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DYNAMICAL ANALYSIS OF A NETWORKED CONTROL SYSTEM

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A new network data transmission strategy was proposed in [Zhang & Chen, 2005], where the resulting nonlinear system was analyzed and the effectiveness of the transmission strategy was demonstrated via simulations. In this paper, we further generalize the results of Zhang and Chen [2005] in the following ways: (1) Construct first-return maps of the nonlinear systems formulated in [Zhang & Chen, 2005] and derive several existence conditions of periodic orbits and study their properties. (2) Formulate the new system as a hybrid system, which will ease the succeeding analysis. (3) Prove that this type of hybrid systems is not structurally stable based on phase transition which can be applied to higher-dimensional cases effortlessly. (4) Simulate a higher-dimensional model with emphasis on their rich dynamics. (5) Study a class of continuous-time hybrid systems as the counterparts of the discrete-time systems discussed above. (6) Propose new controller design methods based on this network data transmission strategy to improve the performance of each individual system and the whole network. We hope that this research and the problems posed here will rouse the interest of researchers in such fields as control, dynamical systems and numerical analysis.

Keywords: Bifurcation; computational complexity; first-return map; hybrid system; networked control system; stability.

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1. Introduction

Consider the networked control system shown in Fig. 1.

It is obvious that connecting various system components via communication media can reduce wiring, ease installation and improve maintenance, among others. As far as distributed control systems are concerned, communication among individual controllers provides each of them with more information so that better control performance can be achieved [Ishii & Francis, 2002]. These advantages endorse a promising future for network control technology in systems engineering and applications.

Unfortunately, since the encoded system output, controller output and other information are transmitted via communication networks shared by many users, data traffic congestion is always unavoidable. This usually gives rise to time delays, packet loss and other undesirable behaviors in control systems. These problems have become a major subject of research in this and several closely related fields. Many network protocols and control strategies have already been proposed to tackle the problems. Loosely speaking, these considerations fall into three categories, which are further discussed as follows.

The first category simply models a networked control system as a control system with bounded time delays. In a series of recent papers [Walsh *et al.*, 1999, 2001, 2002, 2002b; Walsh & Ye, 2001], the try-once-discard (TOD) protocol is proposed and studied intensively, where an upper bound of sensor-to-controller time delays induced by the network is derived, for which exponential stability of the closed-loop system is guaranteed. This idea is further generalized in [Nesic & Teel, 2004] to derive a set of Lyapunov UGES (Uniformly Globally Exponentially Stable) protocols in the L^p framework. In [Yue *et al.*, 2005], assuming bounded time delays and packet dropouts, a robust H_∞ control problem is studied for networked control systems. In general, this approach is quite conservative, as has been widely acknowledged.

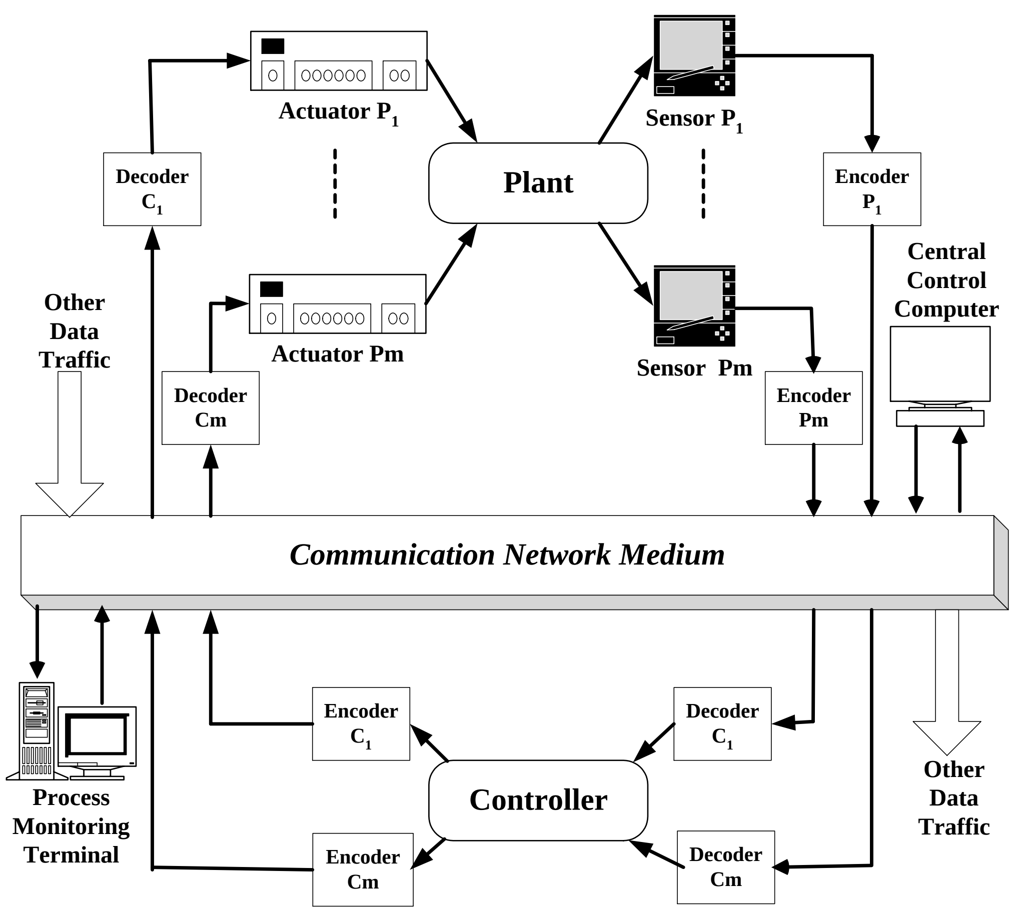

Fig. 1. A standard networked control system.

The second category models network time delays and packet dropouts as random processes, typically Markov chains. In this way, some specific features of these random processes can be utilized to design controllers that guarantee desired system performance. In [Krtolica *et al.*, 1994], a random model of time delays based on Markov chain is established via augmentation. Necessary and sufficient conditions for zero-state mean-square exponential stability have been derived for this system. In [Nilsson *et al.*, 1998], both sensor-to-controller and controller-to-actuator time delays are modeled as independent white-noise with zero mean and unit variance and consequently a (sub)optimal stochastic control problem is studied. Two Matlab toolboxes, Jitterbug and TrueTime, are introduced in [Cervin *et al.*, 2003], based on the principle that networked control systems can be viewed as delayed sampled-data systems with quantization effects. These two toolboxes can be used as experimental platforms for research on real-time dynamical control systems. They can be easily employed to quickly determine how sensitive a control system is to delays, jitters, lost samples, etc.

These two categories of methods deal with network effects passively, i.e. they solely consider the effects of network traffic on the control systems separately, leaving aside the interactions of the control systems and communication network. This latter consideration is very important, which leads to the third category of methodologies. This approach takes into account the tradeoff between data rate and control performance. In order to minimize bandwidth utilization, Goodwin *et al.* [2004] proposed some methods of using quantization to reduce the size of the transmitted data and solved the problem via a moving horizon technique. In [Wong & Brockett, 1999], the effect of quantization error, quantization and propagation time on the containability, a weaker stability concept, of networked control systems is studied. In [Takikonda & Mitter, 2004], the tradeoff of data rate and desirable control objectives is considered with emphasis on observability and stabilizability under communication constraints. A necessary condition is established on the rate for asymptotic observability and stabilizability of a linear discrete-time system. More specifically, the rate must be bigger than the summation of the logarithms of modules of the unstable system poles. Then, these results are further generalized to the study of control over noisy channels in [Takikonda & Mitter, 2004b]. The problem of asymptotic stabilization is considered in [Brockett & Liberzon, 2000], where time-varying quantizers are designed to achieve the stabilization of an unstable system. For the LQG optimal control of an unstable scalar system over an additive white Gaussian noise (AWGN) channel, it is reported [Elia, 2004] that the achievable transmission rate is given by the Bode sensitivity integral formula, thereby establishing the equivalence between feedback stabilization through an analog communication channel and a communication scheme based on feedback which unifies the design of control systems and communication channels.

In this paper, we continue the study of the data transmission strategy proposed in our earlier paper [Zhang & Chen, 2005], where a new network data transmission strategy was proposed to reduce network traffic congestion. By adding constant deadbands to both the controller and the plant shown in Fig. 1, signals will be sent only when it is necessary. By adjusting the deadbands, a tradeoff between control performance and reduction of network data transmission rate can be achieved. The data transmission strategy proposed is suitable for fitting a control network into an integrated communication network composed of control and data networks, to fulfill the need for a new breed geared toward total networking (see [Raji, 1994]). This problem is of course very appealing as depicted by Raji [1994]; and at the same time it is fundamentally important so is listed in Murray *et al.* [2003] as a future direction in control research in an information-rich world: "Current control systems are almost universally based on synchronous, clocked systems, so they require communication networks that guarantee delivery of sensor, actuator, and other signals with a known, fixed delay. Although current control systems are robust to variations that are included in the design process (such as a variation in some aerodynamic coefficient, motor constant or moment of inertia), they are not at all tolerant of (unmodeled) communication delays or dropped or lost sensor or actuator packets. Current control system technology is based on a simple communication architecture: all signals travel over synchronous dedicated links, with known (or worst-case bounded) delays and no packet loss. Small dedicated communication networks can be configured to meet these demanding specifications for control systems, but a very interesting question is whether we can develop a theory and practice for control systems that operate in a distributed, asynchronous, packet-based environment.”

Essentially speaking, under the network data transmission strategy proposed here, in an integrated network composed of data and control networks, it is asked that the network should provide sufficient communication bandwidth upon request of control systems. As a payoff, control systems will save network resources by deliberately dropping packets without degrading system performance severely. This is a crucial tradeoff. On the one hand, control signals are normally time critical, hence the priority should be given to them whenever requested; on the other hand, due to one characteristic of control networks, namely, small packet size but frequent packet flows, it is somewhat troublesome to manage because it demands frequent transmissions. Our scheme aims to relieve this burden for the whole communication network.

As we proceed, readers will find that the simple network transmission data strategy analyzed here gives rise to many unexpected and interesting dynamical phenomena and mathematical problems, which have innocent appearance but are hard to deal with. More specifically, we investigate the following issues: the stability of the control systems under the proposed scheme; existence of periodic orbits by means of first-return maps; bifurcation and phase transition phenomena; computational complexity. Since this research project is oriented toward the study of control problems in a network setting, the effectiveness of the scheme and the corresponding controller design problem will also be addressed after system analysis.

The layout of this paper is as follows: the proposed network protocol is presented in Sec. 2, where its advantages are discussed. The resulting closed-loop system under this network protocol is analyzed in Secs. 3–7. More precisely, Sec. 3 contains a study of a closed-loop system consisting of a scalar plant controlled by a proportional controller with a constant gain, as the simplest case under this framework to provide a detailed analysis. The structural stability of the system is studied in Sec. 4. Section 5 studies the rich dynamics of a higher-dimensional model. The continuous-time counterpart of this type of discrete-time systems is discussed in Sec. 5. The controller design problem is then addressed in Sec. 7. Some concluding remarks, open problems and future research issues are finally posed and discussed in Sec. 8.

2. The Proposed Network Protocol

In [Zhang & Chen, 2005], a new data transmission strategy was proposed, which is briefly reviewed here. Consider the feedback system shown in Fig. 2, where G is a discrete-time system of the form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{1}$$

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^p$, and the reference input $r \in \mathbb{R}^p$, respectively; C is a stabilizing controller:

$$\begin{aligned} x_d(k+1) &= A_d x_d(k) + B_d e(k), \\ u(k) &= C_d x_d(k) + D_d e(k), \\ e(k) &= r(k) - y(k), \end{aligned} \tag{2}$$

with its state $x_d \in \mathbb{R}^{n_c}$. Let $\xi = \begin{bmatrix} x \\ x_d \end{bmatrix}$. Then, the closed-loop system from r to e is described by

$$\begin{aligned} \xi(k+1) &= \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \xi(k) \\ &\quad + \begin{bmatrix} BD_d \\ B_d \end{bmatrix} r(k), \\ e(k) &= \begin{bmatrix} -C & 0 \end{bmatrix} \xi(k) + r(k). \end{aligned} \tag{3}$$

Now, we add some nonlinear constraints on both u and y. Specifically, consider the system shown in Fig. 3. The nonlinear constraint H_1 is defined as follows: for a given $\delta_1 > 0$, let $v(-1) = 0$; and for $k \geq 0$, let

$$\begin{aligned} v(k) &= H_1(u_c(k), v(k-1)) \\ &= \begin{cases} u_c(k), & \text{if } \|u_c(k) - v(k-1)\|_\infty > \delta_1, \\ v(k-1), & \text{otherwise.} \end{cases} \end{aligned} \tag{4}$$

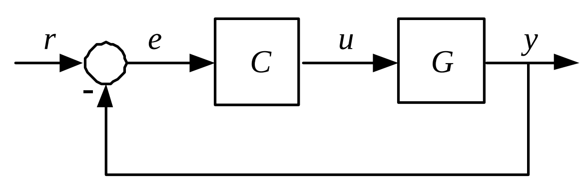

Fig. 2. A typical feedback system.

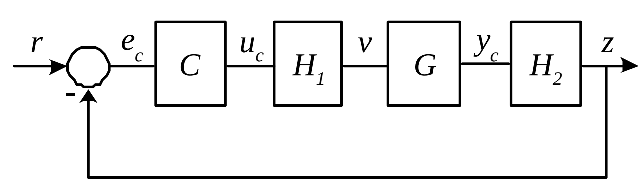

Fig. 3. A constrained feedback system.

Similarly, H_2 is defined as follows: for a given $\delta_2 > 0$, let $z(-1) = 0$; for $k \geq 0$, let

$$z(k) = H_2(y_c(k), z(k-1)) = \begin{cases} y_c(k), & \text{if } \|y_c(k) - z(k-1)\|_\infty > \delta_2, \\ z(k-1), & \text{otherwise.} \end{cases} \quad (5)$$

It can be shown that $\|H_1\|$, the induced norm of H_1, equals 2, and so is $\|H_2\|$.

In [Octanez *et al.*, 2002] *adjustable* deadbands are proposed to reduce network traffics, where the closed-loop system with deadbands is modeled as a perturbed system, with exponential stability followed from that of the original system [Khalil, 1996]. The constraints, δ_1 and δ_2, proposed here are fixed. We have observed [Zhang & Chen, 2005] that the stability of the system shown in Fig. 3 is fairly complicated and only local stability can be obtained. However, the main advantage of fixed deadbands is that it will reduce network traffics more effectively. Furthermore, the stability region can be scaled as large as desired.

For the "constrained" system shown in Fig. 2, let p denote the state of the system G and p_d denote the state of the controller C. Then

$$p(k+1) = Ap(k) + Bv(k),$$
$$y_c(k) = Cp(k),$$

and

$$p_d(k+1) = A_d p_d(k) + B_d e_c(k),$$
$$u_c(k) = C_d p_d(k) + D_d e_c(k),$$
$$e_c(k) = r(k) - z(k).$$

Let $\eta = \begin{bmatrix} p \\ p_d \end{bmatrix}$. Then, the closed-loop system from r to e is

$$\eta(k+1) = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B_d \end{bmatrix} r(k), \quad (6)$$
$$e_c(k) = \begin{bmatrix} -C & 0 \end{bmatrix} \eta(k) + r(k),$$

where v and z are given in Eqs. (4) and (5).

At this point, one can see that in the framework of the communication network containing both data and control networks, this proposed data transmission strategy will provide sufficient communication bandwidth upon request of control networks used by the control systems. As a payoff, the control systems will save some network resources by deliberately dropping packets. This consideration is well tailored to the requirement of control networks in general. On the one hand, control signals are normally time critical, hence the priority should be given to them whenever requested. On the other hand, due to the characteristics of control networks, namely, small packet size but frequent packets flows, it is somewhat troublesome to manage because it demands frequent transmissions. Our scheme aims to relieve this burden for the whole communication network.

3. First-Return Maps

To simplify the following discussions, suppose that the system G in Fig. 2 is a scalar system, the controller C is simply -1, there is no H_2 involved, and $r = 0$. That is, in this section, we consider the following simplified system:

$$x(k+1) = ax(k) + bv(k), \quad (7)$$

with $v(-1) \in \mathbb{R}$, and for $k \geq 0$,

$$v(k) = H(x(k), v(k-1)) := \begin{cases} x(k), & \text{if } |x(k) - v(k-1)| > \delta, \\ v(k-1), & \text{otherwise,} \end{cases} \quad (8)$$

where $|a + b| < 1$ and δ is a positive number.

In [Zhang & Chen, 2005], the system composed of Eqs. (7) and (8) was studied in great detail, where a necessary condition for the existence of periodic orbits was derived. We now generalize it and provide a necessary and sufficient condition and give a characterization of the state space based on it. This analysis is important: To achieve good control, a nonlinear system may be desired to work near an equilibrium point or a limit cycle. In the case that a limit cycle is preferred, this result will reveal under what condition limit cycles may exist and starting from where a trajectory may converge to the desired limit cycle. In the case that an equilibrium is desirable, this result will provide the designer with some information as how to design controllers to prevent trajectories from being stuck into a limit cycle. Hence, this analysis will provide useful insights into the design of control systems under the proposed data transmission strategy. We now investigate this important problem case by case.

3.1. *Case 1:* $0 < a < 1$, $b < 0$

For convenience, we present Fig. 4, where

$$L_o := \left\{ (v_-, x) \in I_a : x = \frac{b}{1-a} v_- \right\},$$

$$L_{\delta+} := \{(v_-, x) \in I_b : x - v_- = \delta\},$$

$$L_{\delta-} := \{(v_-, x) \in I_b : x - v_- = -\delta\}$$

$$L_{1\delta+} := \left\{(v_-, x) \in I_b : x = (a+b)v_-, v_- > \frac{1-|a|}{1-(a+b)}\delta\right\},$$

$$L_{1\delta-} := \left\{(v_-, x) \in I_b : x = (a+b)v_-, v_- < -\frac{1-|a|}{1-(a+b)}\delta\right\},$$

$$L_{Hb+} := \left\{\left(v_-, \frac{-b}{1-|a+b|}\delta\right) : |v_-| \leq \frac{-b}{1-|a+b|}\delta\right\},$$

$$L_{Hb-} := \left\{\left(v_-, \frac{b}{1-|a+b|}\delta\right) : |v_-| \leq \frac{-b}{1-|a+b|}\delta\right\},$$

$$L_{Vb+} := \left\{\left(\frac{-b}{1-|a+b|}\delta, x\right) : |x| \leq \frac{-b}{1-|a+b|}\delta\right\},$$

$$L_{Vb-} := \left\{\left(\frac{b}{1-|a+b|}\delta, x\right) : |x| \leq \frac{-b}{1-|a+b|}\delta\right\},$$

$$L_{Va+} := \left\{\left(\frac{1-|a|}{1-(a+b)}\delta, x\right), : |x| \leq \frac{-b}{1-(a+b)}\delta\right\},$$

$$L_{Va-} := \left\{\left(-\frac{1-|a|}{1-(a+b)}\delta, x\right), : |x| \leq \frac{-b}{1-(a+b)}\delta\right\}.$$

Assume that an initial condition (v_0, x_0) is located on the line $L_{1\delta+}$ satisfying

$$x_0 = (a+b)v_0.$$

Moreover, suppose that the trajectory starting from it does not converge to a fixed point (see [Zhang & Chen, 2005] for details). Hence, the successive iterations are given by

$$x_1 = ax_0 + bv_0 = (a^2 + ab + b)v_0 = \left(a^2 + \sum_{i=0}^{1} a^i b\right) v_0,$$

$$v_1 = v_0,$$

$$x_2 = ax_1 + bv_0 = \left(a^3 + \sum_{i=0}^{2} a^i b\right) v_0,$$

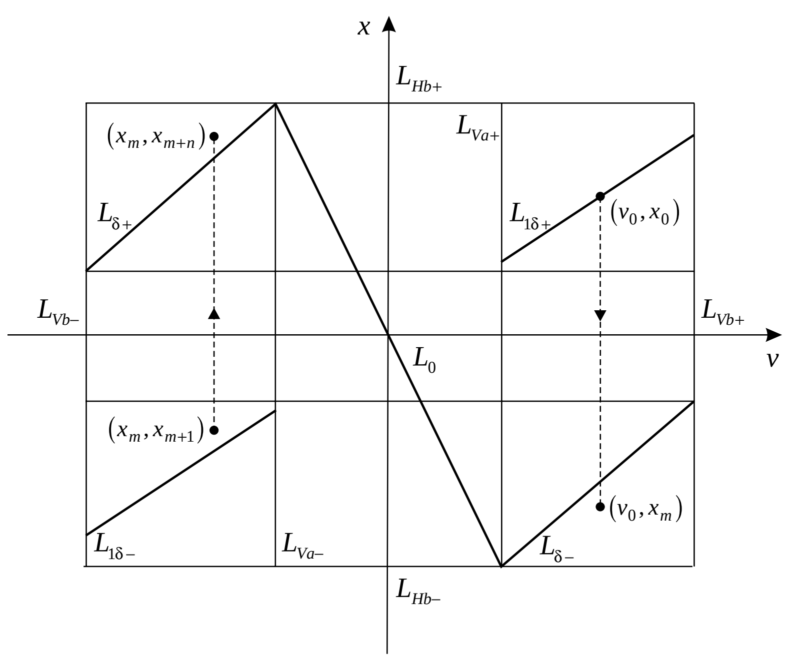

Fig. 4. Diagram for the case with $0 < a < 1$ and $b < 0$.

$$v_2 = v_0,$$
$$\vdots$$
$$x_m = ax_{m-1} + bv_0 = \left(a^{m+1} + \sum_{i=0}^{m} a^i b\right) v_0,$$
$$v_m = v_0.$$

Accordingly,

$$v_m - x_m = \left(1 - a^{m+1} - \sum_{i=0}^{m} a^i b\right) v_0.$$

As indicated in Fig. 4, the orbit moves downward following the line $v = v_0$ on the right part of the region. In this way, there will exist a value of m such that the trajectory crosses the line segment $L_{\delta-}$, i.e.

$$|x_m - v_m| > \delta, \tag{9}$$

and note that such an m always exists. Thus

$$\left(1 - a^{m+1} - \sum_{i=0}^{m} a^i b\right) v_0 > \delta$$
$$\Leftrightarrow \frac{(1-a)\delta}{(1-(a+b))v_0} < 1 - a^{m+1}$$
$$\Leftrightarrow a^{m+1} < 1 - \frac{(1-a)\delta}{(1-(a+b))v_0}$$
$$\Leftrightarrow m > \frac{\ln\left(1 - \frac{(1-a)\delta}{(1-(a+b))v_0}\right)}{\ln a} - 1.$$

Hence, the smallest m is given by

$$m = \left\lceil \frac{\ln\left(1 - \frac{(1-a)\delta}{(1-(a+b))v_0}\right)}{\ln a} \right\rceil - 1, \tag{10}$$

where $\lceil r \rceil$ is the least integer bigger than r. Note that

$$x_{m+1} = (a+b)x_m,$$
$$v_{m+1} = x_m.$$

Since $x_m < 0$, this point is located on the left part of Fig. 4, and also

$$|x_{m+1} - v_{m+1}| < \delta.$$

Hence

$$x_{m+2} = ax_{m+1} + bv_m = \left(a^2 + \sum_{i=0}^{1} a^i b\right) x_m,$$
$$v_{m+2} = x_m,$$
$$\vdots$$
$$x_{m+n} = \left(a^n + \sum_{i=0}^{n-1} a^i b\right) x_m,$$
$$v_{m+n} = x_m,$$
$$x_{m+n} - v_{m+n} = \left(a^n + \sum_{i=0}^{n-1} a^i b - 1\right) x_m.$$

The orbit now moves upward along the line $v = x_m$ (see Fig. 4). Then, there will exist a value of n such that the trajectory crosses $L_{\delta+}$, i.e.

$$|x_{m+n} - v_{m+n}| > \delta.$$

Note that

$$v_{m+n} = x_m < 0.$$

Then

$$\left(a^n + \sum_{i=0}^{n-1} a^i b - 1\right) x_m = \left(a^n + \sum_{i=0}^{n-1} a^i b - 1\right)\left(a^{m+1} + \sum_{i=0}^{m} a^i b\right) v_0 > \delta$$
$$\Leftrightarrow 1 - a^n > \frac{(1-a)\delta}{(a+b-1)\left((1-a^{m+1})\frac{a+b-1}{1-a} + 1\right) v_0}$$
$$\Leftrightarrow n > \frac{\ln\left(1 - \frac{(1-a)\delta}{(a+b-1)\left((1-a^{m+1})\frac{a+b-1}{1-a} + 1\right) v_0}\right)}{\ln a}.$$

Hence, the smallest n is

$$n = \left\lceil \frac{\ln\left(1 - \dfrac{(1-a)\delta}{(a+b-1)\left((1-a^{m+1})\dfrac{a+b-1}{1-a}+1\right)v_0}\right)}{\ln a} \right\rceil. \tag{11}$$

The switching law [Eq. (8)] provokes that the new iteration point

$$x_{m+n+1} = (a+b)x_{m+n},$$
$$v_{m+n+1} = x_{m+n},$$

returns to the zone where the trajectory was originated. In particular, if

$$(v_{m+n+1}, x_{m+n+1}) = (v_0, x_0)$$

then one will get a closed orbit. This motivates us to define the first-return map

$$\varphi : L_1 \to L_1$$
$$v \mapsto \left(a^n + \sum_{i=0}^{n-1} a^i b\right)\left(a^{m+1} + \sum_{i=0}^{m} a^i b\right)v, \tag{12}$$

where L_1 is the projection of $L_{1\delta+}$ onto the v axis, and m and n satisfy Eqs. (10) and (11), respectively.

Definition 1 (Type 1 periodic orbits). A periodic orbit starting from $(v_0, x_0) \in L_{1\delta+}$ is said to be of type 1 if

$$\varphi(v_0) = v_0, \tag{13}$$

where φ is defined by Eq. (12), and m and n satisfy Eqs. (10) and (11), respectively. In this case, the period of this orbit starting from (v_0, x_0) is $m+n+1$.

Remark 1. A periodic orbit is of type 1 if it forms a closed loop right after the first return. There are possibly other periodic orbits that become closed loops after several returns. These orbits can be studied in a similar way, but it is more computationally involved.

The following result follows immediately from the foregoing discussions.

Theorem 1. *The trajectory starting from* (v_0, x_0) *is periodic of type 1 if and only if Eq. (13) holds.*

Actually, we can find all periodic orbits of type 1: If Eq. (13) holds, i.e.

$$\varphi(v_0) = v_0,$$

then

$$\left(a^n + \sum_{i=0}^{n-1} a^i b\right)\left(a^{m+1} + \sum_{i=0}^{m} a^i b\right) = 1,$$

i.e.

$$\left((1-a^n)\frac{a+b-1}{1-a}+1\right) \times \left((1-a^{m+1})\frac{a+b-1}{1-a}+1\right) = 1 \tag{14}$$

for some $m, n > 0$. Given that $0 < a < 1$, $b < 0$ and $|a+b| < 1$, m and n satisfying (14) are both finite. Hence, all periodic orbits of type 1 can be found.

Remark 2. If (v, x) leads to a periodic orbit of type 1, according to Eqs. (10) and (11), there exists a neighborhood of (v, x) on $L_{1\delta+}$ such that each point of which will lead to a periodic orbit of type 1, so all such orbits are together dense.

3.2. *Case 2: a = 1*

Assume that an initial condition (v, x) satisfies

$$x = (1+b)v, v > 0,$$

and also suppose that the orbit starting from it is within the oscillating region. Then

$$x_1 = x + bv = (1+2b)v,$$
$$v_1 = v,$$
$$\vdots$$
$$x_m = x_1 + bv_1 = (1 + (m+1)\,b)\,v,$$
$$v_m = v.$$

Suppose that

$$v_m - x_m = -(m+1)bv > \delta.$$

Then

$$m+1 > \frac{\delta}{(-b)v}.$$

Hence, the least m is given by

$$m = \left\lceil \frac{\delta}{(-b)\,v} \right\rceil - 1.$$

Moreover,

$$\begin{aligned} x_{m+1} &= (1+b)x_m, \\ v_{m+1} &= x_m < 0, \\ &\vdots \\ x_{m+n} &= (1+nb)x_m, \\ v_{m+n} &= x_m. \end{aligned}$$

Suppose that

$$x_{n+m} - v_{n+m} = nbx_m > \delta.$$

Then

$$n > \frac{\delta}{bx_m} = \frac{\delta}{b(1+(m+1)b)v}.$$

Hence, the smallest n is given by

$$n = \left\lceil \frac{\delta}{b(1+(m+1)b)v} \right\rceil.$$

Define

$$\begin{aligned} \varphi &: L_1 \to L_1 \\ v &\mapsto (1+nb)(1+(m+1)b)v, \end{aligned} \tag{15}$$

If

$$\varphi(v) = v,$$

then

$$\frac{1}{m+1} + \frac{1}{n} = -b. \tag{16}$$

Theorem 2. *The trajectory starting from $(v, (1+b)v)$ is periodic of type 1 if and only if v is a fixed point of the first-return map defined in Eq. (15).*

Remark 3. This result is a generalization of Theorem 3 in [Zhang & Chen, 2005], where the condition is only necessary. For example, given $a = 1$ and $b = -1/2$, the origin is the unique invariant set. It is obvious that $p = q = 4$ is a solution to

$$\frac{1}{p} + \frac{1}{q} = -b.$$

However, there are no periodic orbits. This indicates that the necessary condition given by Theorem 3 in [Zhang & Chen, 2005] is not sufficient.

Next we find all periodic orbits of type 1 for the case of $a = 1$.

For convenience, here we use m instead of $m+1$ in Eq. (16). Suppose

$$b = -\frac{q}{p},$$

where $p > 0$, $q > 0$, $\gcd(p, q) = 1$. According to Eq. (16),

$$\frac{1}{m} = -b - \frac{1}{n} = \frac{pn - q}{qn},$$

i.e.

$$m = \frac{qn}{pn - q}.$$

Obviously,

$$m > \frac{q}{p}.$$

Furthermore, m is a decreasing function of n. By symmetry, let

$$n_0 = \left\lceil \frac{q}{p} \right\rceil.$$

Then

$$\left\lceil \frac{q}{p} \right\rceil \le m \le \left\lceil \frac{qn_0}{pn_0 - q} \right\rceil.$$

Similarly,

$$\left\lceil \frac{q}{p} \right\rceil \le n \le \left\lceil \frac{qm_0}{pm_0 - q} \right\rceil,$$

where

$$m_0 = \left\lceil \frac{q}{p} \right\rceil.$$

Based on this analysis and Theorem 2, all periodic orbits of type 1 can be determined.

3.3. *Case 3: $a = -1$ and $|a + b| < 1$*

For this case, each trajectory is an eventually periodic orbit of period 2.

3.4. *Case 4: $a > 1$*

This is similar to the case of $0 < a < 1$. The only difference is

$$m < \frac{\ln\left(1 - \dfrac{(1-a)\delta}{(1-(a+b))v}\right)}{\ln a} - 1,$$

due to $\ln a > 0$. The least m is

$$m = \left\lceil \frac{\ln\left(1 - \dfrac{(1-a)\delta}{(1-(a+b))v}\right)}{\ln a} \right\rceil - 1.$$

The complex dynamics exhibited in this system is due to its nonlinearity induced by switching. This is different from that of a quantized system. The complicated behavior of an unstable quantized scalar system has been extensively studied in [Delchamps, 1988, 1989, 1990] and [Fagnani & Zampieri, 2003], and the MIMO case is addressed in [Fagnani & Zampieri, 2004]. In [Delchamps, 1990], it is mentioned that if the system parameter a is stable, a quantized system may have many fixed points as well as periodic orbits, which are all asymptotically stable. However, for the constrained systems here, almost all trajectories are not periodic orbits. For systems with $a = 1$, periodic orbits are locally stable, which is not the case for a quantized system [Delchamps, 1990]. Given that a is unstable, the ergodicity of the quantized system is investigated in [Delchamps, 1990]. In essence, related results there depend heavily on the affine representation of the system by which the system is piecewise expanding, i.e. the absolute value of the derivative of the piecewise affine map in each interval partitioned naturally by it is greater than 1. Based on this crucial property, the main theorem (Theorem 1) in [Lasota & Yorke, 1973] and then that in [Li & Yorke, 1978] are employed to show that there exists a unique invariant measure under the affine map on which the map is also ergodic. Therefore, ergodicity has been established for scalar unstable quantized systems. However, this is not the case for the system studied here. Though the system is still piecewise linear, it is **singular** with respect to the Lebesgue measure and, furthermore, the derivative of the system map in a certain region is $(a + b)$, whose absolute value is strictly less than 1. Hence, the results in [Lasota & Yorke, 1973] and [Li & Yorke, 1978] are not applicable here. However, by extensive experiments, we strongly believe that the system indeed has the property of ergodicity. This will be left as a conjecture for future research to verify theoretically.

4. Structural Stability

Loosely speaking, a nonlinear system is *structurally stable* if a slight perturbation of its system parameters will not change its phase portrait qualitatively. In [Zhang & Chen, 2005], we proved that given $a = 1$ in system (7), if there are periodic orbits, then the system is not structurally stable. In this section, we will show that the condition of $a = 1$ is actually unnecessary.

We begin with the simplest case, namely, a generic system, whose unique attractor is the line segment of fixed points. Even such a simple case can still be classified into at least two categories. We proved in [Zhang & Chen, 2005] that if the parameters of the system defined by Eqs. (7) and (8) satisfy

$$\frac{1-|a|}{1-(a+b)} > \frac{|b|}{1-|a+b|} \tag{17}$$

with $|a + b| < 1$ and $|a| < 1$, then it is generic. One of its global attracting regions is shown in Fig. 5, where I_o is the line segment of fixed points, which runs from the point $(x_2, x_1) = (-((1-|a|)\delta/(1-(a+b))), (b/(1-a))((1-|a|)\delta/(1-(a+b)))$ on the left to the point $(x_2, x_1) = (((1-|a|)\delta/(1-(a+b)), -(b/(1-a))(((1-|a|)\delta/(1-(a+b)))$ on the right. A trajectory will converge *vertically* to a certain point on I_o. On the other hand, we proved that the system with parameters $a = 3/10$ and $b = -(9/10)$, which do not satisfy Eq. (17), is also generic whose typical trajectories are like that shown in Fig. 6 (the trajectory starting from $(0.005, 0.005)$ around converges to a fixed point close to $(0.0004, 0.004)$ after several oscillations). Till now, we have not found a third type of generic systems. Apparently the first generic system is simpler than the second one. Hence, we first investigate the structural stability of the first type of systems. For convenience, we call such systems type-1 generic systems or generic systems of type 1. Observing that each type-1 generic system has a global attracting region as shown in Fig. 5, thereby we focus on its behavior in this region.

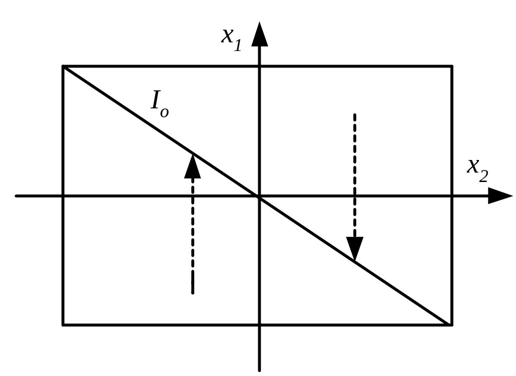

Fig. 5. A global attracting region of a type-1 generic system.

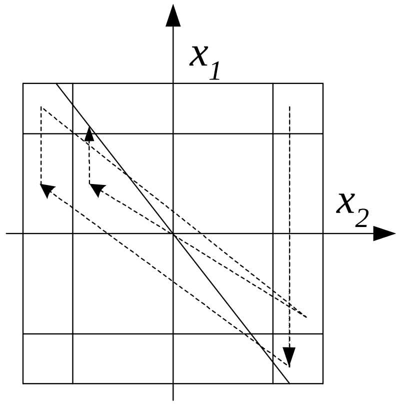

Fig. 6. A global attracting region of the generic system with $a = 0.3$ and $b = -0.9$.

The following result asserts that two generic systems of type 1 are "identical" in the sense of topology:

Proposition 1. *Two type-1 generic systems are homeomorphic, i.e. there exists a bijective map from one to the other which has a continuous inverse.*

Proof. For convenience, define

$$x_2(k) := v(k-1), \quad k \geq 0.$$

Then, the original system defined in Eqs. (7) and (8) is equivalent to

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{cases} \begin{bmatrix} a+b & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, & \text{if } |x_1(k) - x_2(k)| > \delta, \\ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, & \text{otherwise.} \end{cases} \tag{18}$$

Consider the following two type-1 generic systems whose global attracting regions are shown in Fig. 7:

$$\Sigma_1 : x(k+1) = \begin{cases} B_1 x(k), & \text{if } |x_1(k) - x_2(k)| > \delta, \\ A_1 x(k), & \text{otherwise,} \end{cases}$$

$$\Sigma_2 : y(k+1) = \begin{cases} B_2 x(k), & \text{if } |y_1(k) - y_2(k)| > \delta, \\ A_2 x(k), & \text{otherwise,} \end{cases}$$

in which

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad A_i = \begin{bmatrix} a_i & b_i \\ 0 & 1 \end{bmatrix},$$

$$B_i = \begin{bmatrix} a_i + b_i & 0 \\ 1 & 0 \end{bmatrix}, \qquad i = 1, 2.$$

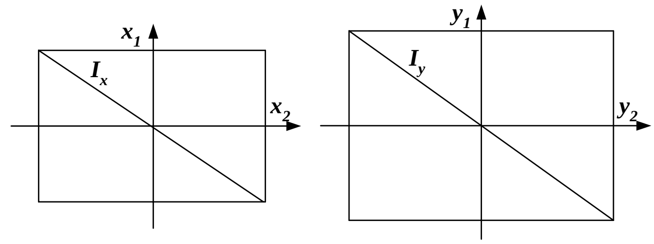

Fig. 7. Global attracting regions of systems Σ_1 and Σ_2.

Next, we define a map h from I_x to I_y by

$$h : I_x \to I_y,$$

$$\left(\frac{b_1}{1-a_1} x_2, x_2\right) \mapsto \left(\frac{b_2}{1-a_2} \frac{\dfrac{1-|a_2|}{1-(a_2+b_2)}}{\dfrac{1-|a_1|}{1-(a_1+b_1)}} x_2, \frac{\dfrac{1-|a_2|}{1-(a_2+b_2)}}{\dfrac{1-|a_1|}{1-(a_1+b_1)}} x_2 \right). \tag{19}$$

Clearly, h is one-to-one, onto and has a continuous inverse.

Next, define

$$\tilde{h} : \Sigma_1 \longrightarrow \Sigma_2,$$

$$(x_1, x_2) \mapsto \left(\frac{\dfrac{b_2}{1-a_2}}{\dfrac{b_1}{1-a_1}} \frac{\dfrac{1-|a_2|}{1-(a_2+b_2)}}{\dfrac{1-|a_1|}{1-(a_1+b_1)}} x_1, \frac{\dfrac{1-|a_2|}{1-(a_2+b_2)}}{\dfrac{1-|a_1|}{1-(a_1+b_1)}} x_2 \right). \tag{20}$$

It is easy to see that the projection of $\tilde{h}$ on I_x is exactly h, and furthermore $\tilde{h}$ is a homeomorphism. Actually system Σ_2 can be obtained by stretching (or contracting) system Σ_1, therefore they are topologically equivalent. ■

Two remarks are in order.

Remark 4. It is hard to apply the foregoing method to other types of generic systems because they may not have such simple global attracting regions.

Remark 5. As can be conjectured, this method is probably not applicable to nongeneric systems that have some complex attractors besides the line segment of fixed points (see [Zhang & Chen, 2005] for details).

Before investigating the structural stability of generic systems, we first discuss their ω-stability. A dynamical system is ω-stable if there exists a homeomorphism from its *nonwandering* set (here it is I_o) to that of the system obtained by perturbing it slightly [Smale, 1967]. Hence, a structurally stable dynamical system is necessarily ω-stable, but the converse may not be true. Because there always exists a homeomorphism between two given line segments, it seems plausible to infer that a generic system is ω-stable. Unfortunately, it is not true. Observe that Proposition 1 holds upon the assumption that two given systems are generic, some other "unusual" types of perturbations may lead to a system that is not generic, thus destroying the ω-stability of generic systems. To that end, a new point of view is required.

Define a family of systems:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{cases} A_1 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, & \text{if } |x_1(k) - x_2(k)| > \delta, \\ A_2 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, & \text{otherwise}, \end{cases} \tag{21}$$

where

$$A_1 = \begin{bmatrix} a+b & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a+\lambda b & (1-\lambda)\,b \\ \lambda & (1-\lambda) \end{bmatrix}. \tag{22}$$

Note that when $\lambda = 1$, $A_2 = A_1$, and that this system is a stable linear system. When $\lambda = 0$, $A_2 = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, giving the system defined by Eq. (18). Hence, by introducing $\lambda \in [0, 1]$, one gets a family of systems.

It is easy to verify the following result:

Theorem 3. *For each $\lambda \in (0, 1]$, system (21) has a unique fixed point $(0, 0)$.*

Consider a perturbation of a system in the form of (18) by choosing a λ sufficiently close to but not equal to zero. Theorem 3 tells us that the new system has a unique fixed point. Clearly there are no homeomorphisms between these two systems since there exist no one-to-one maps from a line segment to a single point. Moreover, a nongeneric system also has a line segment of fixed points. So, we have the following conclusion:

Theorem 4. *System (18) is not ω-stable.*

The following is an immediate consequence.

Corollary 1. *System (18) is not structurally stable.*

Remark 6. The above investigation tells us that the system with $\lambda = 0$ is a rather ill-conditioned one. Will a system with $\lambda \neq 0$ be ω-stable (or even structurally stable): We are convinced that this generally holds, but till now we have not found a proof.

Remark 7. For a generic system, no matter it is of type 1 or not, its nonwandering set is just a line segment of fixed points, therefore its ω-stability is preserved if a perturbation is on a and b, while *not* destroying the structure shown in Eq. (18). Consider the discussion in Sec. 2, the system composed of (7) and (8) is proposed for a new data transmission strategy, hence though the perturbation of a and b is reasonable, the perturbation of the form (21) and (22) induced by λ does not make sense physically. Based on this, we can say that ω-stability is "robust" with respect to uncertainty which is meaningful (the same argument is proposed in [Robbin, 1972] for structural stability). However, it is pretty fragile with respect to such rare uncertainty as that in Eqs. (21) and (22). In other words, it is robust yet fragile. It is argued in [Doyle, 2004] that "robust yet fragile" is the most important property of complex systems.

Remark 8. It follows from the above results that there is a transition process in the family of systems defined by Eqs. (21) and (22) as λ moves from 1 to 0, which has been discussed in our other paper [Zhang *et al.*, 2005].

5. Higher-Order Systems

In this section, we briefly discuss the high-dimensional cases.

Consider the following two-dimensional system:

$$x_1(k+1) = a_1 x_1(k) + b_1 x_2(k),$$
$$x_2(k+1) = a_2 x_2(k) + b_2 v(k),$$

where

$$v(k) = \begin{cases} x_1(k), & \text{if } |x_1(k) - v(k-1)| > \delta, \\ v(k-1), & \text{otherwise}. \end{cases}$$

Introduce a new variable,

$$x_3(k) = v(k-1),$$

and define

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_1 & b_1 & 0 \\ b_2 & a_2 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a_1 & b_1 & 0 \\ 0 & a_2 & b_2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$x(k+1) = \begin{cases} A_1 x(k), & \text{if } |x_1(k) - x_3(k)| > \delta, \\ A_2 x(k), & \text{otherwise.} \end{cases} \tag{23}$$

5.1. *Fixed points and switching surfaces*

Suppose that $(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ is a fixed point of system (23). Then

$$\overline{x}_1 = a_1\overline{x}_1 + b_1\overline{x}_2,$$
$$\overline{x}_2 = a_2\overline{x}_2 + b_2\overline{x}_3,$$
$$\overline{x}_3 = \overline{x}_3.$$

If $a_1 \neq 1$ and $a_2 \neq 1$, then

$$\overline{x}_2 = \frac{b_2}{1-a_2}\overline{x}_3, \quad \overline{x}_1 = \frac{b_1}{1-a_1}\frac{b_2}{1-a_2}\overline{x}_3, \tag{24}$$

where

$$|\overline{x}_3| \leq \frac{\delta}{\left|\dfrac{b_1 b_2}{(1-a_1)(1-a_2)} - 1\right|}. \tag{25}$$

If $a_1 \neq 1$ and $a_2 = 1$, then

$$\overline{x}_3 = 0, \quad \overline{x}_1 = \frac{b_1}{1-a_1}\overline{x}_2 \tag{26}$$

and

$$|\overline{x}_2| \leq \left|\frac{(1-a_1)\,\delta}{b_1}\right|. \tag{27}$$

If $a_1 = 1$ and $a_2 \neq 1$, then

$$\overline{x}_2 = 0, \quad \overline{x}_3 = 0 \tag{28}$$

and

$$|\overline{x}_1| \leq \delta. \tag{29}$$

In all the three cases, fixed points constitute a line segment in $\mathbb{R}^3$. Note that the case of $a_1 = 1$ and $a_2 = 1$ is contained in the case defined by Eqs. (28) and (29).

Next, we consider the first case. Obviously the switching surfaces of this system are

$$x_1 - x_3 = \pm\delta,$$

hence the two end points of the line of fixed points are

$$\pm\left(\frac{k_1}{k_1-1}\delta, \ \frac{b_2}{1-a_2}\frac{k_1}{k_1-1}\delta, \ \frac{1}{k_1-1}\delta\right),$$

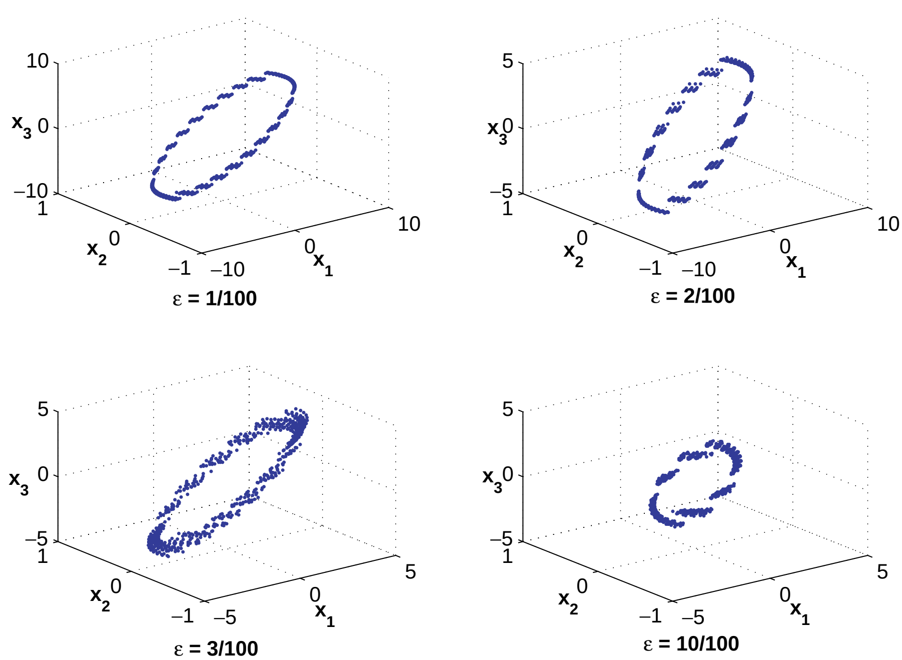

Fig. 8. Attractors in 3-d:I.

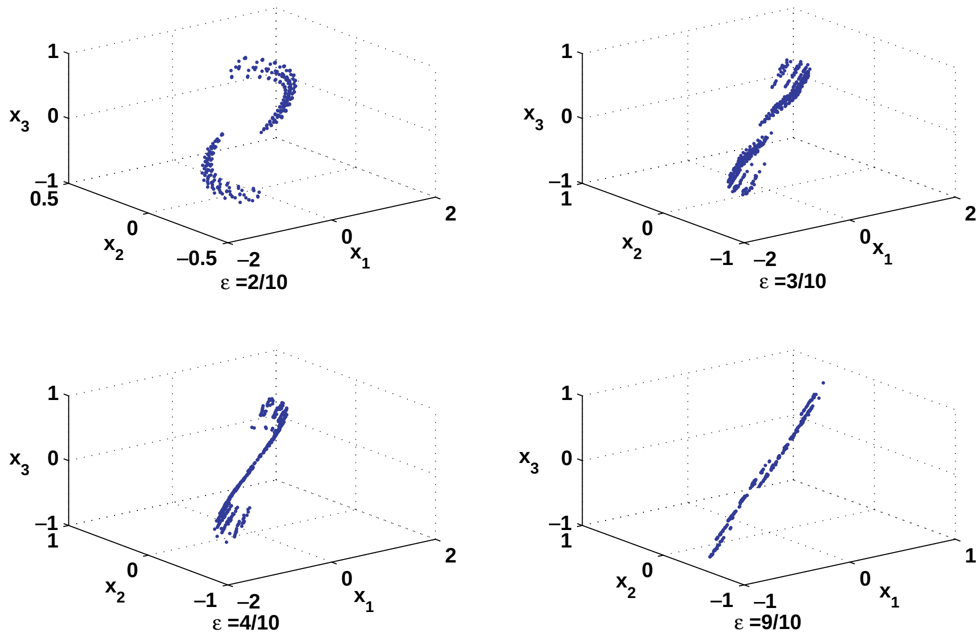

Fig. 9. Attractors in 3-d:II.

where $k_1 = ((b_1/(1-a_1))(b_2/(1-a_2)))$. They are symmetric with respect to the origin.

5.2. *An example*

Based on the analysis in Sec. 4, one can see that system (23) is not structurally stable. In this section, we illustrate the complex behavior of this system. Consider the following system:

$$x(k+1) = \begin{cases} A_1 x(k), & \text{if } |x_1(k) - x_3(k)| > 1, \\ A_2 x(k), & \text{otherwise,} \end{cases} \tag{30}$$

where

$$A_1 = \begin{bmatrix} 1-\epsilon & 1 & 0 \\ -\dfrac{\epsilon}{2} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1-\epsilon & 1 & 0 \\ 0 & 1 & -\dfrac{\epsilon}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

The variations of the trajectory, starting from $(1, 1/10^5, -1)$ as ϵ varies, are plotted in Figs. 8 and 9. One can see the phase transition process vividly from these figures.

Figures 8 and 9 reveal the rich dynamics of a 3-d system governed by the switching law, which will be our future research topic.

6. The Continuous-Time Case

In this section, we study the continuous-time counterpart of the discrete-time system (23). The first motivation is to check the data transmission strategy for analog channels; the second is that discrete-time systems can be regarded as continuous ones if their base frequencies are much bigger than the network data transmission rate [Walsh *et al.*, 2002]; the third is that a system governed by this transmission strategy possesses very rich dynamics, thus it is also interesting in its own right.

6.1. *System setting*

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= a_1 x_1(t) + b_1 x_2(t), \\ \dot{x}_2(t) &= a_2 x_2(t) + b_2 v(t), \end{aligned} \tag{31}$$

where the matrix $A = \begin{pmatrix} a_1 & b_1 \\ b_2 & a_2 \end{pmatrix}$ is stable, and the switching law is given by

$$v(t) = \begin{cases} x_1(t), & \text{if } |x_1(t) - v(t_-)| > \delta, \\ v(t_-), & \text{otherwise,} \end{cases} \tag{32}$$

in which δ is a positive scalar.

As can be observed, the system governed by Eqs. (31) and (32) is the continuous-time counterpart of system (23). Note that the system consists of two first-order ordinary differential equations. Furthermore, if we fix $v(0_-)$ to be 0, then the above system is autonomous. Moreover, due to the switching nature of the system, the vector field of this system may not be continuous for some set of parameters, not to mention differentiability. This suggests that the well-known Poincaré–Bendixson theorem might not be applicable [Hale & Kocak, 1991], i.e. besides equilibria and periodic orbits, the ω-limit set of this system may contain other attractors. This turns out to be true as shown by the following simulations. However, before doing that, let us first state a general result regarding the system composed of Eqs. (31)–(32).

Theorem 5. *The trajectories of the system given by Eqs. (31) and (32) are bounded. Moreover, they converge to the origin as δ tends to zero.*

Its proof is omitted due to space limitation.

6.2. *Simulations*

Consider the simulink model shown in Fig. 10. The system G and the controller C are modeled by the first and the second equations in Eq. (31), respectively. The function FCN, modeling the switching function (32), is defined by

$$f(u) := u_1 + u_2 - u_2 * (|u_1 - u_2| > \delta) - u_1 * (|u_1 - u_2| \le \delta). \tag{33}$$

Thus, by letting

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

one has

$$v := f(u) = \begin{cases} u_1, & \text{if } |u_1 - u_2| > \delta, \\ u_2, & \text{otherwise.} \end{cases} \tag{34}$$

Is the block FCN, the function f, well-defined? It suffices to verify the case at time 0. Firstly, choose $|x_1(0)| < \delta$. Then $u_1 = x_1(0)$. By simulation, it is found that $v = 0$. Secondly, choose $|x_1(0)| > \delta$. Then $u_1 = x_1(0)$. Simulation shows that $v = x_1(0)$. To simplify the notation, denote u_2 at time 0 by $v(0_-)$. Summarizing the above, one has

$$v(0_-) = 0,$$

$$v(0) = \begin{cases} x_1(0), & \text{if } |x_1(0) - v(0_-)| > \delta, \\ v(0_-), & \text{otherwise.} \end{cases}$$

Similarly,

$$v(t) = \begin{cases} x_1(t), & \text{if } |x_1(0) - v(0_-)| > \delta, \\ v(t_-), & \text{otherwise,} \end{cases} \quad \text{for } t > 0,$$

where $v(t_-)$ is either some previous value of the state x_1, say $x_1(t - t_0)$ for some t_0 satisfying $0 < t_0 \le t$, or $v(0_-) = 0$. Hence, the block FCN is well-defined.

Next, we find the equilibria of the system. As expected, the equilibria of the system constitute a line segment, just as in the discrete-time case. The equilibria are given by

$$\Lambda = \left\{ \left(x_1 = \frac{b_1 b_2}{a_1 a_2} v, x_2 = -\frac{b_2}{a_2} v, v \right) : |v| \le \frac{\delta}{\left| 1 - \frac{b_1 b_2}{a_1 a_2} \right|} \right\}. \tag{35}$$

For the system composed of Eqs. (31) and (32), an interesting question is: Given an initial condition $x(0)$, will $x(t)$ settle to a certain equilibrium or converge to a periodic orbit or have more complex behavior? There are two ways of tracking a trajectory $x(t)$: one is to solve Eqs. (31) and (32) directly, and the other is by means of numerical methods. To get an analytic solution, one has to detect the discontinuous points of the right-hand side of Eq. (31). We first show that the number of the discontinuous points within any given time interval is finite.

Start at some time $t_0 \ge 0$ and assume that $(x_1(t_0), x_2(t_0))$ and $v(t_{0-}) = x_1(t_0)$ are given, without loss of generality. Suppose that the first jump of v is at instant $t_0 + T$. To be specific in the following calculation, let $t_0 = 0$. Then

$$x_1(T) = e^{a_1 T} x_1(0) + \int_0^T e^{a_1(T-\tau)} b_1 x_2(\tau) d\tau,$$

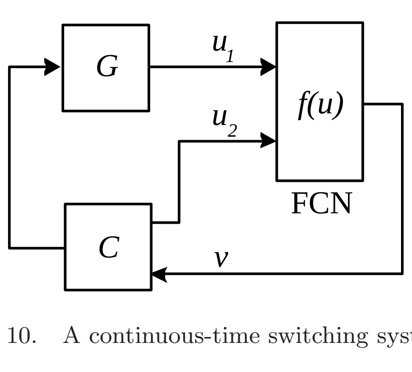

Fig. 10. A continuous-time switching system.

$$x_2(T) = e^{a_2T}x_2(0) + \int_0^T e^{a_2(T-\tau)}b_2x_1(0)d\tau$$
$$= e^{a_2T}x_2(0) + \int_0^T e^{a_2u}dub_2x_1(0).$$

Consequently,

$$x_1(T) - x_1(0)$$
$$= (e^{a_1T} - 1)x_1(0)\left(1 - \frac{b_1b_2}{a_1a_2}\right)$$
$$+ (e^{a_2T} - e^{a_1T})b_1\frac{x_2(0) + \frac{b_2}{a_2}x_1(0)}{a_2 - a_1}. \quad (36)$$

As $T \to 0$, $e^{a_1T} - 1 \to 0$, and $e^{a_2T} - e^{a_1T} \to 0$. Moreover, we have already shown the boundedness of solutions, so there exists a $T^* > 0$ such that

$$|x_1(T) - v(0)| = |x_1(T) - x_1(0)| < \delta \quad (37)$$

for all $T < T^*$. Thus, the finiteness of the number of the discontinuous points within any given time interval is established.

Based on this result, theoretically one can find the analytic solution of the system. However, it is difficult since the condition in Eq. (37) has to be checked all the time to determine the switching time T. Moreover, this process depends on the initial point, $(x_1(t_0), x_2(t_0))$, which is hard due to the impossibility of finding the exact T satisfying $|x_1(T) - v(0)| = \delta$. This problem will be addressed in more details in Sec. 6.7.

Another route to study this type of systems is by means of numerical solutions. In the following, some simulations will be shown to analyze the complexity of the system depicted in Fig. 10. In all the following trajectory figures, the horizontal axes stands for x_1 and the vertical one is x_2.

6.3. *Converging to some fixed point*

Fix those parameters shown in Fig. 10 to be:

$$a_1 = -1, \quad b_1 = 2, \quad a_2 = -2, \quad b_2 = -2,$$

and choose an initial condition $(10, -10)$. Then, we get simulation results shown in Fig. 11. One can see that this trajectory converges to a point specified by Eq. (35), which is close, but not equal, to the origin.

6.4. *Sensitive dependence on initial conditions*

First, fix system parameters as

$$a_1 = 1, \quad b_1 = 2, \quad a_2 = -2, \quad b_2 = -2, \quad \delta = 1, \quad (38)$$

and note that there is an *unstable* pole in the system G. Suppose

$$x_1(0) = 2, \quad x_2(0) = 1,$$

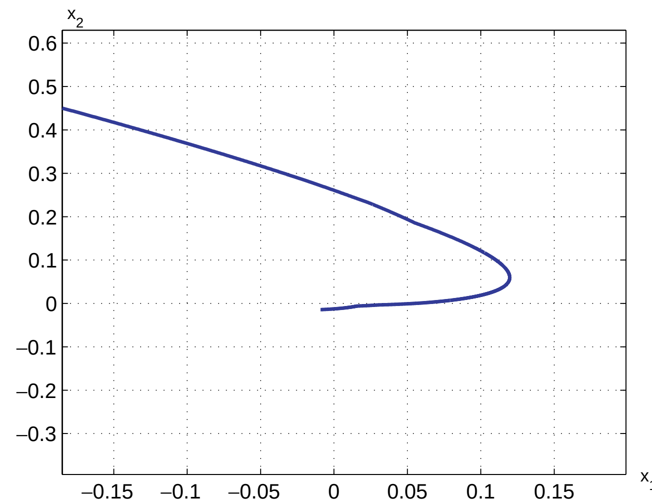

Fig. 11. Converging to an equilibrium other than the origin.

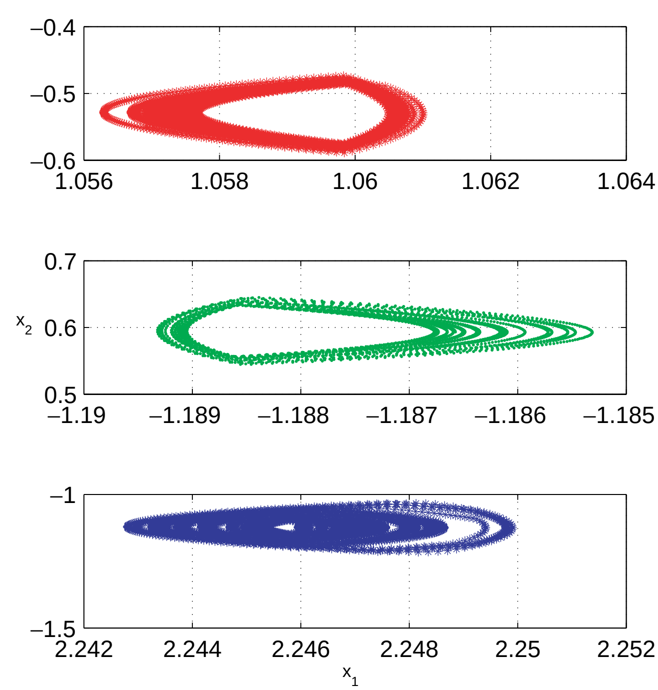

Fig. 12. Sensitive dependence on initial conditions (the horizontal axis is x_1 and the vertical one is for x_2).

and

$$x_1(0) = 2 - 10^{-10}, \quad x_2(0) = 1.$$

Then, we get the simulation result shown in Fig. 12, where the first two are trajectories from those two sets of initial conditions given above and the third one is their difference. Clearly, one can see the sensitive dependence on initial conditions.

6.5. *Coexisting attractors*

Adopt the system parameters as in Eq. (38). Then, we have simulation results shown in Fig. 13, where the largest initial differences of any two such trajectories is 10^{-3}. These attractors are all alike; however, they are located in different positions; that is, they are coexisting attractors.

6.6. *A periodic orbit*

Now choose

$$a_1 = -11, \quad b_1 = \frac{1}{4}, \quad a_2 = 10, \qquad b_2 = \frac{1}{4} - (a_1 - a_2)^2, \quad \delta = 1. \tag{39}$$

Simulations show that most trajectories behave like the one shown in Fig. 14, which is periodic.

Having observed various complex dynamics possessed by the system shown in Fig. 10, one may ask the following question:

> *Is the complexity exhibited by the system due to numerical errors or is the system truly chaotic?*

We received the following warning during our simulations using Simulink: *Block diagram* "*A continuous-time switching system*" *contains 1 algebraic loop*(*s*). This warning is due to the fact that one of the outputs of the function block FCN is its own input. Certainly, this may lead to numerical errors. So, a transport delay is added to rule out this possibility. This consideration leads to the following scheme (Fig. 15):

To correctly implement Eq. (32), the transport delay T must be small enough. Here, it is fixed to be $T = 5 * 10^{-2}$. Suppose that system parameters are given by Eq. (38) and choose two sets of initial conditions, $(2, 1)$ and $(2 - 10^{-6}, 1)$. Then, we get Fig. 16. According to the upper part of this plot, two trajectories almost coincide; however, the lower plot clearly reveals sensitive dependence on initial conditions.

For a sufficiently small transport delay T, many simulations show that the complex attractor is unique, but sensitive dependence on initial conditions still persists. Apparently, this phenomenon needs further investigations.

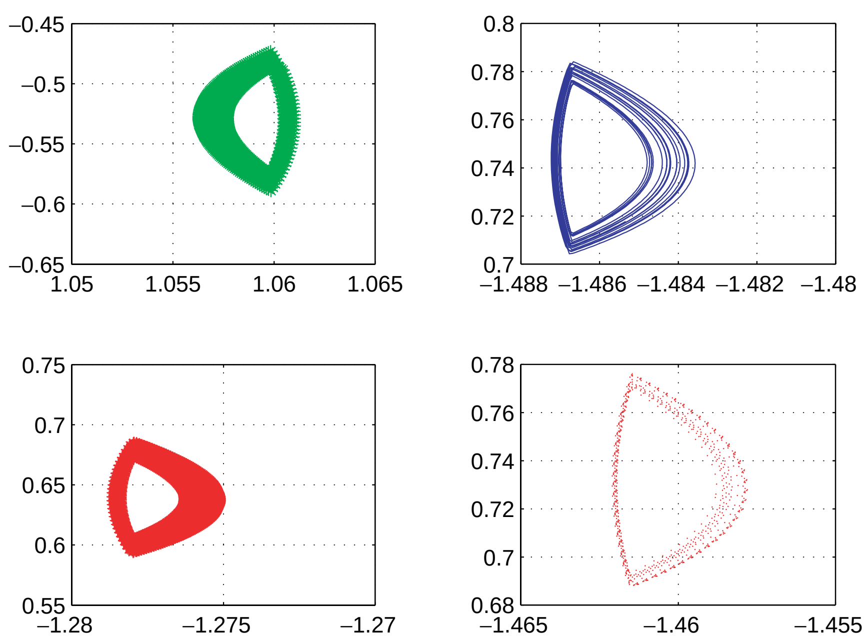

Fig. 13. Several coexisting attractors (the horizontal axis is x_1 and the vertical stands for x_2).

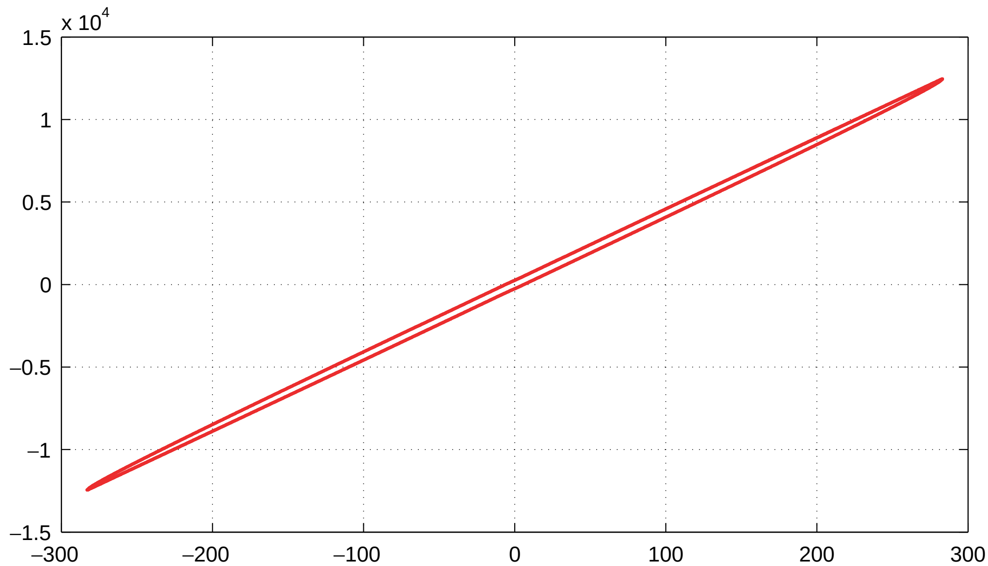

Fig. 14. A periodic orbit (the horizontal axis is x_1 and the vertical one is for x_2).

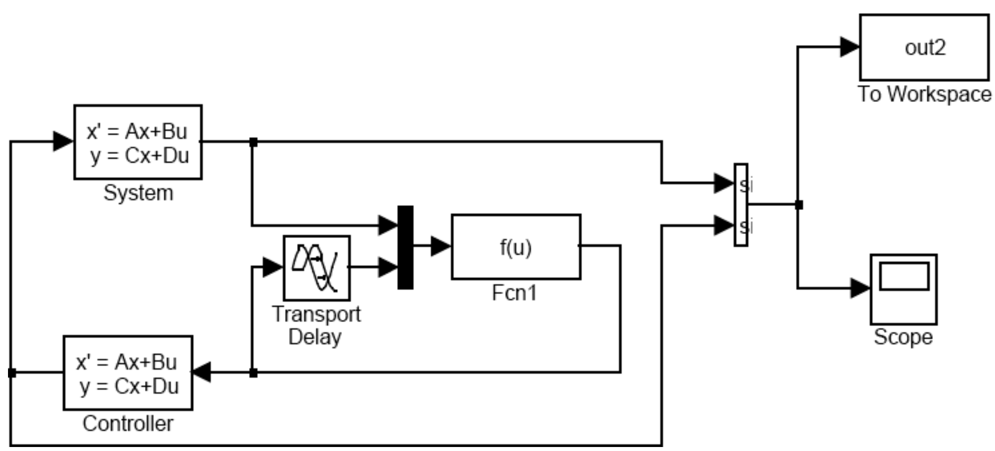

Fig. 15. A modified continuous-time switching system.

6.7. *Computational complexity*

We have visualized some complex behaviors of system (31) and (32), but we have not answered the question posed above. In this section, we study this problem in some detail.

Consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= a_1 x_1(t) + b_1 x_2(t), \\ \dot{x}_2(t) &= a_2 x_2(t) + b_2 p, \end{aligned} \tag{40}$$

where

$$a_1 = 1, \quad b_1 = 2, \quad a_2 = -2, \quad b_2 = -2, \quad \delta = 1,$$

and p is a scalar. At $t = 0$, let

$$x_1(0) = p, \quad x_2(0) = q.$$

Then, a direct calculation gives

$$\begin{aligned} x_1(t) &= 2p - \frac{1}{3}e^t(p - 2q) - \frac{2}{3}e^{-2t}(p + q), \\ x_2(t) &= -p + e^{-2t}(p + q). \end{aligned} \tag{41}$$

Suppose $p \neq 2q$. Then, there exists an instant $t_0 > 0$ such that

$$|p - x_1(t_0)| = 1. \tag{42}$$

Set

$$p = x_1(t_0).$$

And then solve Eqs. (40) starting from $(x_1(t_0), x_2(t_0))$ at time t_0. Repeat this procedure (update p whenever Eq. (42) is satisfied) to get

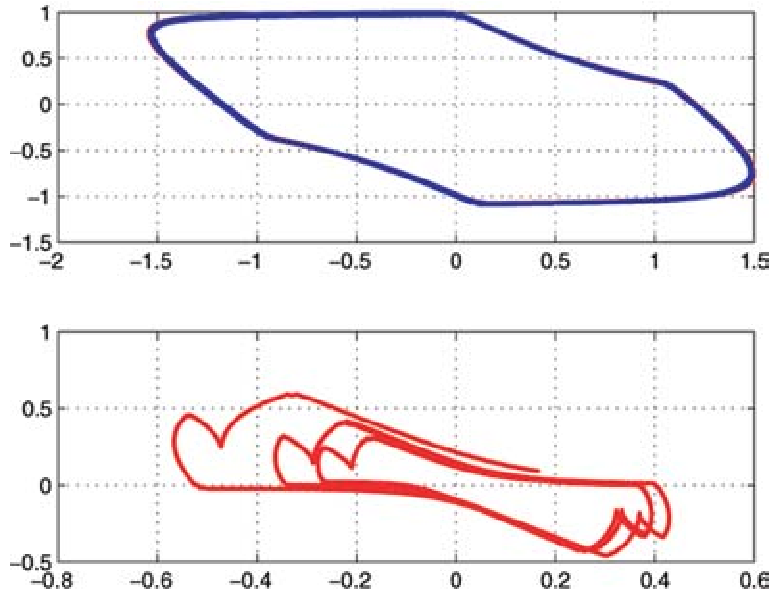

Fig. 16. Sensitive dependence on initial conditions (the horizontal axis is x_1 and the vertical one is for x_2).

an analytic solution of the system starting from $(x_1(0), x_2(0)) = (p, q)$.

Now, it is easy to realize that the complexity may probably be due to the following reasons:

- There is an unstable mode in $x_1(t)$ in Eq. (41).
- It is hard to find the exact switching time, e.g. t_0 in Eq. (42), even numerically. Because of this, numerical errors will accumulate and be exaggerated from time to time by the unstable mode. Further research is required to study the effect of the accumulated errors on the dynamics of the system.
- Sec. 6.6 tells us the first item alone cannot guarantee complex behavior.

7. Control Based on the Network Protocol

Because this research originates from network-based control, in this section we discuss some control problems under this transmission strategy.

Zhang and Chen [2005] concentrating on a scalar case, investiaged chaotic control. Here, we consider a tracking problem: suppose the controller C has been designed for the system G as shown in Fig. 2, so that the output y tracks the reference signal r. How does the nonlinear constraints H_1 and H_2 affect this tracking problem? We begin with a simple example.

Example 1. Consider the following discrete-time system G:

$$\frac{0.005z^{-1} + 0.005z^{-2}}{1 - 2z^{-1} + z^{-2}}.$$

Note that this system is *unstable*. Suppose we have already designed a controller K of the form

$$\frac{37.33 - 33.78z^{-1}}{1 - 0.1111z^{-1}},$$

which achieves step tracking. First, we check the data transmission strategy by simulating the system shown in Fig. 3, with $r \equiv 1$ but without H_2 involved therein. Choose $\delta_1 = 1$. Then, the tracking error is plotted (the $*$ line in Fig. 17). After that, we modify the control law as follows:

$$v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k-1)| > 1, \\ \varepsilon_1 v(k-1) + \varepsilon_2 x(k), & \text{otherwise,} \end{cases}$$

where $\varepsilon_1 \in \mathbb{R}$, $\varepsilon_2 \in \mathbb{R}^{1\times 2}$ are to be determined. Here, we assume that the state x of the system G is available. When there is no transmission from C to G, instead of simply using the previously stored control value $v(k-1)$, $\varepsilon_1 v(k-1) + \varepsilon_2 x(k)$ is used. The reason is that by adjusting ε_1 and ε_2 sensibly we may achieve better control performance. By selecting

$$\varepsilon_1 = 0.86, \quad \varepsilon_2 = [-0.21 \quad 0.21],$$

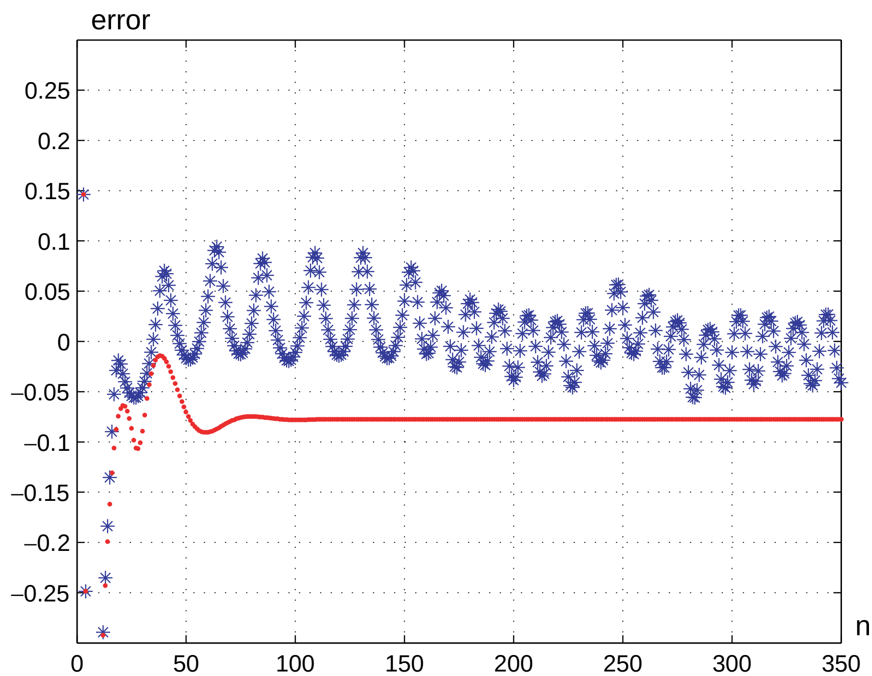

Fig. 17. Tracking error with $\delta_1 = 1$.

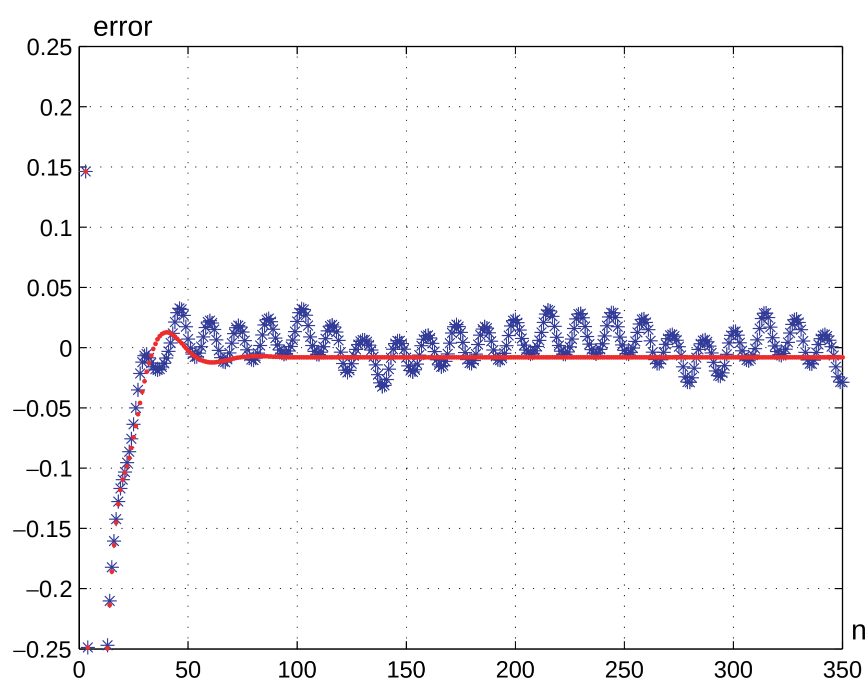

Fig. 18. Tracking error with $\delta_1 = 1/2$.

the tracking error is plotted (the dotted line in Fig. 17). In this simulation, iteration time is 350, so the dropping rate can also be obtained: the former is 85.14% and the latter is 96.57%. Hence, the modified control law is more effective in reducing data traffic. The steady-state error shown in Fig. 17 under the modified control law is around 0.0774. Next, we choose $\delta_1 = 1/2$, and get the result shown in Fig. 18 following the same procedure. The dropping rates are 83.14% for the original and 96.29% for the modified. In this case the steady-state tracking error for the modified system is around 0.0079. So, by modifying the control law, we increased the dropping rate therefore reduced the data traffic, and at the same time improved the performance of the control system.

In the above example, it is shown that for simple control systems it is possible to improve the performance of both the network and the control system by modifying the underlying control law. Clearly, it is more mathematically involved when one confronts a more complex system. In the following, we transform this problem into an optimization problem.

Define $\varsigma = \eta - \xi$, and $e_r := e_c - e$. By subtracting the system in (3) from that in (6), we get

$$\begin{aligned}\varsigma(k+1) &= \breve{A}\varsigma(k) + \begin{bmatrix} B & BD_d \\ 0 & B_d \end{bmatrix} \\ &\quad \times \left(\begin{bmatrix} H_1(u_c(k), v(k-1)) \\ H_2(y_c(k), z(k-1)) \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} \right), \\ e_r(k) &= \begin{bmatrix} -C & 0 \end{bmatrix} \varsigma(k) + z(k) - y_c(k). \end{aligned} \tag{43}$$

Then, the tracking error e_c of system (6) can be obtained via tracking error e of system (3), which is a standard feedback system.

To study the tracking error e_r, we employ the l^∞-norm of the system signals [Bamieh, 2003]. According to (43), we have

$$\begin{aligned}\|\varsigma\|_\infty \le & \left\| \left(z^{-1}I - \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \right)^{-1} \right. \\ & \left. \times \begin{bmatrix} B & BD_d \\ 0 & B_d \end{bmatrix} \right\|_1 \bar{\delta}, \end{aligned} \tag{44}$$

and

$$\begin{aligned}&\|e_r\|_\infty \\ &\le \left\| \begin{bmatrix} -C & 0 \end{bmatrix} \left(z^{-1}I - \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \right)^{-1} \right. \\ &\quad \left. \times \begin{bmatrix} B & BD_d \\ 0 & B_d \end{bmatrix} \right\|_1 \times \left\| \begin{bmatrix} H_1(u_c(k), v(k-1)) \\ H_2(y_c(k), z(k-1)) \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} \right\|_\infty + \delta_2 \\ &\le \bar{\delta} \left\| \begin{bmatrix} -C & 0 \end{bmatrix} \left(z^{-1}I - \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \right)^{-1} \right. \\ &\quad \left. \times \begin{bmatrix} B & BD_d \\ 0 & B_d \end{bmatrix} \right\|_1 + \delta_2. \end{aligned} \tag{45}$$

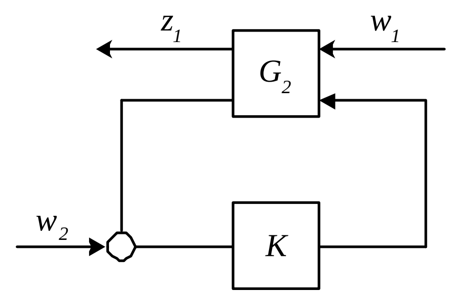

Fig. 19. An ℓ_1 minimization problem.

Remark 9. Equation (45) gives an upper bound of the difference of the tracking error for the systems shown in Figs. 2 and 3. In light of (44) and (45), with δ_1 and δ_2 fixed, minimizing the size of an attractor and the tracking error can be converted to the problem of designing a controller K that achieves step tracking in Fig. 2 and minimizes $\|z_1\|_\infty$ in Fig. 19 simultaneously, with

$$G_2 = \left[\begin{array}{c|cc} A & B & B \\ \hline -C & 0 & 0 \\ -C & 0 & 0 \end{array}\right], \quad K = \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & D_d \end{array}\right],$$
$$\|w_1\|_\infty \le 1, \quad \|w_2\|_\infty \le 1.$$

For this multiple-objective control problem, LMI techniques can be applied. More specifically, by parameterizing all stabilizing controllers, the step tracking problem has an equality constraint for the l^1control problem shown in Fig. 19, which can be modified as an LMI minimization problem (see [Chen & Francis, 1995] and [Chen & Wen, 1995] for more details).

8. Conclusions

In this paper, we have generalized the results of Zhang and Chen [2005] in the following ways:

(i) We have constructed first-return maps of the nonlinear systems in [Zhang & Chen, 2005] and derived existence conditions for periodic orbits and studied their properties.
(ii) We have formulated the involving systems as hybrid systems, and proved that this type of hybrid systems is not structurally stable.
(iii) We have examined higher-dimensional models with detailed studies of the existence of periodic orbits.
(iv) We have investigated a class of continuous-time hybrid systems as the counterparts of the discrete-time systems.

(v) We have proposed new controller design methods based on this network transmission strategy for improving control performance of individual systems as well as the whole network.

Interestingly, one application of this network data transmission strategy is the so-called limited communication control in control and coordination of multiple subsystems. One example is: A single decision maker controls many subsystems over a communication channel of a finite capacity, where the decision maker can control only one subsystem at a time. Let us consider the following situation: Suppose there are several systems sharing a common communication channel, where at each transmission time only one system can send a signal. Is it possible that each subsystem adopts the transmission strategy proposed here so that the whole system can achieve some desired system performance? Note that under the proposed transmission strategy, each system just sends "necessary" signals, leaving communication resources to the others to use. So, if we design the *transmission sequence* carefully, the whole system might perform well. A similar but essentially different problem was discussed in [Hristu & Morgansen, 1999], which is an extension of the work of Brockett [1995]. The problem studied therein is: Given a set of control systems controlled by a single decision maker, which can communicate with only one system at a time, design a communication sequence so that the whole network is asymptotically stable. Using augmentation, this problem can be converted to a mathematical programming problem for which some algorithms are currently available. Here, under the proposed transmission strategy, the communication strategy depends severely on the control systems. Hence the communication sequence depends explicitly on all subsystems, adding more constraints to the design of the communication sequence. This important yet challenging problem will be our future research topic.

Acknowledgments

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