Extremal black hole initial data deformations

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Abstract

We study deformations of axially symmetric initial data for Einstein-Maxwell equations satisfying the time-rotation (t-ϕ) symmetry and containing one asymptotically cylindrical end and one asymptotically flat end. We find that the t-ϕ symmetry implies the existence of a family of deformed data having the same horizon structure. This result allows us to measure how close solutions to Lichnerowicz equation are when arising from nearby free data.

1 Introduction

The observation that wormhole initial data (black hole data having two asymptotically flat ends) rapidly evolve to trumpet initial data (one asymptotically flat and one cylindrical end) (see [21], [20] and references therein) motivated the use of trumpet data to study numerical binary collisions since, in this way, the gauge evolution and the initial noise in wave quantities would be minimized. This then inspired an extensive study of initial data for Einstein equations having cylindrical ends, both from the numerical relativity community [4], [19], [24], [23], [17], [16] and the mathematical relativity side [15], [11], [27], [26], [9], [10]. There seems to be a close relation between the presence of a cylindrical end and certain extremality condition suggested in part, by the behavior of stationary solutions like Kerr-Newman and also by the fact [11] that given a mono-parametric family of conformally flat initial data having a wormhole structure, with given angular momentum and charges, then there exists a singular limit as the parameter goes to zero, where the asymptotic structure changes to trumpet-like and the angular momentum and charges are maximal for given mass. This reinforces the interest in studying initial data with cylindrical ends in an attempt to understand cosmic censorship issues, black hole formation, conical singularities appearing in stationary multi-black hole solutions, etc.

Initial data with more than one cylindrical end, *i.e.* representing many *extremal* black holes, are specially important. They include data for the Majumdar-Papapetrou solution [28], consisting of black holes of the extremal Reissner-Nordström type. It is the only static multi black hole solution of Einstein-Maxwell equations in equilibrium known to us. Moreover one would expect it to be the unique electro-vacuum, stationary solution with disconnected horizon. Nevertheless, the proof of this result and a complete analysis of its stability are lacking. Motivated by these open problems it is our aim here to understand perturbations of electromagnetic fields in initial data for Einstein-Maxwell with cylindrical ends. To us, this is a first step in the study of deformations of the full 4-dimensional Majumdar-Papapetrou solution.

In the past five years there has been increasing interest in developing the mathematical tools appropriate to deal with the problem of initial data with cylindrical ends. In [15] weighted Sobolev spaces were used to prove existence of an extremal solution with one cylindrical end as a special limit of Bowen-York initial data. This was generalized in [11] to conformally flat initial data. Then, Waxenegger *et al* [27] adapted the theorem of sub and super solution on weighted Hölder spaces to prove the same result without invoking the singular extremal limit. On the other hand, in [14], deformations of extreme Kerr black holes were studied. It was proven that for compactly supported perturbations, there exists a unique family of nearby initial data, that have the same horizon structure as extreme Kerr but greater ADM mass. For that result a specific property of extreme Kerr's metric was explicitly used in the proof. In more general terms Chrusciel *et al* [9] have studied solutions to Lichnerowicz equation. They proved existence of vacuum initial data with positive scalar curvature, having a number of asymptotically flat and cylindrical ends. This is an important existence result that extends previous ones by Choquet-Bruhat *et al* [6] to manifolds with cylindrical (or periodic or hyperbolic) ends. Uniqueness of solution however, has not been dealt with in [9] due mainly to the methods used there.

In this article we are interested in electro-vacuum initial data with an asymptotically flat end and one cylindrical end, representing the black hole horizon. We address the problem of how *close* solutions to Lichnerowicz equation are, when they arise from *close* free data. The idea is thus to consider two sets of free data for Lichnerowicz equation that are close in a certain norm, and analyze how close the corresponding initial data found from them are. We choose the free data sets as one being a deformation of the other one. This extends the result of [14] to more general, axially symmetric initial data for Einstein-Maxwell equations having the t-ϕ symmetry. Basically this extra symmetry gives a positivity condition (Yamabe positivity) that replaces the explicit property of Kerr used in [14]. Moreover, we also abandon the vacuum hypothesis, in view of our later study of the Majumdar-Papapetrou solution.

The manuscript is organized as follows: In section 2 we present Einstein constraints and describe in detail the hypotheses we work with, axial symmetry and time-rotation symmetry. We show how they lead to the Lichnerowicz equation and the asymptotic boundary conditions. We present our main result, Theorem 2.1 and discuss its scope and implications afterwards. In section 3 we present the proof of Theorem 2.1.

2 Main result

Consider a 3-dimensional surface $M = \mathbb{R}^3 \setminus \{0\}$. An initial data for the Einstein-Maxwell equations is a set $(M, g_{ij}, K_{ij}, E^i, B^i)$ where g_{ij} is the 3-metric on M, K_{ij} is the extrinsic curvature tensor and E^i, B^i are the electromagnetic fields on M. This set of fields satisfies the constraints on M:

$$R + K^2 - K_{ij}K^{ij} = 2(E_iE^i + B_iB^i) \tag{1}$$

$$D_jK_i^j - D_iK = -2\varepsilon_{ijk}E^jB^k \tag{2}$$

$$D_iE^i = 0, \qquad D_iB^i = 0 \tag{3}$$

where $K = K_{ij}g^{ij}$, D_i, R and ε_{ijk} are respectively the covariant derivative, the curvature scalar and the volume form associated to the metric g_{ij}. For simplicity, we will not consider electromagnetic currents in (2), *i.e.* $j_i = -2\varepsilon_{ijk}E^jB^k = 0$. This is a technical assumption to make the equations and calculations easier, but could be removed without altering the basic results of this article.

We will focus on initial data satisfying the above equations and the following three hypotheses:

H1. Axial symmetry. We consider axially symmetric initial data, namely, we assume that there exists a Killing vector field η tangential to M with complete closed orbits, such that $\mathcal{L}_\eta g_{ij} = 0$, $\mathcal{L}_\eta K_{ij} = 0$, $\mathcal{L}_\eta E^i = 0$, $\mathcal{L}_\eta B^i = 0$. In cylindrical coordinates (ρ, z, ϕ) we write $\eta^i = (\partial_\phi)^i$ and axial symmetry implies in particular that the fields above will not depend on ϕ. Moreover, we will see below that this assumption allows us to write the metric, extrinsic curvature and electromagnetic fields in a simple manner in terms of scalar potentials.

H2. Time-rotation symmetry. Besides axial symmetry we impose a discrete symmetry, namely, time-rotation symmetry. In terms of the initial data and the coordinates associated with the axial symmetry, this means that under the map $\phi \to -\phi$ the initial data map as (see Appendix A)

$$g_{ij} \to g_{ij}, \qquad K_{ij} \to -K_{ij} \tag{4}$$

and

$$E^i \to E^i, \qquad B^i \to B^i. \tag{5}$$

Initial data satisfying this symmetry turn out to be maximal and has been called "momentarily stationary", as this symmetry is to a stationary space-time what time-symmetry is to a static space-time [2], [22], [5]. In the treatment below it will be highlighted why we need this symmetry in order for our equations to be written in a particular form, without being too restrictive as to forbid the consideration of dynamical space-times.

It is important to remark that the time-rotation symmetry implies (see [5]) maximality $K = 0$ and moreover, due to the Hamiltonian constraint (1), also $R \geq 0$. This in turn means that (M, g_{ji}) satisfies the positivity condition

$$\int_M |\partial f|_g^2 + Rf^2 d\mu_g > 0 \tag{6}$$

for all $f \in C_0^\infty$, where ∂ denotes partial derivatives and the norm, curvature scalar and volume element $d\mu_g$ are taken with respect to g. For later purposes, we will say that (M, g_{ji}) satisfying (6) is in the positive Yamabe class $\mathcal{Y}^+$.

H3. Asymptotic structure. The manifold $M = \mathbb{R}^3 \setminus \{0\}$ has an asymptotically flat end and we take the origin to be a cylindrical end. This means [9] that the cylindrical end is identified with the product $\mathbb{R}^+ \times N$, where N is compact and the asymptotic metric is conformal (with bounded conformal factor) to

$$\hat{g} = dx^2 + h + O(e^{-ax}) \tag{7}$$

for a metric h on N and some positive constant a. Moreover we will restrict our study to the topologically spherical case $N = S^2$, and take h to be the standard metric on the unit sphere.

We will approach the constraint equations by using the Conformal Method, [6]. Consider the rescaling

$$g_{ij} = \Phi^4 \tilde{g}_{ij}, \quad K_{ij} = \Phi^{-2}\tilde{K}_{ij}, \quad ,E^i = \Phi^{-6}\tilde{E}^i, \quad B^i = \Phi^{-6}\tilde{B}^i \tag{8}$$

where $\Phi > 0$ and for simplicity, we take the conformal metric to be

$$\tilde{g}_{ij} = e^{2q}(d\rho^2 + dz^2) + \rho^2 d\phi^2 \tag{9}$$

where q does not depend on ϕ. This rescaled conformal metric $\tilde{g}_{ij}$ is not the most general axially symmetric metric satisfying (4) (see eq. (5) in [1]). Nevertheless, it is not difficult to see that the same procedure can be made for that more general metric.

Under this rescaling, the constraints read

$$\tilde{D}_i\tilde{D}^i\Phi = \frac{1}{8}\tilde{R}\Phi - \frac{\tilde{K}_{ij}\tilde{K}^{ij}}{8\Phi^7} - \frac{\tilde{E}_i\tilde{E}^i + \tilde{B}_i\tilde{B}^i}{4\Phi^3} \tag{10}$$

$$\tilde{D}_j\tilde{K}_i^j = 0, \quad \tilde{D}_i\tilde{E}^i = 0, \quad \tilde{D}_i\tilde{B}^i = 0. \tag{11}$$

Here $\tilde{R}$ is the curvature scalar associated to $\tilde{g}_{ij}$

In electro-vacuum and axial symmetry, the fact that M is simply connected implies [8] the existence of potentials ω, ψ and χ given by

$$\tilde{K}^{ij} = \frac{2}{\rho^2}\tilde{S}^{(i}\eta^{j)}, \qquad \tilde{S}^i = \frac{1}{2\rho^2}\tilde{\varepsilon}^{ijk}\eta_j\partial_k\omega, \tag{12}$$

$$\partial_i\chi = F_{ji}\eta^j, \qquad \partial_i\psi = *F_{ji}\eta^j, \tag{13}$$

such that the momentum and Maxwell constraints (11) are automatically satisfied (see [13] for a proof in the momentum case and the appendix B for the electromagnetic case). Here F_{ij} is the 4-dimensional electromagnetic tensor, which can be constructed in the standard way from E^i and B^i (115).

The values of the potentials ω, ψ and χ are constant on each connected component of the symmetry axis $\Gamma := \{\rho = 0\}$ and give the angular momentum J, electric charge Q_E and magnetic charge Q_B respectively [7]:

$$J = \frac{\omega_- - \omega_+}{8}, \qquad Q_E = \frac{\psi_- - \psi_+}{2}, \qquad Q_B = \frac{\chi_- - \chi_+}{2}, \tag{14}$$

where we denote by $\omega_+ := \omega(\rho = 0, z > 0)$, $\omega_- := \omega(\rho = 0, z < 0)$ the values of the function ω on the axis, at positive and negative values of z respectively. Analogous notation holds for the electromagnetic potentials.

In terms of these potentials, the symmetry conditions (4)-(5) translate into the following expressions appearing in the Hamiltonian constraint

$$\tilde{K}_{ij}\tilde{K}^{ij} = e^{-2q}\frac{(\partial\omega)^2}{2\rho^4}, \qquad \tilde{E}_i\tilde{E}^i = e^{-2q}\frac{(\partial\psi)^2}{\rho^2}, \qquad \tilde{B}_i\tilde{B}^i = e^{-2q}\frac{(\partial\chi)^2}{\rho^2}, \tag{15}$$

where the norms are taken with respect to the flat metric on $\mathbb{R}^3$. In general, for data not satisfying the symmetry conditions, a $\geq$ sign holds in the three equations in (15) (see Appendix A).

The scalar curvature in terms of the metric function q is given by

$$\tilde{R} = -2e^{-2q}\Delta_2 q \tag{16}$$

with $\Delta_2 = \partial_\rho^2 + \partial_z^2$.

With these variables, the only non-trivial equation left is the Hamiltonian constraint (1), which takes the form

$$\Delta\Phi = -\frac{\Delta_2 q}{4}\Phi - \frac{(\partial\omega)^2}{16\rho^4\Phi^7} - \frac{(\partial\psi)^2 + (\partial\chi)^2}{4\rho^2\Phi^3}, \tag{17}$$

where $\Delta = \partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2$.

This equation, known as the Lichnerowicz equation, is a non-linear equation for the conformal factor Φ. The set of functions $\mathcal{F} := (q, \omega, \psi, \chi)$ is known as *free data*, and can be freely prescribed, made to satisfy the asymptotic conditions appropriate for the problem at hand. Once Φ is found, we can construct the initial data as follows. From the prescribed function q we have the conformal metric $\tilde{g}_{ij}$ (9). With the obtained conformal factor we calculate the metric g_{ij}, and using the prescribed functions ψ and χ we can calculate E^i (113) and B^i (114). From ω we calculate $\tilde{K}_{ij}$ (12) and rescaling obtain K_{ij}. Therefore we finally have g_{ij}, K_{ij}, E^i and B^i.

Next we investigate the conditions that the functions Φ and q in the metric (9) must satisfy at the cylindrical end. We write g in spherical coordinates (r, θ, ϕ) and make the change $x = -\ln r$,

$$g = r^2\Phi^4[e^{2q}dx^2 + e^{2q}d\theta^2 + \sin^2\theta d\phi]. \tag{18}$$

Thus, by comparison with (7) we obtain that the conditions for the data on the cylindrical end $r \to 0$ are

$$\Phi = O(r^{-1/2}), \qquad q = O(1). \tag{19}$$

In virtue of equation (17) and the regularity near the symmetry axis (see [25]) we obtain conditions for the derivatives of the function q and the potentials on the cylindrical end

$$\Delta_2 q = O(r^{-2}) \tag{20}$$

$$|\partial\omega|^2 = O(r^{-2}\sin^6\theta), \quad |\partial\psi|^2 = O(r^{-2}\sin^2\theta), \quad |\partial\chi|^2 = O(r^{-2}\sin^2\theta). \tag{21}$$

Finally, recall that the Yamabe condition (6) is conformally invariant, and therefore if $\tilde{g}_{ij}$ is conformally related to g_{ij}, then $(M, \tilde{g}_{ij})$ also belongs to $\mathcal{Y}^+$, that is

$$\int_M |\partial f|^2_{\tilde{g}} + \tilde{R} f^2 d\mu_{\tilde{g}} > 0. \tag{22}$$

The question we want to address is the following. Consider two sets of free data $\mathcal{F}_0$ and $\mathcal{F}$ giving rise, through (17), to corresponding conformal factors Φ_0, Φ and thus, to initial data satisfying the hypothesis *H1-H3* above. If the free data are close in certain norm, how close are the data constructed from them? Clearly, this depends mainly on the relative size of the conformal factors. To study this problem we will think of $\mathcal{F}$ as a deformation of the set $\mathcal{F}_0$, then look for a solution to Lichnerowicz equation close to Φ_0 and finally, estimate its relative size.

Assume $(\Phi_0, q_0, \omega_0, \psi_0, \chi_0)$ satisfy (17). Let $|\lambda|$ be a sufficiently small number, take

$$q_0 \to q_0 + \lambda q, \quad \omega_0 \to \omega_0 + \lambda\omega, \quad \psi_0 \to \psi_0 + \lambda\psi, \quad \chi_0 \to \chi_0 + \lambda\chi \tag{23}$$

for appropriate axially symmetric functions q, ω, ψ, χ and write

$$\Phi_0 \to \Phi := \Phi_0 + u. \tag{24}$$

We demand the perturbed function $\Phi = \Phi_0 + u$ to satisfy Lichnerowicz equation (17) and write the resulting equation for u as

$$G(\lambda, u) = 0 \tag{25}$$

with

$$G(\lambda, u) = \Delta u + \frac{\Delta_2 q_0 u}{4} + \frac{\lambda}{4}\Delta_2 q(\Phi_0 + u) + \frac{(\partial\omega_0 + \lambda\partial\omega)^2}{16\rho^4(\Phi_0 + u)^7} - \frac{(\partial\omega_0)^2}{16\rho^4\Phi_0^7} + \\ + \frac{(\partial\psi_0 + \lambda\partial\psi)^2}{4\rho^2(\Phi_0 + u)^3} - \frac{(\partial\psi_0)^2}{4\rho^2\Phi_0^3} + \frac{(\partial\chi_0 + \lambda\partial\chi)^2}{4\rho^2(\Phi_0 + u)^3} - \frac{(\partial\chi_0)^2}{4\rho^2\Phi_0^3}. \tag{26}$$

Clearly, if $\lambda = 0$, we recover the Lichnerowicz equation for the background solution Φ_0 in the form

$$G(0, 0) = 0. \tag{27}$$

Our main result, presented in the next theorem proves that there exists a unique solution u of (25) close to the background $(0, 0)$ for each small enough λ.

The weighted Lebesgue spaces L'^2_δ [3], with weight $\delta \in \mathbb{R}$ are the spaces of measurable functions in $L^2_{loc}(\mathbb{R}^3 \setminus \{0\})$ such that the norms

$$\|u\|_{L'^2_\delta} = \left[\int_{\mathbb{R}^3\setminus\{0\}} |u|^2 r^{-2\delta-3}\right]^{1/2} \tag{28}$$

are finite. As usual the weighted Sobolev spaces H'^k_δ are defined with norms

$$\|u\|_{H'^k_\delta} = \sum_{j=0}^{k} \|D^j u\|_{L'^2_{\delta-j}}. \tag{29}$$

Theorem 2.1. *Let $q, \omega, \psi, \chi \in C_0^\infty(\mathbb{R}^3 \setminus \Gamma)$ be arbitrary smooth axially symmetric functions. Then, there is $\lambda_0 > 0$ such that for all $\lambda \in (-\lambda_0, \lambda_0)$ there exists a solution $u(\lambda) \in H'^2_{-1/2}$ of equation* (25). *The solution $u(\lambda)$ is continuously differentiable in λ and satisfies $\Phi_0 + u(\lambda) > 0$. Moreover, for small λ and small u (in the norm $H'^2_{-1/2}$) the solution $u(\lambda)$ is the unique solution of equation* (25).

Once u is found, we can re-construct the perturbed initial data in the same way as it was explained above.

With this construction, a different wording of Theorem 2.1 can be presented as follows: given initial data $\mathcal{D}^0 := (M, g^0_{ij}, K^0_{ij}, E^{0i}, B^{0i})$ that satisfy hypotheses *H1-H3*, with angular momentum J and electromagnetic charges Q_E and Q_B, there exists a mono-parametric family of initial data sets $(M, g_{ij}(\lambda), K_{ij}(\lambda), E^i(\lambda), B^i(\lambda))$, unique for each λ sufficiently close to zero such that

(i) $g_{ij}(0) = g^0_{ij}$, $K_{ij}(0) = K^0_{ij}$, $E^i(0) = E^{0i}$, $B^i(0) = B^{0i}$. The family is differentiable in λ and it is close to $\mathcal{D}^0$ with respect to an appropriate norm which involves two derivatives.

(ii) The data has the same asymptotic geometry as $\mathcal{D}^0$. The angular momentum, charges and the area of the cylindrical end in the family do not depend on λ, they have the same value as in $\mathcal{D}^0$.

(iii) The data are axially symmetric and time-rotational symmetric.

The λ-dependent initial data is to be constructed from the given functions (23) and the solution $u(\lambda)$ to equation (25). That is, one must solve equations (12)-(13) for $\tilde{K}_{ij}(\lambda)$ and $F_{ij}(\lambda)$ and use (9) to obtain the metric in terms of λ.

Before going to the proof of Theorem 2.1, we want to make some remarks.

- Several known black hole solutions fit into the hypotheses described above. In particular the extreme Bowen-York initial data built in [15], the $\{t = 0\}$ slice in extreme Kerr and extreme Reissner-Nordström black holes. As we explained in the introduction, the results of [14] are of course included in Theorem 2.1 for the vacuum, extreme Kerr case. Moreover, a $\{t = 0\}$ slice of the axially symmetric Majumdar-Papapetrou solution also satisfies the hypotheses, but contains many cylindrical ends (as many as black holes are described). This case is of particular interest for us and will be dealt with in a subsequent paper. The main difficulty that the many ends bring into the problem is the appropriate choice and treatment of the Sobolev spaces involved.

- The method of proof we use not only gives us existence but also *uniqueness* of solution for each λ. Moreover we also obtain an estimate on the perturbed conformal factor in terms of the background one $0 \leq \sqrt{r}\Phi \leq \max(\sqrt{r}\Phi_0) + C_1\sqrt{C_2}/2$ where C_1, C_2 are constants depending on Φ_0 (see eq's. (36), (38)). In turn, this estimation on the conformal factor and the size of λ allows us to control how different the initial data $(M, g_{ij}, K_{ij}, E^i, B^i)$ and $(M, g^0_{ij}, K^0_{ij}, E^{0i}, B^{0i})$ are.

- Axial symmetry is required to define in a well manner the angular momentum of the initial data. Time-rotation symmetry is used to simplify the analysis of the constraint equations as it gives an explicit and simple relation between the fields K_{ij}, E^i, B^i and the potentials. We believe the most important ingredient for our purposes that we obtain from this symmetry is the Yamabe positivity.

- In [14] it was shown that the weighted Sobolev space $H'^2_{-1/2}$ is specially appropriate for the study of small perturbations of solutions to Lichnerowicz equation with a cylindrical end. By small here we mean that the structure of the cylindrical end is unchanged by the perturbation (see [14] for details). This is due to the fact that the background function satisfies $\Phi_0 = O(r^{-1/2})$ asymptotically at the cylindrical end. The perturbation is not meant to change the asymptotic structure of the end, which translates to $u = o(r^{-1/2})$ at the end. This behavior is captured by the $-1/2$ weight in the Sobolev space.

- The compact support away from the symmetry axis of the metric function q is required by the regularity desired on the metric $\tilde{g}$, this guarantees that there will not be a conical singularity on the axis. On the other hand, in view of (14), the compact support of ω, ψ, χ implies that there is no change in the angular momentum and charges of the data. Moreover, the whole horizon structure remains unaltered, in particular the horizon area will be the same as in the background. This can be seen as follows. The horizon area is computed as

$$A = \lim_{r\to 0} \int_{B_r} ds_g \tag{30}$$

where B_r is a coordinate ball of radius r and ds_g is the area element with respect to the metric g_{ij}. This integral can be written as

$$A(\lambda) = \lim_{r\to 0} \int_{B_r} \Phi^4 r^2 e^{q_0+\lambda q} \sin\theta d\theta d\phi = \lim_{r\to 0} \int_{B_r} (\Phi_0 + u)^4 r^2 e^{q_0+\lambda q} \sin\theta d\theta d\phi \tag{31}$$

and using the boundary conditions (19) and $u = o(r^{-1/2})$ we find $A = A_0$.

If one wants to alter, say, the angular momentum, then ω must have a precise asymptotic behavior at $r \to 0$ and the axis $\rho = 0$. We expect that a different treatment will be necessary to deal with this case as it is likely that the perturbed solution $\Phi_0 + u$ will no longer have the same asymptotic behavior, resulting probably in a different character for the end (changing from asymptotically cylindrical to asymptotically flat or giving rise to a naked singularity).

- The condition of positive Yamabe for the background data (M, g_{ij}), (6) does not imply a non-negative conformal scalar curvature, $\tilde{R}_0 \geq 0$ as is assumed in [9]. That is, $\tilde{R}_0$ can attain positive, negative and zero values, we only know that inequality (22) is satisfied. However, if on top of Yamabe positivity, we assume $\tilde{R}_0 \geq 0$, we can estimate how small the deformation parameter λ needs to be in order to guarantee the existence of a new solution. We find

$$\lambda \leq -\frac{\Delta_2 q_0}{\Delta_2 q}. \tag{32}$$

This condition arises from equation (16) and the results in [9]. Note that this does not depend on the size of the ω, ψ, χ functions, but only on the perturbation function q.

3 Proof of main result

Sketch of the proof

The proof uses the Implicit Function theorem to show that there exists a unique solution to (25). We first investigate the appropriate functional spaces where we expect to find the solution. Then prove that the operator G is well defined and continuously differentiable on these spaces. Finally, we prove that the associated linear operator DG is an isomorphism. In this last step we use the Riesz Representation Theorem to find a weak solution and then, a regularity theorem to prove that the weak solution is a strong solution. The Yamabe condition (22) plays a key role in the last parts of the proof, as it serves as the coercivity condition needed for the application of Riesz Theorem.

In this section we use several constants whose exact value is not relevant, we denote them by C_i.

Sobolev spaces and neighborhoods used

We will work with the only non-trivial constraint equation written as (25) and look for a solution u. In [14] the authors deal with an analogous map G, and choose G :

$\mathbb{R}\times H'^2_{-1/2} \to L'^2_{-5/2}$ considering the fall off behavior of the functions involved. As in our case the asymptotic behavior is the same, we choose the same function spaces.

We are considering the map $G : \mathbb{R}\times H'^2_{-1/2} \to L'^2_{-5/2}$, but for a general $u \in H'^2_{-1/2}$ the function $\Phi = \Phi_0 + u$ does not have a definite sign. In order for Φ to be a proper conformal factor we need it to be positive. As we take $\Phi_0 > 0$, then we can conjecture that if u is small enough, then Φ is also going to be positive. There are some subtleties in the problem at hand, as we have a particular behavior at the cylindrical end. Even so, it is possible to prove the conjecture, that is, to show that that there is a neighborhood V of 0 in $H'^2_{-1/2}$ such that

$$\Phi_0 + u > 0. \tag{33}$$

We start by noting that as Φ_0 is a proper conformal factor, then it is positive and bounded away from zero if we remove a neighborhood of the cylindrical end. Approaching the cylindrical end, $\Phi_0 \to r^{-1/2}$ as $r \to 0$, thus we can conclude that there are positive constants C_1, C_2, C_3 and C_4 such that

$$C_1\sqrt{r+C_2} \leq \sqrt{r}\Phi_0 \leq C_3\sqrt{r+C_4}. \tag{34}$$

The argument in [14] carries through. Consider the open ball of radius ξ around the origin in $H'^2_{-1/2}$,

$$V = \{v \in H'^2_{-1/2} : ||v||_{H'^2_{-1/2}} < \xi\}, \tag{35}$$

where $\xi > 0$ is yet to be defined. From Lemma A.1 in [14] we have that if $\|u\|_{H'^2_{-1/2}} < \xi$ then there is a constant C such that

$$\sqrt{r}|u| \leq C\xi. \tag{36}$$

Given Φ_0 satisfying (34) we find

$$\sqrt{r}(\Phi_0 + u) \geq C_1\sqrt{C_2} - C\xi =: C_5, \tag{37}$$

and if we choose ξ such that

$$0 < \xi < \frac{C_1\sqrt{C_2}}{2C} \tag{38}$$

then

$$C_5 > \frac{C_1\sqrt{C_2}}{2} > 0, \tag{39}$$

and therefore

$$\Phi_0 + u > 0. \tag{40}$$

From now on ξ and V are fixed. The factor $1/2$ in the r.h.s. of (38) is a technical requirement needed later to perform some bounds.

The map $G:\mathbb{R}\times V\to L'^2_{-5/2}$ is well defined

To prove that $G:\mathbb{R}\times V\to L'^2_{-5/2}$ is a well defined map we evaluate $\|G\|_{L'^2_{-5/2}}$,

$$\|G(\lambda,u)\|_{L'^2_{-5/2}} \leq \|\Delta u\|_{L'^2_{-5/2}} + \tag{41}$$

$$+ \left\|\frac{\lambda\Delta_2 q}{4}(\Phi_0+u)\right\|_{L'^2_{-5/2}} + \left\|\frac{\lambda\partial\omega(2\partial\omega_0+\lambda\partial\omega)}{16\rho^4(\Phi_0+u)^7}\right\|_{L'^2_{-5/2}} + \tag{42}$$

$$+ \left\|\frac{\lambda\partial\psi(2\partial\psi_0+\lambda\partial\psi)}{4\rho^2(\Phi_0+u)^3}\right\|_{L'^2_{-5/2}} + \left\|\frac{\lambda\partial\chi(2\partial\chi_0+\lambda\partial\chi)}{4\rho^2(\Phi_0+u)^3}\right\|_{L'^2_{-5/2}} + \tag{43}$$

$$+ \left\|\frac{\Delta_2 q_0}{4}u\right\|_{L'^2_{-5/2}} + \left\|\frac{(\partial\omega_0)^2}{16\rho^4}\left[\frac{1}{(\Phi_0+u)^7}-\frac{1}{\Phi_0^7}\right]\right\|_{L'^2_{-5/2}} + \tag{44}$$

$$+ \left\|\frac{(\partial\psi_0)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u)^3}-\frac{1}{\Phi_0^3}\right]\right\|_{L'^2_{-5/2}} + \left\|\frac{(\partial\chi_0)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u)^3}-\frac{1}{\Phi_0^3}\right]\right\|_{L'^2_{-5/2}} \tag{45}$$

The term on the r.h.s. of (41) is bounded by the definition of the $H'^2_{-1/2}$ norm. The terms in (42) and (43) are bounded due to the compact support of q, ω, ψ and χ respectively. The first term in (44) is bounded due to $u\in H'^2_{-1/2}$ and the behavior of q_0 given in (20). The remaining three norms are bounded due to the asymptotic conditions on the background functions (21) together with the inequalities (34) and (37). This can be seen as follows. Use the identity

$$\frac{1}{a^p}-\frac{1}{b^p}=(b-a)\sum_{i=0}^{p-1}a^{i-p}b^{-1-i} \tag{46}$$

to write

$$\frac{1}{\Phi_0^p}-\frac{1}{(\Phi_0+u)^p}=r^{(p+1)/2}uH, \tag{47}$$

where

$$H=\sum_{i=0}^{p-1}[\sqrt{r}(\Phi_0+u)]^{i-p}[\sqrt{r}\Phi_0]^{-1-i}. \tag{48}$$

Using (34) and (37) we see that

$$H\leq C_6, \tag{49}$$

where C_6 is a constant that only depends on previous constants. Using the conditions (21) we can bound for instance

$$\left\|\frac{(\partial\psi_0)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u)^3}-\frac{1}{\Phi_0^3}\right]\right\|_{L'^2_{-5/2}} \leq C_7\left\|\frac{r^{-2}\sin^2\theta}{4\rho^2}(r^2uH)\right\|_{L'^2_{-5/2}} \tag{50}$$

$$=\frac{C_7C_6}{4}\left\|\frac{u}{r^2}\right\|_{L'^2_{-5/2}} \leq \frac{C_7C_6}{4}\|u\|_{L'^2_{-1/2}} \leq \frac{C_7C_6}{4}\|u\|_{H'^2_{-1/2}}. \tag{51}$$

Applying the same argument to the other terms involving ω_0 and χ_0 completes the proof that $||G(\lambda,u)||_{L'^2_{-5/2}}$ is bounded and therefore the map is well-defined.

The map G is continuously differentiable

We now prove that G is differentiable. To propose candidates for $D_1G(\lambda,u)$ and $D_2G(\lambda,u)$ we calculate the directional derivatives

$$D_1G(\lambda,u)[\gamma] := \frac{d}{dt}G(\lambda+t\gamma,u)\bigg|_{t=0}, \tag{52}$$

$$D_2G(\lambda,u)[v] := \frac{d}{dt}G(\lambda,u+tv)\bigg|_{t=0}, \tag{53}$$

obtaining

$$\begin{aligned} D_1G(\lambda,u)[\gamma] &= \left[\frac{\partial\omega(\partial\omega_0+\lambda\partial\omega)}{8\rho^4(\Phi_0+u)^7}+\frac{\Delta_2 q}{4}(\Phi_0+u)\right. \\ &\quad \left.+\frac{\partial\psi(\partial\psi_0+\lambda\partial\psi)}{2\rho^2(\Phi_0+u)^3}+\frac{\partial\chi(\partial\chi_0+\lambda\partial\chi)}{2\rho^2(\Phi_0+u)^3}\right]\gamma, \end{aligned} \tag{54}$$

$$\begin{aligned} D_2G(\lambda,u)[v] &= \Delta v+\left[-\frac{7(\partial\omega_0+\lambda\partial\omega)^2}{16\rho^4(\Phi_0+u)^8}+\frac{\Delta_2 q_0+\lambda\Delta_2 q}{4}\right. \\ &\quad \left.-\frac{3(\partial\psi_0+\lambda\partial\psi)^2}{4\rho^2(\Phi_0+u)^4}-\frac{3(\partial\chi_0+\lambda\partial\chi)^2}{4\rho^2(\Phi_0+u)^4}\right]v. \end{aligned} \tag{55}$$

We show that the operators are bounded. The $L'^2_{-5/2}$ norm of each term inside square brackets in (54) is bounded due to compact support, the conditions (21) and the inequality (37), then

$$\|D_1G(\lambda,u)[\gamma]\|_{L'^2_{-5/2}} \le C_8|\gamma|. \tag{56}$$

For the second operator the proof is a bit more tricky. We have

$$\begin{aligned} \|D_2G(\lambda,u)[v]\|_{L'^2_{-5/2}} &\le \|\Delta v\|_{L'^2_{-5/2}}+\left\|\left[-\frac{7\lambda\partial\omega(2\partial\omega_0+\lambda\partial\omega)}{16\rho^4(\Phi_0+u)^8}-\frac{7(\partial\omega_0)^2}{16\rho^4(\Phi_0+u)^8}+\right.\right. \\ &+ \frac{\Delta_2(q_0+\lambda q)}{4}-\frac{3\lambda\partial\psi(2\partial\psi_0+\lambda\partial\psi)}{4\rho^2(\Phi_0+u)^4}-\frac{3(\partial\psi_0)^2}{4\rho^2(\Phi_0+u)^4}- \\ &- \left.\left.\frac{3\lambda\partial\chi(2\partial\chi_0+\lambda\partial\chi)}{4\rho^2(\Phi_0+u)^4}-\frac{3(\partial\chi_0)^2}{4\rho^2(\Phi_0+u)^4}\right]v\right\|_{L'^2_{-5/2}} \end{aligned} \tag{57}$$

$$\begin{aligned} &= \|\Delta v\|_{L'^2_{-5/2}}+\left\|\left[-\frac{7r^2\lambda\partial\omega(2\partial\omega_0+\lambda\partial\omega)}{16\rho^4(\Phi_0+u)^8}-\frac{7r^2(\partial\omega_0)^2}{16\rho^4(\Phi_0+u)^8}+\right.\right. \\ &\quad +\frac{r^2\Delta_2(q_0+\lambda q)}{4}-\frac{3r^2\lambda\partial\psi(2\partial\psi_0+\lambda\partial\psi)}{4\rho^2(\Phi_0+u)^4}-\frac{3r^2(\partial\psi_0)^2}{4\rho^2(\Phi_0+u)^4}- \\ &\quad -\left.\left.\frac{3r^2\lambda\partial\chi(2\partial\chi_0+\lambda\partial\chi)}{4\rho^2(\Phi_0+u)^4}-\frac{3r^2(\partial\chi_0)^2}{4\rho^2(\Phi_0+u)^4}\right]\frac{v}{r^2}\right\|_{L'^2_{-5/2}} \end{aligned} \tag{58}$$

$$\le \|v\|_{H'^2_{-1/2}}+C_9\|v\|_{L'^2_{-1/2}} \le C_{10}\|v\|_{H'^2_{-5/2}}, \tag{59}$$

where again we have used that $v \in H'^2_{-1/2}$, equations (37), (20) and (21) and the compact support of ω, q, ψ and χ. This proves that the operators $D_1 G$ and $D_2 G$ are bounded.

To show that $D_1 G$ is the partial Fréchet derivative we calculate

$$
\begin{aligned}
& G(\lambda+\gamma,u) - G(\lambda,u) - D_1 G(\lambda,u)[\gamma] && (60)\\
&= \left[\frac{(\partial\omega)^2}{16\rho^4(\Phi_0+u)^7} + \frac{(\partial\psi)^2}{4\rho^2(\Phi_0+u)^3} + \frac{(\partial\chi)^2}{4\rho^2(\Phi_0+u)^3}\right]\gamma^2, && (61)
\end{aligned}
$$

and as ω, ψ and χ have compact support

$$
\|G(\lambda+\gamma,u) - G(\lambda,u) - D_1 G(\lambda,u)[\gamma]\|_{L'^2_{-5/2}} \leq C_{11}|\gamma|^2, \tag{62}
$$

which shows that

$$
\lim_{\gamma\to 0} \frac{\|G(\lambda+\gamma,u) - G(\lambda,u) - D_1 G(\lambda,u)[\gamma]\|_{L'^2_{-5/2}}}{|\gamma|} = 0. \tag{63}
$$

For $D_2 G$ we have

$$
\begin{aligned}
& G(\lambda,u+v) - G(\lambda,u) - D_2 G(\lambda,u)[v] && (64)\\
&= \frac{(\partial\omega_0+\lambda\partial\omega)^2}{16\rho^4}\left[\frac{1}{(\Phi_0+u+v)^7} - \frac{1}{(\Phi_0+u)^7} + \frac{7v}{(\Phi_0+u)^8}\right] && (65)\\
&+ \frac{(\partial\psi_0+\lambda\partial\psi)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u+v)^3} - \frac{1}{(\Phi_0+u)^3} + \frac{3v}{(\Phi_0+u)^4}\right] && (66)\\
&+ \frac{(\partial\chi_0+\lambda\partial\chi)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u+v)^3} - \frac{1}{(\Phi_0+u)^3} + \frac{3v}{(\Phi_0+u)^4}\right] && (67)\\
&= \frac{(\partial\omega_0+\lambda\partial\omega)^2}{16\rho^4} r^{\frac{9}{2}} v^2 H_1 + \left(\frac{(\partial\psi_0+\lambda\partial\psi)^2}{4\rho^2} + \frac{(\partial\chi_0+\lambda\partial\chi)^2}{4\rho^2}\right) r^{\frac{5}{2}} v^2 H_2 && (68)\\
&= \left[\frac{\lambda\partial\omega(2\partial\omega_0+\lambda\partial\omega)}{16\rho^4} r^6 H_1 + \frac{(\partial\omega_0)^2}{16\rho^4} r^6 H_1 + \right. && (69)\\
&+ \frac{\lambda\partial\psi(2\partial\psi_0+\lambda\partial\psi)}{4\rho^2} r^4 H_2 + \frac{(\partial\psi_0)^2}{4\rho^2} r^4 H_2 && (70)\\
&+ \left. \frac{\lambda\partial\chi(2\partial\chi_0+\lambda\partial\chi)}{4\rho^2} r^4 H_2 + \frac{(\partial\chi_0)^2}{4\rho^2} r^4 H_2 \right] \frac{v^2}{r^{\frac{3}{2}}} && (71)
\end{aligned}
$$

where H_1 is as in [14] and satisfies $|H_1| < C_{12}$ and H_2 is given by

$$
H_2 = \frac{1}{[\sqrt{r}(\Phi_0+u+v)]^3[\sqrt{r}(\Phi_0+u)]^4} \sum_{i=0}^{2} C_i[\sqrt{r}(\Phi_0+u)]^{2-i}(\sqrt{r}v)^i, \tag{72}
$$

with C_i numerical constants and satisfy

$$
|H_2| \leq \frac{1}{(C_1\sqrt{r+C_2} - 2C\xi)^7} \sum_{i=0}^{2} |C_i|(C_3\sqrt{r+C_4} + C\xi)^{2-i}(C\xi)^i \leq C_{13}. \tag{73}
$$

Using that ω, ψ and χ have compact support

$$\|G(\lambda,u+v)-G(\lambda,u)-D_2G(\lambda,u)[v]\|_{L'^2_{-5/2}} \leq C_{14}\left\|\frac{v^2}{r^{\frac{3}{2}}}\right\|_{L'^2_{-5/2}} \leq C_{15}\|v\|^2_{H'^2_{-1/2}}, \tag{74}$$

where the last inequality has been calculated in [14]. This proves that D_2G is the Fréchet partial derivative.

The next step is to prove continuity of the derivatives. We compute [1].

$$\|D_1G(\lambda_1,u)[\gamma]-D_1G(\lambda_2,u)[\gamma]\|_{L'^2_{-5/2}} \tag{75}$$

$$= \left\|\left[\frac{(\partial\omega)^2}{8\rho^4(\Phi_0+u)^7}+\frac{(\partial\psi)^2}{2\rho^2(\Phi_0+u)^3}+\frac{(\partial\chi)^2}{2\rho^2(\Phi_0+u)^3}\right]\gamma(\lambda_1-\lambda_2)\right\|_{L'^2_{-5/2}} \tag{76}$$

$$\leq \left[\left\|\frac{(\partial\omega)^2}{8\rho^4(\Phi_0+u)^7}\right\|_{L'^2_{-5/2}}+\left\|\frac{(\partial\psi)^2}{2\rho^2(\Phi_0+u)^3}+\right\|_{L'^2_{-5/2}}\right. \tag{77}$$

$$\left.+\left\|\frac{(\partial\chi)^2}{2\rho^2(\Phi_0+u)^3}+\right\|_{L'^2_{-5/2}}\right]|\gamma|\,|\lambda_1-\lambda_2| \tag{78}$$

$$\leq C_{16}|\gamma|\,|\lambda_1-\lambda_2|, \tag{79}$$

[1]Note that eq (55) in [14] has a typo. It should be a D_2 derivative

where again we used compact support and the bound (37). We also compute

$$
\begin{aligned}
&\|D_2G(\lambda,u_1)[v]-D_2G(\lambda,u_2)[v]\|_{L'^2_{-5/2}}\\
&=\left\|\left\{\frac{7(\partial\omega_0+\lambda\partial\omega)^2}{16\rho^4}\left[\frac{1}{(\Phi_0+u_2)^8}-\frac{1}{(\Phi_0+u_1)^8}\right]\right.\right.\\
&+\frac{3(\partial\psi_0+\lambda\partial\psi)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u_2)^4}-\frac{1}{(\Phi_0+u_1)^4}\right]+\\
&\left.\left.+\frac{3(\partial\chi_0+\lambda\partial\chi)^2}{4\rho^2}\left[\frac{1}{(\Phi_0+u_2)^4}-\frac{1}{(\Phi_0+u_1)^4}\right]\right\}v\right\|_{L'^2_{-5/2}} &(80)\\
&=\left\|\left\{\frac{7(\partial\omega_0+\lambda\partial\omega)^2}{16\rho^4}r^{9/2}H_3+\frac{3(\partial\psi_0+\lambda\partial\psi)^2}{4\rho^2}r^{5/2}H_4\right.\right.\\
&\left.\left.+\frac{3(\partial\chi_0+\lambda\partial\chi)^2}{4\rho^2}r^{5/2}H_4\right\}v(u_2-u_1)\right\|_{L'^2_{-5/2}} &(81)\\
&=\left\|\left\{\frac{7\lambda\partial\omega(2\partial\omega_0+\lambda\partial\omega)}{16\rho^4}r^6H_3+\frac{7(\partial\omega_0)^2}{16\rho^4}r^6H_3+\right.\right.\\
&\frac{3\lambda\partial\psi(2\partial\psi_0+\lambda\partial\psi)}{4\rho^2}r^4H_4+\frac{3(\partial\psi_0)^2}{4\rho^2}r^4H_4+\\
&\left.\left.\frac{3\lambda\partial\chi(2\partial\chi_0+\lambda\partial\chi)}{4\rho^2}r^4H_4+\frac{3(\partial\chi_0)^2}{4\rho^2}r^4H_4\right\}\frac{v(u_2-u_1)}{r^{3/2}}\right\|_{L'^2_{-5/2}} &(82)\\
&\le C_{17}\left\|\frac{v(u_2-u_1)}{r^{\frac{3}{2}}}\right\|_{L'^2_{-5/2}}\le C_{18}\|v\|_{H'^2_{-1/2}}\|u_1-u_2\|_{H'^2_{-1/2}}, &(83)
\end{aligned}
$$

where to go from (80) to (81) we used

$$r^{-9/2}\left(\frac{1}{(\Phi_0+u_1)^8}-\frac{1}{(\Phi_0+u_2)^8}\right)=(u_2-u_1)H_3 \tag{84}$$

and

$$r^{-5/2}\left(\frac{1}{(\Phi_0+u_1)^4}-\frac{1}{(\Phi_0+u_2)^4}\right)=(u_2-u_1)H_4 \tag{85}$$

with

$$H_3:=\sum_{i=0}^{7}(\sqrt{r}(\Phi_0+u_1))^{i-8}(\sqrt{r}(\Phi_0+u_2))^{-1-i}, \tag{86}$$

$$H_4:=\sum_{i=0}^{3}(\sqrt{r}(\Phi_0+u_1))^{i-4}(\sqrt{r}(\Phi_0+u_2))^{-1-i}. \tag{87}$$

Lines (82) are merely a convenient re-writing of (81). To go from (82) to (83) we use the asymptotic conditions on the background functions (21), that ω, ψ and χ have compact support, the bounds

$$|H_3|\le C_{19},\qquad |H_4|\le C_{20} \tag{88}$$

and combined all the constants into C_{17}. Then we have

$$\|D_2G(\lambda,u_1)[v]-D_2G(\lambda,u_2)[v]\|_{L'^2_{-5/2}} \le C_6\|v\|_{H'^2_{-1/2}}\|u_1-u_2\|_{H'^2_{-1/2}} \tag{89}$$

proving that the derivative operator (55) is also continuous.

The map $D_2G(0,0): H'^2_{-1/2} \to L'^2_{-1/2}$ is an isomorphism

Finally, we need to prove that $\mathcal{L} := -D_2G(0,0): H'^2_{-1/2} \to L'^2_{-1/2}$ is an isomorphism. As in [14], we can write

$$D_2G(0,0)[v] = \Delta v - \alpha v, \tag{90}$$

$$\alpha = -\frac{\Delta_2 q_0}{4} + 7\frac{(\partial\omega_0)^2}{16\rho^4\Phi_0^8} + 3\frac{(\partial\psi_0)^2+(\partial\chi_0)^2}{4\rho^2\Phi_0^4}, \tag{91}$$

which can be written as

$$\alpha = hr^{-2} \tag{92}$$

and h is a bounded function in $\mathbb{R}^3$ with $h \in L^2(M)$. In [11] it was proven that when h is positive, the operator (90) is an isomorphism. In general, due to the first term in (91), α is not necessarily positive. However, here is where the Yamabe positivity condition plays a role. We have the following important result.

Lemma 3.1. *Let $(M,\tilde{g}_{ij})$ be in the positive Yamabe class, namely*

$$\int_M |\partial f|^2_{\tilde{g}} + \tilde{R} f^2 d\mu_g > 0 \tag{93}$$

for all $f \in C_0^\infty$, $f \neq 0$, then

$$\int_M |\partial f|^2 + \alpha f^2 d\mu > 0 \tag{94}$$

where α is given in (91) *and the norm and volume element in* (94) *are computed with respect to the flat metric.*

Proof We start with the left hand side of (94)

$$\int_M \left[|\partial f|^2 + \alpha f^2\right] d\mu = \tag{95}$$

$$= \int_M \left[|\partial f|^2 - \frac{\Delta_2 q_0}{4} f^2\right] d\mu + \int_M \left(7\frac{(\partial\omega_0)^2}{16\rho^4\Phi_0^8} + 3\frac{(\partial\psi_0)^2+(\partial\chi_0)^2}{4\rho^2\Phi_0^4}\right) f^2 d\mu \geq \tag{96}$$

$$\geq \int_M \left[e^{-2q_0}|\partial f|^2 - 2e^{-2q_0}\frac{\Delta_2 q_0}{8} f^2\right] e^{2q_0} d\mu = \tag{97}$$

$$= \int_M \left[|\partial f|^2_{\tilde{g}} + \tilde{R} f^2\right] d\mu_{\tilde{g}} > 0, \tag{98}$$

which proves the claim. Note that in order to go from (97) to (98) we have used the background metric $\tilde{g} = e^{2q}(dr^2 + r^2 d\theta^2) + r^2\sin^2\theta d\phi^2$.

□

Theorem 3.2. *The linear map $\mathcal{L}$ defined by*

$$\mathcal{L}u := -\Delta u + \alpha u = f \qquad \text{in } \mathbb{R}^3 \setminus \{0\}, \tag{99}$$

where α is given by (91)-(92) *and satisfies* (94)*, is an isomorphism $H'^2_{-1/2} \to L'^2_{-5/2}$.*

The proof of this result will be given below and departs slightly from [14] because we exploit the symmetry of the weak problem associated to (99) to apply the Riesz Representation theorem instead of Lax-Milgram theorem used in [11]. This is important as we will no longer need to prove the coercivity condition.

We first prove the existence of a weak solution (Lemma 3.3) and then we find it to be regular.

Lemma 3.3. *There exists a unique weak solution* $u \in H'^{1}_{-1/2}$ *of* (99) *for each* $f \in L'^{2}_{-5/2}$.

Proof. For $u, v \in H'^{1}_{-1/2}$, define the bilinear form

$$B[u,v] := \int_{\mathbb{R}^3} \partial u \cdot \partial v + \alpha uv \, d\mu \tag{100}$$

which corresponds to the linear operator $\mathcal{L}$.

Let us check that $B[\ ,\ ]$ satisfies the hypotheses of Riesz Representation theorem (see [18]). We first need to prove that the $B[u,v]$ can be taken as an inner product on $H'^{1}_{-1/2} \times H'^{1}_{-1/2}$. By the Yamabe condition we know that for all $u \not\equiv 0$, $B[u,u] > 0$ and also, by definition, if $u \equiv 0$, then $B[u,u] = 0$. Therefore, the bilinear form is positive definite. Second, it can easily be proven that $B[u,v] = B[v,u]$ and that $B[u, av + cw] = aB[u,v] + bB[u,w]$. Therefore, $B[u,v]$ is an inner product. Next we need to prove that the linear functional $\ell(\cdot) := B[\cdot, v]$ is bounded for all $v \in H'^{1}_{-1/2}$. This is done exactly as in [11]

$$\begin{aligned}
|B[u,v]| &\leq \left| \int \partial u \cdot \partial v d\mu \right| + \left| \int \alpha uv d\mu \right| &(101)\\
&\leq |\partial u|_{L^2} |\partial v|_{L^2} + C|ur^{-1}|_{L^2} |ur^{-1}|_{L^2} &(102)\\
&\leq |\partial u|_{L^2} |\partial v|_{L^2} + C|u|_{L'^2_{-1/2}} |u|_{L'^2_{-1/2}} &(103)\\
&\leq \max\{1, C\} |u|_{H'^1_{-1/2}} |u|_{H'^1_{-1/2}}. &(104)
\end{aligned}$$

Then with these conditions fulfilled, Riesz Representation Theorem states that there exists a unique $u \in H'^{1}_{-1/2}$ such that

$$B[u,v] = \langle f, v \rangle, \qquad \forall v \in H'^{1}_{-1/2}, \tag{105}$$

that is, such that

$$\int_{\mathbb{R}^3} (\mathcal{L}u - f) v dx = 0, \qquad \forall v \in H'^{1}_{-1/2}. \tag{106}$$

Therefore u is the unique weak solution of $\mathcal{L}u = f$. □

Next, we use Lemma A.3 in [11] to prove regularity of solution, namely

Lemma A.3 in [11]. *Let* $f \in L'^{2}_{-5/2}$. *Assume* $u \in H'^{1}_{-1/2}$ *is a weak solution of* $\mathcal{L}u = f$. *Then* $u \in H'^{2}_{-1/2}$.

These two lemmas show that there exists a unique function $u \in H'^{2}_{-1/2}$ which solves equation $-\Delta u + \alpha u = f$ a.e, for each $f \in L'^{2}_{-5/2}$. This, in turn, means that $\mathcal{L} := -\Delta + \alpha$ is an isomorphism $H'^{2}_{-1/2} \to L'^{2}_{-5/2}$, proving Theorem 3.2.

A Time-rotation symmetry

An axially symmetric initial data $(M, g_{ij}, K_{ij}, E^i, B^i)$ has the time-rotation symmetry if, in the coordinates associated with the axial symmetry, under the map $\phi \to -\phi$ the initial data map as

$$g_{ij} \to g_{ij}, \qquad K_{ij} \to -K_{ij}, \qquad E^i \to E^i, \qquad B^i \to B^i. \tag{107}$$

This symmetry on the level of the initial data implies that the development is invariant under the transformation $(t, \phi) \to (-t, -\phi)$ (see [2], [22], [5]).

Using the symmetries it can be concluded that (see [8], [13])

$$\tilde{K}^{ij}\tilde{K}_{ij} = \frac{|D\omega|^2_{\tilde{g}}}{2|\eta|^4_{\tilde{g}}} = e^{-2q}\frac{(\partial\omega)^2}{2\rho^4}. \tag{108}$$

We consider now the electric and magnetic fields. Since the initial data is axially symmetric, the components of the fields in (ρ, z, ϕ) coordinates do not depend in the ϕ coordinate. This means in particular that

$$E^3(\rho, z, -\phi) = E^3(\rho, z, \phi).$$

On the other hand, the discrete symmetry $\phi \to -\phi$ on the initial data implies

$$E^3(\rho, z, -\phi) = -E^3(\rho, z, \phi),$$

and therefore

$$E^3(\rho, z, \phi) = 0. \tag{109}$$

Taking into account that

$$\eta = d\phi, \tag{110}$$

we can write the previous condition in a coordinate invariant way as

$$E^i\eta_i = 0. \tag{111}$$

The condition

$$B^i\eta_i = 0 \tag{112}$$

is proven in an analogous way. We can reconstruct the fields from the potentials, and using (111) and (112) we have

$$E^i = \frac{1}{|\eta|^2}\epsilon^{ijk}\eta_j\partial_k\psi, \tag{113}$$

$$B^i = -\frac{1}{|\eta|^2}\epsilon^{ijk}\eta_j\partial_k\chi. \tag{114}$$

Rescaling and taking the norm we finally arrive at expressions (15)

From the electric and magnetic field we can reconstruct the electromagnetic tensor

$$F_{ij} = 2E_{[i}n_{j]} - B^k n^l \epsilon_{klij}, \tag{115}$$

where n is the normal to the surface M. In terms of the potentials

$$F_{ij} = \frac{1}{|\eta|^2}\left(2\eta_{[i}\partial_{j]}\chi - \eta_k\partial_l\psi\epsilon^{kl}{}_{ij}\right). \tag{116}$$

B Maxwell equations in terms of the potentials

Here we show that the Maxwell equations

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0 \tag{117}$$

are automatically satisfied by any choice of potentials ψ, χ. In terms of the potentials we have (see [8]-[12])

$$E_1 = \frac{1}{\sqrt{g_{33}}}\partial_2\psi, \qquad E_2 = -\frac{1}{\sqrt{g_{33}}}\partial_1\psi, \qquad E_3 = \frac{1}{\sqrt{g_{33}}}\partial_n\chi \tag{118}$$

In components we write

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= g^{ab}\nabla_a E_b = g^{ab}(\partial_a E_b - \Gamma^c_{ab}E_c) && (119)\\
&= g^{11}(\partial_1 E_1 - \Gamma^1_{11}E_1 - \Gamma^2_{11}E_2 - \Gamma^3_{11}E_3) + && (120)\\
&+ g^{22}(\partial_2 E_2 - \Gamma^1_{22}E_1 - \Gamma^2_{22}E_2 - \Gamma^3_{22}E_3) + && (121)\\
&+ g^{33}(\partial_3 E_3 - \Gamma^1_{33}E_1 - \Gamma^2_{33}E_2 - \Gamma^3_{33}E_3) + && (122)\\
& && (123)
\end{aligned}$$

Due to axial symmetry we have $\partial_3 E_3 = 0$, $\Gamma^3_{33} = 0$ and $g_{11} = g_{22}$, which leaves us with

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= g^{11}\partial_2\psi\left[\partial_1(g_{33}^{-1/2}) - g_{33}^{-1/2}(\Gamma^1_{11} + \Gamma^1_{22} + g_{11}g^{33}\Gamma^1_{33})\right] - && (124)\\
&- g^{11}\partial_1\psi\left[\partial_2(g_{33}^{-1/2}) - g_{33}^{-1/2}(\Gamma^2_{11} + \Gamma^2_{22} + g_{11}g^{33}\Gamma^2_{33})\right] - && (125)\\
&- g^{11}g_{33}^{-1/2}\partial_n\chi(\Gamma^3_{11} + \Gamma^3_{22}) && (126)
\end{aligned}$$

Now use that

$$-\Gamma^1_{22} = \Gamma^1_{11} = \frac{1}{2}g^{11}g_{11,1} \tag{127}$$

$$-\Gamma^2_{11} = \Gamma^2_{22} = \frac{1}{2}g^{11}g_{11,2} \tag{128}$$

$$\Gamma^3_{ii} = 0 \tag{129}$$

$$\Gamma^i_{33} = -\frac{1}{2}g^{11}g_{33,i} \tag{130}$$

to obtain

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= g^{11}\partial_2\psi\left[-\frac{1}{2}g_{33}^{-3/2}\partial_1 g_{33} + \frac{1}{2}g_{33}^{-3/2}\partial_1 g_{33}\right] - && (131)\\
&- g^{11}\partial_1\psi\left[-\frac{1}{2}g_{33}^{-3/2}\partial_2 g_{33} + \frac{1}{2}g_{33}^{-3/2}\partial_2 g_{33}\right] = && (132)\\
&= 0. && (133)
\end{aligned}$$

And similarly for $\nabla \cdot \mathbf{B} = 0$. This means that Maxwell constraints are automatically satisfied when the fields are written in terms of the potentials ψ, χ, leaving no equations for the potentials.

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