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INFINITELY MANY MINIMAL CURVES JOINING ARBITRARILY CLOSE POINTS IN A HOMOGENEOUS SPACE OF THE UNITARY GROUP OF A C*-ALGEBRA

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Dedicated to the memory of
Ángel Rafael Larotonda (Pucho).

Abstract. We give an example of a homogeneous space of the unitary group of a C*-algebra which presents a remarkable phenomenon, in its natural Finsler metric there are infinitely many minimal curves joining arbitrarily close points.

1. Introduction

In this paper, we give an example of a homogeneous space of the unitary group of a C*-algebra which presents a remarkable phenomenon. Namely, in its natural Finsler metric there are infinitely many minimal curves joining arbitrarily close points. More precisely the homogeneous space will be called $\mathcal{P}$. The unitary group $\mathcal{U}$ of a C*-algebra $\mathcal{A}$ acts transitively on the left on $\mathcal{P}$. The action is denoted by $L_g\rho$, for $g \in \mathcal{U}$ and $\rho \in \mathcal{P}$. The isotropy $\mathcal{I}_\rho = \{g \in \mathcal{U} \ / \ L_g\rho = \rho\}$ will be the unitary group of a C^*-subalgebra $\mathcal{B} \subset \mathcal{A}$. The Finsler norm in $\mathcal{P}$ is naturally defined by $||X||_\rho = \inf_{b\in\mathcal{B}_{ah}} ||Z + b||$, for $X \in (T\mathcal{P})_\rho$ where $Z \in \mathcal{A}_{ah}$ projects to X in the quotient $\mathcal{A}_{ah}/\mathcal{B}_{ah}$ which is identified to the tangent space $(T\mathcal{P})_\rho$. These definitions and notation are borrowed from [1].

This work is part of a forthcoming paper by the same authors which will contain additional results about minimal vectors. We call an element $Z \in \mathcal{A}_{ah}$ minimal vector if $\|Z\| \leq \|Z + V\|, \quad$ for all $V \in \mathcal{B}_{ah}$.

2. The Minimality Theorem

In [1] the following theorem is proven.

Theorem 2.1. *Let $\mathcal{P}$ be a homogeneous space of the unitary group of a C^*-algebra $\mathcal{A}$. Consider $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$. Suppose that there exists $Z \in \mathcal{A}_{ah}$ which is a minimal vector i.e. $||Z|| = ||X||_\rho$. Then the oneparameter curve $\gamma(t)$ given by $\gamma(t) = L_{e^{tZ}}\rho$ has minimal length in the class of all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2||Z||}$.*

We will use the following notation. Let $\mathcal{A}$ be a unital C*-algebra, and $1 \in \mathcal{B} \subset \mathcal{A}$ a C*-subalgebra. Denote $\mathcal{A}_h$ and $\mathcal{B}_h$ (resp. $\mathcal{A}_{ah}$ and $\mathcal{B}_{ah}$) the sets of selfadjoint (resp. anti-hermitian) elements of $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators acting on $\mathcal{H}$, and $Gl(\mathcal{H})$ the group of invertible operators.

We call an element $Z \in \mathcal{A}_{ah}$ a *minimal vector* if

$$\|Z\| \leq \|Z+V\|, \quad \text{for all } V \in \mathcal{B}_{ah}.$$

Note that since for any operator, $\|Im(X)\| \leq \|X\|$, it follows that $Z \in \mathcal{A}_{ah}$ is minimal if and only if

$$\|Z\| \leq \|Z+B\|, \quad \text{for all } B \in \mathcal{B}.$$

In view of the purpose of this paper stated in the introduction and the previous theorem, to look for minimal curves we have to find minimal vectors and therefore the following theorem is relevant.

Theorem 2.2. *An element $Z \in \mathcal{A}_{ah}$ is minimal if and only if there exists a representation ρ of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and a unit vector ξ such that*

1. $\rho(Z^2)\xi = -\|Z\|^2\xi$.
2. $\langle \rho(Z)\xi, \rho(B)\xi \rangle = 0$ *for all* $B \in \mathcal{B}$.

Proof. The if part is trivial. Suppose that there exist $\rho, \mathcal{H}, \xi$ as above. Then if $B \in \mathcal{B}$,

$$\|Z+B\|^2 \geq \|\rho(Z+B)\xi\|^2 = \|\rho(Z)\xi\|^2 + \|\rho(B)\xi\|^2 \geq \|\rho(Z)\xi\|^2 = -\langle \rho(Z^2)\xi, \xi \rangle = \|Z\|^2.$$

Suppose now that Z is minimal. Denote by $\mathcal{S}$ the closed (real) linear span of $Z^2 + \|Z\|^2 I$ and the operators of the form $ZB - B^*Z$ for all possible $B \in \mathcal{B}$. Note that $Z^2 + \|Z\|^2 I$ is positive and $ZB - B^*Z$ is selfadjoint, i.e. $\mathcal{S} \subset \mathcal{A}_h$.

Denote by $\mathcal{C}$ the cone of positive and invertible elements of $\mathcal{A}$. We claim that the minimality condition implies that $\mathcal{S} \cap \mathcal{C} = \emptyset$. Indeed, otherwise, since $\mathcal{C}$ is open, there would exist $s \in \mathbb{R}$ and $B \in \mathcal{B}$ such that

$$s(Z^2 + \|Z\|^2 I) + ZB - B^*Z \geq rI, \quad \text{with } r > 0.$$

We may suppose that $s > 0$, so that dividing by s we get that for given $B \in \mathcal{B}$, $r > 0$,

$$Z^2 + \|Z\|^2 I + ZB - B^*Z \geq rI. \tag{2.1}$$

Also note that $Z^2 + \|Z\|^2 I \geq 0$, then for $n \geq 1$,

$$n(Z^2 + \|Z\|^2) + ZB - B^*Z \geq Z^2 + \|Z\|^2 I + ZB - B^*Z \geq rI.$$

Or equivalently, dividing by n,

$$Z^2 + \|Z\|^2 I + Z\left(\frac{1}{n}B\right) - \left(\frac{1}{n}B^*\right)Z \geq r'I.$$

In other words, one can find $B \in \mathcal{B}$ with arbitrarily small norm such that inequality 2.1 holds.

This inequality clearly implies that

$$sp(Z^2 + ZB - B^*Z) \subset (-\|Z\|^2, +\infty).$$

On the other hand, since B can be chosen with arbitrarily small norm, and Z^2 is non positive, it is clear that one can choose B in order that $sp(Z^2 + ZB - B^*Z) \subset (-\infty, \|Z\|^2)$. Therefore there exists $B \in \mathcal{B}$ such that $\|Z^2 + ZB - B^*Z\| < \|Z\|^2$.

Let us show that this contradicts the minimality of Z, and thus proves our claim. Indeed, this is stated in lemma 5.3 of [1]:

Lemma 2.3. *If $\|Z+B\| \geq \|Z\|$ for all $B \in \mathcal{B}$, then also $\|Z^2+ZB-B^*Z\| \geq \|Z\|^2$.*

We include its proof. Consider for $t > 0$, $f(t) = Z^2 + \frac{1}{t}((Z+tB)^*(Z+tB) - Z^2)$. Note that $\|f(t)\| \geq \|Z\|^2$. Otherwise $\|f(t)\| < \|Z\|^2$ and then the convex combination $tf(t) + (1-t)Z^2$ has norm strictly smaller than $\|Z\|^2$ for $0 < t < 1$. Note that

$$tf(t) + (1-t)Z^2 = (Z+tB)^*(Z+tB).$$

That is $\|Z+tB\|^2 = \|(Z+tB)^*(Z+tB)\| < \|Z\|^2$, which contradicts the hypothesis, and the lemma is proven, as well as our claim.

We have that $\mathcal{S} \cap \mathcal{C} = \emptyset$, with $\mathcal{S}$ a closed (real) linear submanifold of $\mathcal{A}_h$ and $\mathcal{C}$ open and convex in $\mathcal{A}_h$. By the Hahn-Banach theorem, there exists a bounded linear functional φ_0 in $\mathcal{A}_h$ such that

$$\varphi_0(\mathcal{S}) = 0 \quad \text{and } \varphi_0(\mathcal{C}) > 0.$$

The functional φ_0 has a unique selfadjoint extension to $\mathcal{A}$, let φ be the normalization of this functional. Then clearly φ is a state which vanishes on $\mathcal{S}$. Let $\rho, \mathcal{H}, \xi$ be the GNS triple associated to this state. Note that since $Z^2 + \|Z\|^2 I \in \mathcal{S}$, $\langle \rho(Z^2)\xi, \xi >= \varphi(Z^2) = -\|Z\|^2$, and therefore, by the equality part in the Cauchy-Schwartz inequality, it follows that

$$\rho(Z^2)\xi = -\|Z\|^2\xi.$$

Moreover, $0 = \varphi(ZB - B^*Z + Z^2 + \|Z\|^2 I) = \varphi(ZB - B^*Z)$. Since φ is selfadjoint, this means $Re(\varphi(ZB)) = 0$ for all $B \in \mathcal{B}$. Putting iB in the place of B, one has that in fact $\varphi(ZB) = 0$ for all $B \in \mathcal{B}$. Then,

$$0 = \langle \rho(ZB)\xi, \xi \rangle = \langle \rho(B)\xi, \rho(Z)\xi \rangle,$$

which concludes the proof. □

3. Infinitely many minimal curves joining arbitrarily close points

In this example the homogeneous space $\mathcal{P}$ is the flag manifold of 4-tuples of mutually orthogonal lines in $\mathbb{C}^4$ (1-dimensional complex subspaces). The group of unitary operators in $\mathbb{C}^4$ acts on the left in $\mathcal{P}$ by sending each complex line to its image by the unitary operator (thus preserving the orthogonality of the new 4-tuple complex lines). Consider the canonical flag $p_e = (\mathrm{sp}\{e_1\}, \mathrm{sp}\{e_2\}, \mathrm{sp}\{e_3\}, \mathrm{sp}\{e_4\})$ where $\mathrm{sp}\{e_i\}$ is the complex line spanned by the canonical vector e_i in $\mathbb{C}^4$. The isotropy of the canonical flag p_e is the subgroup of 'diagonal' unitary operators.

We consider now the submanifold $\mathcal{P}_d$ of $\mathcal{P}$ given by

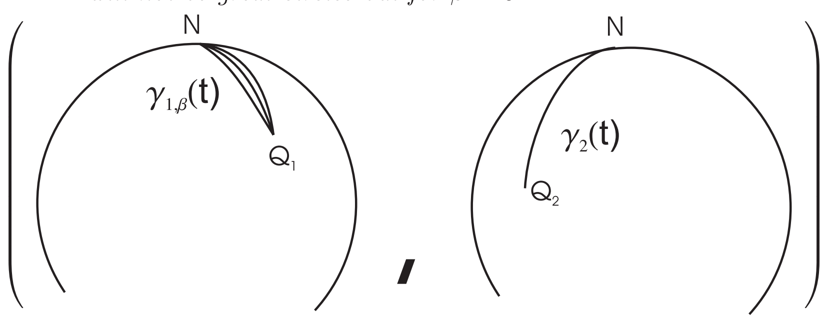
$$\mathcal{P}_d = \{(l_1, l_2, l_3, l_4) \in \mathcal{P} \mid \mathrm{sp}\{l_1, l_2\} = \mathrm{sp}\{e_1, e_2\}\}$$

Notice that $\mathcal{P}_d = \mathcal{W} \times \mathcal{W}$ where $\mathcal{W}$ is the flag manifold of couples of mutually orthogonal 1-dimensional complex lines in $\mathbb{C}^2$. Notice also that an ordered pair of mutually orthogonal 1-dimensional complex lines in $\mathbb{C}^2$ is totally determined by the first complex line of the pair, hence $\mathcal{W} = \mathbb{C}P(2)$. Furthermore $\mathbb{C}P(2) = \mathcal{RS}$, the Riemann Sphere, hence $\mathcal{W}=\mathcal{RS}$.

The minimal curves presented in this example shall be constructed in $\mathcal{P}_d$. For a better geometrical view of those curves we shall identify $\mathcal{RS}$, via stereographic projection, with the unit sphere S^2 in $\mathbb{R}^3$, hence we shall make the identification $\mathcal{P}_d = S^2 \times S^2$.

3.1. **A description of the minimal curves.** Let $\mathcal{N} = (N, N) \in \mathcal{P}_d = S^2 \times S^2$ be the point whose coordinates are both the North Pole, $N \in S^2$. Let $\mathcal{Q} = (Q_1, Q_2) \in S^2 \times S^2$ be any point of $\mathcal{P}_d$ such that Q_1 has higher latitude than Q_2 in S^2 (Q_1 is closer to N than Q_2).

> *We will fix $\mathcal{Q}$ so that Q_2 is above the equator line (and Q_1 is even higher) and present a family of minimal curves $\Gamma_\beta(t) = (\gamma_{1,\beta}(t), \gamma_2(t))$, for $t \in [0,1]$, all joining $\mathcal{N}$ to $\mathcal{Q}$.*
>
> - *The curve $\gamma_2(t)$ in S^2 will trace the smaller arc of the great circle that contains N and Q_2.*
> - *The family of curves $\gamma_{1,\beta}(t)$ will vary continuously with the parameter β.*
> - *Each of the curves of the family $\gamma_{1,\beta}(t)$ will parametrize the smaller arc of some circle in S^2 that joins N to Q_1; the arcs will not be great circles but for $\beta = 0$.*



3.2. **A precise description of the minimal curves.** To present the curves drawn above we give a more manageable description of $\mathcal{P}$. We consider the unitary subgroup $\mathcal{U} = U(4)$ of the C^*-algebra $\mathcal{A} = M_4(\mathbb{C})$ of 4×4 complex matrices, and denote with $\mathcal{B}$ the subalgebra of diagonal matrices in $\mathcal{A}$. The homogeneous space $\mathcal{P}$ is given by the quotient $\mathcal{U}/\mathcal{D}$, where $\mathcal{D} = \mathcal{U} \cap \mathcal{B}$ is the subgroup of the diagonal unitary matrices. The group $\mathcal{U}$ acts on $\mathcal{P}$ (on the left). The tangent space at 1 (the identity class) is the subspace of anti-hermitian matrices in $\mathcal{A}$ with zeroes on the diagonal.

We construct $\mathcal{P}_d \subset \mathcal{P}$ as follows. First consider the subgroup $SU(2) \times SU(2) \subset \mathcal{U}$ of special unitary matrices build with two, 2×2, blocks on the diagonal. We set $\mathcal{P}_d \subset \mathcal{P}$ as the quotient of $SU(2) \times SU(2)$ by the subgroup $\mathcal{D}$ of diagonal special unitary matrices. This submanifold is in itself a product of two copies of the quotient $\mathcal{W}$ of $SU(2)$ by the subgroup of diagonal matrices in $SU(2)$. For the relations among the different groups here mentioned we suggest [2]. We write $\mathcal{P}_d = \mathcal{W} \times \mathcal{W}$ and a point of $\mathcal{P}_d$ is a class (in a quotient) which in itself has two components which are also classes. We shall use the notation $[U] = ([u_1], [u_2]) \in \mathcal{P}_d = \mathcal{W} \times \mathcal{W}$.

The minimal curves starting at $1 \in \mathcal{P}_d$ are of the form $\gamma(t) = \left[e^{tZ}\right]$ where the matrices Z are anti-hermitian matrices with zero trace in $\mathcal{A}$ built with two blocks of anti-hermitian 2×2 matrices on the diagonal (each one with zero trace).

The minimality of the curves is granted by 2.1 for the matrices Z shall be minimal vectors according to theorem 2.2. In fact, we shall consider $Z \in \mathcal{A}_{an}$ of the form

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

where Z_1 and Z_2 are anti-hermitian 2×2 matrices of the form

$$Z_1 = \begin{pmatrix} z\, i & r\,(-\sin(\alpha) + i\, \cos(\alpha)) \\ r(\sin(\alpha) + i\, \cos(\alpha)) & -z\, i \end{pmatrix}, \tag{3.2}$$

and

$$Z_2 = \begin{pmatrix} 0 & w \\ -\overline{w} & 0 \end{pmatrix} \tag{3.3}$$

where $z,\, r, \alpha \in \mathbb{R}$, and $w \in \mathbb{C}$.

The minimality of these matrices Z is assured in the case where $|w|^2 \geq z^2 + r^2$. In such case, $||Z||^2 = |w|^2$ and, in relation to theorem 2.2, just consider the operator representation ρ of the C^*-algebra $\mathcal{A} = M_4(\mathbb{C})$ on $\mathbb{C}^4$, together with the unit vector $\xi = (0, 0, 0, 1) \in \mathbb{C}^4$.

3.2.1. *The two components of the curves in $\mathcal{P}_d$.* The curve $\gamma(t) = \left[e^{tZ}\right] = (\left[e^{tZ_1}\right],\left[e^{tZ_2}\right])$ in $\mathcal{P}_d$ has two components (in $\mathcal{W}$).

We shall regard the Riemann Sphere $\mathcal{RS}$ as the complex plane $\mathbb{C}$ with the point "∞" added. Consider a matrix u in $SU(2)$

$$u = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}, \text{ where } a,\, b \in \mathbb{C} \text{ and } |a|^2 + |b|^2 = 1$$

We consider the mapping L from $SU(2)$ to $\mathcal{RS}$ is given by

$$L(u) = \frac{a}{b}, \text{ if } b \neq 0, \text{ else } L(u) = \infty.$$

It is clear that this mapping induces an explicit diffeomorphism from the quotient of $SU(2)$ by its diagonal matrices to the Riemann Sphere $\mathcal{RS}$.

Consider the unit sphere S^2 in $\mathbb{R}^3$, and let the equatorial plane, $\mathbb{C}$, represent the "finite" part of the Riemann Sphere $\mathcal{RS}$. We set $\varphi : \mathcal{RS} \to S^2$ to be the stereographic projection given as by:

$$\varphi(\zeta) = \left(\frac{2\zeta}{|\zeta|^2+1}, \frac{|\zeta|^2-1}{|\zeta|^2+1}\right) \in \mathbb{C}\times\mathbb{R} = \mathbb{R}^3, \text{ for } \zeta \in \mathbb{C}, \text{ and } \varphi(\infty) = (0,0,1) = N \in S^2 \subset \mathbb{R}^3$$

Notice that in the class $b \neq 0$, if $\zeta = L(u) = \frac{a}{b} \in \mathbb{C}$, then $\varphi(\zeta) = \left(2a\overline{b}\,,\, |a|^2 - |b|^2\right)$. If $b = 0$, then $|a| = 1$, hence $\zeta = L(u) = \infty$, then $\varphi(\zeta) = (0\,,\, 0\,,\, 1)$.

Via a composition of two maps, we define the diffeomorphism Ψ from $\mathcal{W}$ onto S^2: for $[u] = \left[\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}\right] \in \mathcal{W}$ we set,

$$\Psi([u]) = \varphi\left(L(u)\right) = \left(2a\overline{b}\,,\, |a|^2 - |b|^2\right) = \left(2a\overline{b}\,,\, 1 - 2|b|^2\right) \in S^2$$

Considering the curve $q(t) = e^{tZ_1}$ in $SU(2)$ with Z_1 as in (3.2) above, and setting $\lambda = \sqrt{r^2+z^2}$, it can be verified that $L(q(t)) \in \mathcal{RS}$ is given by,

$$L(q(t)) = \frac{z\,(\cos(\alpha)+i\,\sin(\alpha))}{r} + \cot(t\lambda)\frac{\lambda\,(\sin(\alpha)-i\,\cos(\alpha))}{r}, \;\text{ if } t \notin \left\{\frac{k\pi}{\lambda} \mid k\in\mathbb{Z}\right\} \tag{3.4}$$

and,

$$L(q(t)) = \infty, \;\text{ if } t \in \left\{\frac{k\pi}{\lambda} \mid k\in\mathbb{Z}\right\} \tag{3.5}$$

Notice then that $L(q(t))$ parametrizes a straight line l_q in $\mathcal{RS}$. Hence the curve

$$\Psi([q(t)]) = \varphi\,(L(q(t)))$$

is an arc of a circle in S^2 (not necessarily a great circle) contained in the plane in $\mathbb{R}^3$ that contains both the line l_q, in the equatorial plane, and the North Pole N, in S^2. It can be verified that this plane has unit normal vectors given by:

$$\pm\,(\cos(\beta)\cos(\alpha)\,,\,\cos(\beta)\sin(\alpha)\,,\,\sin(\beta))$$

where $\cos(\beta) = \dfrac{r}{\lambda}$, $\sin(\beta) = \dfrac{z}{\lambda}$, with $\lambda = \sqrt{r^2+z^2}$.

3.2.2. *Some observations on the curves $\Psi\left([e^{tZ_1}]\right)$ and $\Psi\left([e^{tZ_2}]\right)$ in $\mathcal{W}$.* Let $\gamma_{1,\beta}(t) = \Psi\left([e^{tZ_1}]\right)$, where $\cos(\beta) = \dfrac{r}{\lambda}$, $\sin(\beta) = \dfrac{z}{\lambda}$, with $\lambda = \sqrt{r^2+z^2}$, and let $\gamma_2(t) = \Psi\left([e^{tZ_2}]\right)$

- In the constructions above, the curve $\gamma_{1,\beta}(t)$ runs over a great circle in S^2 if and only if $\beta = 0$ (equivalently $z = 0$).
- The curve $\gamma_2(t)$ runs over a great circle in S^2 (Z_2 has parameter $z = 0$).
- The curve $\gamma_{1,\beta}(t)$ varies continuously with the parameter β.
- The curve $\gamma_{1,\beta}(t)$ starts at $N \in S^2$ and returns to that point exactly for $t \in \left\{\frac{k\pi}{\lambda} \mid k\in\mathbb{Z}\right\}$.
- The curve $\gamma_{1,\beta}(t)$ has constant speed $2\lambda\cos(\beta)$ in S^2.
- The curve $\gamma_2(t)$ has constant speed $2r$ in S^2.

3.2.3. *The curves $\Gamma_\beta(t) = \left(\Psi\left([e^{tZ_1}]\right),\,\Psi\left([e^{tZ_2}]\right)\right)$ in $\mathcal{P}_d$.* Lets give explicit values of the "parameters" z, α, $r \in \mathbb{R}$ and $w \in \mathbb{C}$ that define Z_1 and Z_2 (according to formulas (3.2) and (3.3)), so that for $t \in [0,1]$, the curves $\gamma_{1,\beta}(t) = \Psi\left([e^{tZ_1}]\right)$ and $\gamma_2(t) = \Psi\left([e^{tZ_2}]\right)$ join the point N to Q_1 and Q_2 respectively.

Suppose that the distances from N to Q_1 and Q_2 in S^2 are $2\phi_1$ and $2\phi_2$ respectively (with $\phi_1 < \phi_2$).

By means of some rotation of the sphere S^2 we may suppose that Q_1 is in the plane generated by $\hat{\jmath}$ and $\hat{k}$, as in figure (1) below, and we have, $Q_1 = (0, \sin(2\phi_1), \cos(2\phi_1))$ and $Q_2 = (\sin(2\phi_2)\cos(\theta_2), \sin(2\phi_2)\sin(\theta_2), \cos(2\phi_2))$.

For Z_2 we set $w = \phi_2\,(-\sin(\theta_2) + i\,\cos(\theta_2))$ so that $\gamma_2(1) = \Psi\left([e^{Z_2}]\right) = Q_2$.

We have to choose the values z, α, $r \in \mathbb{R}$ and $w \in \mathbb{C}$ that define Z_1. This is equivalent to chose β, α, $\lambda \in \mathbb{R}$ via the change of variables given by the equations

$$\cos(\beta) = \frac{r}{\lambda},\;\; \sin(\beta) = \frac{z}{\lambda},\; \text{with } \lambda = \sqrt{r^2+z^2}.$$

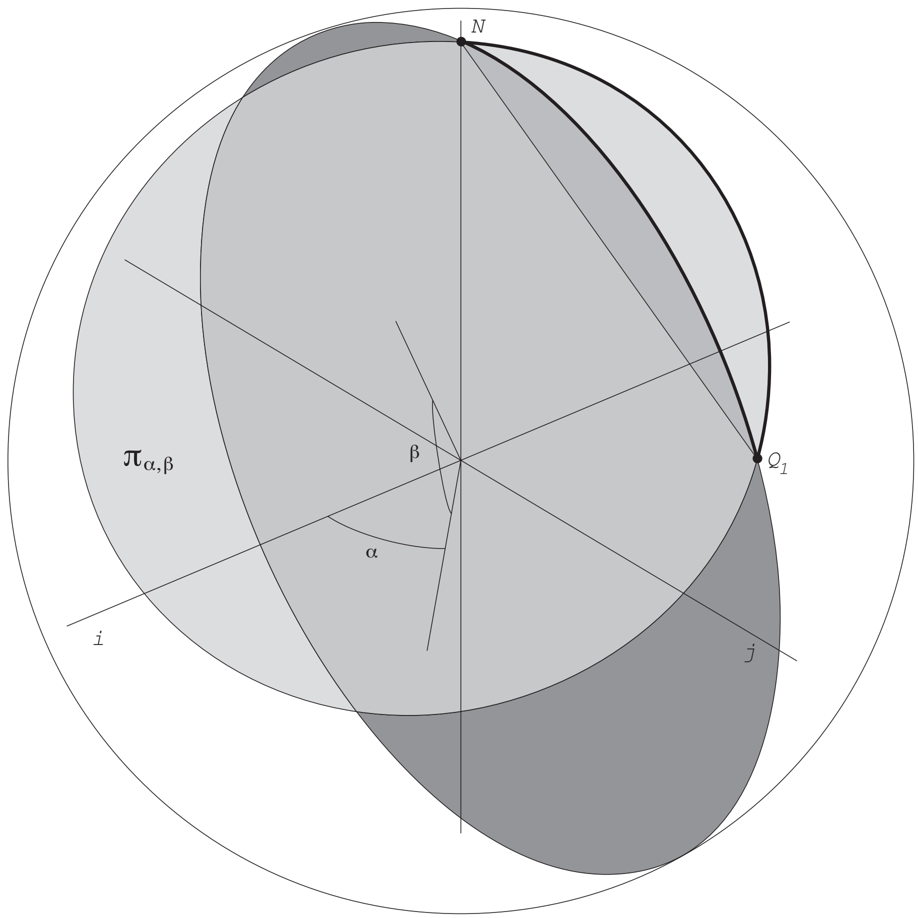


FIGURE 1. We suppose that Q_1 is in the plane generated by ĵ and k̂.

The parameters α and β are shown in figure (1), with the only restriction that the vector

$$\vec{n} = (\cos(\beta)\cos(\alpha)\,,\,\cos(\beta)\sin(\alpha)\,,\,\sin(\beta))$$

is orthogonal to a plane $\pi_{\alpha,\beta}$ that contains N and Q_1.

The parameter λ is determined after choosing α and β so that the short arc joining N and Q_1, in the intersection of the plane $\pi_{\alpha,\beta}$ with the sphere S^2as in figure 1, has length ℓ equal to $2\lambda\cos(\beta)$, from where the value of λ is drawn.

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