orbit does not have norm continuous local cross sections to the unitary group, though it does have local cross sections which are continuous in the quotient topology $U_\mathcal{B}/U_{\mathcal{B}^\varphi}$.

Remark 4.11 The metric $d(\Phi,\Psi) = \|\Phi - \Psi\| + \|supp(\Phi) - supp(\Psi)\|$ can be considered in the whole state space of $\mathcal{L}_{\mathcal{B}}(X)$. In this metric, $\Sigma_{p,X}$ is open. Moreover, any state ψ of $\mathcal{L}_{\mathcal{B}}(X)$ such that $d(\Psi, \Sigma_{p,X}) < 1$, actually lies in $\Sigma_{p,X}$. Indeed, if Φ is a state of $\mathcal{L}_{\mathcal{B}}(X)$, and $d(\Phi, \varphi_x) < 1$ for some $x \in \mathcal{S}_p(X)$ and $\varphi \in \Sigma_p(\mathcal{B})$, then $\|supp(\Phi) - e_x\| < 1$, and therefore $supp(\Phi)$ and e_x are unitarily equivalent. That is $supp(\Phi) = e_y$, with $y = U(x)$ for some unitary U in $\mathcal{L}_{\mathcal{B}}(X)$. Then, by 2.2, there exists $\psi \in \Sigma_p(\mathcal{B})$ such that $\Phi = \psi_y \in \Sigma_{p,X}$.

If $X = \mathcal{B}$ and $\mathcal{B}$ is finite dimensional, then the topology of the d-metric coincides in $\Sigma_{p,\mathcal{B}}$ with the usual norm topology. Indeed, it suffices to see that the map $\varphi_x \mapsto e_x$ is continuous in the norm topology. Since we are in the finite dimensional case, states are represented by positive density matrices with trace 1. Note that the states of the form φ_x have equivalent supports, i.e. their density matrices have kernels with the same dimension. Suppose that a_n is a sequence of positive matrices with trace 1 and $nul(a_n) = k$, converging in norm to the matrix a, also with (a priori) $nul(a) = k$. Then the projections $P_{\ker a_n}$ onto the kernels converge in norm to $P_{\ker a}$. Indeed, we claim that we can find an open interval around zero and an integer n_0 such that for $n \geq n_0$ no eigenvalue of a_n (other than zero) lies inside this interval. Then we would have that $P_{\ker a_n} \to P_{\ker a}$. If there were no such interval, there would exist a sequence $\lambda_n > 0$ such that λ_n is an eigenvalue of a_n and $\lambda_n \to 0$. If q_n is the spectral projection corresponding to λ_n, then $a_n = b_n + \lambda_n q_n$. Then $b_n \to a$, with $nul(a) = k$ and $nul(b_n) < k$, which cannot happen.

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