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Pure pairing modes in trapped fermion systems

Abstract We present numerical predictions for the shape of the pairing fluctuations in harmonically trapped atomic ^{6}Li with two spin projections, based on the fluiddynamical description of cold fermions with pairing interactions. In previous works it has been shown that when the equilibrium of a symmetric mixture is perturbed, the linearised fluiddynamic equations decouple into two sets, one containing the sound mode of fermion superfluids and the other the pairing mode. The latter corresponds to oscillations of the modulus of the complex gap and is driven by the kinetic energy densities of the particles and of the pairs. Assuming proportionality between the heat flux and the energy gradient, the particle kinetic energy undergoes a diffusive behavior and the diffusion parameter is the key parameter for the relaxation time scale. We examine a possible range of values for this parameter and find that the shape of the pairing oscillation is rather insensitive to the precise value of the transport coefficient. Moreover, the pairing fluctuation is largely confined to the center of the trap, and the energy of the pairing mode is consistent with the magnitude of the equilibrium gap.

PACS numbers: 033.75.Ss,05.30.Fk,67.85.Lm

1 Introduction

Recently, we presented a fluiddynamical formulation for the motion of trapped fermions with two spin species, starting from the equations of motion (EOM's)

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of the particle field operators driven by a Hamiltonian that includes a zero range interaction between the different spins. The above EOM's are then applied to the derivation of a coupled system of dynamical equations for the density fields associated to particle number, current and kinetic energies of each species, plus those associated to pair, pair current and pair kinetic energy, within the mean field approach[1]. It is interesting to note that the fluiddynamical description is not a mere reformulation of standard quantum hydrodynamics (see *i.e.*, Ref. 2), where the leading quantities are the particle density and the gradient of the phase of the condensate wave function . The rigorous derivation of the hierarchy of moments of the particle and pair densities, together with the tools to truncate at any desired level, has been reviewed in Ref. 3 (hereafter denoted as I). The EOM's become a closed system called the Extended Superfluid Thomas–Fermi approximation (ESTF), after selecting a local equation of state (EOS) to represent the higher order moment whose microscopic description has been resigned.

In that previous work we have examined the dynamics of fluctuations in homogeneous fermion matter, showing that the fluiddynamical frame contains the characteristic modes put forward by Anderson[4] and Bogoliubov[5], discussed *e.g.*, by Leggett in Ref. 6. These modes are 1) the gapless sound mode of fermion superfluids and 2) the nonpropagating pairing vibration, with energy at zero momentum close to the pairing gap. Since the latter involves the internal dynamics of the pairs, it cannot be reached within the traditional hydrodynamical treatment of superfluids that yields phonons as unique longwavelength excitations.

In general, the fluiddynamical description consists of six coupled equations in partial derivatives and, consequently, is rather complex to apply to large amplitude dynamics of trapped fermions. By contrast, small amplitude oscillations can be boarded at a moderate computational cost, since the linearised EOM's decouple into two sets, each with three equations, one containing the propagation of sound and the other the pairing oscillation. In I we have studied, in addition to the equilibrium densities and gap profiles for an unpolarized fermion system, the nature of the sound mode and its associated new magnitude, the pair current, shown to be clearly different from the superfluid current that derives from the gradient of the superfluid order parameter.

In this work we discuss typical results for the pairing fluctuations described by the second system of EOM's.

2 Abridged formalism of fermion fluiddynamics

Let us briefly recall the derivation of fermion fluiddynamics (FD) as described in detail in I. We propose a zero–temperature grand potential operator for fermions with different spin projections $\sigma = \pm$ interacting with a zero range force, in numbers N_σ subject to harmonic potentials $V_\sigma(\mathbf{r})$

$$
\hat{\Omega} = \int d\mathbf{r} \sum_\sigma \left[-\frac{\hbar^2}{2m} \Psi_\sigma^\dagger(\mathbf{r}) \nabla^2 \Psi_\sigma(\mathbf{r}) + [V_\sigma(\mathbf{r}) - \mu_\sigma] \Psi_\sigma^\dagger(\mathbf{r}) \Psi_\sigma(\mathbf{r}) \right]
$$
$$
+ g \int d\mathbf{r} \, \Psi_+^\dagger(\mathbf{r}) \, \Psi_-^\dagger(\mathbf{r}) \, \Psi_-(\mathbf{r}) \, \Psi_+(\mathbf{r}) \tag{1}
$$

In terms of the *s*–wave scattering length a, the strength of the two body coupling is $g = 4\pi\hbar^2 a/m$.

The one-body density, current and kinetic energy matrices for each fermion species are expectation values in the many–body state,

$$\rho_\sigma(\mathbf{r},\mathbf{r}') = \langle \rho_\sigma(\hat{\mathbf{r}},\mathbf{r}') \rangle = \langle \Psi_\sigma^\dagger(\mathbf{r}')\,\Psi_\sigma(\mathbf{r}) \rangle \tag{2}$$

$$\mathbf{j}_\sigma(\mathbf{r},\mathbf{r}') = \langle \frac{\hbar}{2m\imath}(\nabla - \nabla')\,\hat{\rho}_\sigma(\mathbf{r},\mathbf{r}') \rangle \tag{3}$$

$$\tau_\sigma(\mathbf{r},\mathbf{r}') = \langle \frac{\hbar^2}{2m}\nabla\cdot\nabla'\,\hat{\rho}_\sigma(\mathbf{r},\mathbf{r}') \rangle \tag{4}$$

The pair density is

$$\kappa(\mathbf{r},\mathbf{r}') = \langle \Psi_+(\mathbf{r})\,\Psi_-(\mathbf{r}') \rangle \tag{5}$$

with the pair current and pair kinetic energy operators $\mathbf{j}_\kappa$ and τ_κ defined as above with gradients operating upon $\kappa(\mathbf{r},\mathbf{r}')$. The coupled EOM's for the above matrices are derived starting from

$$\imath\hbar\frac{\partial\Psi_\sigma(\mathbf{r})}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + [V_\sigma(\mathbf{r}) - \mu_\sigma] + g\,\Psi_{-\sigma}^\dagger(\mathbf{r})\,\Psi_{-\sigma}(\mathbf{r}) \right]\Psi_\sigma(\mathbf{r}) \tag{6}$$

and its Hermitian conjugate. In order to reach a description in terms of space–time fields, we select the diagonal terms of the matrices, in other words we take the limit for $\mathbf{r} \longrightarrow \mathbf{r}'$. Keeping in mind that the pair density diverges in this limit, one has to select the regular part $\kappa(\mathbf{r})$ on the diagonal of the matrix and introduce the gap field as $\Delta(\mathbf{r}) = -g\,\kappa(\mathbf{r})$. The FD scheme in terms of the particle, momentum and kinetic energy densities of the fermion species, and of the order parameter κ and its first two moments can be found in I, as well as the criterion to truncate the hierarchy at the kinetic energy level and the evaluation of the unknown higher moment by using a local density approximation. Here we only mention that as in classical hydrodynamics, the EOM for the particle kinetic energy contains the divergence of the heat flow $\mathbf{j}_\tau$. We shall come back to this point in Section 3.

We have analysed the fluctuations in an homogeneous, unpolarized fermion mixture and compared with the earlier predictions derived in the spirit of the Random–Phase Approximation (RPA)[4]. We have shown that the linearised EOM's can be rearranged in terms of a closed block for $\delta\rho$, $\delta\mathbf{j}$ and $\delta\mathbf{j}_\kappa$, containing the sound mode of superfluids $\omega_1 = c_s k$, being $c_s = \sqrt{\rho\,(\partial\mu/\partial\rho)/m}$ the usual sound velocity, and a mode that involves the relative motion within a single pair, plus an enslaved set that contains, in addition to the former, the perturbations of the pair density and the kinetic energies. For $\delta\rho = \delta\tau = 0$, one of the eigenmodes of the second set is a real gapped mode with energy of order Δ. The energy of this pairing vibration is strongly dependent on the particle kinetic energy in equilibrium, which in turn introduces the equilibrium gap to the leading order; this origin explains the impossibility of the traditional hydrodynamics of fermion superfluids as described *e.g.*, in Ref. 2 to account for these massive modes.

3 Pairing vibrations in a trap

When the superfluid is confined by a harmonic trap, the linearised FD EOM's must be solved numerically. Expressing the departure from equilibrium of any field $f(\mathbf{r},t)$ as $\delta f(\mathbf{r},t) = \delta f^+(\mathbf{r})\,e^{\imath\omega t} + \delta f^-(\mathbf{r})\,e^{-\imath\omega t}$, the collective fluctuations in the confined superfluid can be separated as before in density modes and pure pairing fluctuations. In this case, the latter correspond to setting $\delta\rho = \delta\mathbf{j} = \delta\mathbf{j}_\kappa$ equal to zero as well as the equilibrium currents, and linearising around the equilibrium values of the six fields. The resulting eigenvalue equations read

$$\pm\imath\omega\delta\tau^\pm = -\nabla\cdot\delta\mathbf{j}_\tau^\pm + \frac{4g}{m}\mathrm{Im}\left(\kappa\,\delta\tau_\kappa^\pm + \tau_\kappa\,\delta\kappa^\pm\right)$$
$$- \frac{g\hbar}{m}\mathrm{Im}(\kappa^2\nabla^2\delta\kappa^\pm + \nabla^2\kappa\,\delta\kappa^{*\pm}) \tag{7}$$

$$\pm\hbar\omega\,\delta\kappa^\pm = \left(-\frac{\hbar^2}{4m}\nabla^2 + V - \mu\right)\delta\kappa^\pm + 2\,\delta\tau_\kappa^\pm \tag{8}$$

$$\pm\hbar\omega\,\delta\tau_\kappa^\pm = \left(-\frac{\hbar^2}{4m}\nabla^2 + V + g\rho - \mu\right)\delta\tau_\kappa^\pm - g\left(\tau\,\delta\kappa^\pm + \kappa\,\delta\tau^\pm\right)$$
$$- \frac{\hbar^2}{4m}\left[\nabla^2(V + g\rho)\,\delta\kappa^\pm - g\rho\,\nabla^2\delta\kappa^\pm\right] \tag{9}$$

Here ρ, τ, V and μ correspond to the total quantities (summation over both species). Moreover, we choose to substitute the flux contribution $-\nabla\cdot\mathbf{j}_\tau$ entering the dynamics of the particle kinetic energy, where $\mathbf{j}_\tau$ is the kinetic energy current, by a diffusion term of the form $K\nabla^2\tau$, under the assumption that the above current is proportional to the gradient of the kinetic energy, with strength K. Note that K is not the thermal conductivity that relates the heat flux to the gradient of the temperature field in standard transport theory; since the systems here considered are at zero temperature, the origin of difussion is strictly quantal. In our local description, $\kappa(\mathbf{r})$ represents the number of pairs at position $\mathbf{r}$ in the equilibrium cloud at zero temperature, thus $\delta\kappa(\mathbf{r})$ is the number of pairs that are locally excited above the fermi surface. This excitation demands a local diffusive flow of kinetic energy $\delta\mathbf{j}_\tau$, which to lowest order should be linear in the fluctuation of the latter, with a somehow important strength, here represented by K.

A realistic calculation of this coefficient for trapped gases as a function of the scattering length, in a frame equivalent to the well–known one of transport theory at finite temperature, is beyond the scope of our work, so we have examined possible scenarios within a wide range of values of the diffusion parameter. In what follows, every nonfluctuating field in the linearised EOM's has been previously computed in equilibrium in the trapped fluid, by setting the time derivatives in the ESTF EOM's given in I equal to zero. Characteristic profiles for the equilibrium gap and for both kinetic energy densities are shown in Fig. 1 to illustrate the dependence with the scattering length a. It is clear that pairing effects are considerably dampened away as the interaction weakens, while the kinetic energy shows an important decrease at the center of the trap following the behavior of the density profile, and its tail reflects the slight enlargement of the cloud size. These calculations have been performed for a symmetric mixture of N = 1.7 10^4 ^{6}Li atoms in a trap with frequency $\omega_{\mathrm{osc}} = 817$ Hz.
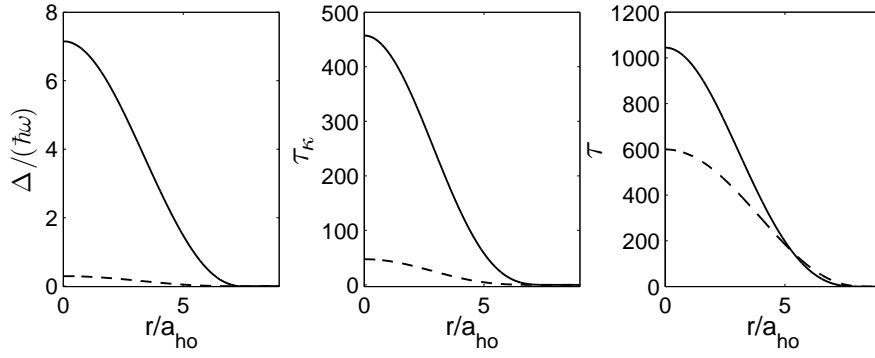
Fig. 1 Equilibrium configuration for Δ, τ_κ and τ for a = -114 nm (solid lines) and -50 nm (dashed lines). $a_{\mathrm{ho}} = [\hbar/(m\omega_{\mathrm{osc}})]^{1/2}$ is the harmonic oscillator size.
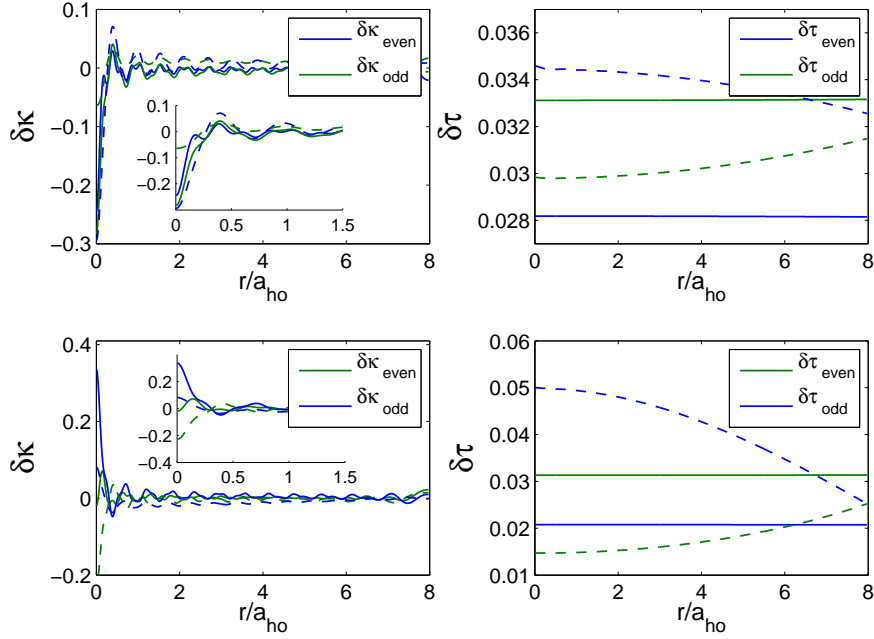

Fig. 2 (color online) Fluctuations of pair density and particle kinetic energy for the two lowest energy levels with zero multipolarity (upper and lower panels, respectively), for diffusion $K = 10^4$ (solid lines) and 10^2 (dashed lines) in units of $\hbar/m$.

All fluctuations are written as $\delta g(\mathbf{r}) = g(r) Y_{lm}(\hat{r})$ with Y_{lm} a spherical harmonic function; the threedimensional eigenvalue equations are then formulated as a set of onedimensional coupled equations for the radial amplitudes of a given multipolarity l. We have chosen to compute, for each physical quantity f, the quantities $\delta f_{\mathrm{even}} = \delta f^+ + \delta f^-$, $\delta f_{\mathrm{odd}} = \delta f^+ - \delta f^-$, that represent straightforwardly the real and imaginary part of δf. In Fig. 2 we show, for a = -114 nm and
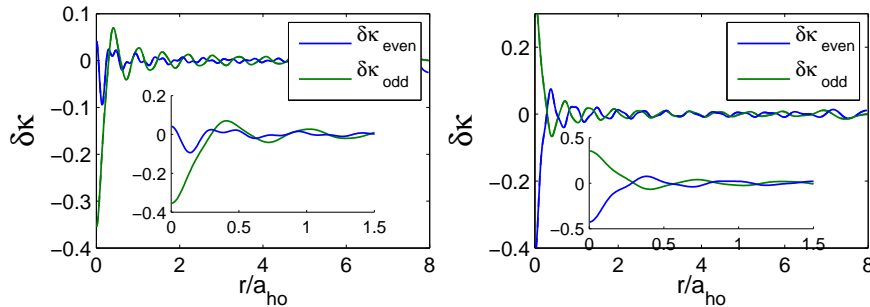
Fig. 3 (color online) Same as Fig 2 with vanishing $\delta\tau$.

for two values of the diffusion K –chosen to qualitatively reproduce the expected behavior of the pairing frequency– the pairing and particle kinetic energy fluctuations for the two lowest energy levels, corresponding to $l = 0$ and with energies $\hbar\omega$ = 0.5 $\hbar\omega_{osc}$, 5.5 $\hbar\omega_{osc}$ for $K = 10^2 \hbar/m$, and 1.1 $\hbar\omega_{osc}$, 1.4 $\hbar\omega_{osc}$ for $K = 10^4 \hbar/m$.

The strong oscillations of the pair density, with sizeable amplitudes near the trap center (see insets), contrast with the smooth behavior of the particle kinetic energy, largely sensitive to the magnitude of the diffusion parameter. However, the shape of the amplitude of the pair fluctuations remains rather independent on K. In other words, the kinetic energy flow affects the eigenfrequencies of modes but it seems to be unimportant in the evolution of the pairs, at least in the small amplitude regime. This statement is more strongly illustrated in Fig. 3, where the the whole kinetic energy fluctuation is removed, without any noticeable modification in the overall aspect of $\delta\kappa$. Moreover, all scenarios show that the pair fluctuations smooth away within the scale of the trap, $r \approx a_{ho}$, whereas the equilibrium profiles occupy the full size of the cloud. On the other hand, the right panel of Fig. 2 shows that $\nabla\delta\tau$ becomes negligible as K increases, indicating that the system reacts to arbitrarily large difussion by smothing away the energy flow, so as to prevent an ultraviolet catastrophe in the spectrum.

Previous results on the pairing modes of the superfluid trapped atomic Fermi gas with attractive interaction in the RPA approximation[7] have shown that for moderate values of the scattering length a and/or of the number of trapped atoms, the value of the lowest pairing mode is of the order of the gap Δ. This result was obtained in the intrashell regime, where Cooper pairs are formed only between atoms with the same radial quantum number, when the coherence length is much shorter than the system size. For our trap configuration and number of particles, we found that for a more negative than -50 nm our system is not in the intrashell regime; although the frequencies provided by our fluiddynamical scheme depend on the exact value of K, for the less interacting systems the lowest pairing mode is consistent with the intrashell model.

Finally, we mention here that the energies of these pairing modes for $a = -114$ nm for not too large values of K are comparable to the energy scale $\hbar\omega_{osc}$. This is due to the relatively high value of a and the presence of the trapping potential, which imposes a clear energy scale for the collective modes of the system. For sizable K –above $10^4\hbar/m$– these energies become comparable to twice the gap magnitude at the trap center.

4 Conclusions

The present study aims at shedding light on the pairing dynamics of a trapped fermi superfluid in the weak coupling regime, in the spirit of mean field theory, here mapped onto a an extended hydrodynamical scheme, that contains the first three moments of the pair density together with those of the particle density. Our approach differs largely from *e.g.*, the study of large amplitude pairing correlations in an homogeneous, unitary fermi gas, where the pairing modes are included[8]. Within our fluiddynamical description we can clarify the role of the particle kinetic energy in an equilibrated cloud, that enters the eigenvalue equation at the same level as the energy eigenvalue (cf. Eqs. (7) to (9)), and the kinetic energy fluctuations, that apparently play a minor role in the pair dynamics, at least for small amplitude oscillations and for the lowest–lying modes. Since the main features of the structure and energetics of the pair fluctuations are reproduced setting the kinetic energy oscillations equal to zero, it is possible to interpret that we are in the presence of an intrinsic, pure pairing mode, that expresses aspects of the internal dynamics of the pairs which are absent in the standard formulation of syperfluid hydrodynamics.

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