Hindawi Publishing Corporation
Journal of Engineering
Volume 2013, Article ID 937596, 7 pages
http://dx.doi.org/10.1155/2013/937596

Research Article

Coupling between Transverse Vibrations and Instability Phenomena of Plates Subjected to In-Plane Loading

D. V. Bambill[1,2] and C. A. Rossit[1,2]

[1] *Department of Engineering, Institute of Applied Mechanics (IMA), Universidad Nacional del Sur (UNS), Alem 1253, B8000CPB Bahía Blanca, Argentina*
[2] *Consejo Nacional de Investigaciones Científicas y Técnicas, (CONICET), Bahía Blanca, Argentina*

Correspondence should be addressed to D. V. Bambill; dbambill@criba.edu.ar

Received 27 November 2012; Accepted 31 January 2013

Academic Editor: Gabriele Milani

As it is known, the problems of free transverse vibrations and instability under in-plane loads of a plate are two different technological situations that have similarities in their approach to elastic solution. In fact, they are two eigenvalue problems in which we analyze the equilibrium situation of the plate in configurations which differ very slightly from the original, undeformed configuration. They are coupled in the event where in-plane forces are applied to the edges of the transversely vibrating plate. The presence of forces can have a significant effect on structural and mechanical performance and should be taken into account in the formulation of the dynamic problem. In this study, distributed forces of linear variation are considered and their influence on the natural frequencies and corresponding normal modes of transverse vibration is analyzed. It also analyzes their impact for the case of vibration control. The forces' magnitude is varied and the first natural frequencies of transverse vibration of rectangular thin plates with different combinations of edge conditions are obtained. The critical values of the forces which cause instability are also obtained. Due to the analytical complexity of the problem under study, the Ritz method is employed. Some numerical examples are presented.

1. Introduction

The transverse-free vibrations and buckling of plates which are subjected to edge loads acting in their middle planes are areas of research which have received a great deal of attention in the past century.

As it was stated experimentally by Hearmon [1] for the case of a beam, bifurcation buckling may be regarded as a special case of the vibration problem, that is, determining the in-plane stresses which cause vibration frequencies to reduce to zero.

Most of the work has dealt with rectangular plates having uniformly distributed in-plane edge loads. In that case, the governing differential equations of motion and equilibrium have constant coefficients, yielding exact solutions for frequencies and buckling loads straightforwardly when two opposite edges of the plates are simply supported.

Many researchers have analyzed both the buckling and vibration of rectangular plates subjected to in-plane stress field. Among them, one can mention Kang and Leissa [2]; Leissa and Kang [3]; Bassily and Dickinson [4]; Dickinson [5]; Kielb and Han [6]; Kaldas and Dickinson [7].

For the linearly varying loading, the governing differential equations have variable coefficients.

Leissa and Kang [3] found exact solutions for the free vibration and buckling problems of the SS-C-SS-C isotropic plate loaded at its simply supported edges by linearly varying in-plane stresses.

They also found the exact solution [8] for the buckling of rectangular plates having linearly varying in-plane loading on two opposite simply supported edges, with different boundary conditions at the other opposite edges.

Within the realm of the classical theory of plates, the case of buckling and vibrations problems for all the possibilities of boundary conditions and linearly varying in-plane forces offers considerable difficulty. This is the reason why it is quite common to make use of the Ritz variational method.

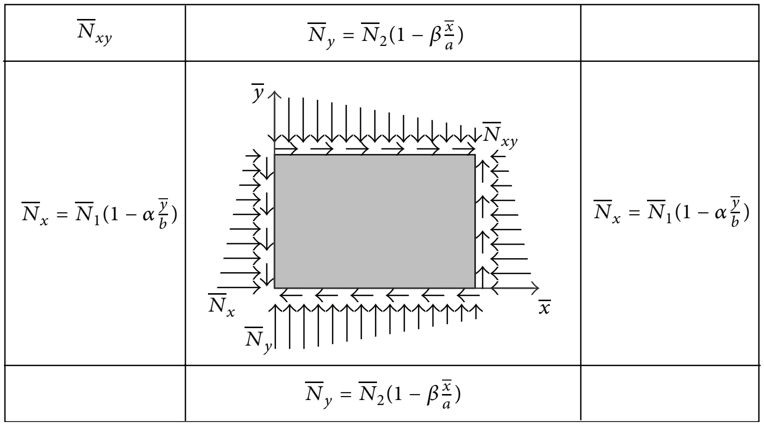

Figure 1: Rectangular vibrating plate subjected to in-plane loads: $\overline{N}_x$, $\overline{N}_y$, and $\overline{N}_{xy}$.

2. Approximate Analytical Solution

In the case of a transversely vibrating, thin, isotropic plate subjected to in-plane forces N_x, N_y, and N_{xy}, (Figure 1 and (5)), the maximum value of the potential energy due to bending deformation is

$$U_{\max} = \frac{1}{2}D\int_A \left[\left(\frac{\partial^2 W}{\partial \overline{x}^2} + \frac{\partial^2 W}{\partial \overline{y}^2}\right)^2 + 2(1-\nu) \times \left(\frac{\partial^2 W}{\partial \overline{x}^2}\frac{\partial^2 W}{\partial \overline{y}^2} - \left(\frac{\partial^2 W}{\partial \overline{x}\partial \overline{y}}\right)^2\right)\right] d\overline{x}\, d\overline{y}, \tag{1}$$

where $W = W(x, y)$ is the deflection amplitude of the middle plane of the plate, D is the well known flexural rigidity $D = Eh^3/12(1-\nu^2)$, E is the Young modulus, and ν is the Poisson coefficient.

While the maximum of the kinetic energy is

$$T_{\max} = \frac{1}{2}\rho\omega^2 \int_A hW^2 d\overline{x}\, d\overline{y}, \tag{2}$$

where ρ is the density of the plate material, ω is the circular frequency, and h is the thickness of the plate.

And the maximum potential energy of the internal stresses caused by the in-plane loading is

$$\tau_N = \frac{1}{2}\int_{A_P}\left(\overline{N}_x\left(\frac{\partial W}{\partial \overline{x}}\right)^2 + \overline{N}_y\left(\frac{\partial W}{\partial \overline{y}}\right)^2 + 2\overline{N}_{xy}\frac{\partial W}{\partial \overline{x}}\frac{\partial W}{\partial \overline{y}}\right) d\overline{x}\, d\overline{y}. \tag{3}$$

The lengths of the sides of the rectangular plate are a in the x direction and b in the y direction. The coordinates are written in the dimensionless form as follows:

$$x = \frac{\overline{x}}{a}, \qquad y = \frac{\overline{y}}{b}. \tag{4}$$

And the in-plane forces are expressed as (Bambill et al. [9]):

$$N_x = \frac{\overline{N}_x b^2}{D}, \qquad N_y = \frac{\overline{N}_y b^2}{D}, \qquad N_{xy} = \frac{\overline{N}_{xy} b^2}{D}. \tag{5}$$

Then, the governing functional of the system is

$$\begin{aligned} J[W] = {} & \frac{1}{2}D\left[\frac{b}{a^3}\int_A N_x\left(\frac{\partial W}{\partial x}\right)^2 dx\, dy + \frac{1}{ab}\int_A N_y\left(\frac{\partial W}{\partial y}\right)^2 dx\, dy + \frac{2}{a^2}\int_A N_{xy}\frac{\partial W}{\partial x}\frac{\partial W}{\partial y} dx\, dy\right] \\ & + \frac{1}{2}D\left[\frac{b}{a^3}\int_A\left(\frac{\partial^2 W}{\partial x^2}\right)^2 dx\, dy + \frac{2\nu}{ab}\int_A \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} dx\, dy + \frac{a}{b^3}\int_A\left(\frac{\partial^2 W}{\partial y^2}\right)^2 dx\, dy + \frac{2(1-\nu)}{ab}\int_A\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2 dx\, dy\right] \\ & - \frac{1}{2}\rho ab\omega^2\int_A hW^2 dx\, dy. \end{aligned} \tag{6}$$

Equation (6) satisfies, if W is the exact solution, the condition:

$$\delta J[W] = 0. \tag{7}$$

TABLE 1: The first six natural frequency coefficients Ω_i for a C-C-F-F plate under general in-plane loading.

	N/N_{crit}				
	0	0.25	0.5	0.75	1
	$\lambda = a/b = 0.75$; $N_{crit} = 5.95168$				
Ω_1	5.39789	4.75442	3.95041	2.84503	0
Ω_2	15.7547	15.2037	14.5835	13.8967	13.1443
Ω_3	23.8207	23.9071	24.0071	24.1145	24.2241
Ω_4	36.3999	36.2826	36.1113	35.7654	35.1888
Ω_5	38.1446	37.5193	36.9334	36.5072	36.2961
Ω_6	59.7605	59.5098	59.2367	58.9439	58.6327
	$\lambda = a/b = 1$; $N_{crit} = 3.69137$				
Ω_1	6.93254	6.06065	4.99441	3.56413	0
Ω_2	23.9780	23.9778	23.6766	23.0314	22.1751
Ω_3	26.6265	26.0033	25.6318	25.5511	25.6238
Ω_4	47.7179	47.4465	47.1607	46.8601	46.5440
Ω_5	62.8830	62.6774	62.3443	61.9069	61.3970
Ω_6	65.6818	65.4396	65.3185	65.2954	65.3384
	$\lambda = a/b = 2$; $N_{crit} = 1.85222$				
Ω_1	17.1515	14.9296	12.2458	8.69504	0
Ω_2	36.4410	36.0647	35.6415	35.1668	34.6357
Ω_3	73.5943	73.8555	74.0906	74.2962	74.4680
Ω_4	91.1002	89.5702	88.0081	86.4144	84.7901
Ω_5	115.363	114.262	113.147	112.021	110.886
Ω_6	131.962	132.267	132.566	132.855	133.130

TABLE 2: The first six natural frequency coefficients Ω_i for a C-C-SS-SS plate under uniform N_x loading ($\alpha = 0$).

$\lambda = a/b = 1$	N/N_{crit}					
$N_{crit} = 61.4151$	0	0.1	0.25	0.5	0.75	1
Ω_1	27.0542	25.7489	23.6368	19.5326	14.0587	0
Ω_2	60.5387	58.4465	54.9733	48.6153	41.2865	32.4139
Ω_3	60.7863	60.1238	59.2827	57.8385	56.3769	54.8441
Ω_4	92.8371	91.4269	89.2672	85.5386	81.6286	77.5081
Ω_5	114.557	112.082	108.152	101.257	93.8564	85.8259
Ω_6	114.704	114.350	113.923	113.207	112.485	111.755

Following the Ritz method, the expression of the deflection of the plate is approximated in the form of a truncated series:

$$W \cong W_a(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} X_m(x) Y_n(y), \tag{8}$$

where $X_m(x)$ and $Y_n(y)$ are the characteristic functions for the normal modes of vibration of beams with end conditions nominally similar to those of the opposite edges of the plate in each coordinate direction [10]. Consequently, they satisfy the essential boundary conditions, as the method requires.

The variational equation (7) is replaced by the homogeneous linear system of equations:

$$\frac{\partial J[W_a]}{\partial A_{ql}} = 0, \quad q = \{1, 2, \ldots M\}; \; l = \{1, 2, \ldots N\}, \tag{9}$$

and using nondimensional variables, becomes

$$\frac{ab}{D} \frac{\partial J[W_a]}{\partial A_{ql}} = 0. \tag{10}$$

Finally, one obtains a homogeneous linear system of equations in terms of the A_{ql}'s parameters.

The nontriviality condition of the system (10) requires the determinant to be zero:

$$\left| S \cdot [\tau_{qlmn}] + [u_{qlmn}] - \Omega^2 \cdot [t_{qlmn}] \right| = 0, \tag{11}$$

where $\Omega_{mn} = \omega_{mn} a^2 \sqrt{\rho h/D}$ are the frequency coefficients.

The elements of the matrices involved in (11) are given by

$$\begin{aligned}
\tau_{qlmn} = {} & \frac{1}{\lambda^2} \int_{A_n} N_x^* \left(\frac{dX_q}{dx} \frac{dX_m}{dx} \right) (Y_l Y_n) \, dx \, dy \\
& + \int_{A_n} N_y^* (X_q X_m) \left(\frac{dY_l}{dy} \frac{dY_n}{dy} \right) dx \, dy \\
& \times \frac{1}{\lambda} \int_{A_n} N_{xy}^* \left(X_q \frac{dX_m}{dx} \frac{dY_l}{dy} Y_n \right. \\
& \left. + \frac{dX_q}{dx} X_m Y_l \frac{dY_n}{dy} \right) dx \, dy,
\end{aligned} \tag{12}$$

where $\lambda = a/b$ is the aspect ratio

$$N_x = S \cdot N_x^*, \quad N_y = S \cdot N_y^*, \quad N_{xy} = S \cdot N_{xy}^*, \tag{13}$$

where S is a factor that indicates the magnitude of the in-plane loading system, regarding the relative value of the forces. Consider

$$\begin{aligned}
u_{qlmn} = {} & \lambda^{-2} \int_A \left(\frac{d^2 X_q}{dx^2} \frac{d^2 X_m}{dx^2} \right) (Y_l Y_n) \, dx \, dy \\
& + \nu \int_A \left[\left(X_q \frac{d^2 X_m}{dx^2} \right) \left(\frac{d^2 Y_l}{dy^2} Y_n \right) \right. \\
& \left. + \left(\frac{d^2 X_q}{dx^2} X_m \right) \left(Y_l \frac{d^2 Y_n}{dy^2} \right) \right] dx \, dy \\
& + \lambda^2 \int_A (X_q X_m) \left(\frac{d^2 Y_l}{dy^2} \frac{d^2 Y_n}{dy^2} \right) dx \, dy \\
& + 2(1 - \nu) \int_A \left(\frac{dX_q}{dx} \frac{dX_m}{dx} \right) \left(\frac{dY_l}{dy} \frac{dY_n}{dy} \right) dx \, dy, \\
t_{qlmn} = {} & \lambda^{-2} \int_A (X_q X_m)(Y_l Y_n) \, dx \, dy.
\end{aligned} \tag{14}$$

As it is known, the condition $\Omega = 0$ in (11) yields the critical value of the in-plane loading.

Table 3: The first six natural frequency coefficients Ω_i for a C-C-SS-SS plate under uniform shear N_{xy} loading.

$\lambda = a/b = 1.5$ $N_{\text{crit}} = 89.8137$	N/N_{crit} 0	0.1	0.25	0.4	0.5	0.75	0.9	1
Ω_1	44.8904	44.7005	43.7991	42.4070	40.2822	32.4323	22.3996	0
Ω_2	76.5451	76.1864	74.3165	70.7817	67.4142	54.4376	41.3590	27.7956
Ω_3	122.319	122.349	122.548	121.465	117.487	104.882	95.5807	88.7020
Ω_4	129.393	128.871	126.129	122.871	123.101	123.415	123.108	122.473
Ω_5	152.529	152.707	153.680	155.033	155.912	157.429	157.633	157.273
Ω_6	202.615	200.101	194.748	188.611	184.103	171.463	163.073	214.018

Table 4: Comparison of nondimensional critical buckling loads $N_{\text{crit}} = \overline{N}_{\text{crit}} b^2/D$ for a SS-C-SS-C plate.

Load (α)	Solution	$\lambda = a/b$ 0.4	0.5	0.6	0.7	0.8	0.9	1.0
	[A] $M = 5$	93.3059	75.9452	69.6553	69.1116	72.0966	77.5543	75.9452
	[A] $M = 10$	93.2555	75.9146	69.6351	69.0972	72.0859	77.5460	75.9146
0	[A] $M = 15$	93.2477	75.9105	69.6323	69.0954	72.0846	77.5450	75.9105
	[A] $M = 20$	93.2476	75.9101	69.6323	69.09531	72.0844	77.5449	75.9101
	[B]	93.247	75.910	69.632	69.095	72.084	77.545	75.910
	[A] $M = 5$	174.533	145.286	134.809	134.624	140.981	152.024	145.286
	[A] $M = 10$	174.395	145.215	134.765	134.593	140.958	152.007	145.215
1	[A] $M = 15$	174.379	145.207	134.760	134.590	140.956	152.005	145.207
	[A] $M = 20$	174.377	145.206	134.760	134.590	140.956	152.005	145.206
	[B]	174.4	145.2	134.8	134.6	141.0	152.0	145.2
	[A] $M = 5$	401.518	392.147	412.162	424.140	401.518	392.143	392.147
	[A] $M = 10$	400.478	391.589	411.812	422.594	400.478	391.398	391.589
2	[A] $M = 15$	400.410	391.548	411.790	422.490	400.410	391.351	391.548
	[A] $M = 20$	400.399	391.548	411.787	422.472	400.399	391.343	391.547
	[B]	400.4	391.5	411.8	422.5	400.4	—	391.5

[A]: present approach with different $M = N$ and [B]: [3].

3. Numerical Evaluations

Hearmon [1] has experimented on a fixed-free strip. Admittedly, the problem is analytically simpler in the case of one-dimensional domains. As an example, let us try with a pinned-pinned transversely vibrating beam, subjected to an axial compressive force P. The expression of the frequency coefficient is

$$\Omega_n = \sqrt{\frac{\rho A_0}{\text{EI}}} L^2 \omega_n = (n\pi)^2 \sqrt{\left[1 - \frac{PL^2}{\text{EI}(n\pi)^2}\right]}; \quad \text{with } n = 1, 2, 3, 4 \ldots, \tag{15}$$

where ρ is the density of the material, A_0 is the cross-section, L is the length, and EI the flexural rigidity of the beam.

All the Euler buckling loads are determined making zero expression (15). For $n = 1$, the critical buckling load of the beam, P_{crit}, is obtained.

Plotting the values $\Omega_n = (n\pi)^2 \sqrt{(1 - (P/n^2 P_{\text{crit}}))}$ of the first three frequency coefficients depending upon the ratio P/P_{crit} yield regular curves as is shown in Figure 2. The presence of the compressive axial load P does not alter the order of the modal shapes of the beam.

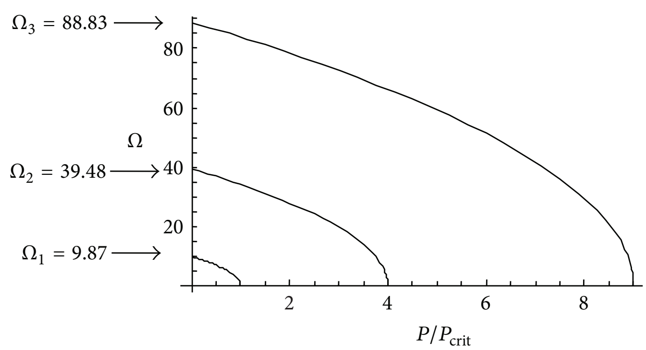

Figure 2: Curves of the frequency coefficients of transverse vibration for a pinned-pinned beam under axial compressive load P.

In the case of a plate, in general, and due to the bidimensional behavior induced by the torsional rigidity, the compressive in-plane load may alter both the order and shape of the modal shapes associated to each natural frequency.

Table 5: Comparison of the first six natural frequency coefficients for a SS-C-SS-C plate under bending moment in x-direction ($\alpha = 2$).

$\lambda = a/b$	N/N_{crit}	Solution	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
0.5	0	[A]	13.6858	23.6465	38.6942	42.5871	51.6767	58.647
		[B]	13.69	23.65	—	42.59	51.67	—
	0.5	[A]	11.4936	24.152	37.9343	38.8757	52.632	58.7048
		[B]	11.49	24.15	37.93	—	52.63	—
	0.95	[A]	3.92178	24.906	27.6496	39.3337	52.6657	58.8581
		[B]	3.926	24.91	27.65	—	52.66	—
1	0	[A]	28.9509	54.7433	69.3271	94.5862	102.217	129.096
		[B]	28.95	54.74	69.33	94.59	102.2	—
	0.5	[A]	27.4647	45.9744	69.6407	87.2215	96.6079	129.164
		[B]	27.47	45.97	69.65	87.22	96.61	—
	0.95	[A]	15.6871	23.3557	46.9793	70.3963	99.624	110.599
		[B]	15.70	23.37	46.96	70.41	99.63	110.6
2	0	[A]	95.2625	115.803	156.357	218.973	254.138	277.308
		[B]	95.26	115.8	156.4	—	254.1	277.3
	0.5	[A]	94.7438	109.859	137.29	183.897	254.214	254.254
		[B]	94.76	109.9	137.3	183.9	254.2	—
	0.95	[A]	62.7485	77.444	93.4166	93.4229	101.133	187.917
		[B]	62.82	77.50	93.46	93.47	—	—

[A]: present approach and [B]: [3].

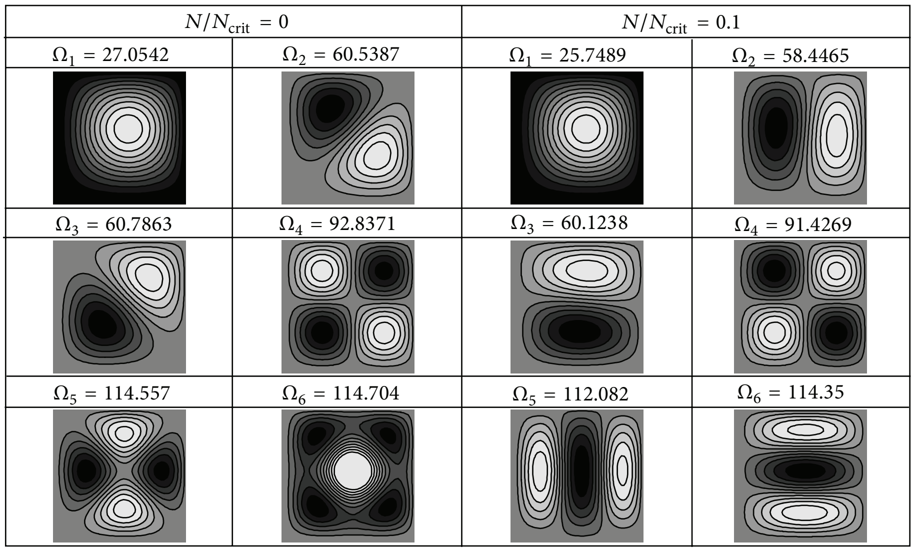

Figure 3: Normal modes of vibration of a square C-C-SS-SS plate under uniform N_x loading ($\lambda = a/b = 1$, $N_{crit} = 61.4151$).

This situation has an important technological signification from the point of view of vibration control.

Certainly, the modal shape of a natural resonant frequency must be known in order to suppress it. In the case of in-plane loading, this shape can be different from the expected one.

Due to the quantity and variability of the parameters involved in the description of the behaviour of these kinds of structures, just a few representative cases will be considered to demonstrate the convenience of the procedure and the importance of the situation.

All the values are determined taking $M = N = 15$ in (8).

Table 1 shows the values of the first natural frequency coefficients for a CCFF plate subjected to a general in-plane loading: linear load in x direction ($\alpha = 2$)—bending moment, constant load in y direction ($\beta = 0$), and constant shear force $N_{xy} = N_1 = N_2 = N$).

In order to show the influence of the in-plane loading, the next two examples are presented.

Table 2 shows the natural frequency coefficients for a C-C-SS-SS square plate under uniform compression in the x direction ($N_x = N$, $\alpha = 0$, and $N_y = N_{xy} = 0$).

Figure 3 shows that a minimal presence of in-plane loading (10% of the critical value) dramatically modifies the

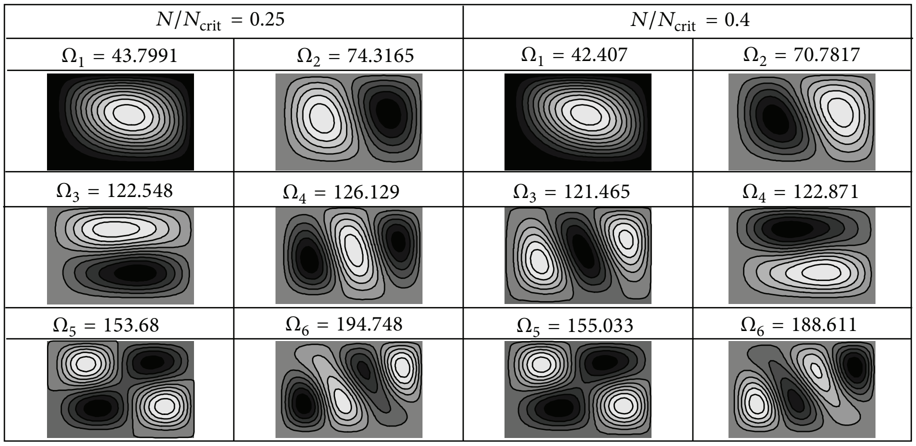

FIGURE 4: Normal modes of vibration of a rectangular C-C-SS-SS plate under uniform N_{xy} loading ($\lambda = a/b = 1.5$, $N_{\text{crit}} = 89.8137$).

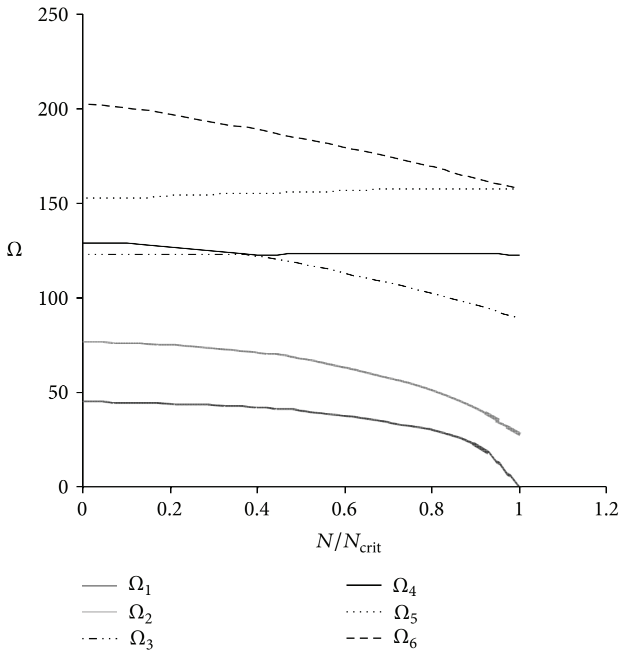

FIGURE 5: The first six natural frequency coefficients Ω_i for a C-C-SS-SS plate under uniform shear N_{xy} loading ($\lambda = a/b = 1.5$, $N_{\text{crit}} = 89.8137$).

mode shapes, while changes in the values of frequencies may not be noticed (0.3% in the sixth frequency). It is important to point out that the small load can be originated by thermal variations and restrictions on plane displacements imposed by the external supports.

Finally, Table 3 shows the results for a rectangular C-C-SS-SS plate subjected to shear in-plane forces.

Figure 4 shows that the third and fourth natural frequencies interchange their normal modes as N_{xy} increases. This situation is noticeable from Figure 5.

This means that for a given value of N_{xy}, between 0.25 and 0.4 of the critical value, there are two normal modes for the same natural frequency (repeated frequency). This is an important point in vibration control, since when repeated frequencies arise in a system, the related vibration mode shape cannot be uniquely determined. Any linear combination of the modes is still valid for the repeated frequency.

In order to evaluate the accuracy of the expounded procedure, comparison is made with the results obtained in [3] for a SS-C-SS-C plate loaded at its simply supported edges by linearly varying in-plane stresses (Tables 4 and 5).

In Table 4, values of N_{crit} are compared for three different cases of the x direction load: constant ($\alpha = 0$), linear with null value at one extreme ($\alpha = 1$), and bending moment ($\alpha = 2$), and different aspect radii of the plate. A convergence study is also made. As it can be seen, taking $M = N = 15$ provides an excellent accuracy from an engineering viewpoint.

4. Conclusions

The classical, variational method of Ritz has been successfully used in the present study to obtain an approximate, yet quite accurate, solution to a difficult elastodynamics problem.

Natural frequencies and mode shapes of transverse vibration are obtained for a meaningful combination of the boundary conditions of a thin plate subjected to general in-plane loads. The critical values of the in-plane forces which cause instability of the plates are also obtained.

The obtained values are the outcome of an algorithm, relatively simple to implement, [11] which allows studying these with only the assistance of a PC.

Additional complexities like orthotropic material characteristics can be taken into account [12].

The agreement with results available in the literature is excellent. Nevertheless, it is also possible to increase the number of terms in the summation on (8) to increase the accuracy.

No claim of originality is made, but it is hoped that the present work draws the attention to the effect that the presence of plane stress state may have on the effectiveness of vibration control on plates.

Acknowledgments

The present work has been sponsored by the Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur, UNS, at the Department of Engineering and by

Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET. The authors wish to thank Dr. D. H. Felix from Universidad Nacional del Sur.

References

[1] R. F. S. Hearmon, "The frequency of vibration and the elastic stability of a fixed-free strip," *British Journal of Applied Physics*, vol. 7, no. 11, pp. 405–407, 1956.

[2] J.-H. Kang and A. W. Leissa, "Vibration and buckling of SS-F-SS-F rectangular plates loaded by in-plane moments," *International Journal of Stability and Dynamics*, vol. 1, no. 4, pp. 527–543, 2001.

[3] A. W. Leissa and J.-H. Kang, "Exact solutions for vibration and buckling of an SS-C-SS-C rectangular plate loaded by linearly varying in-plane stresses," *International Journal of Mechanical Sciences*, vol. 44, no. 9, pp. 1925–1945, 2002.

[4] S. F. Bassily and S. M. Dickinson, "Buckling and lateral vibration of rectangular plates subject to in-plane loads—a Ritz approach," *Journal of Sound and Vibration*, vol. 24, no. 2, pp. 219–239, 1972.

[5] S. M. Dickinson, "The buckling and frequency of flexural vibration of rectangular isotropic and orthotropic plates using Rayleigh's method," *Journal of Sound and Vibration*, vol. 61, no. 1, pp. 1–8, 1978.

[6] R. E. Kielb and L. S. Han, "Vibration and buckling of rectangular plates under in-plane hydrostatic loading," *Journal of Sound and Vibration*, vol. 70, no. 4, pp. 543–555, 1980.

[7] M. M. Kaldas and S. M. Dickinson, "Vibration and buckling calculations for rectangular plates subject to complicated in-plane stress distributions by using numerical integration in a Rayleigh-Ritz analysis," *Journal of Sound and Vibration*, vol. 75, no. 2, pp. 151–162, 1981.

[8] J.-H. Kang and A. W. Leissa, "Exact solutions for the buckling of rectangular plates having linearly varying in-plane loading on two opposite simply supported edges," *International Journal of Solids and Structures*, vol. 42, no. 14, pp. 4220–4238, 2005.

[9] D. V. Bambill, C. A. Rossit, and D. H. Felix, "Comments on 'Buckling behavior of a graphite/epoxy composite plate under parabolic variation of axial loads'," *International Journal of Mechanical Sciences*, vol. 47, no. 9, pp. 1473–1474, 2005.

[10] R. P. Felgar Jr., *Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam*, Circular, no. 14, The University of Texas Publication, 1951.

[11] D. H. Felix, D. V. Bambill, and C. A. Rossit, "Desarrollo de un algoritmo de cálculo para la implementación del método de Rayleigh-Ritz en el cálculo de frecuencias naturales de vibración de placas rectangulares con complejidades diversas," *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería*, vol. 20, no. 2, pp. 123–138, 2004.

[12] D. H. Felix, D. V. Bambill, and C. A. Rossit, "A note on buckling and vibration of clamped orthotopic plates under in-plane loads," *Structural Engineering and Mechanics*, vol. 39, no. 1, pp. 115–123, 2011.