Composition of Fractional Orlicz Maximal Operators and A_1 -weights on Spaces of Homogeneous Type

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Abstract For a Young function Θ with $0 \leq \alpha < 1$, let $M_{\alpha,\Theta}$ be the fractional Orlicz maximal operator defined in the context of the spaces of homogeneous type (X, d, μ) by $M_{\alpha,\Theta}f(x) = \sup_{x \in B} \mu(B)^\alpha \|f\|_{\Theta,B}$, where $\|f\|_{\Theta,B}$ is the mean Luxemburg norm of f on a ball B . When $\alpha = 0$ we simply denote it by M_Θ . In this paper we prove that if Φ and Ψ are two Young functions, there exists a third Young function Θ such that the composition $M_{\alpha,\Psi} \circ M_\Phi$ is pointwise equivalent to $M_{\alpha,\Theta}$. As a consequence we prove that for some Young functions Θ , if $M_{\alpha,\Theta}f < \infty$ a.e. and $\delta \in (0, 1)$ then $(M_{\alpha,\Theta}f)^\delta$ is an A_1 -weight.

Keywords Orlicz maximal function, spaces of homogeneous type, weights

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1 Introduction

Let us consider a space of homogeneous type (X, d, μ) , that is, X is a set endowed with a quasi-distance d such that the balls $B(x, r) = \{y \in X : d(x, y) < r\}$ are open sets, and with a positive measure μ satisfying a doubling condition (we refer to Section 2 for a more complete definition). Given a locally integrable function f on X , let $M_\alpha f$, $0 \leq \alpha < 1$, be the fractional maximal operator defined by

$$M_\alpha f(x) = \sup_{x \in B} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all balls B containing x . If $\alpha = 0$ we get the Hardy–Littlewood maximal operator; in this case, we drop the subscript α . It is known that the following result holds for $\mathcal{M} = M_\alpha$:

$$\text{If } \mathcal{M}f < \infty \text{ a.e. and if } \delta \in (0, 1), \text{ then } (\mathcal{M}f)^\delta \in A_1, \quad (1.1)$$

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where A_1 is the Muckenhoupt class of nonnegative locally integrable functions w such that

$$A_1 : \quad \frac{1}{\mu(B)} \int_B w \, d\mu \leq Cw(x), \quad \text{a.e. } x \in B, \quad (1.2)$$

for all balls B . The proof of this result follows by standard arguments (see [1] for the Euclidean case and [2] for the case $\alpha = 0$), that is, if $\tilde{B}$ is a suitable dilation of B , writing $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$, it is enough to prove that (1.2) holds with w replaced by each $(M_\alpha f_i)^\delta$, $i = 1, 2$. To establish (1.2) for $(M_\alpha f_1)^\delta$ the weak $(1, \frac{1}{1-\alpha})$ type inequality of M_α is applied and, for the other case, there is applied the fact that for any two points x, y belonging B we have

$$M_\alpha(f_2)(y) \leq C M_\alpha(f_2)(x). \quad (1.3)$$

For $0 \leq \alpha < 1$, a generalization of the operator M_α is the fractional Orlicz maximal operator associated with a Young function Φ defined, for each function f on X , by

$$M_{\alpha, \Phi} f(x) = \sup_{x \in B} \mu(B)^\alpha \|f\|_{\Phi, B},$$

where the supremum is taken over all balls B containing x and

$$\|f\|_{\Phi, B} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi \left(\frac{|f(y)|}{\lambda} \right) d\mu(y) \leq 1 \right\} \quad (1.4)$$

is the Φ -mean Luxemburg norm of a function f on a ball B . When $\alpha = 0$ we also drop the subscript α . When $\Phi(t) = t$, $M_{\alpha, \Phi}$ is the fractional maximal operator M_α .

In the last years, weighted inequalities with non-a-priori assumption on the weights have been proved for linear operators like singular integrals, fractional integrals and their commutators by using duality arguments (for the euclidean setting, singular integrals and their commutators see for example [3, 4], for spaces of homogeneous type see [5] for the commutators of singular integrals and [6] for fractional integrals and their commutators). One of the main tools in the proofs of these inequalities is to establish (1.1) for $\mathcal{M} = M_{\alpha, \Phi}$ with suitable Young functions Φ . If $\alpha = 0$ then (1.1) can be proved for $\mathcal{M} = M_\Phi$ and any Young function by using the same arguments described above. However, it is not clear how to prove (1.1) for $\mathcal{M} = M_{\alpha, \Phi}$ and $\alpha \neq 0$ by applying the standard arguments, although it is possible to obtain a result like (1.3) for $M_{\alpha, \Phi}$ (see Lemma 4.2 in [6]) and there is an end-point estimate for this operator for some Young functions (see [7, 8]). We point out that in [6] the authors proved that (1.1) holds for $\mathcal{M} = M_{\alpha, \Phi_k}$ in the special case $\Phi_k(t) = t[\log(e+t)]^k$, $k \in \mathbb{N}$. The proof of this result is based on the fact that M_{α, Φ_k} is equivalent to the composition $M_\alpha(M^k)$ where M^k is the Hardy–Littlewood maximal operator iterated k times.

One of the purposes of this paper is to prove that (1.1) holds for the maximal operators $M_{\alpha, \Phi}$ for more general Young functions Φ . This result will be a consequence of the following type of result: Given two Young functions Ψ and Φ , we shall define a third Young function Θ , such that the composition $M_{\alpha, \Psi} \circ M_\Phi$ is equivalent to the operator $M_{\alpha, \Theta}$. The proof of this last result will be the main point of this article.

Before stating the theorem we shall observe some properties of the maximal functions $M_{\alpha, \Phi}$. Let Φ_1 and Φ_2 be Young functions. We say that Φ_2 dominates Φ_1 at ∞ , and denote it by

$\Phi_1 \prec_\infty \Phi_2$, if there exist $a, b, t_0 > 0$ such that

$$\Phi_1(t) \leq b\Phi_2(at) \quad \text{for all } t \geq t_0.$$

If $\Phi_1 \prec_\infty \Phi_2$ then there exists a constant C , depending on Φ_1 and Φ_2 , such that $\|f\|_{\Phi_1, B} \leq C\|f\|_{\Phi_2, B}$ for all balls B and functions f . Since the constant C is independent of B , we get that $M_{\alpha, \Phi_1} f(x) \leq CM_{\alpha, \Phi_2} f(x)$. We say that Φ_1 is equivalent to Φ_2 at ∞ , and denote it by $\Phi_1 \approx_\infty \Phi_2$, if $\Phi_1 \prec_\infty \Phi_2$ and $\Phi_2 \prec_\infty \Phi_1$. Therefore, if $\Phi_1 \approx_\infty \Phi_2$ then $M_{\alpha, \Phi_1} \approx M_{\alpha, \Phi_2}$, that is, there exist two positive constants C_1 and C_2 such that $C_1 M_{\alpha, \Phi_1} f(x) \leq M_{\alpha, \Phi_2} f(x) \leq C_2 M_{\alpha, \Phi_1} f(x)$. Let Φ be a Young function and define

$$\Phi_0(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ \Phi(t) - \Phi(1), & \text{if } t \geq 1. \end{cases} \quad (1.5)$$

Since Φ_0 is a Young function and $\Phi_0 \approx_\infty \Phi$, it is clear that $M_{\alpha, \Phi_0} \approx M_{\alpha, \Phi}$.

Now we are ready to state our main result.

Theorem 1.1 *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$. Let Ψ and Φ be two Young functions and $0 \leq \alpha < 1$. We define the function*

$$\Theta(t) = \int_0^t \Psi'_0(u) \Phi(t/u) du, \quad (1.6)$$

where Ψ_0 is defined as in (1.5) and Ψ'_0 is the derivative of Ψ_0 . Then, Θ is a Young function and for all Young functions $\bar{\Theta} \approx_\infty \Theta$ we get

$$M_{\alpha, \bar{\Theta}} \approx M_{\alpha, \Psi}(M_\Phi). \quad (1.7)$$

When $X = \mathbb{R}^n$, d is the Euclidean distance, μ is the Lebesgue measure and $\alpha = 0$ the equivalence (1.7) was proved in [9]. As far as we know, the result for the case $\alpha \neq 0$ is new even in the Euclidean case.

In order to show some applications of Theorem 1.1, we introduce the following notation: If $\Phi(t) = t^r$ or $\Phi(t) = t^r(1 + \log^+ t)^\beta$, the fractional Orlicz maximal operators $M_{\alpha, \Phi}$ are respectively written as M_{α, L^r} and $M_{\alpha, L^r(\log L)^\beta}$; if $\alpha = 0$ we simply write M_{L^r} and $M_{L^r(\log L)^\beta}$.

Example 1 Let $p \geq 1$. Applying Theorem 1.1 with $\Psi(t) = t^p$ and $\Phi(t) = t^p(1 + \log^+ t)^\beta$, $\beta \geq 0$, we get

$$M_{\alpha, L^p}(M_{L^p(\log L)^\beta}) \approx M_{\alpha, L^p(\log L)^{\beta+1}}.$$

Notice that if $p = 1$ we get

$$M_\alpha(M_{L(\log L)^\beta}) \approx M_{\alpha, L(\log L)^{\beta+1}}. \quad (1.8)$$

In particular, when $p = 1$ and $\beta = 0$ we get that $M_\alpha(M) \approx M_{\alpha, L(\log L)}$. By induction, using (1.8) and the induction hypothesis with $\alpha = 0$, we easily obtain the known result

$$M_\alpha(M^k) \approx M_{\alpha, L(\log L)^k}, \quad k \in \mathbb{N},$$

where M^k is the iteration of the Hardy–Littlewood maximal operator k times (see Lemma 4.1 in [6]).

Example 2 If $\Psi(t) = t^p$ and $\Phi(t) = t^q$, $p, q \geq 1$ and $p \neq q$, then

$$M_{\alpha, L^p}(M_{L^q}) \approx M_{\alpha, L^r},$$

where $r = \max\{p, q\}$. In particular, for $p > 1$, $M_{\alpha}(M_{L^p}) \approx M_{\alpha, L^p}$.

Example 3 If $\Psi(t) = t$ and $\Phi(t) = t^p(1 + \log^+ t)^\beta$ with $\beta > 0$ and $p > 1$, applying Theorem 1.1 we obtain

$$M_{\alpha}(M_{L^p(\log L)^\beta}) \approx M_{\alpha, L^p(\log L)^\beta}.$$

Example 4 From Example 3 and Example 1 (with $\alpha = 0$) we get

$$M_{\alpha}((M_{L^p})^{k+1}) \approx M_{\alpha, L^p(\log L)^k}, \quad k \in \mathbb{N} \text{ and } p > 1.$$

Notice that if we take $\Psi(t) = t$ in (1.6) we get that for $t > 1$

$$\Theta(t) = \int_1^t \Phi(t/u) du = t \int_1^t \Phi(u) u^{-2} du,$$

or equivalently $\Phi(t) = t\Theta'(t) - \Theta(t)$ for all $t > 1$. Then, it is easy to prove the following corollary of Theorem 1.1.

Corollary 1.2 *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$, and let $0 \leq \alpha < 1$. Let Θ be a Young function which is not equivalent at ∞ to $\eta(t) = t$ and such that there exists a Young function Φ with $\Phi(t) = t\Theta'(t) - \Theta(t)$, for $t > 1$. Then, if $M_{\alpha, \Theta}f < \infty$ a.e. and $\delta \in (0, 1)$, we get that $(M_{\alpha, \Theta}f)^\delta \in A_1$.*

In fact, by Theorem 1.1 we get that $(M_{\alpha, \Theta}f)^\delta \approx [(M_{\alpha}(M_{\Phi}f)]^\delta$, and Corollary 1.2 follows by (1.1) with $\mathcal{M} = M_{\alpha}$.

Remark 1.3 Observe that if $\Theta \approx_{\infty} \eta$ with $\eta(t) = t$, then $M_{\alpha, \Theta} \approx M_{\alpha}$, and the corollary follows by standard arguments.

Remark 1.4 From the above examples we have that (1.1) holds for $\mathcal{M} = M_{\alpha, \Theta}$, where $\Theta(t) = t^p(1 + \log^+ t)^\beta$ for $p > 1$, $\beta \geq 0$ or $p = 1$, $\beta \geq 1$.

The article is organized in the following way: in Section 2 we give some preliminaries results and we prove a reverse inequality of the weak type inequality for the operator M_{Φ} , while Section 3 is devoted to proving Theorem 1.1.

2 Preliminaries and Previous Results

Given a set X , a function $d : X \times X \rightarrow \mathbb{R}_0^+$ is called a quasi-distance on X if the following conditions are satisfied:

- (i) For every x and y in X , $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$,
- (ii) For every x and y in X , $d(x, y) = d(y, x)$,
- (iii) There exists a constant $K \geq 1$ such that

$$d(x, y) \leq K(d(x, z) + d(z, y)) \tag{2.1}$$

for every x, y and z in X . We shall say that two quasi-distances d and d' on X are equivalent if there exist two positive constants c_1 and c_2 such that $c_1 d'(x, y) \leq d(x, y) \leq c_2 d'(x, y)$ for all $x, y \in X$. In particular equivalent quasi-distances induce the same topology on X .

Let μ be a positive measure on the σ -algebra of subsets of X which contains the d -balls $B(x, r) = \{y : d(x, y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty \quad (2.2)$$

holds for all $x \in X$ and $r > 0$.

A structure (X, d, μ) , with d and μ as above, is called a *space of homogeneous type*. The constants K and A in (2.1) and (2.2) will be called the constants of the space.

The balls in a general space of homogeneous type are not necessarily open. Macías and Segovia in [10] proved that there exists a continuous quasi-distance d' equivalent to d , for which every ball is open. In this article we always assume that the quasi-distance d is continuous and the balls are open sets. For a given quasi-distance d , sometimes we write $B_d(x, R)$ to describe the ball centred at x with radius R associated with d .

We shall say that a function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function if there is a nontrivial, non-negative and increasing function ϕ such that $\Phi(t) = \int_0^t \phi(u) du$. Then, Φ is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. For a Young function Φ , the maximal operator M_Φ satisfies the following weak type inequality

$$\mu(\{x \in X : M_\Phi f(x) > \lambda\}) \leq C \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x). \quad (2.3)$$

The proof of the above inequality is similar to that of the $(1, 1)$ -weak type inequality for the Hardy Littlewood maximal operator (see [11]). By standard arguments, it follows from (2.3) that

$$\mu(\{x \in X : M_\Phi f(x) > \lambda\}) \leq C \int_{\{x \in X : |f(x)| > \lambda/2\}} \Phi\left(\frac{2|f(x)|}{\lambda}\right) d\mu(x), \quad (2.4)$$

for some constant C , all $\lambda > 0$ and all measurable function f .

In order to prove a suitable reverse inequality of (2.4) we shall need two results. The first one is a Calderón–Zygmund decomposition with Orlicz norms on a bounded space of homogeneous type. The proof follows the same steps as the one in [12] for the case $\Phi(t) = t$, so we omit it.

Lemma 2.1 *Let (X, d, μ) be a bounded space of homogeneous type, Φ a Young function and f a nonnegative function defined on X . Then, for every $\lambda > \|f\|_{\Phi, X}$, there exists a sequence of disjoint balls, $\{B_i\} = \{B(x_i, r_i)\}$ such that, if $\tilde{B}_i = B(x_i, Cr_i)$, where C is a constant depending only on the constant K in (2.1), then*

- (i) $\|f\|_{\Phi, \tilde{B}_i} \leq \lambda < \|f\|_{\Phi, B_i}$ and
- (ii) $\|f\|_{\Phi, B} \leq \lambda$ for every ball B centered at $x \in X \setminus \bigcup_i \tilde{B}_i$.

Remark 2.2 If (X, d, μ) is a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ we can apply the Lebesgue Differentiation Theorem in (ii) of Lemma 2.1 to obtain that $\Phi(f(x)/\lambda) \leq 1$ for almost every $x \in X \setminus \bigcup_i \tilde{B}_i$.

The second result that we shall be dealing with is the following theorem due to Macías and Segovia [13].

Theorem 2.3 (See [13]) *Let (X, d, μ) be a space of homogeneous type. There exists a quasi-distance δ on X which is equivalent to d such that, for some constant $C > 0$ depending only on*

the constants of the space, if $x \in X$, $0 < r \leq 6K^3R$ and $y \in B_\delta(x, R)$ then

$$\mu(B_\delta(y, r) \cap B_\delta(x, R)) \geq C\mu(B_\delta(y, r)). \quad (2.5)$$

Moreover,

$$\delta(x, y) \leq d(x, y) \leq 3K^2\delta(x, y), \quad (2.6)$$

for every x and y in X .

The balls $B_\delta(x, R)$ endowed with the restrictions of the quasi-distance δ and the measure μ become bounded spaces of homogeneous type with constants K' and A' , satisfying (2.1) and (2.2) respectively, independent of $R > 0$ and $x \in X$.

The above result provides us a quasi-distance δ equivalent to the quasi-distance d of the space with the property that the balls B_δ are spaces of homogeneous type. This property is not necessarily true for the balls B_d .

In the following lemma we state and prove a version of the reverse inequality for M_Φ .

Lemma 2.4 *Let (X, d, μ) be a space of homogeneous type such that the continuous functions are dense in $L^1(X)$ and let δ be the quasi-distance defined in Theorem 2.3. Let $B_\delta = B_\delta(x, R)$ be a fixed ball on X . Then, there exist positive constants C and D , depending only on the constants of the space, such that*

$$\int_{\{y \in B_\delta : \Phi(\frac{f(y)}{\lambda}) > 1\}} \Phi\left(\frac{f(y)}{\lambda}\right) d\mu(y) \leq C\mu(\{y \in B_\delta : M_\Phi f(y) > D\lambda\}),$$

for any $\lambda > \|f\|_{\Phi, B_\delta}$ and all non-negative functions f .

Proof Given a non-negative function f on B_δ and $\lambda > \|f\|_{\Phi, B_\delta}$, we apply Lemma 2.1 (and the corresponding Remark 2.2) to f at the level λ on the space of homogeneous type (B_δ, δ, μ) . That is, there exist a sequence $\{x_i\} \subset B_\delta$ and disjoint δ -balls $S_i = B_\delta(x_i, r_i) \cap B_\delta$ in this space such that if $\tilde{S}_i = B_\delta(x_i, Cr_i) \cap B_\delta$ with C depending only on K , then

- (a) $\|f\|_{\Phi, \tilde{S}_i} \leq \lambda < \|f\|_{\Phi, S_i}$ and
- (b) $\Phi\left(\frac{f(x)}{\lambda}\right) \leq 1$ for almost every $x \in B_\delta \setminus \bigcup_i \tilde{S}_i$.

We start proving that there exists $D > 0$ such that for all i ,

$$S_i \subset \{y \in B_\delta : M_\Phi f(y) > D\lambda\}. \quad (2.7)$$

Notice that, by (2.5) and (2.6) in Theorem 2.3 we get that $\mu(S_i) \geq C_1\mu(B_\delta(x_i, r_i))$, with $C_1 < 1$ and $B_\delta(x_i, r_i) \subset B_d(x_i, 3K^2r_i) \subset B_\delta(x_i, 3K^2r_i)$ respectively. Then $\mu(B_d(x_i, 3K^2r_i)) \leq C_2\mu(B_\delta(x_i, r_i))$, with $C_2 > 1$. Now, since Φ is a convex function

$$\begin{aligned} \frac{1}{\mu(S_i)} \int_{S_i} \Phi\left(\frac{f}{\lambda}\right) d\mu &\leq \frac{1}{C_1\mu(B_\delta(x_i, r_i))} \int_{B_\delta(x_i, r_i)} \Phi\left(\frac{f}{\lambda}\right) d\mu \\ &\leq \frac{C_2}{C_1\mu(B_d(x_i, 3K^2r_i))} \int_{B_d(x_i, 3K^2r_i)} \Phi\left(\frac{f}{\lambda}\right) d\mu \\ &\leq \frac{1}{\mu(B_d(x_i, 3K^2r_i))} \int_{B_d(x_i, 3K^2r_i)} \Phi\left(\frac{C_2f}{C_1\lambda}\right) d\mu, \end{aligned}$$

then, taking $D = \frac{C_1}{C_2}$ and using item (a) we get

$$\lambda < \|f\|_{\Phi, S_i} \leq D^{-1}\|f\|_{\Phi, B_d(x_i, 3K^2r_i)} \leq D^{-1}M_\Phi f(y),$$

for each $y \in S_i$, and we get (2.7).

Finally, by (2.7), (a) and (b) we get

$$\begin{aligned} \mu(\{y \in B_\delta : M_\Phi f(y) > D\lambda\}) &\geq \sum_i \mu(S_i) \geq C \sum_i \mu(\tilde{S}_i) \\ &\geq \sum_i \int_{\tilde{S}_i} \Phi\left(\frac{f}{\lambda}\right) d\mu \geq \int_{\bigcup_i \tilde{S}_i} \Phi\left(\frac{f}{\lambda}\right) d\mu \\ &\geq C \int_{\{y \in B_\delta : \Phi(\frac{f(y)}{\lambda}) > 1\}} \Phi\left(\frac{f}{\lambda}\right) d\mu(y), \end{aligned}$$

as we wished to prove. $\square$

We shall also need the following lemma proved in [6], which is the corresponding result of the inequality (1.3) for $M_{\alpha, \Phi}$.

Lemma 2.5 (See [6]) *Let (X, d, μ) be a space of homogeneous type, $0 \leq \alpha < 1$, Φ a Young function, $B = B(x, R)$ a fixed ball and $\tilde{B} = B(x, 2KR)$. Then, there exists a constant $C > 0$, depending only on the constants of the space, such that*

$$\max\{M_{\alpha, \Phi}(f\chi_{X \setminus \tilde{B}})(y), \mu(B)^\alpha M_\Phi(f\chi_{X \setminus \tilde{B}})(y)\} \leq C \inf_{z \in B} M_{\alpha, \Phi}(f\chi_{X \setminus \tilde{B}})(z),$$

for all $y \in B$.

3 Proof of Theorem 1.1

Without loss of generality we may assume that $\Phi(1) = 1$ and $\Psi(1) = 1$. To prove that Θ is a Young function we proceed as in [14]. In fact, let us assume that $\Phi(t) = \int_0^t \phi(u) du$. Since $\Phi(1) = 1$ we get

$$\Phi(t/u) = 1 + \frac{1}{u} \int_u^t \phi(v/u) dv, \quad \text{for } t \geq u.$$

Replacing this formula in (1.6) and changing the order of integration we get

$$\begin{aligned} \Theta(t) &= \int_0^t \Psi'_0(u) \left[1 + \frac{1}{u} \int_u^t \phi(v/u) dv \right] du \\ &= \int_0^t \Psi'_0(u) du + \int_0^t \left[\int_0^v \Psi'_0(u) \phi(v/u) u^{-1} du \right] dv = \int_0^t \theta(u) du, \end{aligned}$$

with

$$\theta(t) = \Psi'_0(t) + \int_0^t \Psi'_0(u) \phi(t/u) u^{-1} du.$$

It follows that Θ is a Young function, since Ψ'_0 and ϕ are non-negative and Ψ'_0 is increasing.

To prove (1.7), we begin to prove that there exists $C > 0$ such that

$$M_{\alpha, \Theta} f(x) \leq C M_{\alpha, \Psi}(M_\Phi f)(x)$$

for all $x \in X$. Let us assume that $f \geq 0$, and let us fix an $x \in X$ such that $M_{\alpha, \Psi}(M_\Phi f)(x) < \infty$. Let $B = B(z, R)$ be any ball on X such that $x \in B$ and $\tilde{B} = B(z, 3K^2R)$. Notice that it is enough to show that there exists a constant C such that

$$\|f\|_{\Theta, B} \leq C \|M_\Phi f\|_{\Psi, \tilde{B}}. \quad (3.1)$$

Let δ be the quasi-distance equivalent to d defined in Theorem 2.3. If $B_\delta = B_\delta(z, R)$, let $\lambda_0 = \|M_\Phi f\|_{\Psi, B_\delta}$. To prove (3.1) it is enough to show that there exists a constant $C_0 > 1$ such that

$$\frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0 \lambda_0}\right) d\mu(x) \leq 1. \quad (3.2)$$

In fact, from (2.6) we get that $B \subset B_\delta \subset \tilde{B}$. On the other hand, $\mu(\tilde{B}) \leq \tilde{C}\mu(B)$ for some universal constant $\tilde{C} \geq 1$. Since Θ is a convex function, if (3.2) holds then

$$\begin{aligned} \frac{1}{\mu(B)} \int_B \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) &\leq \frac{\mu(\tilde{B})}{\mu(B)} \frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) \\ &\leq \frac{\tilde{C}}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{\tilde{C}C_0\lambda_0}\right) d\mu(x) \\ &\leq \frac{1}{\mu(B_\delta)} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) \leq 1. \end{aligned}$$

Thus,

$$\|f\|_{\Theta, B} \leq \tilde{C}C_0\lambda_0 = \tilde{C}C_0\|M_\Phi f\|_{\Psi, B_\delta} \leq \tilde{C}^2C_0\|M_\Phi f\|_{\Psi, \tilde{B}},$$

and we get inequality (3.1) with $C = \tilde{C}^2C_0$.

Now, by the definition of the function Θ we get that

$$\begin{aligned} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) &= \int_{B_\delta} \int_0^{\frac{f(x)}{C_0\lambda_0}} \Psi'_0(u) \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) du d\mu(x) \\ &= \int_1^\infty \Psi'_0(u) \int_{\{x \in B_\delta: \frac{f(x)}{C_0\lambda_0} > u\}} \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) d\mu(x) du \\ &\leq \int_1^\infty \Psi'_0(u) \int_{\{x \in B_\delta: \Phi(\frac{f(x)}{C_0\lambda_0 u}) > 1\}} \Phi\left(\frac{f(x)}{C_0\lambda_0 u}\right) d\mu(x) du. \end{aligned}$$

Notice that in the last inequality we have used that the Young function Φ is strictly increasing for $t > 1$ (this is a consequence of the convexity and the assumption $\Phi(1) = 1$). Now, let us observe that, since $u > 1$, $C_0\lambda_0 u > C_0\|M_\Phi f\|_{\Psi, B_\delta} \geq C_0\|f\|_{\Phi, \tilde{B}} \geq \frac{C_0}{\tilde{C}}\|f\|_{\Phi, B_\delta}$, where $\tilde{C}$ is such that $\mu(\tilde{B}) \leq \tilde{C}\mu(B)$, then, choosing $C_0 \geq \tilde{C}$ and applying Lemma 2.4 we get that

$$\begin{aligned} \int_{B_\delta} \Theta\left(\frac{f(x)}{C_0\lambda_0}\right) d\mu(x) &\leq C \int_1^\infty \Psi'_0(u) \mu(\{x \in B_\delta : M_\Phi f(x) > DC_0\lambda_0 u\}) du \\ &\leq C \int_{B_\delta} \Psi_0\left(\frac{M_\Phi f(x)}{DC_0\lambda_0}\right) d\mu(x) \\ &\leq C \int_{B_\delta} \Psi\left(\frac{M_\Phi f(x)}{DC_0\lambda_0}\right) d\mu(x) \leq \frac{C\mu(B_\delta)}{DC_0}. \end{aligned}$$

Then, choosing $C_0 \geq \max\{CD^{-1}, \tilde{C}\}$, we clearly obtain (3.2).

Now, we shall prove the other inequality in (1.7), that is, there exists $C > 0$ such that $M_{\alpha, \Psi}(M_\Phi f)(x) \leq CM_{\alpha, \Theta}f(x)$ for all $x \in X$. Let $x \in X$ such that $M_{\alpha, \Theta}f(x) < \infty$. First, we shall show that there exists $C > 0$ such that

$$\|M_\Phi f\|_{\Psi_0, B} \leq C\|f\|_{\Theta, B}, \quad (3.3)$$

for any ball B such that $x \in B$ and for any function f with support in B . By an homogeneous argument, we may assume $\|f\|_{\Theta, B} = 1/2$, that is, $\frac{1}{\mu(B)} \int_B \Theta(2f(x)) d\mu(x) \leq 1$. Now, applying (2.4) we get that

$$\begin{aligned} \int_B \Psi_0(M_\Phi f(x)) d\mu(x) &= \int_0^\infty \Psi'_0(u) \mu(\{x \in B : M_\Phi f(x) > u\}) du \\ &\leq C \int_1^\infty \Psi'_0(u) \left(\int_{\{x \in B : f(x) > u/2\}} \Phi\left(\frac{2f(x)}{u}\right) d\mu(x) \right) du \\ &= C \int_B \left(\int_1^{2f(x)} \Psi'_0(u) \Phi\left(\frac{2f(x)}{u}\right) du \right) d\mu(x) \\ &= C \int_B \Theta(2f(x)) d\mu(x) \leq C\mu(B). \end{aligned}$$

So, we get (3.3) for any f with $\text{supp}(f) \subset B$. For an arbitrary $f \geq 0$, let $x \in X$, $B = B(z, R)$ a ball such that $x \in B$ and $\tilde{B} = B(x, 2KR)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\tilde{B}}$. Then

$$\mu(B)^\alpha \|M_\Phi f\|_{\Psi_0, B} \leq \mu(\tilde{B})^\alpha \|M_\Phi f_1\|_{\Psi_0, \tilde{B}} + \mu(B)^\alpha \|M_\Phi f_2\|_{\Psi_0, B} = \text{I} + \text{II}.$$

By (3.3) we get that

$$\text{I} \leq C\mu(\tilde{B})^\alpha \|f\|_{\Theta, \tilde{B}} \leq M_{\alpha, \Theta} f(x).$$

To estimate II, let us observe that, as in [9], we have that $\Phi \prec_\infty \Theta$. To prove it, notice that there exists $t_0 > 1$ such that $\Psi_0(u) \geq 1$ for all $u \geq t_0$. Then, for $t \geq 2t_0$,

$$\begin{aligned} \Theta(t) &= \int_1^t \Psi'_0(u) \Phi(t/u) du \\ &\geq \int_1^t \Psi_0(u) \Phi'(t/u) t u^{-2} du \\ &\geq \int_{t_0}^t \Phi'(t/u) t u^{-2} du \\ &= \int_1^{t/t_0} \Phi'(v) dv = c_0 \Phi(t/t_0). \end{aligned}$$

Thus, $M_\Phi f(x) \leq CM_\Theta f(x)$. Now, using twice Lemma 2.5 (the first one with $\alpha = 0$), we get that

$$\begin{aligned} \text{II} &\leq \mu(B)^\alpha \inf_{z \in B} \|M_\Phi f_2(z)\|_{\Psi_0, B} \\ &\leq C\mu(B)^\alpha \inf_{z \in B} M_\Phi f_2(z) \\ &\leq C\mu(B)^\alpha \inf_{z \in B} M_\Theta f_2(z) \leq CM_{\alpha, \Theta} f(x). \end{aligned}$$

Finally, putting together the estimates for I and II we are done.

References

- [1] Pérez, C.: Two weighted norm inequalities for Riesz potentials and uniform L^p -weighted Sobolev inequalities. *Indiana Univ. Math. J.*, **39**(1), 31–44 (1990)
- [2] Coifman, R., Rochberg, R.: Another characterization of BMO. *Proc. Amer. Math. Soc.*, **79**, 249–254 (1980)
- [3] Pérez, C.: Weighted norm inequalities for singular integral operators. *J. London Math. Soc.*, **49**, 296–308 (1994)

- [4] Pérez, C.: Sharp estimates for commutators of singular integrals via iterations of the Hardy–Littlewood maximal function. *J. Fourier Anal. Appl.*, **3**(6), 743–756 (1997)
- [5] Pradolini, G., Salinas, O.: Commutators of singular integrals on spaces of homogeneous type. *Czechoslovak Math. J.*, **57**(132), 75–93 (2007)
- [6] Bernardis, A., Hartzstein, S., Pradolini, G.: Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type. *J. Math. Anal. Appl.*, **322**, 825–846 (2006)
- [7] Cruz-Urbe, D., Fiorenza, A.: Endpoint estimates and weighted norm inequalities for commutators of fractional integrals. *Publ. Mat.*, **47**, 103–131 (2003)
- [8] Gorosito, O., Pradolini, G., Salinas, O.: Weighted weak-type estimates for multilinear commutators of fractional integrals on spaces of homogeneous type. *Acta Mathematica Sinica, English Series*, **23**(10), 1813–1826 (2007)
- [9] Carozza, M., Di Napoli, A. P.: Composition of maximal operators. *Publ. Math.*, **40**, 397–409 (1996)
- [10] Macías, R., Segovia, C.: Lipschitz functions on spaces of homogeneous type. *Adv. Math.*, **33**, 257–270 (1979)
- [11] Coifman, R., Weiss, G.: Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math., Vol. 242, Springer-Verlag, New York-Berlin, 1971
- [12] Aimar, H.: Singular integral and approximate identities on space of homogeneous type. *Trans. Amer. Math. Soc.*, **292**, 135–153 (1985)
- [13] Macías, R., Segovia, C.: A well behaved quasi-distance on spaces of homogeneous type. *Trab. Mat. Inst. Argentina Mat.*, **32**, 1–18 (1981)
- [14] Bagby, R., Parsons, J.: Orlicz spaces and rearranged maximal functions. *Math. Nachr.*, **132**, 15–27 (1987)