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PLANAR NORMAL SECTIONS ON ISOPARAMETRIC HYPERSURFACES AND THE INFINITY LAPLACIAN

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Abstract. We present a new characterization of Cartan isoparametric hypersurfaces in terms of properties of the polynomial that determines the algebraic set of planar normal sections on the homogeneous isoparametric hypersurfaces in spheres. We show that Cartan isoparametric hypersurfaces are the only homogeneous isoparametric hypersurfaces in spheres for which the infinity Laplacian of the polynomial that defines the algebraic set of planar normal sections is the polynomial multiplied by the squared norm of the tangent vector.

Since it is required for our work, we also give these polynomials for all homogeneous isoparametric hypersurfaces in spheres.

1. Introduction and previous results

Let M be a compact connected n-dimensional Riemannian manifold and $I : M \longrightarrow \mathbb{R}^{n+k}$ an isometric embedding into the Euclidean space $\mathbb{R}^{n+k}$. Let p be a point in M and consider, in the tangent space $T_p(M)$ to M at p, a unit vector X. We may associate to X (as in [9] and references therein) an affine subspace of $\mathbb{R}^{n+k}$ defined by $\operatorname{Sec}(p,X) = p + \operatorname{Span}\{X, T_p^{\perp}(M)\}$. If U is a small enough neighborhood of p in M, then the intersection $U \cap \operatorname{Sec}(p,X)$ can be considered a C^{∞} regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = X$. This curve is called a *normal section of M at p in the direction of X*. We say that the normal section γ of M at p in the direction of X is *planar* at p if its first three derivatives $\gamma'(0)$, $\gamma''(0)$ and $\gamma'''(0)$ are *linearly dependent.* If M is a compact spherical submanifold in $\mathbb{R}^{n+k}$ (i.e. contained in a sphere of radius r in $\mathbb{R}^{n+k}$), given a point p in M, we shall denote by $\widehat{X_p}[M]$ the algebraic set defined by $\widehat{X_p}[M] = \{X \in T_p(M) : \|X\| = 1, \left(\overline{\nabla}_X\alpha\right)(X,X) = 0\}$ (here $\left(\overline{\nabla}_X\alpha\right)$ is the covariant derivative of the second fundamental form of M in $\mathbb{R}^{n+k}$). This is called *the algebraic set of unit vectors defining planar normal sections* at p on M.

The condition defining the algebraic set $\widehat{X_p}[M]$ can be considered ([9]) as given by certain homogeneous polynomials of degree three defined in the tangent space $T_p(M)$. For isoparametric hypersurfaces in the sphere the behavior of the normal sections is governed by a single polynomial and it is the *infinity Laplacian* of this

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polynomial (cf. Section 2) the tool that we use to obtain a characterization of the Cartan ones among all *homogeneous isoparametric* hypersurfaces in the sphere. Recall that a Cartan isoparametric hypersurface is a homogeneous isoparametric hypersurface of the sphere with three distinct principal curvatures. The reason for restricting to the homogeneous case is that, for these hypersurfaces, the algebraic set $\widehat{X_p}[M]$ is, in some clear sense, "independent" of the point p and this is a desirable property.

It seems to be a strong fact that certain homogeneous isoparametric hypersurfaces in the sphere can be characterized in terms of properties associated to their normal sections. In this sense the present work complements the results in [9].

We shall indicate some necessary notation.

We always identify M with its image by the embedding I. A submanifold of Euclidean space $\mathbb{R}^{n+k}$ is usually called *full*, if it is not included in any affine hyperplane. Let $\langle *, * \rangle$ denote the inner product in $\mathbb{R}^{n+k}$. Let ∇^E be the Euclidean covariant derivative in $\mathbb{R}^{n+k}$ and ∇ the Levi-Civita connection in M associated to the induced metric. We shall say that the submanifold M is *spherical* if it is contained in a sphere of radius r in $\mathbb{R}^{n+k}$, which can be thought centered at the origin. Let α denote the second fundamental form of the embedding in $\mathbb{R}^{n+k}$. We denote by $T_p(M)$ and $T_p(M)^\perp$ the tangent and normal spaces to M at p, respectively. M will be called *extrinsically homogeneous* ([3]) if for any two points $p, q \in M$ there is an isometry g of $\mathbb{R}^{n+k}$ such that $g(M) = M$ and $g(p) = q$.

The definition of the set $\widehat{X_p}[M]$ is based on the following basic fact ([9]).

Proposition 1. *Let M be a spherical compact submanifold. The normal section γ of M at p in the direction of the unitary vector $X \in T_p(M)$, is planar at p if and only if the covariant derivative of the second fundamental form vanishes on the vector $X = \gamma'(0)$. That is, X satisfies the equation*:

$$\left(\overline{\nabla}_X \alpha\right)(X, X) = 0. \qquad \square$$

We must also notice that:

Proposition 2. *If M is spherical and ω_1 is the unitary umbilical vector field on M then for $X, Y, Z \in T_p(M)$ we have*

$$\left\langle \omega_1, \left(\overline{\nabla}_X \alpha\right)(Y, Z) \right\rangle = 0. \qquad \square$$

In order to study the normal sections at p of the compact spherical submanifold M in $\mathbb{R}^{n+k}$, it is convenient to consider the polynomials

$$P_j(X) = \langle \omega_j, (\nabla_X \alpha)(X, X) \rangle, \qquad j = 1, \ldots, k,$$

where $\omega_1, \ldots, \omega_k$ is a basis of the normal space $T_p(M)^\perp$.

The condition in Proposition 1 may be written then as

$$P_j(X) = 0, \qquad j = 1, \ldots, k, \quad \|X\| = 1.$$

1.1. **The polynomials on isoparametric submanifolds.** The embedded submanifold $M^n \subset \mathbb{R}^{n+k}$ as above is said to have constant principal curvatures if, for any parallel normal field $\xi(t)$ along any piecewise differentiable curve in M^n, the eigenvalues of the shape operator $A_{\xi(t)}$ are constant. It is known that the submanifolds with constant principal curvatures are either isoparametric or one of their focal manifolds. For a full isoparametric submanifold M^n of $\mathbb{R}^{n+k}$ the rank is its codimension, namely k. Let M be a compact rank k isoparametric submanifold of $\mathbb{R}^{n+k}$; then M is spherical and we may think that the sphere has center $0 \in \mathbb{R}^{n+k}$ and radius 1. M is a regular level set of an *isoparametric polynomial map* $f : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^k$ which has components $f = (h_1, \dots, h_k)$; usually one takes $M = f^{-1}(0)$. The importance of isoparametric submanifolds for our study is that the gradients $\{\nabla h_j : j = 1, \dots, k\}$ provide a $\nabla^\perp$-parallel frame of the normal bundle of M. We shall use this natural basis of the normal bundle instead of $\omega_1, \dots, \omega_k$.

The following observations ([9]) will be important below.

Proposition 3. *For a compact isoparametric submanifold M of $\mathbf{R}^{n+k}$, the polynomials $P_j(X)$, $j = 1, \dots, k$, are harmonic in $T_p(M)$ for any $p \in M$.* □

Let M be a compact rank k isoparametric submanifold of $\mathbb{R}^{n+k}$. Since the normal bundle of M is globally flat, all shape operators are simultaneously diagonalizable. We have common eigendistributions H_i $(i = 1, \dots, g)$, that is, for any $\xi \in T_p(M)^\perp$

$$A_\xi(X) = \lambda_i(\xi) X, \qquad \forall X \in H_i(p).$$

Each H_i is autoparallel, hence integrable with totally geodesic leaves. Then we have ([9, Corollary 4.3]):

Proposition 4. *Let $e_1, \dots, e_n$ be an orthonormal basis for $T_p(M)$ formed by taking an orthonormal basis in each eigenspace $H_i(p)$, $i = 1, \dots, g$. Then writing $X \in T_p(M)$, $\|X\| = 1$, as $X = \Sigma a_i e_i$, in the polynomials $P_j(X)$, $j = 1, \dots, k$, there are no monomials with two subscripts from the same $H_i(p)$. In particular there are neither cubes nor squares in the polynomials.* □

1.2. **Computing the polynomials.** In the present section we shall see that they can be computed, in a more direct fashion, from the polynomials $(h_1, \dots, h_k)$ defining M.

Proposition 5. [9, Corollary 4.3] *Let $X \in T_p(M)$ be a unit vector. If $\gamma(s)$ is a normal section of M such that $\gamma(0) = p$, $\gamma'(0) = X$, then*

$$P_j(X) = -X\langle \nabla^E_{\gamma'(s)}(\nabla h_j(\gamma(s))), \gamma'(s)\rangle. \qquad \square$$

Since we may take as h_1 the quadratic polynomial defining the unit sphere in $\mathbb{R}^{n+k}$, the algebraic set $\widehat{X_p}(M)$ is defined only by

$$P_j(X) = 0, \quad \|X\| = 1, \quad j = 2, \dots, k.$$

If now M is a compact rank 2 *full* isoparametric submanifold of $\mathbb{R}^{n+2}$, then M is a regular level set of an isoparametric polynomial map $f : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^2$ which has components $f = (h_1, h_2)$. Let p be a point in M; since, as we noticed, we may think that the first polynomial h_1 is the one defining the unit sphere in $\mathbb{R}^{n+2}$ then we have, in this case, only one polynomial defining the algebraic set $\widehat{X_p}(M)$, namely:

$$P_2(X) = 0, \quad \|X\| = 1.$$

The algebraic set of planar normal sections of M at p is then $P_2^{-1}(0)$.

1.3. **Isoparametric hypersurfaces in spheres.** We recall now some well known facts about isoparametric hypersurfaces in spheres.

As we indicated above, an isoparametric hypersurface in the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$ is a level set of the isoparametric polynomial map $f : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^2$, $f = (h_1, h_2)$. It is usual to forget the function h_1 defining the sphere and consider only the restriction h of h_2 to the unit sphere, and so consider M defined as a level set of $h : S^{n+1} \to \mathbb{R}$. If M has g distinct principal curvatures then the principal curvatures have two multiplicities m_1 and m_2 (possibly equal) and the function h has the following important properties (h is usually called the Cartan–Münzner polynomial defining M):

(i) h satisfies

$$\|\nabla h\|^2 = g^2 \|X\|^{2g-2},$$
$$\triangle h = c \|X\|^{g-2},$$

where $c = \frac{g^2(m_2 - m_1)}{2}$.

(ii) h is the restriction to S^{n+1} of a homogeneous polynomial of degree g in $\mathbb{R}^{n+2}$.

(iii) For the restriction h of each homogeneous polynomial of degree g in $\mathbb{R}^{n+2}$ which satisfies (i) and (ii) the level hypersurfaces of h form an isoparametric family.

(iv) If M is an isoparametric hypersurface of S^{n+1} with g distinct principal curvatures, then $g \in \{1, 2, 3, 4, 6\}$.

The number g of distinct principal curvatures for the *homogeneous* isoparametric hypersurfaces obtained in [11] is the same that for *general* isoparametric hypersurfaces. The homogeneous ones are principal orbits of the isotropy representation of certain symmetric spaces.

To obtain our results, we need to indicate which are the polynomials $P_2(X)$ that define the set $\widehat{X_p}(M)$ for each of the homogeneous isoparametric hypersurfaces in spheres. This computation is long and generally quite involved, so we only reproduce here the corresponding results, which is what the present work requires; the full development has been carried out in [8]. The scheme followed in this computation of $P_2(X)$ is the same for all homogeneous isoparametric hypersurfaces. Namely:

(i) Find a convenient basic point $E \in M$. The set $\widehat{X_E}(M)$ is then contained in the unit sphere of $S^{n-1} \subset T_E(M)$.

(ii) Determine the tangent and normal spaces of M at the point E.
(iii) Compute $P_2(X)$ by the formula in the previous proposition.

We identify the hypersurfaces by the degree g.

Case $g = 1$. Here M is an equator in S^{n+1} defined by

$$M = \left\{X \in S^{n+1} : h\left(X\right) = \langle X, v\rangle = 0\right\}$$

for some fixed $v \in S^{n+1}$. For $E \in M$ all normal sections at E are planar so

$$\widehat{X_E}(M) = S^{n-1} \subset T_E\left(M\right).$$

Case $g = 2$. These are the so called *Clifford manifolds*. For p, q natural numbers such that $1 \le p$, $q \le n$, $p + q = n$, they are defined by:

$$M_{p,q} = \left\{X \in S^{n+1} : \sum_{i=1}^{p+1} x_i^2 - \sum_{i=p+2}^{n+2} x_i^2 = 0\right\}.$$

Here again all normal sections are planar so

$$\widehat{X_E}(M) = S^{n-1} \subset T_E\left(M\right).$$

Case $g = 3$. These are called Cartan hypersurfaces. These hypersurfaces denoted by F_R, F_C, F_H and F_O, are the manifolds "complete flags" in the projective planes RP^2, CP^2, HP^2 and OP^2 (real, complex, quaternionic and Cayley), respectively. A complete flag in any of the projective planes is a pair (p, l) where p is a point in the plane and l a line (real, complex, quaternionic or octonionic) containing the point p. Their homogeneous presentation and relevant data are the following:

Hypersurface	$\dim M$	g	m_i
$F_R = SO(3)/(Z_2 \times Z_2)$	3	3	1
$F_C = SU(3)/T^2$	6	3	2
$F_H = \mathrm{Sp}(3)/(\mathrm{Sp}(1))^3$	12	3	4
$F_O = F_4/\,\mathrm{Spin}(8)$	24	3	8

We briefly recall their definition as isoparametric submanifolds. For details see the presentation in [7] or [9].

Let $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. We take as Euclidean ambient space the *real vector space* $U = \{u : \mathrm{tr}(u) = 0\}$, where

$$u = \begin{bmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{bmatrix}, \quad \xi_j \in \mathbb{R},\ x_j \in F.$$

$\overline{x}$ denotes conjugation in F, with the inner product $\langle u, v\rangle = \frac{1}{4}\,\mathrm{tr}\left(uv + vu\right)$. Note that $\dim_{\mathbb{R}}\left(U\right) = 5, 8, 14, 26$ for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Consider now on U

the function defined by

$$f(u) = \left(\frac{\sqrt{3}}{2}\right) t(x_1 x_2 x_3) = \sqrt{3}\,\mathrm{Re}\left((x_1 x_2) x_3\right)$$

and its restriction to the unit sphere $S \subset U$. This is the Cartan–Münzner polynomial. The trilinear function $\mathrm{Re}\left((x_1 x_2) x_3\right)$ has the properties:

- $\mathrm{Re}\left((ab)\,c\right) = \mathrm{Re}\left(a\,(bc)\right)$,
- $\mathrm{Re}\left((ab)\,c\right)$ is invariant by cyclic permutation.

Then we have our four Cartan hypersurfaces M_F (for each $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) defined as:

$$M_F = \{u \in S(U) : f(u) = 0\}.$$

The point $E = \mathrm{diag}(-1, 0, 1)$ is a point in $M_F = f^{-1}(0)$, for all F. It belongs to the subspace $a = \left\{\mathrm{diag}(\xi_1, \xi_2, \xi_3) : \sum_j \xi_j = 0\right\}$.

It is not hard to see that the normal space to all M_F at E is the same for all F, namely $T_E(M_F)^{\perp} = a$, and the tangent spaces at E satisfy $T_E(M_{\mathbb{R}}) \subset T_E(M_{\mathbb{C}}) \subset T_E(M_{\mathbb{H}}) \subset T_E(M_{\mathbb{O}})$ and so the tangent space at E is just the affine subspace

$$T_E(M_F) = E + \left\{ \begin{bmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & x_1 \\ x_2 & \overline{x_1} & 0 \end{bmatrix},\ x_j \in F \right\}.$$

Computing our polynomial $P_2(X)$ by the method described above, one gets:

$$P_2(X) = 3\sqrt{3}t(x_1 x_2 x_3), \qquad t(x_1 x_2 x_3) = 2\,\mathrm{Re}\left((x_1 x_2) x_3\right).$$

For reasons of space in the cases $g = 4, 6$ we limit ourselves to describe the polynomial P_2 with the corresponding notation.

Case $g = 4$**.** In this degree there are four spaces where the Cartan–Münzner polynomial is obtained using *Clifford systems* as defined by Ferus, Karcher and Münzner in [5]. This method is also clearly described in [4]. There are also other two spaces which must be presented differently (see below).

The first three cases can be described in a unified way. They are the principal orbits of the tangential representation of the compact symmetric spaces indicated in the table. The notation is that in [6, p. 518].

U	G	Sym.	$\dim M$	g	$m_i,\ i = 1, \ldots, g$
$SO(n+2), n \geq 3$	$SO(n) \cdot SO(2)$	BDI	$2n-2$	4	$m_1 = m_3 = 1$ $m_2 = m_4 = n-2$
$SU(n+2), n \geq 2$	$S(U(n) \cdot U(2))$	$AIII$	$4n-2$	4	$m_1 = m_3 = 2$ $m_2 = m_4 = 2n-3$
$Sp(n+2), n \geq 4$	$Sp(n) \cdot Sp(2)$	CII	$8n-2$	4	$m_1 = m_3 = 4$ $m_2 = m_4 = 4n-5$

To distinguish the cases we write $M_{\mathbb{R}}$, $M_{\mathbb{C}}$ and $M_{\mathbb{H}}$, associating the field $\mathbb{R}$ to the first case, $\mathbb{C}$ to the second one, and $\mathbb{H}$ (quaternions) to the third. For each of them the corresponding ambient Euclidean space is $\mathbb{R}^{2n}$, $\mathbb{C}^{2n}$ and $\mathbb{H}^{2n}$. It shall be convenient to write again $F=\mathbb{R},\mathbb{C},\mathbb{H}$ so we may write the elements in the ambient spaces in a unified manner as

$$((u_1,u_2\dots,u_n),(v_1,\dots v_{n-1},v_n)),\quad u_j,v_j\in F.$$

The vector E will be the same in all cases, namely

$$\begin{aligned}&E=(A_o,B_o)=((t_1,0,\dots,0),(0,\dots,0,t_2)),\\ &t_1=\cos\theta,\ t_2=\sin\theta,\quad \theta=\pi/8,\end{aligned}$$

with $u_j=0,\ 2\le j\le n$, and $v_k=0,\ 1\le k\le n-1$ (u_1 and v_n are real).

Then the tangent space to M_F at E is

$$T_E(M_F)=\left\{((\alpha,u_2\dots,u_n),(v_1,\dots v_{n-1},\delta))\in F^{2n-1}:u_j,v_k\in F,\ \alpha,\delta\in\Im(F)\right\}$$

(here we write $\Im(F)=0$ for $F=\mathbb{R}$, $\Im(F)=i\mathbb{R}$ for $F=\mathbb{C}$, and $\Im(F)=$ {Pure quaternions} for $F=\mathbb{H}$), and the normal space at E is that of all vectors with first and last components real and zero all the other ones. To be able to write our polynomials we need to fix a notation that can be used in the three cases.

We may denote $X=((\alpha,B),(C,\delta))\in T_E(M)$ by

$$\begin{aligned}&B=(u_2,\dots,u_n),\quad C=(v_1,\dots,v_{n-1}),\quad u_j,v_j\in\mathbb{H},\\ &\alpha=a_1i+a_2j+a_3k,\quad \delta=d_1i+d_2j+d_3k\in\Im(F),\\ &u_s=b_{s,0}+b_{s,1}i+b_{s,2}j+b_{s,3}k,\quad s=2,\dots,n,\\ &v_r=c_{r,0}+c_{r,1}i+c_{r,2}j+c_{r,3}k,\quad r=1,\dots,n-1.\end{aligned}$$

With this notation, in the case $F=\mathbb{H}$ the polynomial may be written as:

$$\begin{aligned}&\frac{1}{96}\mathbb{P}(X)\\ &\quad=(t_1c_{1,0}+t_2b_{n,0})(a_1c_{1,1}+a_2c_{1,2}+a_3c_{1,3}+d_1b_{n,1}+d_2b_{n,2}+d_3b_{n,3})\\ &\qquad+(t_1c_{1,0}+t_2b_{n,0})\sum_{r=2}^{n-1}(b_{r,0}c_{r,0}+b_{r,1}c_{r,1}+b_{r,2}c_{r,2}+b_{r,3}c_{r,3})\\ &\qquad+(-t_1c_{1,1}+t_2b_{n,1})(a_1c_{1,0}-a_2c_{1,3}+a_3c_{1,2}-d_1b_{n,0}+d_2b_{n,3}-d_3b_{n,2})\\ &\qquad+(-t_1c_{1,1}+t_2b_{n,1})\sum_{r=2}^{n-1}(-b_{r,0}c_{r,1}+b_{r,1}c_{r,0}-b_{r,2}c_{r,3}+b_{r,3}c_{r,2})\\ &\qquad+(-t_1c_{1,2}+t_2b_{n,2})(a_1c_{1,3}+a_2c_{1,0}-a_3c_{1,1}-d_1b_{n,3}-d_2b_{n,0}+d_3b_{n,1})\\ &\qquad+(-t_1c_{1,2}+t_2b_{n,2})\sum_{r=2}^{n-1}(-b_{r,0}c_{r,2}+b_{r,1}c_{r,3}+b_{r,2}c_{r,0}-b_{r,3}c_{r,1})\\ &\qquad+(-t_1c_{1,3}+t_2b_{n,3})(-a_1c_{1,2}+a_2c_{1,1}+a_3c_{1,0}+d_1b_{n,2}-d_2b_{n,1}-d_3b_{n,0})\\ &\qquad+(-t_1c_{1,3}+t_2b_{n,3})\sum_{r=2}^{n-1}(-b_{r,0}c_{r,3}-b_{r,1}c_{r,2}+b_{r,2}c_{r,1}+b_{r,3}c_{r,0}).\end{aligned}$$

For $F=\mathbb{C}$ this reduces to

$$\begin{aligned}\frac{1}{96}\mathbb{P}\left(X\right)=&\left(t_1c_{1,0}+t_2b_{n,0}\right)\left(a_1c_{1,1}+d_1b_{n,1}\right)\\&+\left(t_1c_{1,0}+t_2b_{n,0}\right)\sum_{r=2}^{n-1}\left(b_{r,0}c_{r,0}+b_{r,1}c_{r,1}\right)\\&+\left(-t_1c_{1,1}+t_2b_{n,1}\right)\left(a_1c_{1,0}-d_1b_{n,0}\right)\\&+\left(-t_1c_{1,1}+t_2b_{n,1}\right)\sum_{r=2}^{n-1}\left(-b_{r,0}c_{r,1}+b_{r,1}c_{r,0}\right),\end{aligned}$$

and for $F=\mathbb{R}$ we get

$$\frac{1}{96}\mathbb{P}\left(X\right)=\left(t_1c_{1,0}+t_2b_{n,0}\right)\sum_{r=2}^{n-1}b_{r,0}c_{r,0}\,. \tag{1}$$

Case $g=4$**,** $(9,6)$**.** This is an homogeneous submanifold as indicated in [4]. This space M is a principal orbit of the tangential representation of $EIII$.

U	G	sym.	$\dim M$	g	$m_i,\ i=1,\dots,g$
E_6	$\left(SO\left(10\right)\cdot T\right)/Z_4$	$EIII$	30	4	$m_1=m_3=6$ $m_2=m_4=9$

The ambient space is here $\mathbb{H}^8$ of real dimension 32. We adopt the notation

$$\mathbb{H}^8=\left\{\left(\begin{bmatrix}a_1&a_2\\a_3&a_4\end{bmatrix},\begin{bmatrix}b_5&b_6\\b_7&b_8\end{bmatrix}\right):a_s,b_r\in\mathbb{H}\right\}.$$

We take here

$$\begin{gathered}E=\left(A_0,B_0\right)=\left(\begin{bmatrix}t_1&0\\0&0\end{bmatrix},\begin{bmatrix}0&t_6\\0&0\end{bmatrix}\right),\\t_1=\cos\theta,\ t_6=\sin\theta,\quad\theta=\pi/8,\end{gathered}$$

and the tangent space to M at E is

$$T_E\left(M\right)=\left\{\left(\begin{bmatrix}\alpha&a_2\\a_3&a_4\end{bmatrix},\begin{bmatrix}b_5&\beta\\b_7&b_8\end{bmatrix}\right):a_s,b/s\in\mathbb{H};\ \alpha,\beta\text{ pure quaternions}\right\}.$$

Using now the notation

$$\begin{gathered}a_s=u_{s,0}+iu_{s,1}+ju_{s,2}+ku_{s,3},\quad s=2,3,4,\\b_r=v_{r,0}+iv_{r,1}+jv_{r,2}+kv_{r,3},\quad r=5,7,8,\\\langle *,*\rangle\text{ is the scalar product in }\mathbb{H},\end{gathered}$$

the polynomial becomes

$$
\begin{aligned}
\frac{1}{96}\mathbb{P}(X) = {} & (t_1 v_{5,0} + t_6 u_{2,0})\left[\langle \alpha, b_5\rangle + \langle a_2, \beta\rangle + \langle a_3, b_7\rangle + \langle a_4, b_8\rangle\right] \\
& + (-t_1 v_{5,1} + t_6 u_{2,1})\left[\langle \alpha, i b_5\rangle + \langle a_2, i\beta\rangle - \langle a_3, i b_7\rangle - \langle a_4, i b_8\rangle\right] \\
& + (-t_1 v_{5,2} + t_6 u_{2,2})\left[\langle \alpha, j b_5\rangle + \langle a_2, j\beta\rangle - \langle a_3, j b_7\rangle - \langle a_4, j b_8\rangle\right] \\
& + (-t_1 v_{5,3} + t_6 u_{2,3})\left[\langle \alpha, k b_5\rangle + \langle a_2, k\beta\rangle - \langle a_3, k b_7\rangle - \langle a_4, k b_8\rangle\right] \\
& + (t_1 v_{8,0} - t_6 u_{3,0})\left[\langle \alpha, b_8\rangle + \langle a_2, b_7\rangle - \langle a_3, \beta\rangle - \langle a_4, b_5\rangle\right] \\
& + (-t_1 v_{7,1} + t_6 u_{4,1})\left[\langle \alpha, b_7 i\rangle + \langle a_2, b_8 i\rangle + \langle a_3, b_5 i\rangle + \langle a_4, \beta i\rangle\right] \\
& + (-t_1 v_{7,2} + t_6 u_{4,2})\left[\langle \alpha, b_7 j\rangle + \langle a_2, b_8 j\rangle + \langle a_3, b_5 j\rangle + \langle a_4, \beta j\rangle\right] \\
& + (-t_1 v_{7,3} + t_6 u_{4,3})\left[\langle \alpha, b_7 k\rangle + \langle a_2, b_8 k\rangle + \langle a_3, b_5 k\rangle + \langle a_4, \beta k\rangle\right] \\
& + (-t_1 v_{7,0} - t_6 u_{4,0})\left[-\langle \alpha, b_7\rangle + \langle a_2, b_8\rangle + \langle a_3, b_5\rangle - \langle a_4, \beta\rangle\right].
\end{aligned}
$$

The cases $SO(5)$ and $SU(5)$. As mentioned above, there are two homogeneous isoparametric hypersurfaces on the sphere which are of degree $g = 4$ but cannot be described by Clifford systems.

U	G	sym	$\dim M$	g	$m_i,\ i = 1, \dots, g$
$SO(10)$	$SU(5)$	$DIII(5)$	18	4	$m_1 = m_3 = 4$ $m_2 = m_4 = 5$
$SO(5) \cdot SO(5)$	$SO(5)$	$SO(5)$	8	4	$m_i = 2,\ \forall i$

We may call these isoparametric hypersurfaces M_{20} and M_{10}. It is clear that $M_{10} = SO(5)/T^2$. The ambient Euclidean spaces are respectively $\mathbb{R}^{20}$ and $\mathbb{R}^{10}$. We adopt for them the notation

$$
\begin{aligned}
(x_1, x_2, x_3, \dots, x_{10}, y_1, \dots, y_{10}) &\in \mathbb{R}^{20}, \\
(x_1, x_2, x_3, \dots, x_{10}) &\in \mathbb{R}^{10}.
\end{aligned}
$$

We may identify the basic point E with

$$
\begin{aligned}
& E = (t_1, t_2, 0, \dots, 0, 0, \dots, 0) \in \mathbb{R}^{20} \\
& E = (t_1, t_2, 0, \dots, 0) \in \mathbb{R}^{10} \\
& t_1 = \cos\theta, \quad t_2 = \sin\theta, \quad \theta = \pi/8,
\end{aligned}
$$

and the tangent spaces at E with

$$
\begin{aligned}
T_E(M_{20}) &= \{(0, 0, x_3, \dots, x_{10}, y_1, \dots, y_{10}) : x_j, y_k \in \mathbb{R}\}, \\
T_E(M_{10}) &= \{(0, 0, x_3, \dots, x_{10}) : x_j \in \mathbb{R}\}.
\end{aligned}
$$

Let $X=(0,0,x_3,\ldots,x_{10},y_1,\ldots,y_{10})\in T_E\,(M_{20})$ be a tangent vector, then

$$\begin{aligned}\frac{1}{96}\mathbb{P}_{20}(X) &= t_1\left(-y_2x_3y_6-y_2x_6y_3+y_2x_5y_4+y_2x_4y_5\right)\\ &\quad+t_2\left(-y_1x_3y_6-y_1x_6y_3+y_1x_5y_4+y_1x_4y_5\right)\\ &\quad+t_1\left(x_4x_7x_9+x_4y_7y_9+y_4x_7y_9-y_4x_9y_7\right)\\ &\quad+t_1\left(-x_3x_8x_9-x_3y_8y_9-y_3x_8y_9+y_3x_9y_8\right)\\ &\quad+t_1\left(x_6x_7x_{10}+x_6y_7y_{10}+y_6x_7y_{10}-y_6x_{10}y_7\right)\\ &\quad+t_1\left(-x_5x_8x_{10}-x_5y_8y_{10}-y_5x_8y_{10}+y_5x_{10}y_8\right)\\ &\quad+t_2\left(-x_5x_7x_9-x_5y_7y_9-y_5x_9y_7+y_5x_7y_9\right)\\ &\quad+t_2\left(x_3x_7x_{10}+x_3y_7y_{10}+y_3x_{10}y_7-y_3x_7y_{10}\right)\\ &\quad+t_2\left(-x_6x_8x_9-x_6y_8y_9-y_6x_9y_8+y_6x_8y_9\right)\\ &\quad+t_2\left(x_4x_8x_{10}+x_4y_8y_{10}+y_4x_{10}y_8-y_4x_8y_{10}\right).\end{aligned}$$

Now, if $X=(0,0,x_3,\ldots,x_{10})\in T_E\,(M_{10})$, then

$$\begin{aligned}\frac{1}{96}\mathbb{P}_{10}(X) &= t_1\left(x_7x_9x_4+x_7x_{10}x_6-x_8x_3x_9-x_8x_5x_{10}\right)\\ &\quad+t_2\left(-x_7x_9x_5-x_8x_9x_6+x_{10}x_3x_7+x_{10}x_4x_8\right).\end{aligned}$$

It is clear that $\mathbb{P}_{20}$ restricts to $\mathbb{P}_{10}$ if $y_j=0$ $(j=1,\ldots,10)$.

Case $g=6$. In the case $g=6$ there are two types of homogeneous isoparametric hypersurfaces on the sphere which are of dimension 6 and 12 respectively. They may be described as follows: Let $\mathfrak{g}_2$ be the compact simple Lie algebra corresponding to the compact Lie simple group G_2. The non-compact real form of $\mathfrak{g}_2^C$ (which is split) has a Cartan decomposition $\mathfrak{g}_2=\mathfrak{so}\,(4)\oplus\mathfrak{p}$. Following [7] we may identify $\mathfrak{p}$ with $i\mathfrak{p}$ and then identify the split real form and the compact one. With this trick, a maximal abelian subspace $\mathfrak{h}\subset\mathfrak{p}$ is a Cartan subalgebra of $\mathfrak{g}_2$. Let $E\in\mathfrak{h}$ be a *regular* element in $\mathfrak{h}$. Since the set of *restricted roots* coincides with the set of roots of $\mathfrak{g}_2$ we have that the orbits $M_B=\mathrm{Ad}\,(G_2)\,E\subset\mathfrak{g}_2$ and $M_S=\mathrm{Ad}\,(SO\,(4))\,E\subset\mathfrak{p}$ are both *principal orbits* of the corresponding representations and as such they are isoparametric submanifolds of the spheres $\mathbb{S}^{13}\subset\mathfrak{g}_2$ and $\mathbb{S}^7\subset\mathfrak{p}$. Notice that $\dim\,(M_B)=12$ and $\dim\,(M_S)=6$. We take in $\mathfrak{g}_2$ the inner product defined by the opposite of the Killing form and the corresponding inner product in $\mathfrak{p}$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}_2$ with respect to the inner product in $\mathfrak{g}_2$.

We may identify

$$\begin{gathered}T_E\,(M_B)=\mathfrak{m},\qquad T_E\,(M_S)=\mathfrak{m}\cap\mathfrak{p},\\ T_E^{\perp}\,(M_B)=T_E^{\perp}\,(M_S)=\mathfrak{h}.\end{gathered}$$

It is possible to choose an *orthonormal basis* $\{H_j:3\leq j\leq 14\}\subset\mathfrak{m}$ such that $\{H_j:3\leq j\leq 8\}\subset\mathfrak{so}\,(4)$ and $\{H_j:9\leq j\leq 14\}\subset\mathfrak{m}\cap\mathfrak{p}$. Then we may write any tangent vector X to M at E as

$$X = \sum_{j=3}^{14} r_j H_j.$$

With this notation, the polynomial defining the algebraic set of planar normal sections of M_B at E takes the form

$$\begin{aligned}\left(\frac{1}{3}\sqrt{2}\sqrt{3}\right) & P_B(X) \\ &= r_3r_5r_7 + r_3r_6r_8 + r_3r_{11}r_{13} + r_3r_{12}r_{14} + r_4r_{12}r_{13} + r_7r_9r_{11} + r_8r_9r_{12} \\ &\quad + (-r_4r_6r_7 - r_5r_9r_{13} - r_6r_{10}r_{13} - r_6r_9r_{14} - r_7r_{10}r_{12}) \\ &\quad + 3(r_4r_5r_8 + r_5r_{10}r_{14} + r_8r_{10}r_{11} - r_4r_{11}r_{14}) \\ &\quad + \left(\frac{2}{\sqrt{3}}\right)(-r_3r_6r_7 - r_3r_{12}r_{13} - r_6r_9r_{13} + r_7r_9r_{12}),\end{aligned}$$

and by taking equal to zero the variables r_j $(9 \leq j \leq 14)$, we get the polynomial defining the algebraic set of planar normal sections of M_S at E:

$$\left(\frac{1}{3}\sqrt{2}\sqrt{3}\right) P_S(X) = r_3r_5r_7 + r_3r_6r_8 + (-r_4r_6r_7) + \left(\frac{2}{\sqrt{3}}\right)(-r_3r_6r_7) + 3(r_4r_5r_8).$$

2. Study of the infinity Laplacian

2.1. **Definition and property.** Now we show the characterization of hypersurfaces of Cartan in terms of the infinity Laplacian of the polynomials that define the set of planar normal sections. Recall that if $u : U \longrightarrow \mathbb{R}$ is a real smooth function on an open set $U \subseteq \mathbb{R}^n$, the infinity Laplacian ([1], [2]) of u is defined by:

$$\triangle_\infty u = \frac{1}{2}\langle \nabla u, \nabla \|\nabla u\|^2 \rangle.$$

The following fact is useful for studying the infinity Laplacian of the polynomials of normal sections.

Observe that the polynomials that define the set of planar normal sections are defined in the tangent spaces $T_E(M)$ and so it makes sense to study their infinity Laplacian.

Proposition 6. *If $\mathbb{P}(X)$ is a polynomial that defines the set of planar normal sections then $\triangle_\infty \mathbb{P}(X)$ is a homogeneous polynomial of degree 5 which consists of monomials that are of the following three forms:*

$$m_1 = c_1 x_k^3 x_i x_j, \quad m_2 = c_2 x_k^2 x_h x_i x_j, \quad m_3 = c_3 x_k x_r x_h x_i x_j. \ \square \tag{2}$$

2.2. **Harmonic projection operator.** We summarize some facts on the canonical decomposition of the homogeneous polynomials. Let

$$\mathcal{P}^p = \mathcal{P}^p_{m+1} \subset \mathbb{R}[x_0, x_1, \ldots, x_m]$$

denote the real vector space of homogeneous polynomials of degree p, in the $m+1$ variables $x_0, x_1, \ldots, x_m$. Let $\mathcal{H}^p = \mathcal{H}^p_m \subset \mathcal{P}^p$ be the linear subspace of harmonic homogeneous polynomials of degree p. We have the orthogonal decomposition of the vector space of homogeneous polynomials of degree p, ([12, Cap. II, 95–154])

$$\mathcal{P}^p = \bigoplus_{j=0}^{\left[\frac{p}{2}\right]} \mathcal{H}^{p-2j}\rho^{2j}, \tag{3}$$

where $\rho^2 = \|X\|^2 = \sum_{i=0}^{m} x_i^2$. By (3), for $\xi \in \mathcal{P}^p$, there exists a *unique* $\xi_j \in \mathcal{H}^{p-2j}$, $j = 0, \dots, \left[\frac{p}{2}\right]$, such that

$$\xi = \sum_{j=0}^{\left[\frac{p}{2}\right]} \xi_j \rho^{2j}.$$

This is called *the canonical decomposition* of ξ with coefficients ξ_j, $j = 0, \dots, \left[\frac{p}{2}\right]$.

The harmonic projection operator H is defined as the orthogonal projection, $H : \mathcal{P}^p \to \mathcal{H}^p$ (in terms of the powers of the Laplacian) for $\xi \in \mathcal{P}^p$, as follows:

$$H(\xi) = \xi + \sum_{j=1}^{\left[\frac{p}{2}\right]} \frac{(-1)^j (p-1)\dots(p-j)}{j!\lambda_{2(p-1)}\dots\lambda_{2(p-j)}} \triangle^j \xi \rho^{2j},$$

where

$$\lambda_{2(p-1)} = 2(p-1)(2p+m-3),$$
$$\lambda_{2(p-j)} = 2(p-j)(2p-2j+m-1).$$

Remark. In the case $\xi(X) = \triangle_\infty \mathbb{P}(X)$ (where $\mathbb{P}(X)$ is a polynomial that defines the set of planar normal sections) we have

$$\mathcal{P}^5 = \bigoplus_{j=0}^{2} \mathcal{H}^{5-2j}\rho^{2j} = \mathcal{H}^5 \oplus \mathcal{H}^3\rho^2 \oplus \mathcal{H}^1\rho^4,$$

and the projection operator is

$$H(\xi) = \xi - \frac{4}{\lambda_8}\triangle\xi\rho^2 + \frac{6}{\lambda_8\lambda_6}\triangle^2\xi\rho^4$$
$$\lambda_8 = 8(7+m)$$
$$\lambda_6 = 6(5+m).$$

The next proposition reduces further the projection operator.

Proposition 7. *If $\xi(X) = \triangle_\infty \mathbb{P}(X)$ is the infinity Laplacian of the polynomial that defines the set of planar normal sections $\mathbb{P}(X)$ then $\triangle^2\xi = 0$.*

Proof. Since the infinity Laplacian $\xi(X)$ contains monomials in the variables x_k, $k = 1, \dots, n$, of the form (2), and since

$$\Delta^2(\xi) = \sum_{u,v=1}^{n} \frac{\partial^2}{\partial x_u^2}\left(\frac{\partial^2 \xi}{\partial x_v^2}\right),$$

we see that for the monomials of type m_1 we have

$$\frac{\partial^2 m_1}{\partial x_h^2} = \begin{cases} 0, & \text{if } h \neq k, \\ 6c_1 x_k x_i x_j, & \text{if } h = k. \end{cases}$$

Then

$$\frac{\partial^2}{\partial x_u^2}\left(\frac{\partial^2 m_1}{\partial x_h^2}\right) = 0, \qquad \forall u,$$

and it is now clear that the situation is similar for those of types m_2 and m_3. Then $\triangle^2 \xi = 0$. □

Therefore, the projection operator for $\xi = \triangle_\infty \mathbb{P}(X)$ results

$$H(\xi) = \xi - \frac{4}{\lambda_8} \triangle \xi \rho^2,$$
$$\lambda_8 = 8(7+m).$$

2.3. **Infinity Laplacian and its projection operator.** We consider in this section, as an illustration, two examples of infinity Laplacians corresponding to the cases $g = 3$ and $g = 4$.

In the case of Cartan hypersurfaces, the computation of the infinity Laplacian yields

$$\triangle_\infty \mathbb{P}\left(X\right) = k\|X\|^2 \mathbb{P}\left(X\right), \qquad k \in \mathbb{R},\ k \neq 0.$$

Then $\triangle_\infty \mathbb{P}\left(X\right)$ is, essentially, $\mathbb{P}\left(X\right)$ multiplied by the square of the norm of X; and the projection operator is

$$H(\triangle_\infty \mathbb{P}(X)) = 0.$$

For the case $g = 4$ *real* (1), the infinity Laplacian turns out to be

$$\triangle_\infty \mathbb{P}\left(X\right) = 2\left(96\right)^2 \left(\|X\|^2 - \left(t_2 c_{1,0} - t_1 b_{n,0}\right)^2\right) \mathbb{P}\left(X\right),$$

and the projection operator results

$$H(\triangle_\infty \mathbb{P}\left(X\right)) = \left(96\right)^2 \left(\frac{1}{n+2}\|X\|^2 - 2\left(t_2 c_{1,0} - t_1 b_{n,0}\right)^2\right) \mathbb{P}\left(X\right);$$

then, we have

$$H(\triangle_\infty \mathbb{P}\left(X\right)) \neq 0.$$

Then, in this case,

$$\triangle_\infty \mathbb{P}(X) \in \mathcal{H}^5 \oplus \mathcal{H}^3 \rho^2,$$

and $\triangle_\infty \mathbb{P}(X)$ is not the polynomial multiplied by the squared norm of the tangent vector. The following lemma allows us to decide if the projection operator vanishes.

Lemma 1. *If $\xi(X) = \triangle_\infty \mathbb{P}(X)$ contains monomials with all variables to the first power (type m_3 in Proposition 6), then the projection $H(\xi) \neq 0$.* □

2.4. Characterization of Cartan hypersurfaces.

Theorem. *If $\mathbb{P}(X)$ is the polynomial that defines $\widehat{X_p}(M)$, where M is an homogeneous isoparametric hypersurface of S^n, with g distinct principal curvatures, then the infinity Laplacian of $\mathbb{P}(X)$ is*

$$\triangle_\infty \mathbb{P}(X) = k\|X\|^2 \mathbb{P}(X), \quad k \in \mathbb{R},\ k \neq 0,$$

if and only if $g = 3$. □

The proof of the preceding theorem consists in the computation of the infinity Laplacians of the polynomial $\mathbb{P}(X)$, for the eleven homogeneous isoparametric hypersurfaces in spheres.

For the cases $g = 4$ *real* and $g = 6$ (M_S and M_B) (see Section 1.3) a direct computation shows that the harmonic projection operator does not vanish.

For the remaining cases, Lemma 1 shows (easily) that the projection operator does not vanish either, because in all these polynomials it is possible to detect the presence of monomials, of degree five, with all variables to the first power. So, we have checked that only in the case $g = 3$ the harmonic projection operator is zero and the infinity Laplacian is the polynomial multiplied by the squared norm of the tangent vector.

Conclusion. In this way, we have shown that the Cartan isoparametric hypersurfaces are the only homogeneous isoparametric hypersurfaces in spheres for which the infinity Laplacian of the polynomial that defines the algebraic set of planar normal sections, is the polynomial multiplied by the squared norm of the tangent vector.

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