Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 21, NUMBER 4 (2018)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

AN INTEGRAL RELATIONSHIP FOR A FRACTIONAL ONE-PHASE STEFAN PROBLEM

Sabrina Roscani [1], **Domingo Tarzia** [2]

Abstract

A one-dimensional fractional one-phase Stefan problem with a temperature boundary condition at the fixed face is considered by using the Riemann–Liouville derivative. This formulation is more convenient than the one given in Roscani and Santillan (*Fract. Calc. Appl. Anal.*, **16**, No 4 (2013), 802–815) and Tarzia and Ceretani (*Fract. Calc. Appl. Anal.*, **20**, No 2 (2017), 399–421), because it allows us to work with Green's identities (which does not apply when Caputo derivatives are considered). As a main result, an integral relationship between the temperature and the free boundary is obtained which is equivalent to the fractional Stefan condition. Moreover, an exact solution of similarity type expressed in terms of Wright functions is also given.

MSC 2010: Primary 35R35, 26A33, 35C05; Secondary 33E20, 80A22

Key Words and Phrases: Stefan Problem, fractional diffusion equation, Riemann–Liouville derivative, equivalent integral relationship

1. Introduction

The free boundary problems for the one-dimensional diffusion equation are problems linked to the processes of melting and freezing. In these problems the diffusion, considered as a heat flow, is expressed in terms of instantaneous local flow of temperature, and a latent heat-type condition at the interface connecting the velocity of the free boundary and the heat

pp. 901–918 , DOI: 10.1515/fca-2018-0049
DE GRUYTER

flux of the temperatures in both phases. This kind of problems have been widely studied in the last 50 years (see [1, 5, 8, 10, 13, 20, 29, 31, 32]).

In this paper a fractional Stefan problem is considered. That is, a problem governed by a fractional diffusion equation (FDE) with two unknown functions: a two variables function $u = u(x,t)$ and a free boundary $s = s(t)$.

When the order of the FDE is less than 1, the process is called anomalous diffusion and it has been studied by numerous authors [12, 14, 16, 18, 19, 22, 24, 39]. For example it has been observed that the behaviour of proteins is subdiffusive due to molecular crowding [3]. Also, Mainardi studied in [21] the application to the theory of linear viscoelasticity.

Fractional free boundary problems where studied in [2, 4, 15, 17, 28, 35] and it has been observed that growth of frost on a cooled plate can be superdiffusive [30]. Voller et al. [36] state that if we consider an ideal non local flow and we replace it in a heat balance equation, we derive in subdiffusion, modeled by a fractional diffusion equation involving the fractional Caputo derivative in time of order $\alpha \in (0,1)$. Following this idea, we consider the non local flow given by

$$q = -k\, {}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t), \tag{1.1}$$

in which the fractional derivative is the Riemann–Liouville derivative respect on time of order $1-\alpha$ ($\alpha \in (0,1)$) defined by

$${}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{\frac{\partial}{\partial x} u(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau, \qquad \alpha \in (0,1),$$

where Γ is the Gamma function defined by $\Gamma(x) = \int_0^\infty w^{x-1} e^{-w} dw$. Note that the Fourier law

$$q^l(x,t) = -k \frac{\partial}{\partial x} u(x,t) \tag{1.2}$$

that says that the heat flux is proportional to the temperature gradient is recovered when the value $\alpha = 1$ is considered in (1.1).

Now, if we replace equation (1.1) in the heat balance equation, the following fractional diffusion equation is obtained:

$$\frac{\partial}{\partial t} u(x,t) = \lambda^2 \frac{\partial}{\partial x} \left({}_0^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t) \right). \tag{1.3}$$

The classical Stefan condition derived from the Rankine-Hugoniot conditions in a one-phase Stefan problem is given by

$$\frac{d}{dt} s(t) = q^l(x,t)\Big|_{(s(t)^-,t)}, \quad 0 < t \leq T. \tag{1.4}$$

So, replacing the non local flux (1.1) in (1.4) we obtain the following "fractional Stefan condition":

$$\frac{d}{dt}s(t) = -k \; {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\bigg|_{(s(t)^-,t)}, \quad 0 < t \le T. \tag{1.5}$$

Therefore, our problem is given by:

$$\begin{array}{lll}
(i) & \frac{\partial}{\partial t}u(x,t) = \lambda^2 \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\right), & 0 < x < s(t),\, 0 < t < T, \\
(ii) & u(x,0) = f(x), & 0 \le x \le b = s(0), \\
(iii) & u(0,t) = g(t), & 0 < t \le T, \\
(iv) & u(s(t),t) = 0, & 0 < t \le T,\, C \ge 0, \\
(v) & \frac{d}{dt}s(t) = -k \; {}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\big|_{(s(t)^-,t)}, & 0 < t \le T,
\end{array} \tag{1.6}$$

where $0 < \alpha < 1$, f and g are non-negative continuous functions defined in $(0,T]$.

It is interesting to remark that the classical subdiffusion equation is given in terms of the Caputo derivative, that is

$${}_0^{C}D_t^{\alpha}u(x,t) = \lambda^2 \frac{\partial^2}{\partial x^2}u(x,t). \tag{1.7}$$

Equation (1.7) and (1.3) are closed linked (as we will see later). Although equation (1.7) is the preferred choice due to the "blessings" of the Caputo derivative, in this opportunity we have preferred to use equation (1.3) because it is allows us to work with classical Green identities. Recall that the formulations given at the moment for fractional Green identities (see for example [23]) are more complicated than the classical.

In the study of classical Stefan problems, the validity of a so called integral Stefan relationship (which is equivalent to the Stefan condition (1.4)), was demonstrated for temperature, heat flow, and convective condition (a Robin type condition) at the fixed face $x = 0$ in [6], [7] and [33], respectively. This integral relationship is fundamental to obtain results of existence and uniqueness, continuous and monotonous dependence on data and asymptotic behavior of the solution.

The purpose of this work is to obtain this integral condition for the fractional Stefan problem (1.6), predicting future results of uniqueness or asymptotic behavior of the free boundary. This result states that if the pair $\{u,s\}$ is a solution of problem (1.6), then (under certain conditions) the following integral condition for the free boundary $s(t)$ and the function $u(x,t)$ is verified:

$$\begin{aligned}
s^2(t) = b^2 + 2\int_0^t {}^{RL}D_t^{1-\alpha}g(\tau)d\tau + 2\int_0^b zf(z)dz - 2\int_0^{s(t)} zu(z,t)dz \\
-2\int_0^t {}^{RL}D_t^{1-\alpha}u(x,t)\big|_{(s(\tau),\tau)}\,d\tau.
\end{aligned} \tag{1.8}$$

The paper is present as follows. In Section 2 some basic concepts of fractional calculus that will be used later are given. In Section 3 the integral relationship (3.18) is proved by using the Green's Theorem and it is also proved that, under certain hypothesis, this integral relationship is equivalent to the fractional Stefan condition (1.5). Finally we present an exact solution to a particular case of problem (1.6) in terms of the Wright functions and the relationship already obtained is checked.

2. Preliminaries

2.1. Basics of Fractional Calculus.

Definition 2.1. Let $[a,b]\subset\mathbb{R}$ and $\alpha\in\mathbb{R}^+$ be such that $n-1<\alpha\le n$.

(1) If $f\in L^1[a,b]$ we define the *fractional Riemann–Liouville integral of order* α as

$$ {}_aI^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau. $$

(2) If $f\in AC^n[a,b]=\left\{f\in\mathcal{C}^{(n-1)}\,|\,f^{(n-1)}\text{ is absolutely continuous}\right\}$, we define the *fractional Riemann–Liouville derivative of order* α as

$$ {}^{RL}_{\ a}D^\alpha f(t)=\left[D^n\,{}_aI^{n-\alpha}f\right](t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau. $$

(3) If $f\in W^n(a,b)=\left\{f\in\mathcal{C}^n(a,b]\,|\,f^{(n)}\in L^1[a,b]\right\}$ we define the *fractional Caputo derivative of order* α as

$$ {}^{C}_{a}D^\alpha f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau & n-1<\alpha<n,\\ f^{(n)}(t) & \alpha=n.\end{cases} $$

Lemma 2.1. *The following properties involving the fractional integrals and derivatives hold:*

(1) *The fractional Riemann–Liouville derivative operator is a left inverse of the fractional Riemann–Liouville integral of the same order $\alpha\in\mathbb{R}^+$. If $f\in L^1[a,b]$, then*

$$ {}^{RL}_{\ a}D^\alpha\,{}_aI^\alpha f(t)=f(t),\quad a.e. $$

(2) *The fractional Riemann–Liouville integral, in general, is not a left inverse operator of the fractional derivative of Riemann–Liouville.*

In particular, if $0<\alpha<1$, then

$$ {}_aI^\alpha({}^{RL}_{\ a}D^\alpha f)(t)=f(t)-\frac{{}_aI^{1-\alpha}f(a^+)}{\Gamma(\alpha)(t-a)^{1-\alpha}}. $$

(3) *If* $f \in AC^n[a,b]$, *then*

$$ {}_a^{RL}D^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + {}_a^{C}D^{\alpha} f(t). $$

Remark 2.1. It is known that if f and g are functions supported in $[0,\infty)$, then the convolution of f and g is defined by

$$ (f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau. $$

Then if we consider a function f supported in $[0,\infty)$ and χ_α is the locally integrable function defined by

$$ \chi_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{2.1} $$

then we have the following properties:

$$ {}_0I^{\alpha} f(t) = (\chi_\alpha * f)(t), \tag{2.2} $$

$$ {}_0^{RL}D^{\alpha} f(t) = \frac{d^n}{dt^n}(\chi_{n-\alpha} * f)(t), \tag{2.3} $$

$$ {}_0^{C}D^{\alpha} f(t) = (\chi_{n-\alpha}) * \frac{d^n}{dt^n} f(t). \tag{2.4} $$

2.2. **The special functions involved.** In this subsection we give the definitions and properties of the special functions appearing in the explicit solution that will be presented in the next section.

Definition 2.2. For every $z \in \mathbb{C}$, $\rho > -1$ and $\beta \in \mathbb{R}$ the *Wright* function is defined by

$$ W(z;\rho;\beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k+\beta)}. \tag{2.5} $$

The *Mainardi function* [12] is a special case of the Wright function defined by

$$ M_\rho(z) = W(-z,-\rho,1-\rho) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\rho n+1-\rho)}, \quad z \in \mathbb{C},\ \rho < 1. \tag{2.6} $$

Remark 2.2. The function (2.5) was introduced by E.M. Wright at the beginning of the XX Century, and he studied its asymptotic behaviour in [37] and [38]. It is known that:

(1) The Wright function is an entire function if $\rho > -1$.

(2) The derivative of the Wright function can be computed as

$$\frac{\partial}{\partial z}W(z,\rho,\beta)=W(z,\rho,\rho+\beta). \tag{2.7}$$

(3) Some particular cases are:
Gaussian function: $\frac{1}{\sqrt{\pi}}e^{-x^2}=W\left(-2x,-\frac{1}{2},\frac{1}{2}\right)=M_{1/2}(2x)$;
Error function: erf $(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi=1-W\left(-2x,-\frac{1}{2},1\right)$;
and the complementary erf function:

$$\text{erfc }(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}d\xi=W\left(-2x,-\frac{1}{2},1\right).$$

(4)

$$\lim_{x\to\infty}W\left(-x,-\frac{\alpha}{2},\beta\right)=0 \qquad \text{for all } \alpha\in(0,1), \quad \beta>0. \tag{2.8}$$

The next two lemmas were proved in [26].

Lemma 2.2. *If $0<\alpha<1$, then:*

(1) *$M_{\alpha/2}(x)$ is a positive and strictly decreasing positive function in $\mathbb{R}^+$ such that $M_{\alpha/2}(x)<\frac{1}{\Gamma\left(1-\frac{\alpha}{2}\right)}$ for all $x>0$.*
(2) *$W\left(-x,-\frac{\alpha}{2},1\right)$ is a positive and strictly decreasing function in $\mathbb{R}^+$ such that $0<W\left(-x,-\frac{\alpha}{2},1\right)<1$, for all $x>0$.*
(3) *$1-W\left(-x,-\frac{\alpha}{2},1\right)$ is a positive and strictly increasing function in $\mathbb{R}^+$ such that $0<1-W\left(-x,-\frac{\alpha}{2},1\right)<1$, for all $x>0$.*

Lemma 2.3. *If $x\in\mathbb{R}_0^+$ and $\alpha\in(0,1)$ then:*

(1) $\lim\limits_{\alpha\nearrow 1}M_{\alpha/2}(2x)=M_{1/2}(2x)=\frac{e^{-x^2}}{\sqrt{\pi}}$;
(2) $\lim\limits_{\alpha\nearrow 1}\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right]=\text{erf}\,(x)$.

3. The fractional Stefan Problem

Let us now study problem (1.6), which is the main goal of this paper. The following two regions will be considered: $\Omega_T=\{(x,t)/0<x<s(t),\, 0<t\leq T\}$ and $\partial_p\Omega_T=\{(0,t),0<t\leq T\}\cup\{(s(t),t),0<t\leq T\}\cup\{(x,0),0\leq x\leq b\}$, where the latter is called parabolic boundary.

Definition 3.1. A pair $\{u,s\}$ is a solution of problem (1.6) if

(1) u is defined in $[0,b_0]\times[0,T]$ where $b_0:=\max\{s(t),0\leq t\leq T\}$.

(2) $u \in C(D_T) \cap C_x^2(D_T)$, such that $u_x \in AC_t^1((0,T))$ where $AC_t^1((0,T)) := \{f(x,\cdot)\colon f \in AC^1(0,T)$ for every fixed $x \in [0,b_0]\}$.

(3) u is continuous in $D_T \cup \partial_p D_T$ except perhaps at $(0,0)$ and $(b,0)$ where

$$0 \le \liminf_{(x,t)\to(0,0)} u(x,t) \le \limsup_{(x,t)\to(0,0)} u(x,t) < +\infty$$

and

$$0 \le \liminf_{(x,t)\to(b,0)} u(x,t) \le \limsup_{(x,t)\to(b,0)} u(x,t) < +\infty.$$

(4) $s \in C^1(0,T)$.

(5) There exists ${}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\big|_{(s(t),t)}$ for all $t \in (0,T]$.

(6) u and s satisfy (1.6).

Remark 3.1. We request u to be defined in $[0,b_0] \times [0,T]$ since the fractional derivative ${}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)$ involves values $\frac{\partial}{\partial x}u(x,\tau)$ for all τ in $[0,t]$, but we ask u to verify the FDE in the region Ω_T.

Remark 3.2. Similar problems for the fractional derivative in the Caputo sense were studied in [2, 25, 26, 27, 28, 34, 35, 36]. For example, the formulation given in [26] for $\alpha \in (0,1)$ is:

$$\begin{array}{lll}
(i) & {}^C D_t^\alpha u(x,t) = \lambda^2 \frac{\partial}{\partial x^2} u(x,t), & 0 < x < s(t),\ 0 < t < T, \lambda > 0, \\
(ii) & u(x,0) = f(x), & 0 \le x \le b = s(0), \\
(iii) & u(0,t) = g(t), & 0 < t \le T, \\
(iv) & u(s(t),t) = 0, & 0 < t \le T, \\
(v) & {}^C D^\alpha s(t) = -k u_x(s(t),t), & 0 < t \le T.
\end{array} \tag{3.1}$$

We assert that problem (3.1) is similar to problem (1.6) because from Lemma 1 it results that if u is a solution of the fractional diffusion equation for the Caputo derivative $(3.1-i)$, then u verifies the FDE $(1.6-i)$.

However, in general, the converse of the previous statement is not true because the fractional integral of Riemann-Liouville is not the inverse operator of the Riemann–Liouville derivative of equal order.

Also it is worth noting that if we apply the integral operator ${}_0^{RL}D_t^{1-\alpha}$ to both members of equation $(3.1-v)$, we get equation

$$\frac{d}{dt}s(t) = -k\, {}_0^{RL}D_t^{1-\alpha}\left[\frac{\partial}{\partial x}u(s(t),t)\right], \quad 0 < t \le T,$$

which is different to equation (1.5) unless $\alpha = 1$.

3.1. **An exact solution.** Without loss of generality, hereinafter the values $\lambda = 1$ and $k = 1$ are taken.

Consider $b = 0$ and a constant boundary condition in (1.6$-iii$). Namely, let the following fractional Stefan problem be:

$$\begin{array}{lll} (i) & \frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\right), & 0<x<s(t),\ 0<t<T, \\ (ii) & u(0,t)=1, & 0<t\le T, \\ (iii) & u(s(t),t)=0, & 0<t\le T,\ s(0)=0 \\ (iv) & \frac{d}{dt}s(t) = -\,{}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)\Big|_{(s(t),t)}, & 0<t\le T, 0<\alpha<1. \end{array} \tag{3.2}$$

In order to find an exact solution to problem (3.2), the next lemma proved in [11] will be used:

Lemma 3.1. *Let $c(x,t)$ be a solution of the time–fractional diffusion equation for the Caputo derivative $(3.1-i)$ such that:*

$$\text{For every } (x,t), \text{ the function } F(x,t) = \int_x^\infty c(\xi,t)d\xi \text{ is well defined,} \tag{3.3}$$

$$\lim_{x\to\infty}\frac{\partial c}{\partial x}(x,t) = 0, \tag{3.4}$$

$$\left|\frac{\partial}{\partial\tau}c(\xi,\tau)\right| \le g(\xi)\in L^1(x,\infty), \tag{3.5}$$

$$\frac{\frac{\partial}{\partial\tau}c(\xi,\tau)}{(t-\tau)^\alpha} \in L^1((x,\infty)\times(0,t)). \tag{3.6}$$

Then $\int_x^\infty c(\xi,t)d\xi$ is a solution to the time fractional diffusion equation for the Caputo derivative $(3.1-i)$.

Remark 3.3. The factor 2 appearing in the next functions (3.7) and (3.8) was considered with the aim to recover the Gaussian and erf functions when we make $\alpha \nearrow 1$ (according to Lemma 2.3).

Theorem 3.1. *The pair given by*

$$u(x,t) = 1 - \frac{1}{1-W\left(-2\xi,-\frac{\alpha}{2},1\right)}[1-W\left(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1\right)], \tag{3.7}$$

$$s(t) = 2\xi t^{\alpha/2}, \tag{3.8}$$

where ξ is the unique positive solution to the equation

$$2x\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right] = 2xW\left(-2x,-\frac{\alpha}{2},1\right) + W\left(-2x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right), \tag{3.9}$$

is a solution to problem (3.2).

P r o o f. We know that

$$u(x,t) = a + b\left[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] \tag{3.10}$$

is a solution of fractional diffusion equation for the Caputo derivative $(3.1-i)$ for all $a \in \mathbb{R}$, $b \in \mathbb{R}$ (see [19] or [28]). Then, applying Remark 3.2, it results that u is a solution to equation $(3.2-i)$.

From $(3.2-ii)$ we obtain

$$1 = u(0,t) = a + b\left[1 - W\left(0, -\frac{\alpha}{2}, 1\right)\right] = a, \tag{3.11}$$

and from $(3.2-iii)$ we get

$$u(s(t),t) = 1 + b\left[1 - W\left(-\frac{s(t)}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)\right] = 0. \tag{3.12}$$

Note that (3.12) must be verified for all $t > 0$, so we will ask for $s(t)$ to be proportional to $t^{\alpha/2}$, that is to say

$$s(t) = 2\xi t^{\alpha/2} \qquad \text{for some} \quad \xi > 0. \tag{3.13}$$

Replacing (3.13) in (3.12) and taking into account Lemma 2.2 it follows that $b = -k\frac{1}{1-W\left(-2\xi,-\frac{\alpha}{2},1\right)}$ and then (3.7) holds.

With the aim of use the fractional Stefan condition $(3.2-iv)$, the fractional derivative ${}_0^{RL}D_t^{1-\alpha}\left(\frac{\partial}{\partial x}u(x,t)\right)$ must be computed.

From (2.8) and using estimates made in [11], it yields that, for every $x > 0$, $w_1(x,t) = W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)$ is under the assumptions of Lemma 3.1. Clearly, $w_2(x,t) = x$ is a solution to the FDE $(3.2-i)$.

Then, using the linearity of the Caputo derivative [16] and the principle of superposition we can state that the function defined by

$$\begin{aligned} v(x,t) = &-\left[1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right] x \\ &+ \frac{t^{\alpha/2}}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right) \end{aligned} \tag{3.14}$$

is a solution of the FDE such that $\frac{\partial v}{\partial x}(x,t) = -u(x,t)$ for all $x > 0, t > 0$. Hence

$$\begin{aligned} \frac{\partial}{\partial t}v(x,t) = &\ \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}v(x,t)\right) = \frac{\partial}{\partial x}\left({}_0^{RL}D_t^{1-\alpha}(-u(x,t))\right) \\ = &-{}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t), \qquad x > 0,\, t > 0. \end{aligned} \tag{3.15}$$

Differentiating v with respect to the t variable and using (3.15), it yields

$$-{}_0^{RL}D_t^{1-\alpha}\frac{\partial}{\partial x}u(x,t)=\frac{1}{1-W\left(-2\xi,-\frac{\alpha}{2},1\right)}\frac{\alpha}{2}\left[\frac{x}{t}W\left(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1\right)\right.$$
$$\left.+t^{\alpha/2-1}W\left(-\frac{x}{t^{\alpha/2}},-\frac{\alpha}{2},1+\frac{\alpha}{2}\right)\right]. \tag{3.16}$$

Replacing (3.13) and (3.16) into the fractional Stefan condition ($3.2-iv$) it results that ξ must verify the following equation:

$$2x\left[1-W\left(-2x,-\frac{\alpha}{2},1\right)\right]=2xW\left(-2x,-\frac{\alpha}{2},1\right)+W\left(-2x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right). \tag{3.17}$$

Define in $\mathbb{R}_0$ the functions

$$H(x)=x\left[1-W\left(-x,-\frac{\alpha}{2},1\right)\right]$$

and

$$G(x)=xW\left(-x,-\frac{\alpha}{2},1\right)+W\left(-x,-\frac{\alpha}{2},1+\frac{\alpha}{2}\right).$$

From Lemma 2.2, H is an increasing function such that $H(0)=0$. On the other hand, G is a decreasing function in $\mathbb{R}^+$ such that $G(0)=\frac{1}{\Gamma\left(1+\frac{\alpha}{2}\right)}>0$ due to $G'(x)=-xM_{\alpha/2}(x)<0$ for all $x>0$ and $\alpha\in(0,1)$. Then, we can assert that there exists a unique positive solution ξ such that $H(2\xi)=G(2\xi)$. □

3.2. **An integral relationship between u and s.** The next theorem provides an integral relationship between the free boundary s and function u, obtained from the fractional Stefan condition (1.5).

THEOREM 3.2. *Let $\{u,s\}$ be a solution of problem* (1.6) *such that* $\frac{\partial^2}{\partial t\partial x}u(x,t)\in\mathcal{C}^1(\Omega_T)$, $g\in AC^1(0,T)$ *and,* ${}^{RL}D_t^{1-\alpha}g$ *and* ${}^{RL}D_t^{1-\alpha}u(x,t)|_{(s(t),t)}\in L^1(a,b)$. *Then the following integral relationship for the free boundary $s(t)$ and the function $u(x,t)$*

$$s^2(t)=b^2+2\int_0^t{}^{RL}D_t^{1-\alpha}g(\tau)d\tau+2\int_0^b zf(z)dz-2\int_0^{s(t)}zu(z,t)dz$$
$$-2\int_0^t{}^{RL}D_t^{1-\alpha}u(x,t)\big|_{(s(\tau),\tau)}d\tau \tag{3.18}$$

is verified.

P r o o f. Recall the Green identity:

$$\int_{\partial\Omega}Pdt+Qdx=\iint_\Omega\left(Q_t-P_x\right)dA,$$

where Ω is an open simply connected region, $\partial\Omega$ is a positively oriented, piecewise smooth, simple closed curve, and the field $F = (P, Q)$ is $\mathcal{C}^1$ in an open set containing Ω.

Consider the functions P and Q defined by

$$P(x,t) = -x\, {}^{RL}D_t^{1-\alpha} u_x(x,t) + {}^{RL}D_t^{1-\alpha} u(x,t) \tag{3.19}$$

$$Q(x,t) = -x\, u(x,t), \tag{3.20}$$

and the region $\Omega_\epsilon = \left\{(x,\tau) \in \mathbb{R}^2 \,/\, \epsilon < \tau < t, 0 < x < s(\tau)\right\}$, $\epsilon > 0$.

Note that

$${}^{RL}D_t^{1-\alpha} u_x(x,t) = \frac{\partial}{\partial t} I_t^{1-\alpha}(u_x(x,t)) = \frac{\partial}{\partial t}\left(\chi_{1-\alpha}(t) * u_x(x,t)\right),$$

where χ_α (defined in eq. (2.1)) is an L^1_{loc} function. Then the convolution inherits all the properties of u_x and, taking into account that u is under the assumptions of Definition 3.1 and that $\frac{\partial^2}{\partial t \partial x} u(x,t) \in \mathcal{C}^1(\Omega_T)$, the field F has all the regularity required in Ω_ϵ.

Also, due to the regularity of the field F, the derivatives $\partial/\partial x$ and ${}^{RL}D_t^{1-\alpha}$ commutes. Then, applying Green's Theorem and taking into account that u verifies $(1.6 - i)$, we get

$$\begin{aligned}
&\int_{\partial\Omega_\epsilon} P d\tau + Q dx \\
&= \int_{\partial\Omega_\epsilon} \left[-x\, {}^{RL}D_t^{1-\alpha} u_x(x,\tau) + {}^{RL}D_t^{1-\alpha} u(x,\tau)\right] d\tau - x\, u(x,t) dx \\
&= \iint_{\Omega_\epsilon} \Bigg[-x u_t(x,\tau) + {}^{RL}D_t^{1-\alpha} u_x(x,\tau) + x \frac{\partial}{\partial x}\left({}^{RL}D_t^{1-\alpha} u_x(x,\tau)\right) \\
&\qquad\qquad - \frac{\partial}{\partial x}\left({}^{RL}D_t^{1-\alpha} u(x,\tau)\right)\Bigg] d\tau\, dx \\
&= \iint_{\Omega_\epsilon} x \left[{}^{RL}D_t^{1-\alpha} u_{xx}(x,\tau) - u_t(x,\tau)\right] d\tau\, dx = 0.
\end{aligned} \tag{3.21}$$

Consider $\partial\Omega_\epsilon = \partial\Omega_{\epsilon 1} \cup \partial\Omega_{\epsilon 2} \cup \partial\Omega_{\epsilon 3} \cup \partial\Omega_{\epsilon 4}$ where $\partial\Omega_{\epsilon 1} = \{(0,\tau),\, \epsilon \le \tau \le t\}$, $\partial\Omega_{\epsilon 2} = \{(z,\epsilon),\, 0 \le z \le s(\epsilon)\}$, $\partial\Omega_{\epsilon 3} = \{(s(\tau),\tau),\, \epsilon \le \tau \le t\}$ and $\partial\Omega_{\epsilon 4} = \{(z,t),\, 0 \le z \le s(t)\}$. Integrating over the contour $\partial\Omega_\epsilon$ (positively oriented) we get:

$$\int_{\partial\Omega_{\epsilon 1}} P d\tau + Q dx = \int_t^\epsilon {}^{RL}D_t^{1-\alpha} u(0,\tau) d\tau = -\int_\epsilon^t {}^{RL}D_t^{1-\alpha} g(\tau) d\tau, \tag{3.22}$$

$$\int_{\partial\Omega_{\epsilon 2}} P d\tau + Q dx = \int_0^{s(\epsilon)} -z u(z,\epsilon) dz, \tag{3.23}$$

$$\int_{\partial\Omega_{\epsilon 3}} Pd\tau + Qdx = \int_{\epsilon}^{t} \Big[-s(\tau)\ {}_{0}^{RL}D_t^{1-\alpha} \frac{\partial}{\partial x} u(x,t)\Big|_{(s(\tau),\tau)}$$
$$+ {}^{RL}D_t^{1-\alpha}\ u(x,t)|_{(s(\tau),\tau)} - s(\tau)u(s(\tau),\tau)s'(\tau)\Big] d\tau$$
$$= \int_{\epsilon}^{t} s(\tau)s'(\tau)d\tau + \int_{\epsilon}^{t} {}^{RL}D_t^{1-\alpha}\ u(x,t)|_{(s(\tau),\tau)}\, d\tau$$
$$= \frac{s^2(t)}{2} - \frac{s^2(\epsilon)}{2} + \int_{\epsilon}^{t} {}^{RL}D_t^{1-\alpha}\ u(x,t)|_{(s(\tau),\tau)}\, d\tau, \tag{3.24}$$

$$\int_{\partial\Omega_{\epsilon 4}} Pdt + Qdx = \int_{0}^{s(t)} zu(z,t)dz. \tag{3.25}$$

Combining (3.21), (3.22), (3.23), (3.24) and (3.25), it results that

$$\begin{aligned} -\int_{\epsilon}^{t} {}^{RL}D_t^{1-\alpha} g(\tau)d\tau - \int_{0}^{s(\epsilon)} zu(z,\epsilon)dz + \frac{s^2(t)}{2} - \frac{s^2(\epsilon)}{2} \\ + \int_{\epsilon}^{t} {}^{RL}D_t^{1-\alpha}\ u(x,t)|_{(s(\tau),\tau)}\, d\tau + \int_{0}^{s(t)} zu(z,t)dz = 0. \end{aligned} \tag{3.26}$$

Taking the limit when $\epsilon \searrow 0$ we get the integral relationship (3.18), i.e. the thesis holds. □

REMARK 3.4. If we take $\alpha = 1$ in the integral relationship (3.18) we get

$$s^2(t) = b^2 + 2\int_{0}^{t} g(\tau)d\tau + 2\int_{0}^{b} zf(z)dz - 2\int_{0}^{s(t)} zu(z,t)dz - 2\int_{0}^{t} u(s(\tau),\tau)d\tau,$$

and using condition $(1.6-iii)$, it results that

$$s^2(t) = b^2 + 2\int_{0}^{t} g(\tau)d\tau + 2\int_{0}^{b} zf(z)dz - 2\int_{0}^{s(t)} zu(z,t)dz, \tag{3.27}$$

where (3.27) is the classical integral relationship for the free boundary when a classical Stefan problem is considered (see [5], Lemma 17.1.1).

It was also proven in [5] that condition (3.27) is equivalent to the Stefan condition

$$\frac{d}{dt}s(t) = -\frac{\partial}{\partial x}u(s(t),t), \quad \forall\, t > 0. \tag{3.28}$$

Hence, it is natural to wonder if the "fractional Stefan condition" (1.5) and the "fractional integral relationship" (3.18) are equivalent too.

THEOREM 3.3. *Let* $\{u, s\}$ *be a solution of problem* $\{(1.6-i), (1.6-ii), (1.6-iii), (1.6-iv), (3.18)\}$ *such that* $\frac{\partial^2}{\partial t \partial x} u(x,t) \in \mathcal{C}^1(\Omega_T)$, $g \in AC^1(0,T)$,

${}^{RL}D_t^{1-\alpha}g$, ${}^{RL}D_t^{1-\alpha}u(x,t)|_{(s(t),t)} \in L^1(a,b)$ *and* $s(t) > 0$ *for all* $t > 0$. *Then functions* $s = s(t)$ *and* $u = u(x,t)$ *verify the fractional Stefan condition (1.5).*

P r o o f. Reasoning as in Theorem 3.2, we can state that the equalities (3.21), (3.22), (3.23) and (3.24) hold. Then, taking the limit when $\epsilon \searrow 0$ it results that

$$-\int_0^t {}^{RL}D_t^{1-\alpha}g(\tau)d\tau - \int_0^b zf(z)dz - \int_0^t s(\tau)^{RL}D_t^{1-\alpha}\left.\frac{\partial}{\partial x}u(x,t)\right|_{(s(\tau),\tau)} d\tau + \int_0^t {}^{RL}D_t^{1-\alpha}\, u(x,t)|_{(s(\tau),\tau)}\, d\tau + \int_0^{s(t)} zu(z,t)dz = 0. \tag{3.29}$$

Multiplying (3.29) by 2 and using hypothesis (3.18) it yields that

$$-2\int_0^t s(\tau)^{RL}D_t^{1-\alpha}\left.\frac{\partial}{\partial x}u(x,t)\right|_{(s(\tau),\tau)} d\tau = s^2(t) - b^2. \tag{3.30}$$

Differentiating both sides of equation (3.30) whith respect to the $t-$variable and being $s(t) > 0$ for all $t > 0$, the thesis holds. □

REMARK 3.5. The hypothesis $s(t) > 0$ for all $t > 0$ in the previous Theorem is not necessary in the classical Stefan problem. In fact, the Stefan condition (3.28) join with the maximum principle imply that u is a decreasing function of the $x-$variable for every $t > 0$, leading function s to be a non-decreasing function of t.

However this simple tool can not be considered in this case, because decreasing functions may have a positive Riemann-Liouvulle derivative. For example, let $\alpha \in (0,1)$ be and consider $\gamma \in (0,1)$ such that $0 < \alpha - \gamma$. Function $f(t) = t^{-\gamma}$, $t > 0$ is a decreasing function in $\mathbb{R}^+$ and

$${}^{RL}D^{1-\alpha}(t^{-\gamma}) = \frac{\Gamma(-\gamma+1)}{\Gamma(-\gamma-(1-\alpha)+1)}t^{-\gamma+\alpha-1} = \frac{\Gamma(-\gamma+1)}{\Gamma(\alpha-\gamma)}t^{-\gamma+\alpha-1} > 0$$

for all $t > 0$.

3.3. **Example.** The solution (3.7), (3.8) to problem (3.2) given in Theorem 3.1 verify the integral relationship (3.18). In this case, $g(t) = 1$ for all $t > 0$, $b = 0$, $s(t) = 2\xi t^{\alpha/2}$ and $u(x,t) = 1 - \frac{1}{1-W\left(-2\xi,-\frac{\alpha}{2},1\right)}[1 - W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)]$ where ξ is the unique solution to equation (3.9).

The fractional derivative of Riemann–Liouville of a constant is easy to compute (see [24]) and it is given by

$$ {}^{RL}D_t^{1-\alpha}1 = \frac{t^{0-(1-\alpha)}}{\Gamma(0-(1-\alpha)+1)} = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{3.31} $$

Integrating (3.31) from 0 to t, and using the Gamma function property $z\Gamma(z) = \Gamma(z+1)$, we have

$$ \int_0^t {}^{RL}D_t^{1-\alpha}1 d\tau = \frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.32} $$

Applying two times Lemma 3.1 to function $W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)$, we get that function $w(x,t) = W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1+\alpha\right) t^{\alpha}$ is a solution to the fractional diffusion equation for the Caputo derivative $(3.1-i)$ and therefore is a solution to $(1.6-i)$ such that $\frac{\partial^2}{\partial x^2} w(x,t) = W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right)$. Then

$$ \frac{\partial}{\partial t} w(x,t) = {}^{RL}D_t^{1-\alpha} \frac{\partial^2}{\partial x^2} w(x,t) = {}^{RL}D_t^{1-\alpha} W\left(-\frac{x}{t^{\alpha/2}}, -\frac{\alpha}{2}, 1\right). \tag{3.33} $$

Using (3.33) and the linearity of the Riemann-Liouville derivative, it results that

$$ \begin{aligned} {}^{RL}D_t^{1-\alpha} u(x,t) = {}^{RL}D_t^{1-\alpha} & \left(1 - \frac{1}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) \\ & + \frac{1}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} \frac{\partial}{\partial t} w(x,t). \end{aligned} \tag{3.34} $$

Hence

$$ \begin{aligned} {}^{RL}D_t^{1-\alpha} u(x,t)\big|_{(2\xi t^{\alpha/2}, t)} &= \left(1 - \frac{1}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \\ \frac{t^{\alpha-1}}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} & \left[W\left(-2\xi, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right)\alpha\xi + W\left(-2\xi, -\frac{\alpha}{2}, 1+\alpha\right)\alpha\right]. \end{aligned} \tag{3.35} $$

Integrating (3.35) from 0 to t,

$$ \begin{aligned} \int_0^t {}^{RL}D_t^{1-\alpha} u(x,t)\big|_{(2\xi \tau^{\alpha/2}, \tau)} d\tau &= \left(1 - \frac{1}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right) \frac{t^{\alpha}}{\Gamma(\alpha+1)} \\ \times \frac{t^{\alpha}}{1-W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} & \left[W\left(-2\xi, -\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right)\xi + W\left(-2\xi, -\frac{\alpha}{2}, 1+\alpha\right)\right]. \end{aligned} \tag{3.36} $$

Integrating by parts, the next computation follows:

$$\int_0^{2\xi t^{\alpha/2}} zu(z,t)dz = 2\xi^2 t^\alpha \left(1 - \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right)$$
$$- \frac{W\left(-2\xi, -\frac{\alpha}{2}, 1+\alpha\right)}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} t^\alpha - \frac{2\xi t^\alpha}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} W\left(-2\xi, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)$$
$$+ \frac{t^\alpha}{\left(1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)\right)\Gamma(1+\alpha)}. \quad (3.37)$$

Taking into account (3.32), (3.36) and (3.37), it follows that

$$2\int_0^t {}^{RL}D_t^{1-\alpha} 1 d\tau - 2\int_0^t {}^{RL}D_t^{1-\alpha} u(x,t)\big|_{(2\xi\tau^{\alpha/2},\tau)} - 2\int_0^{2\xi t^{\alpha/2}} zu(z,t)dz$$
$$= -4\xi^2 t^\alpha + \frac{4\xi^2 t^\alpha}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} + \frac{2\xi t^\alpha W\left(-2\xi, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}$$
$$= 2\xi t^\alpha \left[-2\xi + \frac{2\xi W\left(-2\xi, -\frac{\alpha}{2}, 1 + \frac{\alpha}{2}\right)}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)} + \frac{1}{1 - W\left(-2\xi, -\frac{\alpha}{2}, 1\right)}\right]. \quad (3.38)$$

Finally, taking into account that ξ verify equation (3.9), it results that

$$2\int_0^t {}^{RL}D_t^{1-\alpha} 1 d\tau - 2\int_0^t {}^{RL}D_t^{1-\alpha} u(x,t)\big|_{(2\xi\tau^{\alpha/2},\tau)} - 2\int_0^{2\xi t^{\alpha/2}} zu(z,t)dz$$
$$= 4\xi^2 t^\alpha = s^2(t), \quad (3.39)$$

and the fractional integral relationship (3.18) is satisfied.

4. Conclusions

We have considered a fractional Stefan problem (1.6) by using the fractional Riemann–Liouville derivative. We have obtained an integral relationship between the fractional temperature and the fractional free boundary, which is equivalent to the fractional Stefan condition. We have also shown an exact solution to the fractional Stefan problem (3.2), which verifies the integral relationship (3.18).

Acknowledgments

The present work has been sponsored by the Projects PIP N° 0275 from CONICET–Univ. Austral, and by AFOSR–SOARD Grant FA9550 – 14 – 1 – 0122.

References

[1] V. Alexiades, A. D. Solomon, *Mathematical Modelling of Melting and Freezing Processes*. Hemisphere, Washington (1993).

[2] C. Atkinson, Moving bounary problems for time fractional and composition dependent diffusion. *Fract. Calc. Appl. Anal.* **15**, No 2 (2012), 207–221; DOI: 10.2478/s13540-012-0015-2; https://www.degruyter.com/view/j/fca.2012.15.issue-2/issue-files/fca.2012.15.issue-2.xml.

[3] D. Banks, C. Fradin, Anomalous diffusion of proteins due to molecular crowding. *Biophys. J.* **89**, No 5 (2005), 2960–2971.

[4] M. Blasiak, M. Klimek, Numerical solution of the one phase 1D fractional Stefan problem using the front fixing method. *Math. Methods Appl.* **38**, No 15 (2015), 3214–3228.

[5] J. R. Cannon, *The One–Dimensional Heat Equation*. Cambridge University Press, Cambridge (1984).

[6] J. R. Cannon, C. D. Hill, M. Primicerio, The one–phase Stefan problem with temperature–boundary conditions. *Arch. Ration. Mech. Anal.* **39** (1970), 270–274.

[7] J. R. Cannon, M. Primicerio, Remarks on the one–phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary. *J. Math. Anal. Appl.* **35** (1971), 361–373.

[8] J. Crank, *Free and Moving Boundary Problems*. Clarendon Press, Oxford (1984).

[9] A. N. Ceretani, D. A. Tarzia, Determination of two unknown thermal coefficients through an inverse one-phase fractional Stefan problem. *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 399–421; DOI: 10.1515/fca-2017-0021; https://www.degruyter.com/view/j/fca.2017.20.issue-2/issue-files/fca.2017.20.issue-2.xml.

[10] C. M. Elliott, J. R. Ockendon, *Weak and Variational Methods for Moving Boundary Problems*. Pitman, London (1982).

[11] D. Goos, G. Reyero, S. Roscani, E. Santillan Marcus, On the initial–boundary–value problem for the time–fractional diffusion equation on the real positive semiaxis. *Int. J. Difference Equ.* **2015** (2015), Article ID 439419, 1–14.

[12] R. Gorenflo, Y. Luchko, F. Mainardi, Analytical properties and applications of the Wright function. *Fract. Calc. Appl. Anal.* **2**, No 4 (1999), 383–414.

[13] S. C. Gupta, *The Classical Stefan Problem. Basic Concepts, Modelling and Analysis*. Elsevier, Amsterdam (2003).

[14] A. A. Gusev, U. W. Suter, Dynamics of small molecules in dense polymers subject to thermal motion. *J. Chem. Phys.* **99** (1993), 2228–2234.

[15] L. Jinyi, X. Mingyu, Some exact solutions to stefan problems with fractional differential equations. *J. Math. Anal. Appl.* **351** (2009), 536–542.

[16] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Vol. 204 of North-Holland Mathematics Studies, Elsevier, Amsterdam (2006).

[17] X. Li, M. Xu, X. Jiang, Homotopy perturbation method to time–fractional diffusion equation with a moving boundary condition. *Appl. Math. Comput.* **208** (2009), 434–439.

[18] Y. Luchko, Some uniqueness and existence results for the initial-boundary–value problems for the generalized time–fractional diffusion equation. *Comput. Math. Appl.* **59** (2010), 1766–1772.

[19] Y. Luchko, F. Mainardi, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4**, No 2 (2001), 153–192.

[20] V. J. Lunardini, *Heat Transfer with Freezing and Thawing*. Elsevier, London (1991).

[21] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, London (2010).

[22] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* **339** (2000), 1–77.

[23] T. Odzijewicz, A. B. Malinowska, D. F. M. Torres, Green's theorem for generalized fractional derivatives. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 64–75; DOI: 10.2478/s13540-013-0005-z; https://www.degruyter.com/view/j/fca.2013.16.issue-1/issue-files/fca.2013.16.issue-1.xml.

[24] I. Podlubny, *Fractional Differential Equations*. Vol. 198 of Math. in Sci. and Engin., Academic Press, San Diego (1999).

[25] S. Roscani, Hopf lemma for the fractional diffusion operator and its application to a fractional free–boundary problem. *J. Math. Anal. Appl.* **434** (2016), 125–135.

[26] S. Roscani, E. Santillan Marcus, Two equivalen Stefan's problems for the time–fractional diffusion equation. *Fract. Calc. Appl. Anal.* **16**, No 4 (2013), 802–815; DOI: 10.2478/s13540-013-0050-7; https://www.degruyter.com/view/j/fca.2013.16.issue-4/issue-files/fca.2013.16.issue-4.xml.

[27] S. Roscani, E. Santillan Marcus, A new equivalence of Stefan's problems for the time–fractional–diffusion equation. *Fract. Calc. Appl. Anal.* **17**, No 2 (2014), 371–381; DOI: 10.2478/s13540-014-0175-3; https://www.degruyter.com/view/j/fca.2014.17.issue-2/issue-files/fca.2014.17.issue-2.xml.

[28] S. Roscani, D. Tarzia, A generalized neumann solution for the two-phase fractional Lame-Clapeyron-Stefan problem. *Adv. Math. Sci. Appl.* **24**, No 2 (2014), 237–249.

[29] L. I. Rubinstein, *The Stefan Problem.* Transl. of Mathematical Monographs, 27, Amer. Math. Soc., Providence (1971).
[30] Y. X. Tao, R. W. Besant, K. S. Rezkallah, A mathematical model for predicting the densification and growth of frost on a flat plate. *Int. J. Heat Mass Transfer* **36** (1993), 353–363.
[31] D. A. Tarzia, A bibliography on moving–free boundary problems for the heat diffusion equation. The Stefan and related problems. *MAT–Serie A* **2** (2000), 1–297.
[32] D. A. Tarzia, *Explicit and Approximated Solutions for Heat and Mass Transfer Problems with a Moving Interface*, Chapter 20. In: Advanced Topics in Mass Transfer, Mohamed El-Amin. InTech Open Access Publisher, Rijeka (2011).
[33] D. A. Tarzia and C. V. Turner, The asynptotic behavior for the one-phase Stefan problem with a convective boundary condition. *Appl. Math. Lett.* **9** (1996), 21–24.
[34] C. J. Vogl, M. J. Miksis, S. H. Davis, Moving boundary problems governed by anomalous diffusion. In: *Proc. Math. Phys. Eng. Sci.* **468** (2012), 3348–3369.
[35] V. R. Voller, Fractional stefan problems. *Int. J. Heat Mass Transfer* **74** (2014), 269–277.
[36] V. R. Voller, F. Falcini, R. Garra, Fractional Stefan problems exhibiting lumped and distributed latent–heat memory effects. *Phys. Rev. E* **87** (2013), # 042401.
[37] E. M. Wright, The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc. (2)* **38** (1943), 257–270.
[38] E. M. Wright, The generalized Bessel function of order greater than one. *Q. J. Math.*, Ser. **11** (1940), 36–48.
[39] S. B. Yuste, E. Abad, K. Lindenberg, Reaction-subdiffusion model of morphogen gradient formation. *Phys. Rev. E* **82** (2010), # 061123.

[1,2] *CONICET - Depto. Matemática, FCE, Univ. Austral*
Paraguay 1950, S2000FZF, Rosario, ARGENTINA

e-mail: sabrinaroscani@gmail.com
e-mail: dtarzia@austral.edu.ar *Received: May 26, 2017*

Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **21**, No 4 (2018), pp. 901–918,
DOI: 10.1515/fca-2018-0049 , at https://www.degruyter.com/view/j/fca