A Hamilton-Jacobi Theory for general dynamical systems and integrability by quadratures in symplectic and Poisson manifolds

Sergio Grillo
Instituto Balseiro, Universidad Nacional de Cuyo and CONICET
Av. Bustillo 9500, San Carlos de Bariloche
R8402AGP, República Argentina
sergiog@cab.cnea.gov.ar

Edith Padrón
ULL-CSIC Geometría Diferencial y Mecánica Geométrica
Departamento de Matemáticas, Estadística e IO
Universidad de la Laguna, La Laguna, Tenerife, Canary Islands, Spain
mepadron@ull.edu.es

April 7, 2018

Abstract

In this paper we develope, in a geometric framework, a Hamilton-Jacobi Theory for general dynamical systems. Such a theory contains the classical theory for Hamiltonian systems on a cotangent bundle and recent developments in the framework of general symplectic, Poisson and almost-Poisson manifolds (including some approaches to a Hamilton-Jacobi theory for nonholonomic systems). Given a dynamical system, we show that every complete solution of its related Hamilton-Jacobi Equation (HJE) gives rise to a set of first integrals, and vice versa. From that, and in the context of symplectic and Poisson manifolds, a deep connection between the HJE and the (non)commutative integrability notion, and consequently the integrability by quadratures, is stablished. Moreover, in the same context, we find conditions on the complete solutions of the HJE that also ensures integrability by quadratures, but they are weaker than those related to the (non)commutative integrability. Examples are developed along all the paper in order to illustrate the theoretical results.

1 Introduction

In the context of Classical Mechanics, the *standard* or *classical* (time-independent) Hamilton-Jacobi Theory was designed to construct, for a given Hamiltonian system on a cotangent bundle T^*Q, local coordinates such that the Hamilton equations expressed on these coordinates adopt a very simple form. Here, by *simple* we mean that such equations can be integrated by *quadratures* (i.e. its solutions can be given in terms of primitives and inverses of known functions). The fundamental tool of the theory is the so-called (time-independent) *Hamilton-Jacobi equation (HJE)* for a Hamiltonian function $H : T^*Q \to \mathbb{R}$. The problem is to find a function W on Q, known as the *characteristic Hamilton function*, such that the equation (the classical HJE)

$$H\left(q, \frac{\partial W}{\partial q}\right) = \text{constant}$$

is satisfied (see for example [1]). When we are given a family of solutions $\{W_\lambda\}_{\lambda\in\Lambda}$ such that the square matrix $\partial^2 W_\lambda \big/ \partial q \partial \lambda$ is non-degenerated, with Λ an open subset of $\mathbb{R}^{\dim Q}$, the above mentioned coordinates can be constructed. More precisely, a type 2 canonical transformation (see [17]) can be defined from the functions W_λ's such that the equations of motions of the system become, under such a transformation, simple enough to be solved by quadratures.

In modern terms (see Ref. [10]), the classical HJE reads $d(H \circ \sigma) = 0$, and its unknown is a closed 1-form $\sigma : Q \to T^*Q$. If a solution σ is given for a Hamiltonian function H, the problem of finding the integral curves of its Hamiltonian vector field X_H (w.r.t. the canonical symplectic structure on T^*Q), with initial conditions along the image of σ, reduces to find the integral curves of the vector field $X_H^\sigma := (\pi_Q)_* \circ X_H \circ \sigma$ on Q, where $\pi_Q : T^*Q \to Q$ is the cotangent fibration. If one wants to find all the integral curves of X_H, i.e. the solutions of the equations of motions of the system for every initial condition, it is necessary to introduce the notion of a *complete solution*: a surjective local diffeomorphism $\Sigma : Q \times \Lambda \to T^*Q$, with Λ a manifold, such that $\sigma_\lambda := \Sigma(\lambda, \cdot)$ is a solution of the classical HJE for each $\lambda \in \Lambda$. In terms of Σ, around every point of T^*Q, a local coordinate chart can be constructed such that the equations of motion are easily solved. We can see that through the well-known connection between the classical Hamilton-Jacobi Theory and the notion of *commutative integrability*. Let us describe such a connection (see, for example, [10]). From every complete solution $\Sigma : Q \times \Lambda \to T^*Q$, we can construct (unless locally) a Lagrangian fibration $F : T^*Q \to \Lambda$ (transverse to π_Q) such that the image of X_H lives inside $\operatorname{Ker} F_*$. In other terms, if Λ is an open subset of $\mathbb{R}^n$ (with $n = \dim Q$), we can build up n first integrals $f_i : T^*Q \to \mathbb{R}$ (the components of F) which are independent and in involution with respect to the canonical Poisson bracket. Conversely, from every Lagrangian fibration $F : T^*Q \to \Lambda$ (transverse to π_Q) such that $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$, a complete solution of the HJE can be constructed (unless locally). We recall that, if such a fibration is given for a Hamiltonian system, the Arnold-Liouville Theorem [4] establishes that the system in question is integrable by quadratures.

In the last few years, several generalized versions of the classical HJE have been developed for Hamiltonian systems on different contexts: on general symplectic, Poisson and almost-Poisson manifolds, and also on Lie algebroids over vector bundles. The resulting Hamilton-Jacobi theories were applied to nonholonomic systems, time-dependent Hamiltonian systems, reduced systems by symmetries and systems with external forces [5, 10, 11, 23, 24]. In all of these contexts, a fibration $\Pi : M \to N$ (i.e. a surjective submersion) is defined on the phase space M of each system, the solutions of the generalized HJE are sections $\sigma : N \to M$ of such a fibration, and the complete solutions are local diffeomorphisms $\Sigma : N \times \Lambda \to M$ such that $\sigma_\lambda := \Sigma(\lambda, \cdot)$ is a solution of the HJE for each $\lambda \in \Lambda$. This clearly extends the classical situation, where the involved fibration is the cotangent projection $\pi_Q : T^*Q \to Q$ of a manifold Q. Unfortunately, no connection between complete solutions and some kind of *exact solvability* (as the integrability by quadratures) has been given for any of those generalized versions of the HJE (see for instance the Ref. [6]).

This paper is the first of a series of papers in which we shall further extend the previously mentioned Hamilton-Jacobi theories to general dynamical systems on a fibered phase space. We are mainly interested in the connection between complete solutions and exact solvability. In the present paper, as a first step, we shall focus on (time-independent) Hamiltonian systems on general Poisson manifolds. In forthcoming papers, we shall address the case of Hamiltonian systems with external forces (including Hamiltonian systems with constraints) and time-dependent Hamiltonian systems.

One of the contribution of this paper is to show, in the context of general dynamical systems, that there exists a *duality* between complete solutions and first integrals, extending similar results that appear in the literature. This enable us to establish, in the particular context of Hamiltonian systems on Poisson manifolds, a deep connection between *(non)commutative integrability* and certain subclasses of complete solutions.

Recall that a Hamiltonian system on a Poisson manifold M, with Hamiltonian function H, is a *noncommutative* *integrable* system (see [14, 15] for symplectic manifolds and [16, 26] for Poisson ones) if a fibration $F : M \to \Lambda$ such that:

1. $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$ (F defines first integrals for the system),
2. $\operatorname{Ker} F_* \subset (\operatorname{Ker} F_*)^{\perp}$, i.e. F is isotropic,
3. $(\operatorname{Ker} F_*)^{\perp}$ is integrable (F has a polar),

can be exhibited. In particular, the system is commutative integrable if in addition $\operatorname{Ker} F_* = (\operatorname{Ker} F_*)^{\perp}$, i.e. F is Lagrangian, as we have said above. By $\perp$ we are denoting the Poisson orthogonal. To be more precise, if Ξ is the Poisson bi-vector on M and $\Xi^{\sharp} : T^*M \to TM$ is its related linear bundle map, then $(\operatorname{Ker} F_*)^{\perp} := \Xi^{\sharp}\left((\operatorname{Ker} F_*)^0\right)$. All of these systems, among other things, are integrable by quadratures.

Another contribution of the paper (which can be seen as a first application of our extended Hamilton-Jacobi Theory) is to show that conditions 1 and 2 listed above are enough in order to ensure integrability by quadratures of Hamiltonian systems on Poisson manifolds. That is to say, the integrability of $(\operatorname{Ker} F_*)^{\perp}$ is not needed for that purpose. Moreover, we show that condition 2 can be replaced by a weaker one: $\operatorname{Ker} F_* \cap \operatorname{Im} \Xi^{\sharp} \subset (\operatorname{Ker} F_*)^{\perp}$, which we call *weak isotropy* (together some regularity assumptions about the symplectic leaves of Ξ). It is worth mentioning that the proof of this result was done mainly in terms of complete solutions (instead of first integrals). We think that, in spite of the duality between complete solutions and first integrals, it would have been very hard to anticipate the mentioned result by working with first integrals only.

The paper is organized as follows. In Section 2, given a dynamical system (M, X) equipped with a fibration $\Pi : M \to N$, being M and N smooth manifolds and X a vector field on M, we introduce the notion of Π-Hamilton-Jacobi equation (Π-HJE) for (M, X). The unknown of such an equation is a section $\sigma : N \to M$ of Π. The Π-HJE is defined in such a way that, if σ is a solution, then the vector field X restricts to the closed submanifold $\operatorname{Im} \sigma \subset M$ determined by the image of σ. For the case in which M is a Leibniz manifold[1] and X is a Hamiltonian vector field, we give characterizations of the corresponding Π-HJEs in terms of (co)isotropic fibrations. In particular, when M is a symplectic manifold, we recover the well-known results for the classical HJE (some of them contained in Ref. [10]). Moreover, for Poisson and almost-Poisson manifolds, we recover the results of [24] and related works. In Section 3, we define the complete solutions $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X), following the same ideas as in the previous works on the subject (see the above discussion). In particular, for each $\lambda \in \Lambda$, the map $\sigma_\lambda := \Sigma(\lambda, \cdot) : N \to M$ must be a solution of the Π-HJE. This tool allows us to obtain any integral curve of X from the integral curves of the vector fields $X^{\sigma_\lambda} := \Pi_* \circ X \circ \sigma_\lambda$ on N. In the Leibniz scenario, several characterizations and results about the complete solutions are given. In particular, in the symplectic setting, we show that, if the fibration Π and each submanifold $\operatorname{Im} \sigma_\lambda$ are Lagrangian, then Darboux coordinates around every point of M, or equivalently local canonical transformations, can be constructed in such a way that the solutions of the Hamilton equations can be obtained by quadratures. This extends the fundamental essence of the classical theory (valid for cotangent symplectic manifolds and the cotangent fibration) to general symplectic manifolds and general fibrations. Moreover, by choosing different Lagrangian fibrations Π, we show that the mentioned canonical transformations can be taken of type 1, 2, 3 or 4 (instead of type 2 only, as in the classical situation). We illustrate that with simple examples. Section 4 is devoted to analyze the *duality* between complete solutions and first integrals. More precisely, we show that given a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for a dynamical system (M, X), for each $m \in M$ there exists an open neighborhood U of m and a fibration $F : U \to F(U) \subset \Lambda$ (transverse to Π)

[1] These manifolds were introduced in [18] and they are used in the context of generalized nonholonomic systems, gradient dynamical systems, in the study of the interaction between non linear oscillators and the energy exchange between them, and in the modeling of certain dissipative phenomena [28].

such that $\operatorname{Im} X|_U \in \operatorname{Ker} F_*$. Reciprocally, given a fibration $F : M \to \Lambda$ (transverse to Π) such that $\operatorname{Im} X \in \operatorname{Ker} F_*$, for each $m \in M$ there exists an open neighborhood U of m such that $(F, \Pi)|_U$ is invertible and its inverse defines a complete solution of the $\Pi|_U$-HJE for $(U, X|_U)$. Then, based on that duality, we prove an existence theorem for complete solutions and, on the other hand, we establish a deep connection between complete solutions of the Π-HJE and noncommutative integrability on Poisson manifolds [16, 26], i.e. a connection between complete solutions and integrability by quadratures. Finally, in the last section of the paper, we show for Hamiltonian systems on Poisson manifolds that, in order to ensures integrability by quadratures, the above listed conditions 1-3 that define a noncommutative system can be drastically weakened, as explained in the previous paragraph.

We assume that the reader is familiar with the basic concepts of Differential Geometry (see [8, 20, 27]), and with the basic ideas related to the Lagrangian systems, Hamiltonian systems, Symplectic Geometry and Poisson Geometry in the context of Geometric Mechanics (see [1, 4, 25]). We shall work in the smooth (i.e. C^∞) category, focusing exclusively on finite-dimensional smooth manifolds.

2 The Hamilton-Jacobi equation for a fibration

In this section we introduce the notion of *Hamilton-Jacobi equation* in the context of general dynamical systems on a fibered phase space. We study the consequences of having a solution of such an equation, focusing on how it can help to find the trajectories of the system under consideration. We show that our theory extends those developed for Hamiltonian systems on symplectic, Poisson and almost-Poisson manifolds (including the nonholonomic systems). In particular, in the framework of Hamiltonian systems on a cotangent bundle, we show that our equation reduces to the classical Hamilton-Jacobi equation when the cotangent fibration is considered.

2.1 Definition and basic properties

Let us consider a dynamical system (M, X), where M is a smooth manifold (the *phase space*) and $X \in \mathfrak{X}(M)$ (i.e. X is a vector field on M). Let N be another smooth manifold and $\Pi : M \to N$ a fibration, i.e. a surjective submersion (*ipso facto* an open map). Given an open subset $U \subset M$, we denote by $\Pi|_U : U \to \Pi(U)$ the restricted fibration and $X|_U : U \to TU$ the corresponding vector field obtained by restricting X to U.

Definition 2.1 *We shall call the equation*

$$\sigma_* \circ \Pi_* \circ X \circ \sigma = X \circ \sigma \tag{1}$$

*for a section $\sigma : N \to M$ of Π (*ipso facto* a closed map), the Π-**Hamilton-Jacobi equation** (Π-**HJE**) **for** (M, X). If σ satisfies such an equation, we shall say that σ is a **(global) solution of the** Π-**HJE for** (M, X). On the other hand, given an open subset $U \subset M$, we shall say that a map $\sigma : \Pi(U) \to U$ is a **local solution of the** Π-**HJE for** (M, X) **on** U if σ is a solution of the $\Pi|_U$-HJE for $(U, X|_U)$.*

Remark 2.2 *Given an open subset $V \subset N$, if a local section $\sigma : V \to M$ of Π satisfies (1) along V, then σ defines (by co-restriction) a local solution of the Π-HJE for (M, X) on every U such that $\sigma(V) \subset U \subset \Pi^{-1}(V)$ (*ipso facto* $\Pi(U) = V$). Reciprocally, given an open subset $U \subset M$, if $\sigma : \Pi(U) \to U$ is a local solution of the Π-HJE for (M, X) on U, then σ defines a local section with domain $V := \Pi(U)$ which satisfies Eq. (1) along V.*

Example 2.3 *Suppose that $M = \mathbb{R}^d$, $N = \mathbb{R}^k$, with $k < d$, and $\Pi : \mathbb{R}^d \to \mathbb{R}^k$ is the projection onto the first k components of $\mathbb{R}^d$. A section $\sigma : \mathbb{R}^k \to \mathbb{R}^d$ of Π is a map of the form*

$$\sigma : n \in \mathbb{R}^k \longmapsto (n, \hat{\sigma}(n)) \in \mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^l,$$

with $l=d-k$*. On the other hand, identifying* $T\mathbb{R}^d$ *with* $\mathbb{R}^d\times\mathbb{R}^d$*, a vector field* $X\in\mathfrak{X}\left(\mathbb{R}^d\right)$ *can be described as a map*

$$X : m\in\mathbb{R}^d\longmapsto\left(m,\hat{X}\left(m\right)\right)\in\mathbb{R}^d\times\mathbb{R}^d.$$

Under this notation, it is easy to see that the Π*-HJE for* (M,X) *is given by*

$$\sum_{i=1}^{k}\frac{\partial\hat{\sigma}_j\left(n\right)}{\partial n_i}\hat{X}_i\left(n,\hat{\sigma}\left(n\right)\right)=\hat{X}_{j+k}\left(n,\hat{\sigma}\left(n\right)\right),\quad j=1,...,l, \tag{2}$$

where n_i*,* $\hat{\sigma}_i$ *and* $\hat{X}_i$ *are the* i*-th components of* n*,* $\hat{\sigma}$ *and* $\hat{X}$*, respectively.*

It is clear that, for general manifolds M and N, if we fix local coordinates of M and N adapted[2] to Π, the local expression of (1) is exactly given by the Eq. (2). In particular, the local expression of the Π-HJE for (M,X) is a system of $l:=\dim M-\dim N$ first order partial differential equations (PDE) with l unknowns. So, a sufficient condition for the existence of local solutions can be obtained from the characteristic curve method. More precisely, in the notation of the above example, if the vector field X satisfies

$$\left(\hat{X}_1\left(m\right),...,\hat{X}_k\left(m\right)\right)\neq 0,\quad\text{i.e.}\quad X\left(m\right)\notin\operatorname{Ker}\Pi_{*,m}, \tag{3}$$

for some $m\in M$, it can be shown that there exists an open neighborhood $U\subset M$ of m and a local solution of the Π-HJE for (M,X) on U. In Section 4.3.1, we shall prove this fact as a particular case of a more general result.

It is worth mentioning that the aim of this paper is not to study the existence problem for global and local solutions of the Π-HJE (because they use to exist around almost any point -see Section 4.3.1-), but to investigate what we can say about *integrability* when a solution of such an equation is given to us. By integrability we mean some kind of *exact solvability*, like the so-called *solvability by quadratures*, i.e. the possibility of finding an explicit expression of each trajectory of the system by solving linear equations, using the inverse function theorem and integrating known functions: *the quadratures*.

Since the local solutions for (M,X) are exactly the global solutions for each $(U,X|_U)$, from now on, we shall mainly focus, just for simplicity, on the global ones.

Let $\sigma:N\to M$ be a solution of the Π-HJE for (M,X). Since σ is a section of Π, then, by definition,

$$\Pi\circ\sigma=id_N\quad\text{and}\quad\sigma\circ\Pi|_{\operatorname{Im}\sigma}=id_{\operatorname{Im}\sigma}. \tag{4}$$

Consequently, σ is an injective immersion and a closed map, so it is an embedding. Then $\operatorname{Im}\sigma\subset M$ is a closed regular submanifold and N is diffeomorphic to $\operatorname{Im}\sigma$. Also,

$$\operatorname{Ker}\Pi_*|_{\operatorname{Im}\sigma}\oplus\operatorname{Im}\sigma_*=T_{\operatorname{Im}\sigma}M. \tag{5}$$

These observations give rise to the following characterization of the solutions of the Π-HJE.

Proposition 2.4 *A section* $\sigma:N\to M$ *of* Π *is a solution of the* Π*-HJE for* (M,X) *if and only if .*

$$\operatorname{Im}\left(X|_{\operatorname{Im}\sigma}\right)\subset T\operatorname{Im}\sigma, \tag{6}$$

i.e. X *restricts to* $\operatorname{Im}\sigma$*.*

[2]By *adapted to* Π, we mean local coordinates on M and N such that the local expression of Π is the projection on the k-first coordinates, being $k:=\dim N$.

Proof. Given $m \in \operatorname{Im}\sigma$, we can write $m = \sigma(n)$ for a unique $n \in N$. It follows from Eq. (1) that

$$X(m) = X(\sigma(n)) = \sigma_{*,n} \circ \Pi_{*,\sigma(n)}(X(\sigma(n))) \in T_{\sigma(n)} \operatorname{Im}\sigma = T_m \operatorname{Im}\sigma,$$

so X restricts to $\operatorname{Im}\sigma$. Reciprocally, suppose that $\sigma : N \to M$ is a section of Π such that Eq. (6) holds. Combining this last equation with (4), we have that

$$\sigma_{*,n}\left[\Pi_{*,\sigma(n)}(X(\sigma(n)))\right] = \sigma_{*,n}\left[\left(\Pi|_{\operatorname{Im}\sigma}\right)_{*,\sigma(n)}(X(\sigma(n)))\right] = X(\sigma(n)), \quad \forall n \in N,$$

i.e. σ satisfies the Eq. (1). □

Since $\operatorname{Im}\sigma$ is closed, above result says, as it is well-known, that $\operatorname{Im}\sigma$ is an X-**invariant submanifold** of M. In other words, an integral curve $\Gamma : I \to M$ of X, with I an open interval, intersects $\operatorname{Im}\sigma$ if and only if $\operatorname{Im}\Gamma \subset \operatorname{Im}\sigma$. Also, Eq. (1) for σ says exactly that the vector fields $X \in \mathfrak{X}(M)$ and

$$X^\sigma := \Pi_* \circ X \circ \sigma \in \mathfrak{X}(N) \tag{7}$$

are σ-related, i.e.

$$X \circ \sigma = \sigma_* \circ X^\sigma. \tag{8}$$

Then, if γ is an integral curve of X^σ, the curve $\Gamma := \sigma \circ \gamma$ is an integral curve of X (intersecting $\operatorname{Im}\sigma$). Moreover, every integral curve of X intersecting $\operatorname{Im}\sigma$ can be obtained in this way, as we prove below.

Theorem 2.5 *Let $\sigma : N \to M$ be a solution of the Π-HJE for (M, X). If $\Gamma : I \to M$ is an integral curve of X intersecting $\operatorname{Im}\sigma$, then we can write $\Gamma = \sigma \circ \gamma$ for a unique integral curve γ of X^σ.*

Proof. The X-invariance of $\operatorname{Im}\sigma$ ensures that $\operatorname{Im}\Gamma \subset \operatorname{Im}\sigma$, and consequently $\Pi(\Gamma(t)) = \Pi|_{\operatorname{Im}\sigma}(\Gamma(t))$, for all $t \in I$. So, the second part of Eq. (4) implies the identity $\sigma \circ \Pi \circ \Gamma = \Gamma$. Defining $\gamma := \Pi \circ \Gamma$, we have that $\sigma \circ \gamma = \Gamma$ and from (1) that

$$\gamma'(t) = \Pi_*(\Gamma'(t)) = \Pi_* \circ X(\Gamma(t)) = \Pi_* \circ X \circ \sigma \circ \gamma(t) = X^\sigma(\gamma(t)),$$

$\forall t \in I$, i.e. γ is an integral curve of X^σ. The injectivity of σ ensures that such a curve γ is unique. □

Remark 2.6 *It is easy to see in Example 2.3 that the vector field $X^\sigma \in \mathfrak{X}(N)$ is given by*

$$X^\sigma(n) = \left(n, \hat{X}_1(\sigma(n)), ..., \hat{X}_k(\sigma(n))\right).$$

So, in order to find a trajectory of (M, X) starting in $\operatorname{Im}\sigma$, instead of solving the system of d first order ordinary differential equations (ODE)

$$\dot{m}_i(t) = \hat{X}_i(m(t)), \quad i = 1, ..., d,$$

for a curve $m : I \to M$, being $I \subset \mathbb{R}$ an open interval, it is enough to solve the system of k first order ODEs

$$\dot{n}_i(t) = \hat{X}_i(\sigma(n(t))), \quad i = 1, ..., k,$$

for $n : I \to N$, and then define $m(t) = \sigma(n(t))$.

2.2 The symplectic scenario

In the case of a Hamiltonian system defined on a symplectic manifold (M,ω), we have other characterizations of the extended HJE. As usual, given $m \in M$ and a subspace $V \subset T_m M$, we denote by $V^\omega \subset T_m M$ the symplectic orthogonal of V, and we say that V is **isotropic** (resp. **co-isotropic**) if $V \subset V^\omega$ (resp. $V^\omega \subset V$), and **Lagrangian** if $V = V^\omega$. Also, by $\omega^\flat : TM \to T^*M$ we denote the vector bundle isomorphism given by $\langle \omega^\flat(X), Y \rangle = \omega(X,Y)$, and by $\omega^\sharp : T^*M \to TM$ we denote its inverse.

Definition 2.7 *We shall say that a submersion $\Pi : M \to N$ is **isotropic** (resp. **co-isotropic**) if each fiber of $\operatorname{Ker}\Pi_*$ is isotropic (resp. co-isotropic), i.e.*

$$\operatorname{Ker}\Pi_* \subset (\operatorname{Ker}\Pi_*)^\omega, \tag{9}$$

*(resp. $(\operatorname{Ker}\Pi_*)^\omega \subset \operatorname{Ker}\Pi_*$). Π is **Lagrangian** if it is both isotropic and co-isotropic. We shall say that a section $\sigma : N \to M$ of Π is **isotropic** (resp. **co-isotropic**) if each fiber of $\operatorname{Im}\sigma_*$ is isotropic (resp. co-isotropic), i.e.*

$$\operatorname{Im}\sigma_* \subset (\operatorname{Im}\sigma_*)^\omega$$

*(resp. $(\operatorname{Im}\sigma_*)^\omega \subset \operatorname{Im}\sigma_*$), and that σ is **Lagrangian** if it is both isotropic and co-isotropic.*

It is easy to show that σ is isotropic if and only if $\sigma^*\omega = 0$. Note that isotropy (resp. co-isotropy) condition on σ implies that $2\dim N \leq \dim M$ (resp. $\dim M \leq 2\dim N$).

Theorem 2.8 *Consider a symplectic manifold (M,ω), a function $H : M \to \mathbb{R}$, its Hamiltonian vector field X_H w.r.t. ω, a fibration $\Pi : M \to N$ and a section $\sigma : N \to M$ of Π.*

1. *If σ is a solution of the Π-HJE for (M, X_H), then*

$$d(H\circ\sigma) = i_{X_H^\sigma}\sigma^*\omega, \tag{10}$$

   being X_H^σ the vector field on N defined as in (7), for $X = X_H$.

2. *If in addition σ is isotropic, then*

$$d(H\circ\sigma) = 0. \tag{11}$$

3. *On the other hand, if σ satisfies (10), then*

$$\operatorname{Im}(X_H\circ\sigma - \sigma_*\circ X_H^\sigma) \subset (\operatorname{Ker}\Pi_*) \cap (\operatorname{Im}\sigma_*)^\omega.$$

Proof. (1) If σ is a solution of the Π-HJE for (M, X_H), then [see Eqs. (7) and (8)]

$$X_H\circ\sigma = \sigma_*\circ X_H^\sigma. \tag{12}$$

Using that $X_H = \omega^\sharp \circ dH$, for all $n\in N$ and $y \in T_n N$, we have that

$$\begin{aligned}\omega\left(X_H(\sigma(n)), \sigma_{*,n}(y)\right) &= \langle dH(\sigma(n)), \sigma_{*,n}(y)\rangle \\ &= \langle \sigma^* dH(n), y\rangle = \langle d(H\circ\sigma)(n), y\rangle.\end{aligned} \tag{13}$$

On the other hand,

$$\omega\left(\sigma_{*,n}(X_H^\sigma(n)), \sigma_{*,n}(y)\right) = \sigma^*\omega\left(X_H^\sigma(n), y\right). \tag{14}$$

Using (12) on the first members of (13) and (14), we deduce (10).

(2) The isotropy condition for σ means that $\sigma^*\omega = 0$. Then, Eq. (10) translates to (11).

(3) Given a section σ of Π, it is clear that [recall Eqs. (4) and (7)] $\Pi_* \circ (X_H \circ \sigma - \sigma_* \circ X_H^\sigma) = 0$, i.e.

$$\mathrm{Im}\,(X_H \circ \sigma - \sigma_* \circ X_H^\sigma) \subset \mathrm{Ker}\,\Pi_*.$$

On the other hand, if σ satisfies (10), from (13) and (14) we have that

$$\omega\left(X_H(\sigma(n)) - \sigma_{*,n}(X_H^\sigma(n)), \sigma_{*,n}(y)\right) = 0$$

for all $n \in N$ and $y \in T_n N$, i.e. $\mathrm{Im}\,(X_H \circ \sigma - \sigma_* \circ X_H^\sigma) \subset (\mathrm{Im}\,\sigma_*)^\omega$, and the Theorem is proved. □

Let us emphasize that, in general, we can not ensure that the Π-HJE for a Hamiltonian system on a symplectic manifold is equivalent to Eq. (10). Some conditions that enable us to do that are given below.

Corollary 2.9 *Under the conditions of Theorem 2.8, if in addition the fibration Π is isotropic (or in particular if Π is Lagrangian), we have that σ is a solution of the Π-HJE for (M, X_H) if and only if (10) holds.*

Proof. The first implication was shown in the point (1) of Theorem 2.8. We have to show the converse. Since Π is isotropic, then [see Eqs. (5) and (9)], for all $m \in \mathrm{Im}\,\sigma$,

$$\begin{aligned}\mathrm{Ker}\,\Pi_{*,m} \cap \left(\mathrm{Im}\,\sigma_{*,\Pi(m)}\right)^\omega &\subset (\mathrm{Ker}\,\Pi_{*,m})^\omega \cap \left(\mathrm{Im}\,\sigma_{*,\Pi(m)}\right)^\omega \\ &= \left(\mathrm{Ker}\,\Pi_{*,m} + \mathrm{Im}\,\sigma_{*,\Pi(m)}\right)^\omega = (T_m M)^\omega = 0,\end{aligned}$$

and accordingly [see the item (3) of Theorem 2.8] $X_H \circ \sigma - \sigma_* \circ X_H^\sigma = 0$, i.e. σ is a solution of the Π-HJE for (M, X_H) [see Eq. (8) for $X = X_H$]. □

Example 2.10 *Consider a Lagrangian system on a manifold Q with regular Lagrangian function $L : TQ \to \mathbb{R}$, and let $E_L : TQ \to \mathbb{R}$ be the* energy function *associated to L (see [1, 4] and references therein). Regularity of L implies that its* Cartan 2-form ω_L *is a symplectic form on TQ, and consequently the trajectories of the system are given by the integral curves of the vector field*

$$X_L = \omega_L^\sharp(dE_L). \tag{15}$$

So, we have the dynamical system (TQ, X_L). Consider the canonical projection $\tau_Q : TQ \to Q$. The τ_Q-HJE for (TQ, X_L) is given by

$$\sigma_* \circ (\tau_Q)_* \circ X_L \circ \sigma = X_L \circ \sigma,$$

being $\sigma : Q \to TQ$ a section of τ_Q [note that $\sigma \in \mathfrak{X}(Q)$]. As it is well-known, X_L is a section of $(\tau_Q)_*$, i.e. X_L is a second order vector field*, then*

$$X_L^\sigma := (\tau_Q)_* \circ X_L \circ \sigma = \sigma, \tag{16}$$

and accordingly the τ_Q-HJE for (TQ, X_L) reduces to

$$\sigma_* \circ \sigma = X_L \circ \sigma. \tag{17}$$

Last equation was introduced in Ref. [10] under the name of generalized Lagrangian Hamilton-Jacobi problem*. Since τ_Q is a Lagrangian fibration, Corollary 2.9 says that Eq.* (17) *is equivalent to $d(E_L \circ \sigma) = i_\sigma \sigma^* \omega_L$ [recall Eq.* (16)*], as it was also shown in [10].*

The following result tells us what happen in the case when the section σ is co-isotropic or Lagrangian.

Corollary 2.11 *Under the conditions of Theorem 2.8, if in addition*

1. *σ is co-isotropic, then σ is a solution of the Π-HJE for (M, X_H) if and only if $d(H\circ\sigma) = i_{X_H^\sigma}\sigma^*\omega$.*

2. *σ is Lagrangian, then σ is a solution of the Π-HJE for (M, X_H) if and only if $d(H\circ\sigma) = 0$.*

Proof. (1) As in the previous corollary, we must show that, for all $m \in \operatorname{Im}\sigma$,

$$\operatorname{Ker}\Pi_{*,m} \cap \left(\operatorname{Im}\sigma_{*,\Pi(m)}\right)^\omega = 0.$$

From (5), it follows that

$$\left(\operatorname{Ker}\Pi_*|_{\operatorname{Im}\sigma}\right)^\omega \oplus \left(\operatorname{Im}\sigma_*\right)^\omega = T_{\operatorname{Im}\sigma}M.$$

So, using the co-isotropy of σ, for all $m \in \operatorname{Im}\sigma$,

$$\begin{aligned}\operatorname{Ker}\Pi_{*,m} \cap \left(\operatorname{Im}\sigma_{*,\Pi(m)}\right)^\omega &\subset \operatorname{Ker}\Pi_{*,m} \cap \operatorname{Im}\sigma_{*,\Pi(m)} \\ &= \left(\left(\operatorname{Ker}\Pi_{*,m}\right)^\omega + \left(\operatorname{Im}\sigma_{*,\Pi(m)}\right)^\omega\right)^\omega = \left(T_m M\right)^\omega = 0,\end{aligned}$$

as we wanted to show.

(2) Combining the point (1) of this corollary and the point (2) of Theorem 2.8, the result easily follows. □

Example 2.12 *Coming back to Example 2.10, if we ask σ to be isotropic w.r.t. ω_L, then σ is Lagrangian, for dimensional reasons, and consequently it must satisfy $d(E_L\circ\sigma) = 0$ and $\sigma^*\omega_L = 0$. These equations were called the* Lagrangian Hamilton-Jacobi problem *in Ref. [10].*

2.2.1 The standard Hamilton-Jacobi Theory

In this section we shall see how to recover the classical HJE. Suppose that $M = T^*Q$ for some manifold Q, $\omega = \omega_Q = d\theta$, where θ is the Liouville 1-form (i.e. ω is the canonical 2-form on T^*Q), $H : T^*Q \to \mathbb{R}$ is a function and $\Pi = \pi_Q : T^*Q \to Q$ is the canonical cotangent projection. Note that the sections of π_Q are the 1-forms on Q [i.e. they belong to $\Omega^1(Q)$], π_Q is a Lagrangian fibration, and[3]

$$\pi_{Q*}\circ X_H = \mathbb{F}H. \tag{19}$$

Then, the π_Q-HJE for (T^*Q, X_H) is an equation for a 1-form $\sigma \in \Omega^1(Q)$ and it reads

$$\sigma_*\circ\mathbb{F}H\circ\sigma = X_H\circ\sigma. \tag{20}$$

Moreover, $X_H^\sigma = \mathbb{F}H\circ\sigma$ and, since π_Q is Lagrangian, Corollary 2.9 says that (20) is equivalent to [see Eq. (10)]

$$d(H\circ\sigma) = i_{\mathbb{F}H\circ\sigma}\sigma^*\omega.$$

Recall that $\sigma^*\theta = \sigma$ for every $\sigma \in \Omega^1(Q)$, and consequently

$$\sigma^*\omega = \sigma^* d\theta = d\sigma^*\theta = d\sigma. \tag{21}$$

Hence, the π_Q-HJE for (T^*Q, X_H) can be written

$$d(H\circ\sigma) = i_{\mathbb{F}H\circ\sigma}d\sigma. \tag{22}$$

In Ref. [10], this last equation was called *generalized Hamiltonian Hamilton-Jacobi problem.*

[3] Given a function $f : T^*Q \to \mathbb{R}$, by $\mathbb{F}f : T^*Q \to TQ$ we are denoting the *fiber derivative of* f, given by

$$\langle\alpha, \mathbb{F}f(\beta)\rangle = \left.\frac{d}{ds}\right|_0 f(\alpha + s\,\beta), \quad \forall q \in Q, \quad \forall \alpha, \beta \in T_q^*Q. \tag{18}$$

Remark 2.13 *Another extension of the Hamilton-Jacobi Theory was presented in Reference [21]. Its related (time-independent) Hamilton-Jacobi equation is given precisely by the Eq.* (22) *(see the Equation* 1.6 *of that reference). It was called* Lamb's equation *in [21].*

The *classical* case is obtained when we look for a solution σ of (22) such that $d\sigma = 0$. In this case, Eq. (22) translates to[4]

$$d(H \circ \sigma) = 0 \quad \text{and} \quad d\sigma = 0, \tag{23}$$

which defines the so-called *Hamiltonian Hamilton-Jacobi problem* of Ref. [10].

Definition 2.14 *From now on, we shall mean by* ***standard*** *(resp.* ***standard Lagrangian or classical) situation*** *the case in which $M = T^*Q$ for some manifold Q, ω is its canonical symplectic form, $X = X_H$ for some function $H : T^*Q \to \mathbb{R}$, $N = Q$ and $\Pi = \pi_Q$ (resp. and $d\sigma = 0$). And we shall refer to* (22) *[resp.* (23)*] as the* ***standard*** *(resp.* ***standard Lagrangian or classical) HJE****.*

Every solution of (23) is locally given by a primitive function $W : V \to \mathbb{R}$, i.e. $\sigma|_V = dW$, where V is an open submanifold of Q. The function W is usually called **characteristic Hamilton function**.

2.2.2 Some examples: standard and nonstandard equations

Let us take $M := \mathbb{R}^k \times \mathbb{R}^k$ and the symplectic structure $\omega := dq^i \wedge dp_i$ (summation over repeated indices is assumed from now on), being (q, p) the natural global coordinates of $\mathbb{R}^k \times \mathbb{R}^k$. Note that $\theta = p_i \, dq^i$. Fix $N = \mathbb{R}^k$ and consider the submersions $\mathfrak{p}_r : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, $r = 1, 2$, given by the projectors on the first and second $\mathbb{R}^k$ factors, respectively. It is clear that $\operatorname{Ker}(\mathfrak{p}_r)_*$ is Lagrangian for $r = 1, 2$. As a consequence, given any function $H : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, the $\mathfrak{p}_r$-HJE for $\left(\mathbb{R}^k \times \mathbb{R}^k, X_H\right)$ is given by

$$d(H \circ \sigma) = i_{X_H^{\sigma,r}} \sigma^* \omega, \tag{24}$$

where $\sigma : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k$ is a section of $\mathfrak{p}_r$ and $X_H^{\sigma,r} = (\mathfrak{p}_r)_* \circ X_H \circ \sigma$. It is also clear that the $r = 1$ case corresponds to the standard HJE [see Eq. (22)]. In fact, under the identifications mentioned above, $X_H^{\sigma,1} = \mathbb{F}H \circ \sigma$ and $\sigma^*\omega = d\sigma$, so Eq. (24) reduces to Eq. (22). Writing $\sigma(q) = (q, \hat{\sigma}(q))$, it is easy to see that Eq. (24) results

$$\underbrace{\frac{\partial H(q, \hat{\sigma}(q))}{\partial q^i}}_{d(H\circ\sigma)} = \underbrace{\left(\frac{\partial \hat{\sigma}_j(q)}{\partial q^i} - \frac{\partial \hat{\sigma}_i(q)}{\partial q^j}\right)}_{\sigma^*\omega = d\sigma} \underbrace{\frac{\partial H}{\partial p_j}(q, \hat{\sigma}(q))}_{X_H^{\sigma,1} = \mathbb{F}H\circ\sigma}, \qquad i = 1, ..., k, \tag{25}$$

where $\hat{\sigma}_i$ is the i-th component of $\hat{\sigma}$.

Remark 2.15 *Since we can write* $X_H(q, p) = \left(q, p, \hat{X}_H(q, p)\right)$ *with*

$$\hat{X}_H(q, p) = \left(\frac{\partial H}{\partial p_1}(q, p), ..., \frac{\partial H}{\partial p_k}(q, p), -\frac{\partial H}{\partial q^1}(q, p), ..., -\frac{\partial H}{\partial q^k}(q, p)\right),$$

Eq. (25) *can be derived from the PDEs* (2) *for $d = 2l = 2k$ and*

$$\hat{X}_j(q, p) = \begin{cases} \dfrac{\partial H}{\partial p_j}(q, p), & 1 \leq j \leq k, \\[2ex] -\dfrac{\partial H}{\partial q^{j-k}}(q, p), & k + 1 \leq j \leq 2k. \end{cases}$$

[4] This also can be seen in the following way. The condition $d\sigma = 0$ says exactly that σ is a Lagrangian section. Thus, according to Corollary 2.11, the π_Q-HJE (20) reduces precisely to (23).

For the $r=2$ case, writing $\sigma(p) = (\hat{\sigma}(p), p)$, Eq. (24) is given by

$$\underbrace{\frac{\partial H(\hat{\sigma}(p), p)}{\partial p_i}}_{d(H\circ\sigma)} = \underbrace{\left(\frac{\partial \hat{\sigma}^j(p)}{\partial p_i} - \frac{\partial \hat{\sigma}^i(p)}{\partial p_j}\right)}_{\sigma^*\omega} \underbrace{\frac{\partial H}{\partial q^j}(\hat{\sigma}(p), p)}_{X_H^{\sigma,2}}, \qquad i = 1, ..., k. \tag{26}$$

Note that isotropy condition $\sigma^*\omega = 0$ reads

$$\partial_i \hat{\sigma}^j - \partial_j \hat{\sigma}^i = 0 \tag{27}$$

in both cases. Eq. (26) is an example of a nonstandard HJE. Let us focus on the $k=1$ case. The Equations (25) and (26) translate to

$$\frac{\partial H(q, \hat{\sigma}(q))}{\partial q} = 0 \quad \text{and} \quad \frac{\partial H(\hat{\sigma}(p), p)}{\partial p} = 0,$$

respectively, since the second member is identically zero in both of the equations. Suppose that $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(q, p) = \frac{1}{2}\left(p^2 + f(q)\right),$$

for some function $f : \mathbb{R} \to \mathbb{R}$. Then, (25) and (26) reduce to the ODEs

$$\frac{1}{2}\left[\left(\hat{\sigma}^2(q)\right)' + f'(q)\right] = 0 \quad \text{and} \quad p + \frac{1}{2}\left[f(\hat{\sigma}(p))\right]' = 0.$$

For the first one, the solutions are given by the formulae

$$\hat{\sigma}^+_\lambda(q) = \sqrt{\lambda - f(q)} \quad \text{and} \quad \hat{\sigma}^-_\lambda(q) = -\sqrt{\lambda - f(q)}, \quad \lambda \in \mathbb{R};$$

while for the second one, if f is monotonic in some open interval, the solutions are given by

$$\hat{\sigma}_\lambda(p) = f^{-1}\left(\lambda - p^2\right), \quad \lambda \in \mathbb{R}.$$

Remark 2.16 *Since we are working on a* 1*-dimensional manifold, functions* $\hat{\sigma}_\lambda$ *above always have a primitive* W_λ.

Of course, the Equations (25) and (26) (and consequently their solutions) coincide when $f(q) = \frac{1}{2}q^2$: the 1-*dimensional harmonic oscillator*.

2.3 The Poisson, almost-Poisson and Leibniz scenarios

Given a manifold M, a *Leibniz structure* on M is a $(0,2)$-tensor $\Xi : T^*M \times T^*M \to \mathbb{R}$ (see Ref. [28]). If Ξ is anti-symmetric, then Ξ is an *almost-Poisson structure* [7, 9, 29], and if, in addition,

$$\Xi(d[\Xi(df, dg)], dh) + \Xi(d[\Xi(dh, df)], dg) + \Xi(d[\Xi(dg, dh)], df) = 0,$$

for all $f, g, h \in C^\infty(M)$, i.e. the *Jacobi identity* holds, then Ξ is a Poisson structure. The pair (M, Ξ) is called a Leibniz, an almost-Poisson and a Poisson manifold, respectively. In any case, we shall define the morphisms of vector bundles $\Xi^\sharp, \Xi^{\sharp op} : T^*M \to TM$ by the formulae

$$\left\langle \alpha, \Xi^\sharp(\beta)\right\rangle = \Xi(\alpha, \beta) \quad \text{and} \quad \left\langle \alpha, \Xi^{\sharp op}(\beta)\right\rangle = -\Xi(\beta, \alpha).$$

Note that $\Xi^{\sharp *} = -\Xi^{\sharp op}$, and consequently

$$\operatorname{Ker} \Xi^\sharp = \left(\operatorname{Im} \Xi^{\sharp op}\right)^0. \tag{28}$$

Of course, $\Xi^\sharp = \Xi^{\sharp op}$ for almost-Poisson and Poisson structures. Also, for a Poisson manifold, $\operatorname{Im} \Xi^\sharp = TM$ (i.e. the Poisson manifold is *transitive*) if and only if there exists a symplectic form ω on M such that $\Xi^\sharp = \omega^\sharp$.

Consider a Leibniz manifold (M, Ξ), a function $H : M \to \mathbb{R}$ and the related vector field $X_H = \Xi^\sharp \circ dH$. The dynamical system (M, X_H) is called a *Hamiltonian system in a Leibniz manifold* or simply a *Leibniz system*. Examples of Leibniz systems are given by the *generalized nonholonomic systems* with linear constraints (see [12] and references therein; see also the examples appearing in Ref. [28]). Standard nonholonomic systems correspond to the case in which Ξ is anti-symmetric, i.e. an almost Poisson structure (see again Refs. [7, 9, 29]).

Remark 2.17 *The manifold M corresponding to a generalized and a standard nonholonomic system is given by a co-distribution*[5] $D \subset T^*Q$ *on some manifold Q: the* constraint co-distribution*. So, the restriction* $\Pi := \pi_Q|_D : D \to Q$ *is a natural fibration for this kind of systems.*

Given a fibration $\Pi : M \to N$, we shall briefly study the Π-HJE for a Leibniz system (M, X_H). In particular, we shall show that, in our setting, the Hamilton-Jacobi Theory developed in [24] for almost-Poisson systems is recovered.

Fix a Leibniz manifold (M, Ξ) and a point $m \in M$. We shall say that a subspace $V \subset T_m M$ is **weakly isotropic** (resp. **co-isotropic**) if

$$V \cap \operatorname{Im} \Xi^\sharp \subset \Xi^\sharp \left(V^0\right) \tag{29}$$

(resp. $\Xi^\sharp \left(V^0\right) \subset V$), and **weakly Lagrangian** if $V \cap \operatorname{Im} \Xi^\sharp = \Xi^\sharp \left(V^0\right)$. On the other hand, V is **isotropic** if it is weakly isotropic and

$$V \subset \operatorname{Im} \Xi^{\sharp op}, \tag{30}$$

and **Lagrangian** if it is isotropic and co-isotropic.[6] In parallel with Definition 2.7, we have the next one.

Definition 2.18 *Given a fibration $\Pi : M \to N$ and a section $\sigma : N \to M$ of Π, we shall say that Π (resp. σ) is **(weakly) isotropic**, **co-isotropic** or **(weakly) Lagrangian** if so is each fiber of $\operatorname{Ker} \Pi_*$ (resp. $\operatorname{Im} \sigma_*$).*

Note that, when Ξ is an almost-Poisson structure, $\sigma : N \to M$ is isotropic if and only if

$$\operatorname{Im} \sigma_* \subset \Xi^\sharp \left[\left(\operatorname{Im} \sigma_*\right)^0\right],$$

since $\operatorname{Im} \Xi^\sharp = \operatorname{Im} \Xi^{\sharp op}$. Of course, if (M, Ξ) is a symplectic manifold, since $\operatorname{Im} \Xi^\sharp = TM$, the above notions of weakly isotropic (resp. weakly Lagrangian) and isotropic (resp. Lagrangian) coincide with that given at the beginning of Section 2.2.

Proposition 2.19 *Fix a Leibniz manifold (M, Ξ), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a section $\sigma : N \to M$ of Π.*

1. *If σ is a solution of the Π-HJE for (M, X_H) and σ is weakly isotropic, then*

$$\operatorname{Im}\left(dH|_{\operatorname{Im}\sigma}\right) \subset \left(\operatorname{Im} \sigma_* \cap \operatorname{Im} \Xi^{\sharp op}\right)^0. \tag{31}$$

   If in addition σ is isotropic, then

$$d\left(H \circ \sigma\right) = 0. \tag{32}$$

2. *On the other hand, if σ satisfies Eq. (31) [or Eq. (32)] and σ is co-isotropic, then σ is a solution of the Π-HJE for (M, X_H).*

[5] By a co-distribution on Q we mean the canonical dual notion of a regular distribution of constant rank.

[6] Our notion of weakly isotropic (resp. weakly Lagrangian), corresponds to the notion of isotropic (resp. Lagrangian) of Ref. [24].

Proof. (1) Suppose that a solution σ of the Π-HJE for (M, X_H) is given. According to Proposition 2.4, this is equivalent to $\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) \subset T\operatorname{Im}\sigma = \operatorname{Im}\sigma_*$. Since in addition

$$\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) = \operatorname{Im}\left(\Xi^\sharp \circ dH|_{\operatorname{Im}\sigma}\right) \subset \operatorname{Im}\Xi^\sharp,$$

then σ is a solution of the Π-HJE if and only if

$$\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) \subset \operatorname{Im}\sigma_* \cap \operatorname{Im}\Xi^\sharp. \tag{33}$$

On the other hand, since $\operatorname{Im}\sigma_* \cap \operatorname{Im}\Xi^\sharp \subset \Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right]$ [see (29)], Eq. (33) implies that

$$\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) \subset \Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right],$$

i.e. $\Xi^\sharp(dH(m)) = X_H(m) \in \Xi^\sharp\left[(\operatorname{Im}\sigma_{*,m})^0\right]$ for all $m \in \operatorname{Im}\sigma$. Then,

$$dH(m) \in (\operatorname{Im}\sigma_{*,m})^0 + \operatorname{Ker}\Xi^\sharp_m = \left(\operatorname{Im}\sigma_{*,m} \cap \left(\operatorname{Ker}\Xi^\sharp_m\right)^0\right)^0,$$

or using Eq. (28),

$$dH(m) \in \left(\operatorname{Im}\sigma_{*,m} \cap \operatorname{Im}\Xi^{\sharp op}_m\right)^0, \quad \forall m \in \operatorname{Im}\sigma.$$

This is precisely Eq. (31). If in addition $\operatorname{Im}\sigma_* \cap \operatorname{Im}\Xi^{\sharp op} = \operatorname{Im}\sigma_*$ [see Eq. (30)], the last equation implies that $\sigma^*_m(dH(m)) = 0$ for all $m \in \operatorname{Im}\sigma$, i.e. $d(H \circ \sigma) = 0$, as we wanted to show.

(2) Applying $\Xi^\sharp$ to (31) we obtain that

$$\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) \subset \Xi^\sharp\left[\left(\operatorname{Im}\sigma_* \cap \operatorname{Im}\Xi^{\sharp op}\right)^0\right] = \Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right] + \Xi^\sharp\left[\left(\operatorname{Im}\Xi^{\sharp op}\right)^0\right] = \Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right],$$

where we have used Eq. (28) in the last step. Thus,

$$\operatorname{Im}(X_H|_{\operatorname{Im}\sigma}) \subset \Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right].$$

The same we would obtain from Eq. (32). Since in addition $\Xi^\sharp\left[(\operatorname{Im}\sigma_*)^0\right] \subset \operatorname{Im}\sigma_*$ (i.e. σ is co-isotropic), the Equation (33) follows immediately from the above one, i.e. σ is a solution of the Π-HJE for (M, X_H). □

Eq. (31) is exactly the Hamilton-Jacobi equation that appears in Ref. [24] for almost-Poisson systems with a fibration Π (see Theorem 2.3 of Ref. [24]). Eq. (32), on the other hand, is just the equation found in the same reference for the systems defined on a symplectic manifold (see Theorem 2.4 of [24]): the classical HJE. The next result, which is an immediate consequence of the above proposition, shows that the mentioned equations are equivalent to our Π-HJE for weakly Lagrangian and Lagrangian sections, respectively.

Corollary 2.20 *Under the conditions of the above proposition,*

1. *if σ is weakly Lagrangian, then σ is a solution of the Π-HJE for (M, X_H) if and only if Eq. (31) holds.*
2. *if σ is Lagrangian, then σ is a solution of the Π-HJE for (M, X_H) if and only if Eq. (32) holds.*

Consider a nonholonomic system with constraint co-distribution $D \subset T^*Q$ (see Remark 2.17). The Hamilton-Jacobi equation obtained in Ref. [24] for this system (seen as an almost-Poisson system), and for the natural fibration $\Pi := \pi_Q|_D : D \to Q$, coincides with the so-called *nonholonomic Hamilton-Jacobi equation* of Refs. [11, 23, 5]. Then, based on the last corollary, we can conclude that the nonholonomic Hamilton-Jacobi equation is precisely our $\pi_Q|_D$-HJE (restricted to weakly Lagrangian and Lagrangian sections).

3 Complete solutions

Fix again a dynamical system (M, X) and a fibration $\Pi : M \to N$. Following previous works on the subject (see for instance Ref. [10]), in this section we shall introduce and study the notion of a complete solution of the Π-HJE for (M, X). We shall see that this tool can be used to construct local coordinates in which the equation of motions of the system can be substantially simplified. Moreover, we shall see, in the context of Hamiltonian systems on symplectic manifolds, that for certain complete solutions and certain fibrations Π, the trajectories of X can be obtained up to quadratures (as in the classical situation). This result will be extended to Poisson manifolds (and to more general fibrations Π) in the last section of the paper.

3.1 Definition and basic properties

Let Λ be an l-manifold with $l := \dim M - \dim N$.

Definition 3.1 *We shall say that* $\Sigma : N \times \Lambda \to M$ *is a* ***complete solution of the*** Π-***HJE*** *for* (M, X) *if*

T1 Σ *is surjective,*

T2 Σ *is a local diffeomorphism (*ipso facto *an open map),*

T3 *for each* $\lambda \in \Lambda$*, the map*

$$\sigma_\lambda := \Sigma(\cdot, \lambda) : n \in N \longmapsto \Sigma(n, \lambda) \in M \tag{34}$$

is a solution of the Π-*HJE for* (M, X)*, i.e. is a section of* Π *solving* (1)*. Each map* σ_λ *will be called* ***partial solution****.*

Given an open subset $U \subset M$*, we shall say that* $\Sigma : \Pi(U) \times \Lambda \to U$ *is a* ***local complete solution on*** U ***of the*** Π-***HJE*** *for* (M, X) *if it is a complete solution of the* $\Pi|_U$-*HJE for* $(U, X|_U)$ *(recall Definition 2.1).*

It is easy to show that, if the surjectivity condition (i.e. condition $T1$) on $\Sigma : N \times \Lambda \to M$ is not fulfilled, then Σ still defines (by co-restriction) a global complete solution, provided we change the phase space M of the system by the open submanifold $\Sigma(N \times \Lambda) \subset M$.

Example 3.2 *Going back to Example 2.3, suppose that a family of solutions* $\hat{\sigma}_\lambda$ *of the Eq.* (2)*, with* $\lambda \in \mathbb{R}^l$*, is given. It is easy to show that*

$$\Sigma : (n, \lambda) \in \mathbb{R}^k \times \mathbb{R}^l \longmapsto (n, \hat{\sigma}_\lambda(n)) \in \mathbb{R}^k \times \mathbb{R}^l$$

is a local diffeomorphism if and only if the $l \times l$ *matrix with coefficients* $\partial (\hat{\sigma}_\lambda)_i / \partial \lambda_j$ *is non-degenerate for all* (n, λ)*. In such a case, according to above discussion,* Σ *defines a global complete solution with* $\Lambda = \mathbb{R}^l$ *(taking the phase space equal to the image of* Σ*).*

Existence conditions of complete solutions will be studied in Section 4.3.1. Now, let us study some consequences of having a complete solution Σ. First, we shall focus on the properties $T1$ and $T3$ of Σ. Let σ_λ be a partial solution, and define

$$M_\lambda := \operatorname{Im} \sigma_\lambda \quad \text{and} \quad X^{\sigma_\lambda} := \Pi_* \circ X \circ \sigma_\lambda \in \mathfrak{X}(N). \tag{35}$$

According to the discussions we made in Section 2.1, each M_λ is a closed regular submanifold diffeomorphic to N, the vector field $X \in \mathfrak{X}(M)$ restricts to M_λ (see Proposition 2.4), and X and X^{σ_λ} are σ_λ-related. On the other hand, since Σ is surjective,

$$M = \Sigma(N \times \Lambda) = \bigcup_{\lambda \in \Lambda} \sigma_\lambda(N) = \bigcup_{\lambda \in \Lambda} M_\lambda.$$

From this point, using Theorem 2.5, it can be shown that all of the trajectories of X can be constructed from those of the vector fields X^{σ_λ}'s. We shall see that below, in terms of the following characterization of the complete solutions. Denote by $p_N : N \times \Lambda \to N$ and $p_\Lambda : N \times \Lambda \to \Lambda$ the canonical projections.

Proposition 3.3 *A surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$ is a complete solution of the Π-HJE for (M, X) if and only if*

$$\Pi \circ \Sigma = p_N \quad \text{and} \quad \Sigma_* \circ X^\Sigma = X \circ \Sigma, \tag{36}$$

being $X^\Sigma \in \mathfrak{X}(N \times \Lambda)$ the unique vector field on $N \times \Lambda$ satisfying

$$(p_N)_* \circ X^\Sigma = \Pi_* \circ X \circ \Sigma \quad \text{and} \quad (p_\Lambda)_* \circ X^\Sigma = 0. \tag{37}$$

In particular, the fields X and X^Σ are Σ-related.

Proof. Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE with partial solutions $\sigma_\lambda : N \to M$. Since each σ_λ is a section of Π, then

$$\Pi \circ \Sigma (n, \lambda) = \Pi \circ \sigma_\lambda (n) = n = p_N (n, \lambda),$$

from which the first part of (36) follows. Let us prove the second one. First, note that the vector field satisfying (37) is given on (n, λ) by

$$X^\Sigma (n, \lambda) = (\Pi_* \circ X \circ \Sigma (n, \lambda), 0).$$

Then, using Eq. (1) for each σ_λ, we have that

$$\begin{aligned} \Sigma_* \circ X^\Sigma (n, \lambda) &= \Sigma_{*,(n,\lambda)} (\Pi_* \circ X \circ \Sigma (n, \lambda), 0) = (\sigma_\lambda)_{*,n} (\Pi_* \circ X \circ \sigma_\lambda (n)) \\ &= (\sigma_\lambda)_* \circ \Pi_* \circ X \circ \sigma_\lambda (n) = X \circ \sigma_\lambda (n) = X \circ \Sigma (n, \lambda), \end{aligned}$$

and the second part of (36) is obtained. The converse is left to the reader. □

Using (35) and (37), it is easy to see that

$$X^\Sigma (n, \lambda) = (\Pi_* \circ X \circ \Sigma (n, \lambda), 0) = (X^{\sigma_\lambda} (n), 0), \quad \forall (n, \lambda) \in N \times \Lambda. \tag{38}$$

Then, any integral curve of X^Σ is of the form $t \mapsto (\gamma(t), \lambda)$, for some $\lambda \in \Lambda$ and some integral curve γ of X^{σ_λ}. On the other hand, according to the last proposition, X and X^Σ are Σ-related. Consequently, given an integral curve γ of X^{σ_λ},

$$\Gamma(t) = \Sigma(\gamma(t), \lambda) = \sigma_\lambda (\gamma(t)) \tag{39}$$

is an integral curve of X. Moreover, using the surjectivity of Σ and Theorem 2.5, every trajectory of (M, X) can be obtained in that way. More precisely,

Theorem 3.4 *Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE for (M, X). For every integral curve $\Gamma : I \to M$ of X there exists $\lambda \in \Lambda$ such that we can write $\Gamma = \sigma_\lambda \circ \gamma$ for a unique integral curve γ of X^{σ_λ}.*

Let us now exploit the condition $T2$, i.e. the fact that Σ is a local diffeomorphism. It is clear that, for every $m \in M$ there exist $(n, \lambda) \in N \times \Lambda$ and open charts (U, ψ), (V_N, ψ_N) and $(V_\Lambda, \psi_\Lambda)$ of M, N and Λ, respectively, such that $\Sigma(n, \lambda) = m$, $n \in V_N$, $\lambda \in V_\Lambda$, $\Sigma(V_N \times V_\Lambda) \subset U$ and

$$\Sigma|_{V_N \times V_\Lambda} : V_N \times V_\Lambda \to U$$

is a diffeomorphism.

Proposition 3.5 *If $\Sigma : N \times \Lambda \to M$ is a complete solution of the Π-HJE for (M, X), then for every $m \in M$ there exist an open neighborhood U of m and coordinates $(n_1, ..., n_{d-l}, \lambda_1, ..., \lambda_l) : U \to \mathbb{R}^d$, with $d := \dim M$, such that*

$$X|_U = \sum\nolimits_{i=1}^{d-l} f_i \frac{\partial}{\partial n_i}$$

for some functions $f_i : U \to \mathbb{R}$.

Proof. Given $m \in M$, consider the local charts (V_N, ψ_N) and $(V_\Lambda, \psi_\Lambda)$ and the open subset U described previously. Then, U and the map

$$\Phi := (\psi_N \times \psi_\Lambda) \circ \left(\Sigma|_{V_N \times V_\Lambda}\right)^{-1} : U \to \mathbb{R}^{d-l} \times \mathbb{R}^l = \mathbb{R}^d$$

define a local chart of M. Fix $u \in U$ and let $(n, \lambda) \in V_N \times V_\Lambda$ such that $u = \Sigma(n, \lambda) \in U$. Then, using (38) and the fact that X and X^Σ are Σ-related, we have that

$$\begin{aligned}\Phi_* (X(u)) &= (\psi_N \times \psi_\Lambda)_* \circ \left(\Sigma|_{V_N \times V_\Lambda}\right)_*^{-1} (X(u)) = (\psi_N \times \psi_\Lambda)_* \left[X^\Sigma \left(\Sigma|_{V_N \times V_\Lambda}\right)_*^{-1} (u)\right] \\ &= (\psi_N \times \psi_\Lambda)_* \left[X^\Sigma (n, \lambda)\right] = (\psi_N \times \psi_\Lambda)_* (X^{\sigma_\lambda}(n), 0) = ((\psi_N)_* (X^{\sigma_\lambda}(n)), 0),\end{aligned}$$

from which the proposition immediately follows. □

Given a local chart as in proposition above, it is clear that the equations of motion for X along U read

$$\dot{n}_i(t) = f_i(n(t), \lambda(t)) \quad \text{and} \quad \dot{\lambda}_j(t) = 0, \tag{40}$$

for $i = 1, ..., d-l$ and $j = 1, ..., l$. So, as we have seen in Remark 2.6, in order to find the trajectories of X, we only need to solve $d-l$ equations, instead of d. In particular, if $d - l = 1$, it is easy to see that the Equations (40) can be solved up to the quadrature

$$t = \int_{n_0}^{n(t)} \frac{ds}{f_1(s, \lambda_0)}, \quad \text{with} \quad n_0 := n(0) \quad \text{and} \quad \lambda_0 := (\lambda_1(0), ..., \lambda_{d-1}(0)). \tag{41}$$

Here, we have supposed that $f_1(s, \lambda_0)$ is not zero in a certain neighborhood of (n_0, λ_0). If $d - l > 1$, we shall see in Section 5 that, in the symplectic and Poisson contexts, certain complete solutions enable us to find coordinates $(n_1, ..., n_{d-l})$ for N such that above equations can be solved by quadratures too. A special situation is described in Section 3.3, where the mentioned coordinates are Darboux coordinates.

3.2 The Leibniz and symplectic scenarios

In the context of Leibniz systems (see Section 2.3), we have a very simple characterization of the complete solutions. Let us first note that, if (M, Ξ) is a Leibniz (resp. almost-Poisson and Poisson) manifold and $\Phi : P \to M$ is a local diffeomorphism, then the assignment

$$(\alpha, \beta) \mapsto \left\langle \alpha, \Phi_{*,p}^{-1} \circ \Xi_{\Phi(p)}^\sharp \circ \left(\Phi_p^*\right)^{-1} (\beta) \right\rangle, \quad \forall p \in P \text{ and } \forall \alpha, \beta \in T_p^* P, \tag{42}$$

defines a Leibniz (resp. almost-Poisson and Poisson) structure on P. We shall denote it $\Phi^*\Xi$. On the other hand, in the particular case of a symplectic manifold (M, ω), its clear that the pull-back $\Phi^*\omega \in \Omega^2(P)$ is closed and non-degenerated, so $\Phi^*\omega$ is a symplectic form on P.

Theorem 3.6 *Consider a Leibniz (resp. almost-Poisson, Poisson and symplectic) manifold (M, Ξ), a function $H : M \to \mathbb{R}$ and a fibration $\Pi : M \to N$. Then, a surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$ is a complete solution of the Π-HJE for (M, X_H) if and only if $\Pi \circ \Sigma = p_N$ and $X_H^\Sigma \in \mathfrak{X}(N \times \Lambda)$ [see (37)] is the Hamiltonian vector field of $H \circ \Sigma$ w.r.t. the Leibniz (resp. almost-Poisson, Poisson and symplectic) structure $\Sigma^*\Xi$, i.e.*

$$X_H^\Sigma = (\Sigma^*\Xi)^\sharp (\Sigma^* dH). \tag{43}$$

Proof. We have to show that the second part of (36) for $X = X_H$ is equivalent to (43). Since $X_H = \Xi^\sharp \circ dH$, Eq. (36) says that

$$\Sigma_{*,(n,\lambda)} \left(X_H^\Sigma (n,\lambda) \right) = \Xi^\sharp_{\Sigma(n,\lambda)} \left(dH \left(\Sigma \left(n,\lambda \right) \right) \right),$$

and since $dH\left(\Sigma\left(n,\lambda\right)\right) = \left(\Sigma^*_{(n,\lambda)}\right)^{-1} \left(d\left(H\circ\Sigma\right)\left(n,\lambda\right)\right)$, the Theorem immediately follows from the definition of $\Sigma^*\Xi$ [see Eq. (42)]. $\square$

So, every complete solution Σ for a Leibniz system (M, X_H), with Leibniz structure Ξ, is a Leibniz map between (M,Ξ) and $(N \times \Lambda, \Sigma^*\Xi)$, and the related dynamical system $\left(N \times \Lambda, X_H^\Sigma\right)$ is also a Leibniz system, with Hamiltonian function $H \circ \Sigma$. The same is true if we replace "Leibniz" by "almost-Poisson," "Poisson" and "symplectic."

Definition 3.7 *We shall say that a map $\Sigma : N \times \Lambda \to M$ is **(weakly) isotropic**, **co-isotropic** or **(weakly) Lagrangian** if so is each map $\sigma_\lambda := \Sigma(\cdot,\lambda)$ (see Definition 2.18), for all $\lambda \in \Lambda$.*

The next result is immediate, and will be useful later.

Proposition 3.8 *Statements below are equivalent:*

1. *Σ is (weakly) (co)isotropic;*
2. *the subspaces $T_m M_\lambda \subset T_m M$ are (weakly) (co)isotropic, for all $m \in M$ and $\lambda \in \Lambda$;*
3. *the subspaces $T_n N \times 0_\lambda$ are (weakly) (co)isotropic w.r.t. the Leibniz structure $\Sigma^*\Xi$, for all $n \in N$ and $\lambda \in \Lambda$.*

And for the fibration Π, it follows similarly that,

Proposition 3.9 *Π is (weakly) (co)isotropic if and only if the subspaces $0_n \times T_\lambda \Lambda$ are (weakly) (co)isotropic w.r.t. the Leibniz structure $\Sigma^*\Xi$, for all $n \in N$ and $\lambda \in \Lambda$.*

For isotropic and Lagrangian complete solutions Σ, Proposition 2.19 and Corollary 2.20 imply the following results.

Proposition 3.10 *Consider a Leibniz manifold (M,Ξ), a function $H : M \to \mathbb{R}$ and a fibration $\Pi : M \to N$. Assume that N is connected.*

1. *If $\Sigma : N \times \Lambda \to M$ is an isotropic solution of the Π-HJE for (M, X_H), then there exists a unique function $h : \Lambda \to \mathbb{R}$ such that*
$$H \circ \Sigma = h \circ p_\Lambda. \tag{44}$$
2. *A Lagrangian surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$ is a complete solution of the Π-HJE for (M, X_H) if and only if $\Pi \circ \Sigma = p_N$ and Eq. (44) holds for a unique function $h : \Lambda \to \mathbb{R}$.*

In the case of Hamiltonian systems on symplectic manifolds, the characterization given by Theorem 3.6 can be slightly re-formulated. (Compare to Theorem 2.8).

Theorem 3.11 *Consider a symplectic manifold (M,ω), a function $H : M \to \mathbb{R}$ and a fibration $\Pi : M \to N$. Then a surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$ is a complete solution of the Π-HJE for (M, X_H) if and only if $\Pi \circ \Sigma = p_N$ and [see Eq. (37)]*

$$d\left(H \circ \Sigma\right) = i_{X_H^\Sigma} \Sigma^*\omega. \tag{45}$$

In particular, for all $\lambda \in \Lambda$ [see Eq. (35)],

$$d\left(H \circ \sigma_\lambda\right) = i_{X_H^{\sigma_\lambda}} \sigma_\lambda^*\omega. \tag{46}$$

Proof. Since $\Sigma^*\omega$ is a symplectic form on $N \times \Lambda$, we can write Eq. (45) as

$$\Sigma^* dH = (\Sigma^*\omega)^\flat \left(X_H^\Sigma\right), \tag{47}$$

or equivalently $(\Sigma^*\omega)^\sharp (\Sigma^* dH) = X_H^\Sigma$. So, the first affirmation follows from the previous theorem. To show the second affirmation, note that Eq. (47) implies that

$$\langle \Sigma^* dH, Y \rangle = (\Sigma^*\omega) \left(X_H^\Sigma, Y\right),$$

for all $Y \in \mathfrak{X}(N \times \Lambda)$. So, if we take $Y = (y, 0)$ with $y \in \mathfrak{X}(N)$, Eq. (46) follows form the above one. $\square$

Remark 3.12 *According to the Corollaries 2.9 and 2.11, if* Π *is isotropic or if* Σ *is co-isotropic, then* (45) *is equivalent to* (46)*, for all* $\lambda \in \Lambda$.

Concluding, given a symplectic manifold (M, ω), every complete solution Σ for a Hamiltonian system (M, X_H) is a local symplectomorphism (i.e. a local diffeomorphism and a symplectic map) between (M, ω) and $(N \times \Lambda, \Sigma^*\omega)$, and the related dynamical system $\left(N \times \Lambda, X_H^\Sigma\right)$ is also a Hamiltonian system (with Hamiltonian function $H \circ \Sigma$).

3.3 Lagrangian fibrations and related canonical transformations

Let us continue working with a symplectic manifold (M, ω), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H). We shall see, under certain hypothesis, that Σ defines Darboux coordinates on which the equations of motion can be easily solved, as it is well-know in the standard Lagrangian situation (see Definition 2.14).

Remark 3.13 *Recall that giving Darboux coordinates* $\psi : U \subset M \to \mathbb{R}^k \times \mathbb{R}^k$*, with* $2k = \dim M$*, it is the same as giving a canonical transformation, i.e. a symplectomorphism from* U *to* $T^*\mathbb{R}^k$ *(with its canonical symplectic structure). We just need to identify* $T^*\mathbb{R}^k$ *with* $\mathbb{R}^k \times \mathbb{R}^k$ *in the usual way.*

3.3.1 A related local symplectomorphism

Assume that $\omega = d\theta$, for some 1-form θ, and that N is simply-connected. Let us also assume that Σ is isotropic, i.e. $\sigma_\lambda^*\omega = 0$ for all λ. Then

$$0 = \sigma_\lambda^*\omega = \sigma_\lambda^* d\theta = d\sigma_\lambda^*\theta,$$

and accordingly, since N is simply-connected, $\sigma_\lambda^*\theta$ is exact for each λ. This implies that there exists a function $W_\lambda : N \to \mathbb{R}$ such that $dW_\lambda = \sigma_\lambda^*\theta$.

Remark 3.14 *Each function* W_λ *can be seen as a generalization of the idea of a characteristic Hamilton function, presented in the classical setting.*

In turn, the family of functions W_λ's gives rise to a function $W : N \times \Lambda \to \mathbb{R}$ satisfying

$$\langle (\Sigma^*\theta - dW)(n, \lambda), (y, 0) \rangle = \langle \sigma_\lambda^*\theta(n) - dW_\lambda(n), y \rangle = 0$$

for all $(n, \lambda) \in N \times \Lambda$ and $y \in T_n N$. Now, let us define $\varphi : N \times \Lambda \to T^*\Lambda$ by the formula

$$\langle \varphi(n, \lambda), z \rangle = \langle (\Sigma^*\theta - dW)(n, \lambda), (0, z) \rangle, \qquad z \in T_\lambda \Lambda. \tag{48}$$

Proposition 3.15 *Under the previous conditions, the map* φ *is an immersion, and if* Σ *is Lagrangian, then* φ *is a local diffeomorphism.*

Proof. In order to prove the first affirmation, because of the form of φ, it is enough to prove that each map

$$\varphi_\lambda := \varphi(\cdot,\lambda) : N \to T^*_\lambda \Lambda$$

is an immersion. Given $n \in N$ and $x \in T_n N$, and identifying (as usual) the tangent of the linear space $T^*_\lambda \Lambda$ with itself, it can be shown that

$$\left\langle (\varphi_\lambda)_{*,n}(x), z \right\rangle = d(\Sigma^*\theta - dW)((x,0),(0,z)),$$

for all $z \in T_\lambda \Lambda$. But $d(\Sigma^*\theta - dW) = \Sigma^*\omega$, so

$$\left\langle (\varphi_\lambda)_{*,n}(x), z \right\rangle = \Sigma^*\omega((x,0),(0,z)).$$

Accordingly, if $(\varphi_\lambda)_{*,n}(x) = 0$, then

$$\Sigma^*\omega((x,0),(0,z)) = 0, \quad \forall (0,z) \in 0_n \times T_\lambda \Lambda. \tag{49}$$

On the other hand, since Σ is isotropic, each subspace $T_n N \times 0_\lambda$ is isotropic w.r.t. $\Sigma^*\omega$ (see Proposition 3.8), what implies that

$$\Sigma^*\omega((x,0),(y,0)) = 0, \quad \forall (y,0) \in T_n N \times 0_\lambda. \tag{50}$$

Combining (49) and (50), it follows that

$$\Sigma^*\omega((x,0),(y,z)) = 0, \quad \forall (y,z) \in T_n N \times T_\lambda \Lambda.$$

Finally, since $\Sigma^*\omega$ is non-degenerated, then x must vanishes. This proves that φ is an immersion. It is clear that, if Σ is Lagrangian, then, for dimensional reasons, φ is a local diffeomorphism. □

Proposition 3.16 *Under the previous conditions, if in addition* Π *is isotropic (and consequently* Π *and* Σ *are Lagrangian), then* $\varphi : N \times \Lambda \to T^*\Lambda$ *is a local symplectomorphism, i.e.* $\varphi^*\omega_\Lambda = \Sigma^*\omega$*, being* ω_Λ *the canonical symplectic structure on* $T^*\Lambda$*.*

Proof. Let us fix a coordinate chart $\psi : V \subset \Lambda \to \psi(V) \subset \mathbb{R}^k$ for Λ. Note that $\left(\pi_\Lambda^{-1}(V), (\psi^*)^{-1}\right)$ is a Darboux coordinate chart for $(T^*\Lambda, \omega_\Lambda)$. Let us write

$$\psi(\lambda) = \left(\lambda^1, ..., \lambda^k\right) \quad \text{and} \quad (\psi^*)^{-1}(z) = \left(\lambda^1, ..., \lambda^k, \alpha_1, ..., \alpha_k\right)$$

for all $\lambda \in V$ and $z \in T^*_\lambda V$. In this notation, we have that

$$\omega_\Lambda\left(\frac{\partial}{\partial\lambda^i}, \frac{\partial}{\partial\lambda^j}\right) = \omega_\Lambda\left(\frac{\partial}{\partial\alpha_i}, \frac{\partial}{\partial\alpha_j}\right) = 0 \quad \text{and} \quad \omega_\Lambda\left(\frac{\partial}{\partial\lambda^i}, \frac{\partial}{\partial\alpha_j}\right) = \delta_i^j,$$

for all $i, j = 1, ..., k$. Also, since $d(\Sigma^*\theta - dW) = \Sigma^*\omega$, it can be shown that

$$(\psi \circ \varphi)_*\left(0, \frac{\partial}{\partial\lambda^i}\right) = (\mathbf{e}_i, \mathbf{a}_i),$$

where $\mathbf{e}_i \in \mathbb{R}^k$ is the i-th canonical vector and each $\mathbf{a}_i \in \mathbb{R}^k$ has j-th component

$$(\mathbf{a}_i)_j = \Sigma^*\omega\left(\left(0, \frac{\partial}{\partial\lambda^i}\right), \left(0, \frac{\partial}{\partial\lambda^j}\right)\right),$$

and, for all $x \in TN$,

$$(\psi \circ \varphi)_*(x,0) = (0, \mathbf{b}(x)),$$

where each $\mathbf{b}(x) \in \mathbb{R}^k$ has j-th component

$$\mathbf{b}_j(x) = \Sigma^* \omega \left((x,0), \left(0, \frac{\partial}{\partial \lambda^j}\right) \right).$$

Using that Π is Lagrangian, we have that $\mathbf{a}_i = 0$ for all i (see Proposition 3.9), and consequently,

$$\varphi^* \omega_\Lambda \left(\left(0, \frac{\partial}{\partial \lambda^i}\right), \left(0, \frac{\partial}{\partial \lambda^j}\right) \right) = \omega_\Lambda \left(\varphi_* \left(0, \frac{\partial}{\partial \lambda^i}\right), \varphi_* \left(0, \frac{\partial}{\partial \lambda^j}\right) \right) = (\mathbf{e}_i, 0)^t \cdot \mathbb{J} \cdot (\mathbf{e}_j, 0)$$

$$= 0 = \Sigma^* \omega \left(\left(0, \frac{\partial}{\partial \lambda^i}\right), \left(0, \frac{\partial}{\partial \lambda^j}\right) \right),$$

where "t" indicates transposition and $\mathbb{J}$ is the real $(2k \times 2k)$-matrix

$$\mathbb{J} := \begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}, \tag{51}$$

where $\mathbb{I}$ is the identity $(k \times k)$-matrix. On the other hand,

$$\begin{aligned} \varphi^* \omega_\Lambda ((x,0), (y,0)) &= \omega_\Lambda (\varphi_* (x,0), \varphi_* (y,0)) = (0, \mathbf{b}(x))^t \cdot \mathbb{J} \cdot (0, \mathbf{b}(y)) \\ &= 0 = \Sigma^* \omega ((x,0), (y,0)), \end{aligned}$$

for all $x, y \in TN$, where in the last equality we have used that Σ is isotropic. Finally,

$$\varphi^* \omega_\Lambda \left((x,0), \left(0, \frac{\partial}{\partial \lambda^j}\right) \right) = (0, \mathbf{b}(x))^t \cdot \mathbb{J} \cdot (\mathbf{e}_j, 0) = \mathbf{b}_j(x) = \Sigma^* \omega \left((x,0), \left(0, \frac{\partial}{\partial \lambda^j}\right) \right),$$

and the proposition is proved. □

Above results are also valid if we do not assume that $\omega = d\theta$ and N is connected, but at a local level (i.e. around every $m \in M$). This is justified by the following reasoning:

Let (M, ω) be a symplectic manifold, $\Pi : M \to N$ a fibration and $\Sigma : N \times \Lambda \to M$ a surjective local diffeomorphism such that $\Pi \circ \Sigma = p_N$. Then, for each $m \in M$ there exist an open neighborhood U of m and a submanifold $\Lambda_1 \subset \Lambda$ such that (1) $i^* \omega = d\theta$ for some $\theta \in \Omega^1(U)$, with $i : U \to M$ the canonical inclusion, (2) $\Pi(U)$ is simply connected, and (3) Σ restricted to $\Pi(U) \times \Lambda_1$ is a surjection onto U. In particular, if Σ is a (resp. (co)isotropic) complete solution of the Π-HJE for some dynamical system (M, X), then Σ restricted to $\Pi(U) \times \Lambda_1$ is a (resp. (co)isotropic) complete solution of the $\Pi|_U$-HJE for $(U, X|_U)$.

3.3.2 Special Darboux coordinates

Since a complete solution Σ is a local symplectomorphism and, according to the previous results, so is φ, for every Darboux coordinate chart Ψ on $T^*\Lambda$ (that can be composed with φ) we have a Darboux coordinate chart Φ on M giving, roughly speaking, by the formula

$$\Phi := \Psi \circ \varphi \circ \Sigma^{-1}. \tag{52}$$

What is even more interesting is that, in such charts, the expression of the equations of motion for (M, X_H) are particularly simple. Note that

$$\dim M = 2 \dim N = 2 \dim \Lambda = 2k.$$

Theorem 3.17 *Consider a symplectic manifold (M, ω), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H). If Π and Σ are Lagrangian, then for each $m_0 \in M$ there exist a Darboux coordinate chart*

$$\Phi : m \in U \mapsto (\lambda, \alpha) \in \mathbb{R}^k \times \mathbb{R}^k$$

of M and a coordinate chart $\psi : V_\Lambda \to \mathbb{R}^k$ of Λ such that $m_0 \in U$ and

$$[\Phi_* \circ (X_H|_U)] (\lambda, \alpha) = \left(0, -\nabla \left(h \circ \psi^{-1}\right)(\lambda)\right), \quad \forall (\lambda, \alpha) \in \Phi(U),$$

for some function $h : V_\Lambda \to \mathbb{R}$.

Proof. Again, for simplicity, we shall assume that $\omega = d\theta$ and that N is simply-connected (*ipso facto* connected). If this is not true, we must restrict ourselves to appropriate open subsets of M, N and Λ, as we explained at the end of Section 3.3.1. This enable us to use Propositions 3.15 and 3.16.

Given $m_0 \in M$, consider an open neighborhood $U \subset M$ of m_0, an open subset $V_N \subset N$, and a local chart (V_Λ, ψ) of Λ, with $\psi : V_\Lambda \to \psi(V_\Lambda) \subset \mathbb{R}^k$, such that $\Sigma_{|V_N \times V_\Lambda} : V_N \times V_\Lambda \to U$ is a diffeomorphism. Now, shrink V_N and V_Λ, if necessary, in such a way that

$$\varphi|_{V_N \times V_\Lambda} : V_N \times V_\Lambda \to \varphi(V_N \times V_\Lambda) \subset T^*\Lambda$$

is a diffeomorphism. Note that $\varphi(V_N \times V_\Lambda) \subset T^*V_\Lambda$. Consider the map

$$\Phi := (\psi^*)^{-1} \circ \varphi|_{V_N \times V_\Lambda} \circ \left(\Sigma|_{V_N \times V_\Lambda}\right)^{-1} : U \to \Phi(U) \subset \mathbb{R}^k \times \mathbb{R}^k.$$

Since $(\psi^*)^{-1}$ defines Darboux coordinates on $T^*\Lambda$ and the maps $\varphi|_{V_N \times V_\Lambda}$ and $\left(\Sigma|_{V_N \times V_\Lambda}\right)^{-1}$ are symplectomorphisms, then Φ defines Darboux coordinates on M, or equivalently, Φ defines a symplectomorphism between U and $T^*\mathbb{R}^k$ (see Remark 3.13). Writing $(\psi^*)^{-1}(z) = (\lambda, \alpha)$ for all $\lambda \in V_\Lambda$ and $z \in T^*_\lambda V_\Lambda$, this implies that

$$\Phi_* \circ (X_H|_U) = X_{H \circ \Phi^{-1}} = \mathbb{J}^{-1} \cdot \begin{pmatrix} \partial \left(H \circ \Phi^{-1}\right) / \partial \lambda \\ \partial \left(H \circ \Phi^{-1}\right) / \partial \alpha \end{pmatrix},$$

where $\mathbb{J}$ is given by Eq. (51). On the other hand, since N is connected and Σ is a Lagrangian complete solution, it follows from Proposition 3.10 that $H \circ \Sigma = h \circ p_\Lambda$ for a unique function $h : \Lambda \to \mathbb{R}$. As a consequence,

$$\begin{aligned} H \circ \Phi^{-1} &= H \circ \Sigma|_{V_N \times V_\Lambda} \circ \left(\varphi|_{V_N \times V_\Lambda}\right)^{-1} \circ \psi^*|_{\Phi(U)} \\ &= h \circ p_\Lambda|_{V_N \times V_\Lambda} \circ \left(\varphi|_{V_N \times V_\Lambda}\right)^{-1} \circ \psi^*|_{\Phi(U)} = h \circ \pi_\Lambda \circ \psi^*|_{\Phi(U)}. \end{aligned}$$

So, for every pair $(\lambda, \alpha) \in \Phi(U)$, we have that

$$H \circ \Phi^{-1}(\lambda, \alpha) = h \circ \pi_\Lambda \circ \psi^*(\lambda, \alpha) = h \circ \psi^{-1}(\lambda),$$

and accordingly

$$\Phi_* \circ (X_H|_U) = \mathbb{J}^{-1} \cdot \begin{pmatrix} \partial \left(h \circ \psi^{-1}\right) / \partial \lambda \\ 0 \end{pmatrix} = \left(0, -\frac{\partial \left(h \circ \psi^{-1}\right)}{\partial \lambda}\right). \quad \square$$

The equations of motion on the coordinates $(\lambda, \alpha) = \left(\lambda^1, ..., \lambda^k, \alpha_1, ..., \alpha_k\right)$, defined in the last theorem, are

$$\dot{\alpha}_i(t) = -\frac{\partial \left(h \circ \psi^{-1}\right)}{\partial \lambda^i}(\lambda(t)) \quad \text{and} \quad \dot{\lambda}^i(t) = 0, \qquad i = 1, ..., k, \tag{53}$$

which can be easily integrated [compare with the general situation, given by Eq. (40)]. Note that, to arrive at the last equations, we need to construct (at least locally) the function W and the 1-form θ, which give rise to the map φ [see Eq. (48)]. According to the Poincaré Lemma, they can be found by quadratures. So, we can conclude that, given a Lagrangian fibration Π, if a Lagrangian solution of the Π-HJE for (M, X_H) is known, then (M, X_H) can be integrated by quadratures. Weaker conditions for integrability by quadratures will be given in Section 5.1.

3.4 Some examples: type 1 to 4 generating functions

Let us illustrate previous results with simple examples. Let us go back to Section 2.2.2, and concentrate first on the $r=1$ case, i.e. in the standard situation. Suppose that a family of (local) solutions $\sigma_\lambda : q \mapsto \sigma_\lambda(q) = (q, \hat{\sigma}_\lambda(q))$ is given for the $\mathfrak{p}_1$-HJE, with $\lambda \in \mathbb{R}^k$, such that the square matrix with coefficients $\partial(\hat{\sigma}_\lambda)_i / \partial\lambda_j$ is non-degenerate for all (q,λ). Then, $\Sigma(q,\lambda) := (q, \hat{\sigma}_\lambda(q))$ is a local diffeomorphism, and consequently Σ defines a (local) complete solution of the $\mathfrak{p}_1$-HJE.

Example 3.18 *Take $k=1$ and*

$$H(q,p) = \frac{1}{2}\left(p^2 + q\right).$$

According to the results of Section 2.2.2 for $f(q) = q$, the solutions of the $\mathfrak{p}_1$-HJE are

$$\hat{\sigma}^+_\lambda(q) = \sqrt{\lambda - q} \quad \text{and} \quad \hat{\sigma}^-_\lambda(q) = -\sqrt{\lambda - q}, \quad \lambda \in \mathbb{R}.$$

As a consequence, for every $a \in \mathbb{R}$, we have the local complete solutions

$$\Sigma^\pm : (-\infty, a) \times (a, \infty) \to \mathbb{R}^2$$

given by

$$\Sigma^+(q,\lambda) = \left(q, \sqrt{\lambda - q}\right) \quad \text{and} \quad \Sigma^-(q,\lambda) = \left(q, -\sqrt{\lambda - q}\right).$$

They are local diffeomorphisms since

$$\frac{\partial}{\partial\lambda}\left(\pm\sqrt{\lambda - q}\right) = \mp\frac{1}{2\sqrt{\lambda - q}} \neq 0.$$

In addition, suppose that each solution σ_λ is isotropic (and consequently Lagrangian), i.e. $d\sigma_\lambda = 0$ for each λ [see Eq. (21)]. Then, there exists a family of functions W_λ such that

$$(\hat{\sigma}_\lambda)_i(q) = \frac{\partial W_\lambda(q)}{\partial q^i}, \quad \frac{\partial H(q, \partial W_\lambda(q)/\partial q)}{\partial q^i} = 0, \quad i = 1, ...k, \tag{54}$$

[see Eq. (25)] and the matrix with coefficients $\dfrac{\partial^2 W_\lambda(q)}{\partial\lambda_j \partial q^i}$ is non-degenerate. In this situation, as it is well-known, the formulae

$$p_i = \frac{\partial W_\lambda(q)}{\partial q^i} \quad \text{and} \quad \alpha^i = -\frac{\partial W_\lambda(q)}{\partial\lambda_i}$$

define two canonical transformations (i.e. a change of coordinates between Darboux coordinates)

$$(q,p) \longmapsto (Q,P) := (\lambda, \alpha) \tag{55}$$

and

$$(q,p) \longmapsto (Q,P) := (-\alpha, \lambda), \tag{56}$$

and the function $W(q,\lambda) = W_\lambda(q)$ plays the role of a *generating function* of type 1 and type 2 (see Ref. [17]), respectively. Moreover, Eq. (54) implies that the related Hamiltonians $K_1(Q,P)$ and $K_2(Q,P)$ only depend on λ, i.e. $K_1 = K_1(Q)$ and $K_2 = K_2(P)$. This makes the integration of the equation of motion, in the new Darboux coordinates (Q,P), a trivial task. For instance, for the type 2 case, the equations of motion translate to [compare to Eq. (53)]

$$\dot{Q}^i(t) = \frac{\partial K_2}{\partial P_i}(P(t)), \quad \dot{P}_i(t) = 0,$$

so, the solutions are

$$\left(Q^i(t), P_i(t)\right) = \left(\frac{\partial K_2}{\partial P_i}(P(0))\, t + Q^i(0), P_i(0)\right).$$

Let us analyze the previous construction in terms of the map φ and the canonical transformation Φ [see Eqs. (48) and (52)] defined in the previous section. Since

$$\theta\left(\Sigma\left(q,\lambda\right)\right)=\theta\left(q,\hat{\sigma}_{\lambda}\left(q\right)\right)=\left(\hat{\sigma}_{\lambda}\right)_{i}\left(q\right)\,dq^{i},$$

then (omitting the dependence on q and λ)

$$\left\langle \sigma_{\lambda}^{*}\theta,\frac{\partial}{\partial q^{i}}\right\rangle =\left\langle \theta,\left(\mathbf{e}_{i},\frac{\partial\hat{\sigma}_{\lambda}}{\partial q^{i}}\right)\right\rangle =\left(\hat{\sigma}_{\lambda}\right)_{i},$$

where $\mathbf{e}_i$ is the i-th vector of the canonical basis of $\mathbb{R}^k$, and

$$\left\langle \sigma_{\lambda}^{*}\theta,\frac{\partial}{\partial\lambda_{i}}\right\rangle =\left\langle \theta,\left(0,\frac{\partial\hat{\sigma}_{\lambda}}{\partial\lambda_{i}}\right)\right\rangle =0.$$

Accordingly,

$$\varphi\left(q,\lambda\right)=\left\langle \sigma_{\lambda}^{*}\theta-dW_{\lambda},\frac{\partial}{\partial\lambda_{i}}\right\rangle d\lambda_{i}=-\frac{\partial W_{\lambda}}{\partial\lambda_{i}}\,d\lambda_{i}=\left(\lambda,-\frac{\partial W_{\lambda}}{\partial\lambda}\right),$$

with W_λ such that $\left(\hat{\sigma}_{\lambda}\right)_{i}=\partial W_{\lambda}/\partial q^{i}$. Thus, for $(q,p)=\Sigma(q,\lambda)$, the map $\Phi:=\Psi\circ\varphi\circ\Sigma^{-1}$ [recall Eq. (52)] is given by

$$\Phi\left(q,p\right)=\Psi\circ\varphi\left(q,\lambda\right)=\Psi\left(\lambda,-\frac{\partial W_{\lambda}}{\partial\lambda}\right)=:\left(Q,P\right),$$

So, if we take $\Psi=\Psi_1$ or $\Psi=\Psi_2$, with

$$\Psi_{1}\left(\lambda,\alpha\right):=\left(\lambda,\alpha\right)\qquad\text{and}\qquad\Psi_{2}\left(\lambda,\alpha\right):=\left(-\alpha,\lambda\right),\tag{57}$$

we have the canonical transformations (55) or (56), respectively.

Consider now the $\mathfrak{p}_2$-HJE. If we have a family of (local) solutions $\sigma_\lambda : p\mapsto\sigma_\lambda(p)=(\hat{\sigma}_\lambda(p),p)$, with $\lambda\in\mathbb{R}^k$, such that the matrix with coefficients $\partial\left(\hat{\sigma}_{\lambda}\right)^{i}\big/\partial\lambda_{j}$ is non-degenerate for all (p,λ), then $\Sigma(p,\lambda):=(\hat{\sigma}_\lambda(p),p)$ is a local complete solution of the $\mathfrak{p}_2$-HJE. Again, suppose in addition each σ_λ is Lagrangian. According to (27), this also means that there exist functions $\widetilde{W}_\lambda$ such that

$$\left(\hat{\sigma}_{\lambda}\right)^{i}\left(p\right)=\frac{\partial\widetilde{W}_{\lambda}\left(p\right)}{\partial p_{i}},\qquad\frac{\partial H\left(\partial\widetilde{W}_{\lambda}\left(p\right)\big/\partial p,p\right)}{\partial p_{i}}=0,\quad i=1,...k,\tag{58}$$

and the numbers $\dfrac{\partial^{2}\widetilde{W}_{\lambda}\left(p\right)}{\partial\lambda_{j}\partial p_{i}}$ define a non-degenerate matrix. In this situation, the formulae

$$q^{i}=-\frac{\partial\widetilde{W}_{\lambda}\left(p\right)}{\partial p_{i}}\quad\text{and}\quad\alpha_{i}=-\frac{\partial\widetilde{W}_{\lambda}\left(p\right)}{\partial\lambda^{i}}$$

define the canonical transformations

$$\left(q,p\right)\longmapsto\left(Q,P\right):=\left(\lambda,\alpha\right)\quad\text{and}\quad\left(q,p\right)\longmapsto\left(Q,P\right):=\left(\alpha,-\lambda\right),\tag{59}$$

and the function $\widetilde{W}(p,\lambda)=\widetilde{W}_\lambda(p)$ plays the role of a type 3 and type 4 generating function, respectively. Again, because of Eq. (58), the integration of the equations of motion in the new coordinates is trivial.

Example 3.19 *Recalling again the results of Section 2.2.2, we have for $f(q)=q$ that the solutions of the $\mathfrak{p}_2$-HJE are $\hat{\sigma}_\lambda(p)=\lambda-p^2$, $\lambda\in\mathbb{R}$. Since $\frac{\partial}{\partial\lambda}\left(\lambda-p^2\right)=1\neq0$, such solutions give rise to a global complete solution $\Sigma:\mathbb{R}^2\to\mathbb{R}^2$ of the $\mathfrak{p}_2$-HJE given by $\Sigma(p,\lambda)=\left(\lambda-p^2,p\right)$. On the other hand, since*

$$\hat{\sigma}_{\lambda}\left(p\right)=\frac{\partial\widetilde{W}_{\lambda}\left(p\right)}{\partial p}\qquad\textit{with}\quad\widetilde{W}_{\lambda}\left(p\right)=\lambda p-p^{3}/3,$$

the function $\widetilde{W}:(p,\lambda)\in\mathbb{R}^2\mapsto\lambda p-p^3/3\in\mathbb{R}$ defines a type 3 and a type 4 generating function.

Let us construct the maps φ and Φ for this case. Since

$$\left\langle \sigma_\lambda^* \theta, \frac{\partial}{\partial p_i} \right\rangle = \left\langle \theta, \left(\frac{\partial \hat{\sigma}_\lambda}{\partial p_i}, \mathbf{e}_i \right) \right\rangle = p_j \, \frac{\partial \left(\hat{\sigma}_\lambda \right)^j}{\partial p_i}$$

and

$$\left\langle \sigma_\lambda^* \theta, \frac{\partial}{\partial \lambda_i} \right\rangle = \left\langle \theta, \left(\frac{\partial \hat{\sigma}_\lambda}{\partial \lambda_i}, 0 \right) \right\rangle = p_j \, \frac{\partial \left(\hat{\sigma}_\lambda \right)^j}{\partial \lambda_i},$$

then

$$\varphi \left(p, \lambda \right) = \left\langle \sigma_\lambda^* \theta - dW_\lambda, \frac{\partial}{\partial \lambda_i} \right\rangle d\lambda_i = \left(p_j \, \frac{\partial \left(\hat{\sigma}_\lambda \right)^j}{\partial \lambda_i} - \frac{\partial W_\lambda}{\partial \lambda_i} \right) d\lambda_i = \left(\lambda, p_j \, \frac{\partial \left(\hat{\sigma}_\lambda \right)^j}{\partial \lambda} - \frac{\partial W_\lambda}{\partial \lambda} \right)$$

$$= \left(\lambda, \partial \left(p_j \, \left(\hat{\sigma}_\lambda \right)^j - W_\lambda \right) \Big/ \partial \lambda \right),$$

with W_λ such that

$$p_j \, \frac{\partial \left(\hat{\sigma}_\lambda \right)^j}{\partial p_i} = \frac{\partial W_\lambda}{\partial p_i},$$

or equivalently $\left(\hat{\sigma}_\lambda \right)^i = \partial \left(p_j \, \left(\hat{\sigma}_\lambda \right)^j - W_\lambda \right) \Big/ \partial p_i$. And for $(q, p) = \Sigma \left(p, \lambda \right)$,

$$\Phi \left(q, p \right) = \Psi \circ \varphi \left(p, \lambda \right) = \Psi \left(\lambda, \partial \left(p_j \, \left(\hat{\sigma}_\lambda \right)^j - W_\lambda \right) \Big/ \partial \lambda \right) =: \left(Q, P \right).$$

Then, defining $\widetilde{W}_\lambda := p_j \, \left(\hat{\sigma}_\lambda \right)^j - W_\lambda$ [see the first part of (58)], the map Φ corresponding to $\Psi = \Psi_1$ and $\Psi = \Psi_2$ [see (57)] give rise just to the first and the second canonical transformation of (59), respectively. In the table below, a dictionary relating the classical terminology and the canonical transformations constructed in the previous section is presented.

Type 1 :	$r = 1,$	$\Psi = \Psi_1$
Type 2 :	$r = 1,$	$\Psi = \Psi_2$
Type 3 :	$r = 2,$	$\Psi = \Psi_1$
Type 4 :	$r = 2,$	$\Psi = \Psi_2$

4 The complete solution - first integral's *duality*

It is well-known that, in the context of the standard Hamilton-Jacobi Theory, complete solutions and first integrals are closely related (see for example [10]). We show in this section that the same is true in our extended setting. More precisely, we show that, given a dynamical system (M, X) and a fibration Π, every complete solution of the Π-HJE for (M, X) gives rise, around each $m \in M$, to a set of $l := \dim M - \dim N$ local first integrals of X, and reciprocally, every set of l first integrals of X (*transversal* to Π) gives rise, around each $m \in M$, to a local complete solution of the Π-HJE for (M, X). Using these results, we give a sufficient condition for existence of complete solutions for any dynamical system and any fibration. At the end of the section we shall study the connection between complete solutions and the notions of *commutative* and *non-commutative integrability.*

4.1 From complete solutions to first integrals

4.1.1 The related local momentum maps

Consider three manifolds M, N and Λ, and a surjective local diffeomorphism $\Sigma : N \times \Lambda \to M$. For every $m \in M$, consider a point $(n, \lambda) \in N \times \Lambda$ and open neighborhoods $U \subset M$, $V_N \subset N$ and $V_\Lambda \subset \Lambda$ of m, n and λ, respectively, such that $\Sigma \left(n, \lambda \right) = m$ and $\Sigma_{|V_N \times V_\Lambda} : V_N \times V_\Lambda \to U$ is a diffeomorphism. Define

$$\pi := p_N|_{V_N \times V_\Lambda} \circ \left(\Sigma|_{V_N \times V_\Lambda} \right)^{-1} \quad \text{and} \quad F := p_\Lambda|_{V_N \times V_\Lambda} \circ \left(\Sigma|_{V_N \times V_\Lambda} \right)^{-1}. \tag{60}$$

It is clear that they are submersions and

$$\pi(U) = V_N \quad \text{and} \quad F(U) = V_\Lambda. \tag{61}$$

Then, given $m \in M$, there exists an open neighborhood $U \subset M$ of m and fibrations $\pi : U \to \pi(U) \subset N$ and $F : U \to F(U) \subset \Lambda$ such that $\Sigma|_{\pi(U)\times F(U)} : \pi(U) \times F(U) \to U$ is a diffeomorphism and

$$(\pi, F) = \left(\Sigma|_{\pi(U)\times F(U)}\right)^{-1},$$

according to (60) and (61). The fact that (π, F) is a diffeomorphism implies that

$$TU = \operatorname{Ker}\pi_* \oplus \operatorname{Ker}F_*,$$

as stated in the lemma below.

Lemma 4.1 *Consider three finite dimensional manifolds M, N and Λ, and two submersions $h : M \to N$ and $g : M \to \Lambda$. Then, (h, g) is a local diffeomorphism if and only if $TM = \operatorname{Ker}h_* \oplus \operatorname{Ker}g_*$. We shall say in this case that h and g are **transverse**.*

Proof. Given $m \in M$ and $x \in T_m M$, since

$$(h, g)_{*,m}(x) = (h_{*,m}, g_{*,m})(x) = (h_{*,m}(x), g_{*,m}(x)),$$

it is clear that $(h, g)_{*,m}$ is injective if and only if $\operatorname{Ker}h_{*,m} \cap \operatorname{Ker}g_{*,m} = \{0\}$. On the other hand, since h and g are submersions, it is easy to prove, just by counting dimensions, that $(h, g)_{*,m}$ is surjective if and only if

$$\dim M = \dim(\operatorname{Ker}h_{*,m}) + \dim(\operatorname{Ker}g_{*,m}).$$

This completes the proof. □

Suppose now that $\Pi \circ \Sigma = p_N$ for some fibration $\Pi : M \to N$. It is easy to show that, for any triple (U, π, F) as above, we have that $\pi = \Pi|_U$. This means that, for every $m \in M$, there exists an open neighborhood $U \subset M$ of m and a fibration $F : U \to F(U)$ such that

$$\Sigma|_{\Pi(U)\times F(U)} : \Pi(U) \times F(U) \to U$$

is a diffeomorphism and $(\Pi|_U, F) = \left(\Sigma|_{\Pi(U)\times F(U)}\right)^{-1}$. In particular,

$$F = p_\Lambda|_{\Pi(U)\times F(U)} \circ \left(\Sigma|_{\Pi(U)\times F(U)}\right)^{-1}. \tag{62}$$

Note that, if Σ is a global diffeomorphism, we can take $U := M$ and $F := p_\Lambda \circ \Sigma^{-1}$.

Remark 4.2 *If N is a compact manifold, using standard techniques, U can be taken such that $\Pi(U) = N$. In other words, there exist U and F such that $\Sigma|_{N\times F(U)} : N \times F(U) \to U$ is a diffeomorphism and $(\Pi|_U, F) = \left(\Sigma|_{N\times F(U)}\right)^{-1}$.*

Finally, assume that Σ is a complete solution of the Π-HJE for some dynamical system (M, X). Let us show that, in this case, for any pair (U, F) as given above, we have that $X|_U \in \operatorname{Ker}F_*$. Because of the definition of X^Σ [see Eq. (37)],

$$(p_\Lambda)_*\left(X^\Sigma(n, \lambda)\right) = 0, \quad \forall (n, \lambda) \in N \times \Lambda.$$

On the other hand, it follows from Eqs. (36) and (62),

$$F_*\left(X\left(\Sigma\left(n,\lambda\right)\right)\right)=F_*\left(\Sigma_*\left(X^{\Sigma}\left(n,\lambda\right)\right)\right)=\left(p_\Lambda\right)_*\left(X^{\Sigma}\left(n,\lambda\right)\right)=0$$

for all $(n,\lambda)\in\Pi(U)\times F(U)$. So, the wanted result follows from the fact that $U=\Sigma\left(\Pi(U)\times F(U)\right)$. Summing up, we have proved the next theorem.

Theorem 4.3 *Let (M,X) be a dynamical system, $\Pi:M\to N$ a fibration and $\Sigma:N\times\Lambda\to M$ a complete solution of the Π-HJE for (M,X). Then, for every $m\in M$ there exist an open neighborhood U of m and a fibration $F:U\to F(U)\subset\Lambda$ such that*

$$\operatorname{Im}\left.X\right|_U\subset\operatorname{Ker}F_*$$

and F is transverse to $\left.\Pi\right|_U$, i.e.

$$TU=\operatorname{Ker}\left(\left.\Pi\right|_U\right)_*\oplus\operatorname{Ker}F_*.$$

Moreover, U and F can be taken such that

1. *$\Sigma\left(\Pi(U)\times F(U)\right)\subset U$,*
2. *$\left.\Sigma\right|_{\Pi(U)\times F(U)}$ is a global diffeomorphism onto U,*
3. *and*

$$F\circ\left.\Sigma\right|_{\Pi(U)\times F(U)}=\left.p_\Lambda\right|_{\Pi(U)\times F(U)}. \tag{63}$$

In the context of the previous Theorem, suppose that Λ is an open subset of $\mathbb{R}^l$. If we call f_i to each component of F, the fact that $\operatorname{Im}\left.X\right|_U\subset\operatorname{Ker}F_*$ is equivalent to $\langle df_i,\left.X\right|_U\rangle=0$ for all $i=1,...,l$, i.e. each f_i is a first integral for $\left.X\right|_U$. We can see that from another point of view. We know from Theorem 3.4 that every integral curve $\Gamma:I\to M$ of X is given by the formula $\Gamma(t)=\Sigma\left(\gamma(t),\lambda\right)$ for some $\lambda\in\Lambda$ and some integral curve γ of X^{σ_λ} [see (35)]. Taking an appropriate pair U and F as in the previous theorem, Eq. (63) implies that

$$F\left(\Gamma(t)\right)=F\left(\Sigma\left(\gamma(t),\lambda\right)\right)=p_\Lambda\left(\gamma(t),\lambda\right)=\lambda,$$

for all t such that $\Gamma(t)\in U$. Concluding, related to any complete solution $\Sigma:N\times\Lambda\to M$ of the Π-HJE for the dynamical system (M,X), we have $\dim\Lambda$ (local) constant of motion for (M,X). This motives us to the next definition.

Definition 4.4 *Let (M,X) be a dynamical system, $\Pi:M\to N$ a fibration and $\Sigma:N\times\Lambda\to M$ a complete solution of the Π-HJE for (M,X). A pair (U,F), with U an open submanifold of M and $F:U\to F(U)\subset\Lambda$ a fibration satisfying the items 1, 2 and 3 of the last theorem, will be called a* ***local momentum map*** *of (M,X)* ***related*** *to Σ. If Σ is a global diffeomorphism, we shall call $F:=p_\Lambda\circ\Sigma^{-1}:M\to\Lambda$ the* ***global momentum map*** *of (M,X)* ***related*** *to Σ.*

It is clear that a local momentum map (U,F) of (M,X) related to Σ is the global momentum map $F:U\to F(U)$ of $(U,\left.X\right|_U)$ related to $\left.\Sigma\right|_{\Pi(U)\times F(U)}$.

Remark 4.5 *Given a pair (U,F) as in definition above, the level surfaces of F are*

$$F^{-1}(\lambda)=M_\lambda\cap U,\quad\forall\lambda\in F(U),$$

where $M_\lambda=\Sigma\left(N\times\{\lambda\}\right)=\operatorname{Im}\sigma_\lambda$ [see (35)]. Of course, if Σ is a global diffeomorphism, we have for the global momentum map that $F^{-1}(\lambda)=M_\lambda$, $\forall\lambda\in\Lambda$. In any case,

$$\operatorname{Ker}F_{*,m}=T_m\left(M_{F(m)}\right) \tag{64}$$

for all m in the domain of F. On the other hand, if N is compact (see Remark 4.2), the pairs (U, F) can be chosen such that

$$F^{-1}(\lambda) = M_\lambda, \quad \forall \lambda \in F(U).$$

It is well-known that, when the number of first integrals is $l = d-1$, the system can be integrated by quadratures. This can be deduced from the last theorem and a comment we made at the end of the Section 3.1 [see Eq. (41)].

4.1.2 Some commutation relations

Consider a Leibniz manifold (M, Ξ) (see Section 2.3), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H). Consider also a local momentum map (U, F) related to Σ, and assume that Λ, and consequently $F(U)$, is an open subset of $\mathbb{R}^l$. Denote by f_i the i-th component of F. We shall study conditions on Σ under which the functions f_i's are in involution. We are mainly interested on the symplectic and the Poisson cases, because of the connection between involutivity and integrability that exists in those cases. The next proposition is an immediate consequence of Proposition 3.8 and the Eq. (64).

Proposition 4.6 *Consider a Leibniz manifold (M, Ξ), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H). Σ is (weakly) (co)isotropic if and only if, for each local momentum map (U, F) related to Σ, the fibration $F : U \to F(U)$ is (weakly) (co)isotropic. And if Σ is a global diffeomorphism, Σ is (weakly) (co)isotropic if and only if the global momentum map related to Σ is (weakly) (co)isotropic.*

Since $\operatorname{Ker} F_* = \langle df_1, ..., df_l \rangle^0$ and $\Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right] = \langle X_{f_1}, ..., X_{f_l} \rangle$, from the identities

$$\{f_i, f_j\} = \Xi(df_i, df_j) = \left\langle df_i, \Xi^\sharp(df_j) \right\rangle = \left\langle df_i, X_{f_j} \right\rangle,$$

we have that the functions f_i's are in involution if and only if $\Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right] \subset \operatorname{Ker} F_*$. Using Proposition 4.6, this is true if Σ is co-isotropic, and in particular if it is (weakly) Lagrangian. (This result includes the analogous ones obtained in Refs. [11] and [24] for almost-Poisson manifolds.) In the case of symplectic manifolds, unless in some special cases, we can deduce that from the way in which the brackets $\{f_i, f_j\}$ depend on the partial solutions σ_λ's. From now on, we shall assume that $\Xi^\sharp = \omega^\sharp$ for some symplectic form ω on M.

Proposition 4.7 *Under above notation, if Π is isotropic w.r.t. ω [recall Eq. (9)], then*

$$\{f_i, f_j\}(m) = \left(\sigma_{F(m)}\right)^* \omega\left(\Pi_{*,m}\left(X_{f_i}(m)\right), \Pi_{*,m}\left(X_{f_j}(m)\right)\right), \quad \forall m \in U, \tag{65}$$

for all $i, j = 1, ..., l$. In particular, in the standard situation (see Definition 2.14),

$$\{f_i, f_j\}(m) = \left(\sigma_{F(m)}\right)^* \omega\left(\mathbb{F}f_i(m), \mathbb{F}f_j(m)\right), \quad \forall m \in U.$$

Proof. To simplify the notation, assume for simplicity that $U = M$. Given $m \in M$,

$$X_{f_i}(m) - \left(\sigma_{F(m)}\right)_{*,\Pi(m)} \circ \Pi_{*,m}\left(X_{f_i}(m)\right) \in \operatorname{Ker} \Pi_{*,m},$$

for all $i = 1, ..., l$, so

$$\omega\left(X_{f_i}(m) - \left(\sigma_{F(m)}\right)_{*,\Pi(m)} \circ \Pi_{*,m}\left(X_{f_i}(m)\right), X_{f_j}(m) - \left(\sigma_{F(m)}\right)_{*,\Pi(m)} \circ \Pi_{*,m}\left(X_{f_j}(m)\right)\right) = 0,$$

for all $i, j = 1, ..., l$. Note that, since $F \circ \Sigma(n, \lambda) = \lambda$, i.e. $F \circ \sigma_\lambda : N \to \mathbb{R}$ is a constant function, the same is true for each component $f_i \circ \sigma_\lambda$. This ensures that

$$\omega\left(X_{f_i}(m), \left(\sigma_{F(m)}\right)_{*,\Pi(m)} \circ \Pi_{*,m}\left(X_{f_j}(m)\right)\right) = \left\langle df_i(m), \left(\sigma_{F(m)}\right)_{*,\Pi(m)} \circ \Pi_{*,m}\left(X_{f_j}(m)\right)\right\rangle =$$

$$= \left\langle \left(\sigma_{F(m)}\right)^*_{\Pi(m)}\left(df_i(m)\right), \Pi_{*,m}\left(X_{f_j}(m)\right)\right\rangle = \left\langle d\left(f_i \circ \sigma_{F(m)}\right)(\Pi(m)), \Pi_{*,m}\left(X_{f_j}(m)\right)\right\rangle = 0,$$

from which the first part of the proposition easily follows. For the standard situation, recall Eq. (19). $\square$

Let us characterize the case in which the functions f_i's are in involution (unless when Π is an isotropic fibration).

Proposition 4.8 *Under the conditions of the last proposition, the functions f_i's are in involution if and only if the partial solutions σ_λ's are Lagrangian for all $\lambda \in F(U)$. In the standard situation, the functions f_i's are in involution if and only if each 1-form σ_λ is closed. In any case, Π and F must also be Lagrangian.*

Proof. If the functions f_i's are in involution, then F is co-isotropic, i.e.

$$\omega^\sharp \left[(\operatorname{Ker} F_*)^0 \right] = \langle X_{f_1}, ..., X_{f_l} \rangle \subset \operatorname{Ker} F_*.$$

So, taking into account that F is transverse to Π, it is easy to show that the subset of vectors

$$\{\Pi_{*,m}(X_{f_1}(m)), ..., \Pi_{*,m}(X_{f_l}(m))\} \subset T_{\Pi(m)}N$$

is l.i. (linearly independent). Using Eq. (65), the last observation implies that $\sigma_\lambda^* \omega = 0$ (i.e. σ_λ is isotropic) for all $\lambda \in F(U)$. But if σ_λ is isotropic, since the same is assumed for Π, then both Π and σ_λ must be Lagrangian (because the dimensions of $\operatorname{Ker} \Pi_*$ and $\operatorname{Im}(\sigma_\lambda)_*$ are complementary). This proves the first affirmation. The converse follows immediately from Eq. (65). For the standard situation, recall Eq. (21) of Section 2.2.1. $\square$

It is clear that the last proposition establishes a deep connection between Lagrangian complete solutions and *commutative* (or *Liouville-Arnold*) *integrability* [4]. In the standard situation, it recovers the following well-know result: *closedness of partial solutions is equivalent to the involutivity of the related first integrals*. In Section 4.3.3, we shall extend the above connection to general Poisson manifolds, and even to the notion of *non-commutative* (or *Mischenko-Fomenko*) *integrability* [26].

4.2 From first integrals to complete solutions

In this section, we are going to show a result which can be seen as a converse of Theorem 4.3. Let us fix a dynamical system (M, X) and a fibration $\Pi : M \to N$.

Theorem 4.9 *Let $l := \dim M - \dim N$, Λ an l-manifold and $F : M \to \Lambda$ a fibration such that*

$$\operatorname{Im} X \subset \operatorname{Ker} F_*$$

and F and Π are transverse, i.e.

$$TM = \operatorname{Ker} \Pi_* \oplus \operatorname{Ker} F_*. \tag{66}$$

Then, for every $m \in M$, there exists an open neighborhood U of m such that $(\Pi, F)|_U : U \to \Pi(U) \times F(U)$ is a diffeormorphism and

$$\Sigma := \left((\Pi, F)|_U \right)^{-1}$$

is a complete solution of the $\Pi|_U$-HJE for $(U, X|_U)$. Moreover, $F|_U$ is the global momentum map related to Σ.

Proof. Since Π and F are submersions and condition (66) holds, we know from Lemma 4.1 that (Π, F) is a local diffeomorphism. Let $U \subset M$ such that

$$(\Pi, F)|_U : U \to \Pi(U) \times F(U)$$

is a diffeomorphism, and define $\Sigma := \left((\Pi, F)|_U \right)^{-1} : \Pi(U) \times F(U) \to U$. We have to prove that Σ is a complete solution of the $\Pi|_U$-HJE for $(U, X|_U)$. To do that, we shall show that (36) holds for Σ along $\Pi(U) \times F(U)$. Let us fix $(n,\lambda) \in \Pi(U) \times F(U)$. Then $(\Pi, F)(\Sigma(n,\lambda)) = (n,\lambda)$ and, applying p_N on both sides, $\Pi(\Sigma(n,\lambda)) = n$, which proves the first part of (36). On the other hand, the condition $\operatorname{Im} X \subset \operatorname{Ker} F_*$ implies that

$$(\Pi_*, F_*)(X(\Sigma(n,\lambda))) = (\Pi_*(X(\Sigma(n,\lambda))), 0) = X^{\Sigma}(n,\lambda).$$

Thus, $\Sigma_*\left(X^{\Sigma}(n,\lambda)\right) = \Sigma_*\left[(\Pi_*, F_*)(X(\Sigma(n,\lambda)))\right] = X(\Sigma(n,\lambda))$, which implies the second part of (36). □

This theorem says that, given a set of first integrals for a dynamical system (M, X), defined by a fibration $F : M \to \Lambda$ transverse to Π, we can construct local complete solutions of the Π-HJE for (M, X) around every point $m \in M$.

Example 4.10 *Suppose that $M = T^*Q$, $N = Q$, $\Pi = \pi_Q$ and $\Lambda \subset \mathbb{R}^l$ is an open subset. Let us call f_i the components of the fibration $F : T^*Q \to \Lambda$. Then, transversality condition* (66) *in this case means that, for all $m \in T^*Q$, the set of fiber derivatives [recall Eq.* (18)*] $\{\mathbb{F}f_1(m), ..., \mathbb{F}f_l(m)\} \subset T_{\pi_Q(m)}Q$ is linearly independent.*

If no fibration Π is considered *a priori*, we have the next result.

Theorem 4.11 *Let Λ be an l-manifold, with $l \leq d$, and $F : M \to \Lambda$ a fibration such that $\operatorname{Im} X \subset \operatorname{Ker} F_*$. Then, for every $m \in M$, there exist an open neighborhood U of m and a fibration $\pi : U \to \pi(U) \subset \mathbb{R}^{d-l}$ such that*

$$(\pi, F|_U) : U \to \pi(U) \times F(U)$$

is diffeormorphism and $\Sigma := (\pi, F|_U)^{-1}$ is a complete solution of the π-HJE for $(U, X|_U)$. Moreover, $F|_U$ is the global momentum map related to Σ.

Proof. Consider, around a point $m \in M$, coordinates charts adapted of M and Λ adapted to the fibration F, i.e. charts $\varphi : U \subset M \to \varphi(U) \subset \mathbb{R}^d$ and $\psi : \Lambda_1 \subset \Lambda \to \psi(\Lambda_1) \subset \mathbb{R}^l$ such that $m \in U$, $F(U) \subset \Lambda_1$ and, for all $(x_1, ..., x_d) \in \varphi(U)$,

$$\psi\left[F\left(\varphi^{-1}(x_1, ..., x_d)\right)\right] = (x_{d-l+1}, ..., x_d).$$

Defining $\pi : U \to \mathbb{R}^{d-l}$ such that

$$\pi\left(\varphi^{-1}(x_1, ..., x_d)\right) = (x_1, ..., x_{d-l}),$$

it is clear that

$$TU = \operatorname{Ker} \pi_* \oplus \operatorname{Ker}(F|_U)_*.$$

The rest is a consequence of the last theorem applied to $(U, X|_U)$, π and $F|_U$. □

In other words, related to a set of first integrals of a dynamical system (M, X), we always have, unless locally, a complete solution of the π-HJE for (M, X), for some fibration π.

Remark 4.12 *It is worth mentioning that, given a fibration F, its adapted charts can be found by quadratures from F (actually, by linear calculations only). Then, the fibration π defined in the last proposition, can be found by quadratures too.*

4.3 Some applications of the duality

Fix again a dynamical system (M, X) and a fibration $\Pi : M \to N$. We shall apply the previous results about the duality between complete solutions and first integrals in different situations.

4.3.1 Existence of (local) complete solutions

A sufficient condition for existence of complete solutions will be presented in this section. As above, write $l = \dim M - \dim N$ and $d = \dim M$, and note that $\dim(\operatorname{Ker}\Pi_*) = l$.

Proposition 4.13 *Suppose that $X(m) \notin \operatorname{Ker}\Pi_*$ on some point $m \in M$. Then, there exists an open neighborhood U of m and a fibration $F : U \to F(U) \subset \mathbb{R}^l$ such that*

$$\operatorname{Im} X|_U \subset \operatorname{Ker} F_* \quad \textit{and} \quad TU = \operatorname{Ker}(\Pi|_U)_* \oplus \operatorname{Ker} F_*.$$

In order to prove that, we need the following Lemma.

Lemma 4.14 *Let V be an n-dimensional space, $S \subset V$ a k-dimensional subspace and $B = \{v_1, ..., v_n\}$ a basis of V. Suppose that $v_1 \notin S$. Then, there exists $B_1 \subset B$ containing v_1 such that B_1 is a basis of a complement of S.*

Proof. Let us consider the quotient space V/S and the classes $[v_i]$'s. Since B is a basis of V, the elements $[v_i] \in V/S$ generate V/S. It is well-known that, since $[v_1] \neq 0$ (because $v_1 \notin S$), we can extract from the set of generators $\{[v_1], ..., [v_n]\}$ a basis of V/S containing $[v_1]$. Let $\{[v_{i_1}], ..., [v_{i_k}]\}$, with $v_{i_1} = v_1$, such a basis. Then, $B_1 := \{v_{i_1}, ..., v_{i_k}\}$ is the subset we are looking for. □

Proof of Proposition 4.13. Since $X(m) \notin \operatorname{Ker}\Pi_*$, and consequently $X(m) \neq 0$, using the Rectification Theorem (see, for instance [3]) for vector fields we know that there exists an open neighborhood U of m and a chart

$$\varphi = (\varphi_1, ..., \varphi_d) : U \to \varphi(U) \subset \mathbb{R}^d$$

such that

$$X|_U = \frac{\partial}{\partial \varphi_1}.$$

Applying the previous Lemma to $V = T_m M$, $S = \operatorname{Ker}\Pi_{*,m}$ and $v_i = \partial/\partial\varphi_i(m)$, we can construct a complement of $\operatorname{Ker}\Pi_{*,m}$, generated by some of the vectors $\partial/\partial\varphi_i(m)$, and containing $\partial/\partial\varphi_1(m) = X|_U(m)$. Since $\dim(\operatorname{Ker}\Pi_{*,m}) = l$, we shall need $d - l$ of such vectors, including $\partial/\partial\varphi_1(m)$. Let us re-order the components of φ in such a way that those vectors are

$$\frac{\partial}{\partial \varphi_1}(m), ..., \frac{\partial}{\partial \varphi_{d-l}}(m).$$

Then,

$$\left\langle \frac{\partial}{\partial \varphi_1}(m), ..., \frac{\partial}{\partial \varphi_{d-l}}(m) \right\rangle \cap \operatorname{Ker}\Pi_{*,m} = \{0\}.$$

By continuity, shrinking U if it is necessary, previous condition is true not only for m, but for all the points inside U. Now, define $F : U \to \mathbb{R}^l$ such that

$$F\left(\varphi^{-1}(x_1, ..., x_d)\right) = (x_{d-l+1}, ..., x_d).$$

It is easy to see that the vector fields $\partial/\partial\varphi_1, ..., \partial/\partial\varphi_{d-l}$ define a basis for $\operatorname{Ker} F_*$. In particular,

$$\operatorname{Im} X|_U = \operatorname{Im}\frac{\partial}{\partial \varphi_1} \subset \operatorname{Ker} F_*$$

and $\dim(\operatorname{Ker} F_*) = d - l$. On the other hand, $\operatorname{Ker}(\Pi|_U)_* \cap \operatorname{Ker} F_* = \{0\}$. All that proves the proposition. □

Combining Theorem 4.9 and the last proposition, the next result is immediate.

Theorem 4.15 *Suppose that, on some point $m \in M$,*

$$X(m) \notin \operatorname{Ker} \Pi_* . \tag{67}$$

Then, there exists an open neighborhood U of m, an l-manifold Λ, with $l = \dim M - \dim N$, and a complete solution $\Sigma : \Pi(U) \times \Lambda \to U$ of the $\Pi|_U$-HJE for $(U, X|_U)$.

Note that the condition (67) is exactly the condition that we gave in Section 2.1 in order to ensure the existence of a partial solution [see Eq. (3)].

Example 4.16 *Consider the standard situation and assume that H is simple, i.e. of* kinetic plus potential terms *form. Since $(\pi_Q)_* \circ X_H = \mathbb{F}H$, and $\mathbb{F}H$ is non-degenerate, the last theorem ensures the existence of complete solutions around every point of T^*Q, except for the null co-distribution.*

4.3.2 Restriction of a complete solutions to an invariant submanifold

Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE for (M, X), and let S be an X-invariant submanifold of M. We shall denote by $X|_S : S \to TS$ the corresponding vector field obtained by the restriction of X to S. To take advantage of both Σ and S (for finding the trajectories of X intersecting S), it would be useful to "restrict" somehow Σ to S, in order to obtain a (local) complete solution for the dynamical system $(S, X|_S)$. Assuming certain regularity conditions, we can make such a restriction. To state that result precisely, consider the next definition.

Definition 4.17 *Under the previous notation, we shall say that Σ* ***restricts to*** *S if for every $s_0 \in S$ there exists an open neighborhood $R \subset S$ of s_0 and a submanifold $\Lambda_1 \subset \Lambda$ such that $\Pi(R) \subset N$ is a submanifold, $\Pi|_R : R \to \Pi(R)$ is a fibration and*

$$\Sigma|_{\Pi(R)\times\Lambda_1} : \Pi(R) \times \Lambda_1 \to R$$

is a complete solution of the $\Pi|_R$-HJE for $(R, X|_R)$. We shall call $\Sigma|_{\Pi(R)\times\Lambda_1}$ a ***local restriction of*** *Σ to S* ***around*** *s_0.*

Now, the announced result.

Proposition 4.18 *Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE for (M, X), and let S be an X-invariant submanifold of M. Suppose that:*

1. *for every local momentum map (U, F) related to Σ, $F|_{S\cap U} : S \cap U \to F(U)$ has constant rank r_1;*
2. *the restriction $\Pi|_S : S \to N$ has constant rank r_2;*
3. *$r_1 + r_2 = \dim S$.*

Then, Σ restricts to S.

Proof. Given $s_0 \in S$, consider a local momentum map (U, F) related to Σ (see Definition 4.4) such that $s_0 \in U$. Recall that (see Theorem 4.3)

$$\operatorname{Im} X|_U \subset \operatorname{Ker} F_* \tag{68}$$

and

$$TU = \operatorname{Ker} (\Pi|_U)_* \oplus \operatorname{Ker} F_* . \tag{69}$$

Since, for all $s \in S \cap U$,

$$\operatorname{Ker}\left(\Pi|_{S\cap U}\right)_{*,s} = \operatorname{Ker}\Pi_{*,s} \cap T_sS \quad \text{and} \quad \operatorname{Ker}\left(F|_{S\cap U}\right)_{*,s} = \operatorname{Ker}F_{*,s} \cap T_sS, \tag{70}$$

then

$$\operatorname{Ker}\left(\Pi|_{S\cap U}\right)_{*,s} \cap \operatorname{Ker}\left(F|_{S\cap U}\right)_{*,s} = 0, \tag{71}$$

and using the condition 3 of the proposition, it is clear that

$$T_sS = \operatorname{Ker}\left(\Pi|_{S\cap U}\right)_{*,s} \oplus \operatorname{Ker}\left(F|_{S\cap U}\right)_{*,s}, \quad \forall s \in S \cap U. \tag{72}$$

Also, using the conditions 1 and 2, from the Constant Rank Theorem it follows that, for any point of $S \cap U$, and in particular for the point $s_0 \in S \cap U$ given above, there exists an open neighborhood $V \subset S \cap U$ of s_0 in S such that $\Pi(V)$ and $F(V)$ are submanifolds of N and Λ, respectively, and $\Pi|_V : V \to \Pi(V)$ and $F|_V : V \to F(V)$ are fibrations. Moreover, from Eq. (72) it follows that

$$TV = \operatorname{Ker}\left(\Pi|_V\right)_* \oplus \operatorname{Ker}\left(F|_V\right)_*.$$

On the other hand, using Eq. (68) and the fact that S is X-invariant,

$$\operatorname{Im}\left(X|_V\right) \subset \operatorname{Ker}F_* \cap TV = \operatorname{Ker}\left(F|_V\right)_*.$$

Then, Theorem 4.9 ensures that $\left(\Pi|_V, F|_V\right)$ is a local diffeomorphism and that there exists an open neighborhood $R \subset V$ of s_0 such that $\left(\Pi|_R, F|_R\right) : R \to \Pi(R) \times F(R)$ is a global diffeomorphism and

$$\Sigma_R := \left(\Pi|_R, F|_R\right)^{-1} : \Pi(R) \times F(R) \to R$$

is a solution of the $\Pi|_R$-HJE for $(R, X|_R)$. Of course, $\Sigma_R = \Sigma|_{\Pi(R)\times F(R)}$. Choosing $\Lambda_1 := F(R)$, it follows that Σ_R is a local restriction of Σ to S around s_0. $\square$

Corollary 4.19 *Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE for (M, X), and let S be an X-invariant submanifold of M. If for every momentum map (U, F) related to Σ we have that $\operatorname{Ker}F_{*,s} \subset T_sS$ for all $s \in S \cap U$, then Σ restricts to S.*

Proof. It is enough to show that the items 1, 2 and 3 of the last proposition are fulfilled. In fact, if such condition above holds, the second part of (70) implies that $\operatorname{Ker}\left(F|_{S\cap U}\right)_{*,s} = \operatorname{Ker}F_{*,s}$. So, $F|_{S\cap U}$ has constant rank, say r_1, and the item 1 follows. Using basic linear algebra and (69),

$$\left(\operatorname{Ker}\Pi_{*,s} \cap T_sS\right) + \operatorname{Ker}F_{*,s} = \left(\operatorname{Ker}\Pi_{*,s} + \operatorname{Ker}F_{*,s}\right) \cap T_sS = T_sS,$$

then, from (71) and the fact that $\operatorname{Ker}\left(\Pi|_{S\cap U}\right)_{*,s} = \operatorname{Ker}\Pi_{*,s} \cap T_sS$, we finally have Eq. (72). This implies that the rank of $\Pi|_{S\cap U}$ is constant, say r_2, and $r_1 + r_2 = \dim S$, i.e. the items 2 and 3 follow. $\square$

Suppose now that we have an X-invariant foliation $\mathcal{S}$ of M (i.e. a foliation such that each one of its leaves is X-invariant). Denote by $T\mathcal{S}$ the related distribution. From previous results, the next proposition is straightforward.

Proposition 4.20 *Let $\Sigma : N \times \Lambda \to M$ be a complete solution of the Π-HJE for (M, X), and let $\mathcal{S}$ be an X-invariant foliation of M. If, for every momentum map (U, F) related to Σ, we have that $\operatorname{Ker}F_* \subset T\mathcal{S}$, or for every leaf S*

1. *$\operatorname{Ker}\Pi_* \cap TS$ (resp. $\operatorname{Ker}F_* \cap TS$) is a constant rank distribution along S (resp. along $U \cap S$),*
2. *$\dim\left(\operatorname{Ker}\Pi_* \cap TS\right) + \dim\left(\operatorname{Ker}F_* \cap TS\right) = \dim S$,*

then Σ restricts to each leaf of $\mathcal{S}$.

We shall apply this result in the last section, in the context of Poisson manifolds, where $\mathcal{S}$ will be the foliation given by the symplectic leaves of M.

4.3.3 Non-commutative integrability on Poisson manifolds

Fix a Poisson manifold (M, Ξ) and a function $H : M \to \mathbb{R}$. We give below a definition of the commutative and noncommutative integrability notions (on Poisson manifolds) in terms that we find more appropriate for this paper.[7]

Definition 4.21 *The Hamiltonian system (M, X_H) is **non-commutative integrable (NCI)** by means of F if a isotropic fibration $F : M \to \Lambda$, such that $X_H \in \operatorname{Ker} F_*$ and $\Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right]$ is integrable, can be exhibited. If in addition F is Lagrangian, (M, X_H) is **commutative integrable (CI)** by means of F.*

Under conditions above, the system (M, X_H) can be integrated by quadratures. This fact was proved in [2] for the commutative case. A similar proof can be applied to the noncommutative scenario, provided a number $\dim(\operatorname{Ker} F_*)$ of components of F are in involution with all of them. In a forthcoming paper, we shall further extend the proof to the general case.

Some authors include in the definition of NCI and CI systems one more requirement: F has compact and connected leaves. In such a case, beside integrability by quadratures, *action-angle* like coordinates can be found for such systems (see Refs. [22] and [16]). We do not analyze this case here.

Using the duality between complete solutions and first integrals (see Sections 4.1 and 4.2), a deep connection between our Hamilton-Jacobi Theory and noncommutative integrability on Poisson manifolds can be easily established (extending our results obtained at the end of Section 4.1.2, in the restricted context of symplectic manifolds and commutative integrability). We only need Proposition 4.6 and the next result, which is an immediate consequence of the Remark 4.5. Consider a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for system (M, X_H).

Proposition 4.22 *$(\Sigma^*\Xi)^\sharp \left((TN \times 0)^0 \right)$ is integrable if and only if so is $\Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right]$ for all pair (U, F).*

Now, the mentioned connection for the general noncommutative case.

Theorem 4.23 *Consider a Poisson manifold (M, Ξ), a function $H : M \to \mathbb{R}$ and a fibration $\Pi : M \to N$.*

1. *Suppose that an isotropic complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H), with*

$$(\Sigma^*\Xi)^\sharp \left((TN \times 0)^0 \right)$$

integrable, can be exhibited. Then, for every $m \in M$, there exists an open neighborhood U of m such that $(U, X_H|_U)$ is NCI.

2. *If (M, X_H) is NCI by means of F, with F and Π transversal, then, for each $m \in M$, there exists an open neighborhood U of m such that an isotropic complete solution Σ of the $\Pi|_U$-HJE for $(U, X_H|_U)$, with*

$$(\Sigma^*\Xi)^\sharp \left((\Pi_* (TU) \times 0)^0 \right)$$

integrable, can be exhibited.

3. *If (M, X_H) is NCI, then, for each $m \in M$, there exists an open neighborhood U of m and a fibration $\pi : U \to \pi(U)$ such that an isotropic complete solution Σ of the π-HJE for $(U, X_H|_U)$, with*

$$(\Sigma^*\Xi)^\sharp \left((\pi_* (TU) \times 0)^0 \right)$$

integrable, can be exhibited.

[7] Our definition corresponds to the so-called *abstract (non)commutative integrability* of Ref. [22], but with a *momentum map*, as defined in [16]. Nevertheless, note that we are not asking that some of the components of the momentum map are in involution with all of them.

Proof. (1) Theorem 4.3 ensures the existence of an open neighborhood U of m and that a fibration $F : U \to F(U)$, such that $\operatorname{Im}(X_H|_U) \subset \operatorname{Ker} F_*$, can be constructed from Σ. On the other hand, Propositions 4.6 and 4.22 ensure that F is isotropic with $\Xi^\sharp\left[(\operatorname{Ker} F_*)^0\right]$ integrable. Then, $(U, X_H|_U)$ is NCI by means of F.

(2) Since $\operatorname{Im}(X_H) \subset \operatorname{Ker} F_*$ and F and Π are transverse, Theorem 4.9 ensures the existence of U and the possibility of constructing Σ from F; and since F is isotropic and $\Xi^\sharp\left[(\operatorname{Ker} F_*)^0\right]$ is integrable, Propositions 4.6 and 4.22 ensure that the same is true for Σ and $(\Sigma^*\Xi)^\sharp\left((\Pi_*(TU)\times 0)^0\right)$.

(3) We must proceed as in the previous item (2), but considering Theorem 4.11 instead of Theorem 4.9. □

Remark 4.24 *Kozlov has presented in [21] an alternative HJE (see Remark 2.13) and a connection of that equation with the non-commutative integrability in symplectic manifolds. In contrast to our theory, the first integrals of the system (by means of which the system is integrable) are not completely determined by the solutions of the Kozlov's HJE.*

For completeness, let us express the previous result for the commutative case.

Theorem 4.25 *Consider a Poisson manifold (M, Ξ), a function $H : M \to \mathbb{R}$ and a fibration $\Pi : M \to N$.*

1. *Suppose that a Lagrangian complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H) can be exhibited. Then for every $m \in M$ there exists an open neighborhood U of m such that $(U, X_H|_U)$ is CI.*
2. *If (M, X_H) is CI by means of F, with F and Π transversal, then for each $m \in M$ there exists an open neighborhood U of m such that a Lagrangian complete solution Σ of the $\Pi|_U$-HJE for $(U, X_H|_U)$ can be exhibited.*
3. *If (M, X_H) is CI, then for each $m \in M$ there exists an open neighborhood U of m and a fibration $\pi : U \to \pi(U)$ such that a Lagrangian complete solution Σ of the π-HJE for $(U, X_H|_U)$ can be exhibited.*

Remark 4.26 *If in Definition 4.21 we ask F to have compact and connected leaves, previous theorems are also true, provided we assume that N is compact and connected (see the Remark 4.5).*

5 Weakly isotropic complete solutions and integrability by quadratures

Theorems 4.23 and 4.25 say, among other things, that given a Hamiltonian system (M, X_H) on a Poisson manifold (M, Ξ) fibered by Π, if an "appropriate" complete solution of the Π-HJE is known for (M, X_H), then (M, X_H) is integrable by quadratures. There, by "appropriate" we mean that Σ is isotropic and the distribution $(\Sigma^*\Xi)^\sharp\left((TN\times 0)^0\right)$ is integrable. We shall show in this section that only the isotropy condition is needed to ensure integrability by quadratures.

5.1 The case of symplectic manifolds

Consider a symplectic manifold (M, ω), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X_H). As in Section 3.3.1, let us assume for simplicity that $\omega = d\theta$, N is simply connected (*ipso facto* connected), and Σ is isotropic. Recall that, under these assumptions, $H \circ \Sigma = h \circ p_\Lambda$ for a unique function $h : \Lambda \to \mathbb{R}$, and there exists a function $W : N \times \Lambda \to \mathbb{R}$ (that can be found by quadratures) satisfying

$$\langle (\Sigma^*\theta - dW)(n, \lambda), (y, 0)\rangle = 0 \tag{73}$$

for all $(n,\lambda) \in N \times \Lambda$ and $y \in T_n N$. Since $d\Sigma^*\theta = d(\Sigma^*\theta - dW)$, using the Cartan formula $i_Y \circ d + d \circ i_Y = L_Y$ for $Y := X_H^\Sigma$, Eq. (73) and the fact $\operatorname{Im}\left(X_H^\Sigma\right) \subset TN \times 0$, it follows that

$$i_{X_H^\Sigma} d\Sigma^*\theta = L_{X_H^\Sigma}(\Sigma^*\theta - dW).$$

Combining the last equation and the identity $d(H \circ \Sigma) = p_\Lambda^* dh$, Eq. (45) reads

$$p_\Lambda^* dh = L_{X_H^\Sigma}(\Sigma^*\theta - dW). \tag{74}$$

Summing up, we have proved the following result.

Proposition 5.1 *Consider a symplectic manifold* (M,ω)*, a function* $H : M \to \mathbb{R}$*, a fibration* $\Pi : M \to N$ *and a complete solution* Σ *of the* Π*-HJE for* (M, X_H)*. If* $\omega = d\theta$*,* N *is simply connected, and* Σ *is isotropic, then there exist functions* $h : \Lambda \to \mathbb{R}$ *and* $W : N \times \Lambda \to \mathbb{R}$ *(which can be given by quadratures), such that Eq. (74) holds.*

We are going to find a set of functional equations for the integral curves of X_H^Σ. Let us consider the immersion $\varphi : N \times \Lambda \to T^*\Lambda$ defined in (48). Since the curves of X_H^Σ are of the form $(\gamma(t), \lambda)$, with $\lambda \in \Lambda$ and γ an integral curve of $X_H^{\sigma_\lambda}$, let us fix some λ and focus on the immersion $\varphi_\lambda : N \to T_\lambda^*\Lambda$. Recall that the later is given by

$$\langle \varphi_\lambda(n), z\rangle = \langle (\Sigma^*\theta - dW)(n,\lambda), (0,z)\rangle, \quad \forall z \in T_\lambda \Lambda. \tag{75}$$

Proposition 5.2 *Given* $\lambda \in \Lambda$ *and an integral curve* $\gamma : I \to N$ *of* $X_H^{\sigma_\lambda}$*, there exist unique covectors* $\alpha_\lambda, \beta_\lambda \in T_\lambda^*\Lambda$ *such that*

$$\varphi_\lambda(\gamma(t)) = \alpha_\lambda\, t + \beta_\lambda, \quad \forall t \in I.$$

Proof. Consider a vector field $z \in \mathfrak{X}(\Lambda)$ and the related vector field $Z = (0,z) \in \mathfrak{X}(N \times \Lambda)$. Since $\operatorname{Im} X_H^\Sigma \subset TN \times 0$, it is easy to see that $\operatorname{Im}\left[Z, X_H^\Sigma\right] \subset TN \times 0$. Using the identity

$$i_{\left[Z, X_H^\Sigma\right]} = i_Z \circ L_{X_H^\Sigma} - L_{X_H^\Sigma} \circ i_Z,$$

and the fact that $i_Y(\Sigma^*\theta - dW) = 0$ for all Y such that $\operatorname{Im} Y \subset TN \times 0$ [see Eq. (73)], it follows that

$$i_Z\left[L_{X_H^\Sigma}(\Sigma^*\theta - dW)\right] = L_{X_H^\Sigma}\left[i_Z(\Sigma^*\theta - dW)\right].$$

Then, applying i_Z on both sides of (74), we obtain

$$\langle dh, z\rangle = L_{X_H^\Sigma}\langle \Sigma^*\theta - dW, (0,z)\rangle.$$

Evaluating on the curve $(\gamma(t), \lambda)$, we have

$$\begin{aligned}\langle dh(\lambda), z(\lambda)\rangle &= \frac{d}{dt}\langle (\Sigma^*\theta - dW)(\gamma(t),\lambda), (0, z(\lambda))\rangle \\ &= \frac{d}{dt}\langle \varphi_\lambda(\gamma(t)), z(\lambda)\rangle.\end{aligned}$$

Assume for simplicity that $0 \in I$. Then, integrating the equation above between 0 and t,

$$\langle \varphi_\lambda(\gamma(t)), z(\lambda)\rangle = \langle dh(\lambda), z(\lambda)\rangle\, t + \langle \varphi_\lambda(\gamma(0)), z(\lambda)\rangle,$$

i.e.

$$\varphi_\lambda(\gamma(t)) = dh(\lambda)\, t + \varphi_\lambda(\gamma(0)), \tag{76}$$

which proves the proposition. □

Equation (76) defines the functional equations for γ that we mentioned above. The Inverse Function Theorem for the immersion φ_λ ensures that such functional equations can be solved for $\gamma(t)$. On the other hand, since each φ_λ is constructed from Σ by quadratures (i.e. through the function W), we can conclude that the dynamical system (M, X_H) can be solved by quadratures. Note that this is also true in spite of ω is not exact or N is not simply connected. In fact, as we argue at the end of Section 3.3.1, such conditions are always true at a local level, and consequently we can write a functional equation like (76) around every point of M. Concluding,

Theorem 5.3 *Consider a symplectic manifold* (M, ω)*, a function* $H : M \to \mathbb{R}$*, a fibration* $\Pi : M \to N$ *and a complete solution* Σ *of the* Π*-HJE for* (M, X_H)*. If* Σ *is isotropic, then* (M, X_H) *can be solved by quadratures.*

In terms of first integrals, using Theorem 4.11 and Proposition 4.6 (and recalling the Remark 4.12), we have the next result.

Theorem 5.4 *Consider a symplectic manifold* (M, ω)*, a function* $H : M \to \mathbb{R}$ *and a fibration* $F : M \to \Lambda$ *such that* $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$*. If* F *is isotropic, then* (M, X_H) *can be solved by quadratures.*

Theorem above gives an affirmative answer to the question formulated in [30] (page 6), if by "integrable" we mean some kind of *exact solvability.*

5.2 An illustrative example

On the symplectic space $T^*\mathbb{R}^4 \cong \mathbb{R}^8$ with its canonical structure ω, consider the Hamiltonian function $H : \mathbb{R}^8 \to \mathbb{R}$ given by

$$H(q, p) = (p_2 - q^2)(p_4 - q^4),$$

where $(q, p) = (q^1, q^2, q^3, q^4, p_1, p_2, p_3, p_4)$ are the standard Darboux coordinates on $T^*\mathbb{R}^4$. Fixing $N := \mathbb{R}^3$,

$$\Pi : \mathbb{R}^8 \to \mathbb{R}^3 : (q, p) \longmapsto (p_2, p_4, q^3)$$

and $\Lambda := \mathbb{R}^5$, it can be shown that the diffeomorphism $\Sigma : \mathbb{R}^3 \times \mathbb{R}^5 \to \mathbb{R}^8$ given by

$$\Sigma(n, \lambda) = (\lambda_1, \lambda_2 + n_1, n_3, \lambda_3 + n_2, \lambda_4, n_1, \lambda_5 - \lambda_4 n_3, n_2),$$

where $(n, \lambda) := (n_1, n_2, n_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, is a complete solution of the Π-HJE for the Hamiltonian system $(\mathbb{R}^8, X_H)$. The momentum map related to Σ is

$$F : \mathbb{R}^8 \to \mathbb{R}^5 : (q, p) \longmapsto (q^1, q^2 - p_2, q^4 - p_4, p_1, q^3 p_1 + p_3),$$

which is an isotropic submersion. In fact,

$$\ker F_* = \left\langle \frac{\partial}{\partial q^2} + \frac{\partial}{\partial p_2}, \frac{\partial}{\partial q^4} + \frac{\partial}{\partial p_4}, \frac{\partial}{\partial q^3} - p_1 \frac{\partial}{\partial p_3} \right\rangle \subseteq (\ker F_*)^\omega.$$

This implies that Σ is isotropic too.

Remark 5.5 *Since the vector fields* $\dfrac{\partial}{\partial q^3} - p_1 \dfrac{\partial}{\partial p_3}$ *and* $\dfrac{\partial}{\partial p_1}$ *belong to* $(\ker F_*)^\omega$*, but*

$$\left[\frac{\partial}{\partial q^3} - p_1 \frac{\partial}{\partial p_3}, \frac{\partial}{\partial p_1} \right] = \frac{\partial}{\partial p_3} \notin (\ker F_*)^\omega,$$

we have that $(\ker F_*)^\omega$ is **not** integrable.

Regarding the functions $h : \mathbb{R}^5 \to \mathbb{R}$ and $W : \mathbb{R}^3 \times \mathbb{R}^5 \to \mathbb{R}$, we have from the equation $H \circ \Sigma = h \circ p_\Lambda$ that $h(\lambda) = \lambda_2 \lambda_3$, and it is easy to see that

$$W(n, \lambda) := \frac{n_1^2 + n_2^2}{2} + \left(\lambda_3 n_2 + \lambda_2 n_1 - \frac{1}{2}\lambda_4 n_3^2\right)$$

satisfies (73). Then, the immersion $\varphi_\lambda : \mathbb{R}^3 \to T_\lambda^* \mathbb{R}^5 \cong \mathbb{R}^5$ defined in (75) is just

$$\varphi_\lambda(n) = \left(0, -n_1, -n_2, \lambda_1 + \frac{1}{2} n_3^2, n_3\right), \tag{77}$$

and, using (76) and the fact that

$$dh(\lambda) = (0, \lambda_3, \lambda_2, 0, 0),$$

it transforms the integral curve $\gamma(t) = (n_1(t), n_2(t), n_3(t))$ of $X_H^{\sigma_\lambda}$ with initial condition $\left(n_1^0, n_3^0, n_3^0\right)$ into the expression

$$\varphi_\lambda(\gamma(t)) = (0, \lambda_3, \lambda_2, 0, 0)\ t + \left(0, -n_1^0, -n_2^0, \lambda_1 + \frac{1}{2}\left(n_3^0\right)^2, n_3^0\right). \tag{78}$$

This implies that, combining (77) and (78),

$$n_1(t) = n_1^0 - \lambda_3\ t, \quad n_2(t) = n_2^0 - \lambda_2\ t \quad \text{and} \quad n_3(t) = n_3^0.$$

Consequently, using Eq. (39), any curve $\Gamma(t) = (q(t), p(t))$ of X_H is given by

$$(q(t), p(t)) = \left(\lambda_1, \lambda_2 + n_1^0 - \lambda_3\ t, n_3^0, \lambda_3 + n_2^0 - \lambda_2\ t, \lambda_4, n_1^0 - \lambda_3\ t, \lambda_5 - \lambda_4 n_3^0, n_2^0 - \lambda_2\ t\right),$$

for a certain 8-uple $\left(n_1^0, n_3^0, n_3^0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\right) \in \mathbb{R}^8$.

Of course, the equations of motion related to H can be integrated without any change of coordinates. We did it in this way just to illustrate the procedure developed in the previous sections.

5.3 The case of Poisson manifolds

Let (M, Ξ) be a Poisson manifold. Recall that $\operatorname{Im} \Xi^\sharp$ is an integrable distribution and its maximal connected integral manifolds S are symplectic manifolds (S, ω): the *symplectic leaves*. Denote by $\mathcal{S}$ the related (symplectic) foliation, i.e. $\operatorname{Im} \Xi^\sharp = T\mathcal{S}$. Let us fix a symplectic leaf (S, ω) and indicate by $i : S \to M$ the inclusion map. For each $m \in S$,

$$T_m S = \operatorname{Im} \Xi_m^\sharp = \left(\operatorname{Ker} \Xi_m^\sharp\right)^0$$

and $\omega\left(\Xi^\sharp(\alpha), \Xi^\sharp(\beta)\right) = \Xi(\alpha, \beta)$, $\forall \alpha, \beta \in T_m^* M$. Consequently,

$$\Xi^\sharp(\alpha) = i_{*,m}\left[\omega^\sharp\left(i_m^*(\alpha)\right)\right], \quad \forall \alpha \in T_m^* M. \tag{79}$$

Fix also a function $H : M \to \mathbb{R}$ and consider its Hamiltonian vector field X_H w.r.t. Ξ. For all $m \in S$,

$$X_H(m) = \Xi^\sharp(dH(m)) = i_{*,m}\left[\omega^\sharp\left(i_m^*(dH)\right)\right] = i_{*,m}\left[\left(\omega^\sharp \circ i^* dH\right)(m)\right],$$

so, defining $H_S := H \circ i : S \to \mathbb{R}$, we have that $X_H \circ i = i_* \circ X_{H_S}$, where $X_{H_S} = \omega^\sharp \circ dH_S \in \mathfrak{X}(S)$ (the Hamiltonian vector field w.r.t. ω and with Hamiltonian function H_S). This says, in particular, that S is X_H-invariant and, moreover, the trajectories of X_H intersecting S coincide with the trajectories of X_{H_S}.

Remark 5.6 *Thus, if we known the trajectories of each Hamiltonian system (S, X_{H_S}), for every symplectic leaf S, then we shall know the trajectories of (M, X_H).*

From now on, we shall assume that we know (each leaf S of) the symplectic foliation $\mathcal{S}$. Let $\Pi : M \to N$ be a fibration and $\Sigma : N \times \Lambda \to M$ a complete solution of the Π-HJE for (M, X_H). According to the discussion we made in Section 4.3.2, in order to take advantage of both the complete solution Σ and the X_H-invariant foliation $\mathcal{S}$, we need that Σ restricts to each leaf S of $\mathcal{S}$ (see Definition 4.17). In addition, using Theorem 5.3, if each local restriction of Σ to a given leaf S is isotropic w.r.t. the symplectic structure ω on S, the Hamiltonian system (S, X_{H_S}) can be integrated by quadratures. And from above remark, if this is true for all S, then (M, X_H) is integrable by quadratures. Let us study sufficient conditions for this to happen.

Proposition 5.7 *Consider a Poisson manifold (M, Ξ), a function $H : M \to \mathbb{R}$, a fibration $\Pi : M \to N$ and a complete solution Σ of the Π-HJE for (M, X_H). If Σ is isotropic, then Σ restricts to each leaf of its symplectic foliation $\mathcal{S}$.*

Proof. If Σ is isotropic (recall Definition 3.7), all the local momentum maps (U, F) related to Σ are also isotropic (see Proposition 4.6), and consequently $\operatorname{Ker} F_* \subset \operatorname{Im} \Xi^\sharp = T\mathcal{S}$. So, the hypothesis of the Proposition 4.20 hold. □

Proposition 5.8 *Under the same conditions of previous proposition, assume that Σ is weakly isotropic and restricts to a given symplectic leaf S. Then, for every $s \in S$ there exists an isotropic local restriction of Σ to S around s (w.r.t. the symplectic structure ω on S).*

Proof. From Proposition 4.6, Σ is weakly isotropic if and only if every local momentum map (U, F) satisfies

$$\operatorname{Ker} F_* \cap \operatorname{Im} \Xi^\sharp \subset \Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right]. \tag{80}$$

Since Σ restricts to S, given $s \in S \cap U$, consider an open neighborhood $R \subset S$ of s, a submanifold $\Lambda_1 \subset \Lambda$ and a local restriction $\Sigma|_{\Pi(R)\times\Lambda_1} : \Pi(R) \times \Lambda_1 \to R$ of Σ to S around s. Shrink R and Λ_1, if it is necessary, to ensure that $R \subset S \cap U$ and $\Sigma|_{\Pi(R)\times\Lambda_1}$ is a global diffeomorphism. It is clear that

$$F|_R = p_\Lambda|_{\Pi(R)\times\Lambda_1} \circ \left(\Sigma|_{\Pi(R)\times\Lambda_1} \right)^{-1}, \qquad F(R) = \Lambda_1$$

and $F|_R : R \to \Lambda_1$ is the global momentum map for the system $(R, X_H|_R)$ related to $\Sigma|_{\Pi(R)\times\Lambda_1}$. Regarding Proposition 4.6 again (but for global momentum maps), if we show that $F|_R$ is isotropic, then so is $\Sigma|_{\Pi(R)\times\Lambda_1}$. Let $i : R \to U$ be the inclusion and write $F|_R = F \circ i$. Since

$$\begin{aligned} i_* \left(\operatorname{Ker} (F|_R)_{*,r} \right) &= i_* \left(\operatorname{Ker} (F \circ i)_{*,r} \right) = \operatorname{Ker} F_{*,r} \cap i_* (T_r R) = \operatorname{Ker} F_{*,r} \cap \operatorname{Im} \Xi^\sharp_r \\ &\subset \Xi^\sharp \left[(\operatorname{Ker} F_{*,r})^0 \right], \end{aligned}$$

for all $r \in R$ [see Eq. (80)], and

$$\begin{aligned} \Xi^\sharp \left[(\operatorname{Ker} F_*)^0 \right] &= \Xi^\sharp \left[(\operatorname{Ker} F_*)^0 + \operatorname{Ker} \Xi^\sharp \right] = \Xi^\sharp \left[\left(\operatorname{Ker} F_* \cap \left(\operatorname{Ker} \Xi^\sharp \right)^0 \right)^0 \right] \\ &= \Xi^\sharp \left[\left(\operatorname{Ker} F_* \cap \operatorname{Im} \Xi^\sharp \right)^0 \right], \end{aligned}$$

then

$$\begin{aligned} i_* \left(\operatorname{Ker} (F|_R)_{*,r} \right) &\subset \Xi^\sharp \left[(\operatorname{Ker} F_{*,r})^0 \right] = \Xi^\sharp \left[\left(\operatorname{Ker} F_{*,r} \cap \operatorname{Im} \Xi^\sharp_r \right)^0 \right] \\ &= \Xi^\sharp \left[\left(i_* \left(\operatorname{Ker} (F|_R)_{*,r} \right) \right)^0 \right], \qquad \forall r \in R. \end{aligned} \tag{81}$$

On the other hand, from Eq. (79) we have that[8]

$$\begin{aligned}\Xi^\sharp\left[\left(i_{*,r}\left(\operatorname{Ker}\left(F|_R\right)_{*,r}\right)\right)^0\right] &= \Xi^\sharp\left[\left(i_r^*\right)^{-1}\left(\left(\operatorname{Ker}\left(F|_R\right)_{*,r}\right)^0\right)\right] \\ &= i_*\left[\omega^\sharp\left[i_r^*\left(\left(i_r^*\right)^{-1}\left(\left(\operatorname{Ker}\left(F|_R\right)_{*,r}\right)^0\right)\right)\right]\right] \\ &\subset i_*\left[\omega^\sharp\left[\left(\operatorname{Ker}\left(F|_R\right)_{*,r}\right)^0\right]\right]. \end{aligned} \tag{82}$$

As a consequence, combining (81) and (82),

$$i_*\left(\operatorname{Ker}\left(F|_R\right)_*\right) \subset i_*\left[\omega^\sharp\left(\left(\operatorname{Ker}\left(F|_R\right)_*\right)^0\right)\right],$$

and from the injectivity of i_* we have that $F|_R$ is isotropic w.r.t. ω. □

Now, we are ready to the main result of this section.

Theorem 5.9 *Consider a Poisson manifold (M,Ξ), its symplectic foliation $\mathcal{S}$, a function $H: M \to \mathbb{R}$, a fibration $\Pi: M \to N$ and a complete solution Σ of the Π-HJE for (M, X_H). If Σ is weakly isotropic and for every leaf S of $\mathcal{S}$ we have that*

1. *$\operatorname{Ker}\Pi_* \cap TS$ (resp. $\operatorname{Ker}F_* \cap TS$) is a constant rank distribution along S (resp. $S \cap U$),*
2. *$\dim(\operatorname{Ker}\Pi_* \cap TS) + \dim(\operatorname{Ker}F_* \cap TS) = \dim S$,*

then (M, X_H) can be solved by quadratures.

Proof. According to the Propositions 4.20 and 5.8, the conditions above ensure that there exists an isotropic local restriction of Σ to every S and around every $s \in S$. Then, by Theorem 5.3, each Hamiltonian system (S, X_{H_S}) can be integrated by quadratures, and from Remark 5.6 the same is true for (M, X_H). □

Combining the Propositions 5.7 and 5.8 and the last theorem, we have the following corollary.

Corollary 5.10 *Consider a Poisson manifold (M,Ξ), its symplectic foliation $\mathcal{S}$, a function $H: M \to \mathbb{R}$, a fibration $\Pi: M \to N$ and a complete solution Σ of the Π-HJE for (M, X_H). If Σ is isotropic, then (M, X_H) can be solved by quadratures.*

In terms of first integrals, we have the next result and its immediate corollary.

Theorem 5.11 *Consider a Poisson manifold (M,Ξ), its symplectic foliation $\mathcal{S}$, a function $H: M \to \mathbb{R}$ and a fibration $F: M \to \Lambda$ such that $\operatorname{Im}X_H \subset \operatorname{Ker}F_*$. If F is weakly isotropic and for every leaf S of $\mathcal{S}$ we have that $\operatorname{Ker}F_* \cap TS$ is a constant rank distribution along S, then (M, X_H) can be solved by quadratures.*

Proof. Fix a leaf S and a point $s \in S$. Since $\operatorname{Ker}F_* \cap TS$ is a constant rank distribution along S, then $F|_S$ has constant rank. As a consequence, as we saw in the proof of Proposition 4.18, there exists an open neighborhood $V \subset S$ of s such that $F(V) \subset \Lambda$ is a submanifold, $F|_V: V \to F(V)$ is a fibration and

$$\operatorname{Im}\left(X_{H_S}|_V\right) = \operatorname{Im}\left(X_H|_V\right) \subset \operatorname{Ker}\left(F|_V\right)_*.$$

On the other hand, since F is weakly isotropic, we can prove as in Proposition 5.8 [see Eqs. (81) and (82)] that $F|_V$ is isotropic w.r.t. the symplectic structure ω of S. Then, the present theorem follows from Theorem 5.4 and the Remark 5.6. □

[8]Given two sets A and B and a function $G: A \to B$, recall that $G\left(G^{-1}(S)\right) \subset S$ for every subset $S \subset B$. Also, if A and B are vector spaces, then $(G(S))^0 = (G^*)^{-1}\left(S^0\right)$, for every subspace $S \subset A$.

Corollary 5.12 *Consider a Poisson manifold (M, Ξ), its symplectic foliation $\mathcal{S}$, a function $H : M \to \mathbb{R}$ and a fibration $F : M \to \Lambda$ such that* $\operatorname{Im} X_H \subset \operatorname{Ker} F_*$. *If F is isotropic, then (M, X_H) can be solved by quadratures.*

Concluding, given a Hamiltonian system on a Poisson manifold, in order to ensure integrability by quadratures it is enough to have a weakly isotropic set of first integrals (and an additional regularity condition), provided the symplectic foliation of the manifold is known. Note that we are not asking for the Poisson structure to have constant rank.

6 Final comments and future work

We have defined a Hamilton-Jacobi equation (HJE) for general dynamical systems on fibered phase spaces. We have focused on the consequences of having a partial or a complete solution of such an equation, for a given dynamical system, and, in particular, on the possibility of integrating by quadratures the equations of motion of the system in question. We have shown that the resulting Hamilton-Jacobi Theory is an extension of those developed for Hamiltonian systems on symplectic, Poisson and almost-Poisson manifolds. We have also shown, in the context of general dynamical systems, that there exists a deep connection between first integrals and the complete solutions of our HJE. This enabled us, in the context of Poisson manifolds, to identify properties of the complete solutions that give rise to non-commutative integrability. Also, a new method to integrate by quadratures a Hamiltonian system on a Poisson manifold have been developed. It is worth mentioned that, such a method, can be applied under weaker assumptions than those considered in the notion of non-commutative integrability.

This paper contains our first steps in the study of this new Hamilton-Jacobi Theory. In future works, we are planning to address the following problems.

Action-angle coordinates. We want to study the relationship between the generalized action-angle coordinates, which appears in a non-commutative integrable system on a Poisson manifold, and the immersion $\varphi : N \times \Lambda \to T^*\Lambda$ defined in Eq. (48) for isotropic complete solutions.

Hamiltonian systems with external forces and constrained systems. Given a manifold Q, a Hamiltonian system with external forces on Q is a dynamical system (T^*Q, X) with $X = X_H + Y$ for some vertical vector field $Y \in \mathfrak{X}(T^*Q)$ (i.e. $\operatorname{Im} Y \subset \operatorname{Ker} (\pi_Q)_*$): *the external force.* Such a force can represent a constraint force. So, the nonholonomic and generalized nonholonomic systems are particular examples. The complete version of the Π-HJE for (T^*Q, X) [see the second part of Eq. (36)], and for any fibration $\Pi : T^*Q \to N$, can be written

$$i_{X^\Sigma} \Sigma^* \omega = \Sigma^* (dH + \beta), \quad \text{with} \;\; \beta := \omega^\flat (Y).$$

If we ask, for instance, that Σ is isotropic and $(*)$ $\operatorname{Ker}(\Sigma^*\beta), \operatorname{Ker}(\Sigma^* d\beta) \subset TN \times 0$, it can be shown that a procedure analogous to that realized in Section 5.1 can be done for (T^*Q, X). Consequently, the system (T^*Q, X) can be integrated by quadratures. We are studying weaker conditions than $(*)$, and trying to apply them to particular examples of nonholonomic systems.

Time-dependent systems, time-dependent HJE, co-symplectic and contact manifolds. Cosymplectic and contact manifolds are the counterpart of the symplectic manifolds in odd dimension. The cosymplectic geometry plays an important role in the geometric description of the time-dependent Mechanics. On the other hand, the Thermodynamics admits a geometric formulation in terms of a certain contact structure on the phase space of the system. We shall study the relationship between the integration of the equations which determine the dynamics in both cases (cosymplectic and contact) and the Hamilton-Jacobi Theory. In the case of the cosymplectic structures, since they are Poisson structures, part of the theory developed in this paper can be considered.

Symmetries and reconstruction. Consider a Lie group G and a G-invariant dynamical[9] system (M, X) such that $N := M/G$ is a manifold and the canonical projection $\Pi : M \to N$ is a fibration. Let $Y \in \mathfrak{X}(N)$ be the reduced field, i.e. the unique vector field on N such that $\Pi_* \circ X = Y \circ \Pi$. It can be shown that, for every complete solution $\Sigma : N \times \Lambda \to M$ of the Π-HJE for (M, X), we have that $X^{\Sigma}(n, \lambda) = (Y(n), 0)$ for all n, λ. Then, each integral curve $\Gamma : I \to M$ of X can be obtained from an integral curve $\gamma : I \to N$ of the reduced field Y by the formula $\Gamma(t) = \Sigma(\gamma(t), \lambda)$, for some $\lambda \in \Lambda$. This means that the Π-HJE gives rise to a *reconstruction process*. We want to further study this fact.

7 Acknowledgements

This work was partially supported by Ministerio de Economía y Competitividad (Spain) grants MTM2012-34478 (E. Padrón) and the European Community IRSES-project GEOMECH-246981 (S. Grillo and E. Padrón). S. Grillo thanks CONICET for its financial support.

References

[1] R. Abraham, J.E. Marsden, *Foundation of Mechanics*, New York, Benjaming Cummings (1985).

[2] M. Adler, P. van Moerbeke, and P. Vanhaecke. *Algebraic integrability, Painlev e geometry and Lie algebras*, volume 47 of Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, 2004.

[3] V.I. Arnold, Ordinary differential equations, MIT Press, (1973) Translated from the Russian by Richard A. Silverman.

[4] V.I. Arnold, *Mathematical Models in Classical Mechanics*, Berlin, Springer-Verlag (1978).

[5] P. Balseiro, J.C. Marrero, D. Martín de Diego, E. Padrón, *A unified framework for Mechanics, Hamilton-Jacobi equation and applications*, Nonlinearity, **23**, 8 (2010), 1887–1918.

[6] L. Bates, S. Fasso, N. Sansonetto, *The Hamilton-Jacobi equation, integrability, and nonholonomic systems*, Journal of Geometric Mechanics **6**, 4 (2014), 441-449.

[7] L. Bates, J. Sniatycki, *Nonholonomic reduction*, Reports on Math. Phys. **32** (1993), 99-115.

[8] W.M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, New York, Academic Press (1985).

[9] F. Cantrijn, M. de León, D. Martín de Diego, *On almost-Poisson structures in nonholonomic mechanics*, Nonlinearity **12** (1999), 721-737.

[10] J. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. Muñoz-Lecanda, N. Roman-Roy, *Geometric Hamilton-Jacobi theory*, International Journal of Geometric Methods in Modern Physics **3**, 7 (2006), 1417-1458.

[11] J. Cariñena, X. Gràcia, G. Marmo, E. Martínez, M. Muñoz-Lecanda, N. Roman-Roy, *Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems*, International Journal of Geometric Methods in Modern Physics **7**, 3 (2010), 431-454.

[9] This means that there exists an action $\rho : G \times M \to M$ such that $(\rho_g)_* \circ X = X \circ \rho_g$, for all $g \in G$.

[12] H. Cendra, S. Grillo, *Generalized nonholonomic mechanics, servomechanisms and related brackets*, J. Math. Phys. **47** (2006), 2209-38.

[13] M. Crampin, *On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics*, J. Phys. A **14** (1981), 2567–2575.

[14] J.J. Duistermaat: *On global action-angle coordinates,* Comm. Pure Appl. Math. **33** (1980), 687–706.

[15] F. Fassò, *Superintegrable Hamiltonian Systems: Geometry and Perturbations*, Acta Applicandae Mathematicae **87** (2005), 93–121.

[16] R. Fernández, C. Laurent-Gengoux, P. Vanhaecke, *Global action-angle variables for non-commutative integrable systems*, arXiv:1503.00084 (2015).

[17] H. Goldstein, *Classical Mechanics*, Addison-Wesley (1951).

[18] J. Grabowski, P. Urbanski, *Algebroid-general differential calculus on vector bundles*, J. Geom. Phys. **31** (1999), 111-141.

[19] J. Klein, *Opérateurs différentielles sur les varietés presque tangentes*, C.R. Math. Acad. Sci. Paris **257** (1963), 2392–2394.

[20] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, New York, John Wiley & Son (1963).

[21] V. Kozlov, *An extension of the Hamilton-Jacobi method*, J. Appl. Maths Mechs **60**, 6 (1996), 911-920.

[22] C. Laurent-Gengoux, E. Miranda, P. Vanhaecke, Action-angle coordinates for integrable systems on Poisson manifolds, IMRN **13**, 8 (2011)1839-1869.

[23] M. de León, J.C. Marrero, D. Martín de Diego: *Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic Mechanics*, J. Geom. Mech. **2** (2010) 159–198.

[24] M. de León, D. Martín de Diego, M. Vaquero: *A Hamilton-Jacobi theory on Poisson manifolds*, Journal of Geometric Mechanics **6**, 1 (2014) 121-140.

[25] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry,* New York, Springer-Verlag (1994).

[26] A.S. Mischenko, A.T. Fomenko: *Generalized Liouville Methods of integration of Hamiltonian systems,* Funct. Anal. Appl., **12**, 2 (1978) 113–1978.

[27] J.E. Marsden, T.S. Ratiu, *Manifolds, Tensor Analysis and Applications*, New York, Springer-Verlag (2001).

[28] J.-P. Ortega, V. Planas-Bielsa, *Dynamics on Leibniz manifolds*, J. Geom. Phys. **52** (2004), 1-27.

[29] A.J. van der Schaft, B.M. Maschke, *On the Hamiltonian formulation of nonholonomic mechanical systems*, Rep. Math. Phys. **34** (1994), 225-233.

[30] N. T. Zung, *Reduction and Integrability*, arXiv:math/0201087v2 [math.DS] (2002).