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Zamolodchikov relations and Liouville hierarchy in $SL(2,R)_k$ WZNW model

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Abstract

We study the connection between Zamolodchikov operator-valued relations in Liouville field theory and in the $SL(2,\mathbb{R})_k$ WZNW model. In particular, the classical relations in $SL(2,\mathbb{R})_k$ can be formulated as a classical Liouville hierarchy in terms of the isotopic coordinates, and their covariance is easily understood in the framework of the AdS_3/CFT_2 correspondence. Conversely, we find a closed expression for the classical Liouville decoupling operators in terms of the so-called uniformizing Schwarzian operators and show that the associated uniformizing parameter plays the same role as the isotopic coordinates in $SL(2,\mathbb{R})_k$. The solutions of the jth classical decoupling equation in the WZNW model span a spin j reducible representation of $SL(2,\mathbb{R})$. Likewise, we show that in Liouville theory solutions of the classical decoupling equations span spin j representations of $SL(2,\mathbb{R})$, which is interpreted as the isometry group of the hyperbolic upper half-plane.

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1. Introduction

In [1] Al. Zamolodchikov proved the existence of a set of operator-valued relations in Liouville field theory (LFT). There is one such relation for every degenerate primary field, which is labelled by a pair of positive integers (m,n). These relations correspond to a higher order generalization of the Liouville equations of motion, and at the classical level ($n=1, b^2\to 0$), they can be written as

$$D_m\bar{D}_m\big[\varphi e^{\frac{1-m}{2}\varphi}\big]=B_m^{(c)}e^{\frac{1+m}{2}\varphi}, \tag{1.1}$$

where $B_m^{(c)}=(-2)^{1-m}M^m m!(m-1)!$ are classical Zamolodchikov coefficients and

$$D_m\big[e^{\frac{1-m}{2}\varphi}\big]=\bar{D}_m\big[e^{\frac{1-m}{2}\varphi}\big]=0. \tag{1.2}$$

The linear differential operators D_m can be schematically written as

$$D_m=\partial_z^m+\Gamma^m, \tag{1.3}$$

where $\Gamma^m=\sum_{k=0}^{m-2}d_k^{(m)}\partial_z^k$ with $d_k^{(m)}$ polynomials in the classical Liouville stress tensor

$$T=-\frac{1}{2}(\partial_z\varphi)^2+\partial_z^2\varphi \tag{1.4}$$

and its derivatives.[1]

In [2], two of us derived an infinite set of operator-valued relations which hold for degenerate representations of the $\widehat{sl(2)}_k$ Kac–Moody algebra and which are similar to those found by Zamolodchikov for the Virasoro degenerate representations in LFT mentioned above.[2] In the classical limit, which corresponds to $k\to\infty$, these relations are equivalent to

$$\partial_x^m\partial_{\bar{x}}^m\tilde{\Phi}_m=-m!(m-1)!\Phi_{-m}, \tag{1.5}$$

where

$$\Phi_{2j+1}(z|x)=\frac{2j+1}{\pi}\big(\left|x-x_0(z)\right|^2e^{\phi(z)}+e^{-\phi(z)}\big)^{2j} \tag{1.6}$$

are functions on the homogeneous space $SL(2,\mathbb{C})/SU(2)=\mathbb{H}_3^+$, the Euclidean version of AdS_3, and

$$\tilde{\Phi}_m(z|x)=\frac{m}{\pi}\big(\left|x-x_0(z)\right|^2e^{\phi(z)}+e^{-\phi(z)}\big)^{m-1}\ln\big(\left|x-x_0(z)\right|^2e^{\phi(z)}+e^{-\phi(z)}\big). \tag{1.7}$$

[1] The operators D_m have scaling dimension m. Hence, since $T^{(j)}\equiv\partial_z^jT$ has dimension $2+j$, $d_k^{(m)}$ is a sum of terms of the form $\prod_{i=1}^{l}T^{(j_i)}$ with $\sum_{i=1}^{l}j_i+2l+k=m$.

[2] Zamolodchikov's proof of the hierarchy of operator-valued relations possesses some general, model-independent features. In particular, it is plausible that it could be applied with appropriate modifications to "any" CFT which involves a continuous spectrum and degenerate states. In order to determine the Zamolodchikov coefficients, however, it is necessary to know the explicit form of the three-point functions.

Furthermore, Eq. (1.5) are in one-to-one correspondence with the decoupling equations for null states in the Kac–Moody Verma module

$$\partial_{\bar{x}}^m \Phi_m = \partial_x^m \Phi_m = 0. \tag{1.8}$$

The meaning of these decoupling equations is that the fields Φ_m transform in a finite-dimensional spin $j = \frac{m-1}{2}$ representation of $SL(2,\mathbb{R})$. This is encoded in the fact that $\Phi_{2j+1}(z|x)$ is a polynomial of order $2j$ in x.

It was also observed in [2] that in terms of $\varphi(z|x) \equiv -2\ln(\frac{\pi}{2}\Phi_2)$ the first equation in (1.5) can be rewritten as

$$\partial_x \partial_{\bar{x}} \varphi(z|x) = -2e^{\varphi(z|x)}, \tag{1.9}$$

which is the Liouville equation (with the "wrong sign") in the $SL(2,\mathbb{R})$-isospin variables $(x,\bar{x})$. This is interesting since in the context of the AdS_3/CFT_2 correspondence $(x,\bar{x})$ are the variables of the Boundary CFT [3,4].

Eqs. (1.1), (1.5) and (1.9) show a manifest parallelism between the Zamolodchikov hierarchy of equations in LFT and the similar one in the $SL(2,\mathbb{R})_k$ WZNW model. This raises the question as to whether there is a more precise correspondence between the two. Furthermore, while on the WZNW side the decoupling operator is simply ∂_x^m, the general form of D_m involves quite complicated expressions of the classical energy–momentum tensor [1]. We will work out the details of the connection between (1.1) and (1.5) by analyzing the geometrical meaning of the entities involved. The principal element appearing in the discussion turns out to be the uniformization problem of Riemann surfaces, which includes the $SL(2,\mathbb{R})$ group as a basic element and the classical Liouville equation naturally appears in this framework.

1.1. Uniformizing Schwarzian operators

In this paper, we will first show that the classical decoupling operators D_m in LFT correspond to *uniformizing Schwarzian operators* (USO) $S_\tau^{(m)}$ introduced in [5]. Such operators correspond to a particular kind of covariantized mth derivative. These operators are a particular kind of the so-called "Bol operators", independently rediscovered in [6] in the framework of the KdV equation formulated on Riemann surfaces. A step in [5] has been the use of the polymorphic vector field $1/\tau'$, with τ the inverse of the uniformizing map, as covariantizing vector field. This leads to new structures involving uniformization theory and covariant operators. Another important property of such operators is that they have a compact form: this will enable us to answer the question about the existence of a closed and generic explicit form for these differential operators besides the iterative computation at the classical level presented in Section 2 of Ref. [1]. We will see that

$$D_m = \mathcal{S}_\tau^{(m)} = \tau'^{(m-1)/2} \underbrace{\partial_z \tau'^{-1} \cdots \partial_z \tau'^{-1} \partial_z}_{m \text{ derivatives}} \tau'^{(m-1)/2}, \tag{1.10}$$

where τ is the inverse of the uniformizing map.

1.2. The $PSL(2,\mathbb{C})$ gauge invariance of $\mathcal{S}_\tau^{(m)}$

To connect the USO $\mathcal{S}_\tau^{(m)}$ to the higher equations of motion of the Liouville theory, one first notes that the $\mathcal{S}_\tau^{(m)}$ are invariant under $PSL(2,\mathbb{C})$ transformations of τ. Furthermore, following [5], one observes that the Poincaré metric

$$e^{\varphi} = \frac{|\tau'|^2}{(\operatorname{Im}\tau)^2} \tag{1.11}$$

can be seen as $\tau'(z)$ after a $PSL(2,\mathbb{C})$ transformation of τ. In particular, since under a $PSL(2,\mathbb{C})$ transformation

$$\tau \to \gamma\cdot\tau \equiv \frac{A(\bar z)\tau + B(\bar z)}{C(\bar z)\tau + D(\bar z)}, \tag{1.12}$$

one has $\tau' \to \tau'/(C\tau+D)^2$, it follows that for

$$C = \frac{1}{2i\bar\tau'^{1/2}}, \qquad D = -\frac{\bar\tau}{2i\bar\tau'^{1/2}}, \tag{1.13}$$

τ' is transformed into the Poincaré metric

$$e^{\varphi} = \partial_z(\gamma\cdot\tau), \tag{1.14}$$

which is equivalent to $e^{\varphi} = \partial_{\bar z}(\bar\gamma\cdot\bar\tau)$. Since the Poincaré metric is invariant under $PSL(2,\mathbb{R})$ transformations of τ, it follows that γ in (1.14) can be replaced by the product $\gamma\mu$ with μ an arbitrary element of $PSL(2,\mathbb{R})$ with constant entries.

It is interesting to observe that even if the above $PSL(2,\mathbb{C})$ transformations depend on the point $\bar z$ through $\bar\tau$ and $\bar\tau'$, they commute with ∂_z just like a constant.[3] It follows that the operator $\mathcal{S}_\tau^{(m)}$, which is invariant under $PSL(2,\mathbb{C})$ fractional transformations of τ, remains invariant also under such a point dependent transformation. Therefore, the global $PSL(2,\mathbb{C})$ symmetry of $\mathcal{S}_\tau^{(m)}$ extends to a local $PSL(2,\mathbb{C})$ symmetry, and can be seen as an anti-holomorphic gauge invariance. As a consequence τ' in $\mathcal{S}_\tau^{(m)}$ can be replaced by e^{φ}.

Note that by the Liouville equation $\partial_z\varphi_{\bar z} = e^{\varphi}/2$, we have that (1.14) implies $\gamma\cdot\tau = 2\varphi_{\bar z} + f(\bar z)$, where $f(\bar z)$ is any solution of $\partial_z f = 0$. This means that $\varphi_{\bar z} \equiv \partial_{\bar z}\varphi$ itself is a local $PSL(2,\mathbb{C})$ transformation of τ. Actually, we have

$$\varphi_{\bar z} = \frac{\bar\tau''}{\bar\tau'} + 2\frac{\bar\tau'}{\tau-\bar\tau}, \tag{1.15}$$

which has the form

$$\tau \to \frac{A(\bar z)\tau + B(\bar z)}{C(\bar z)\tau + D(\bar z)} = \varphi_{\bar z}. \tag{1.16}$$

[3] Observe that local univalence of τ implies that τ' never vanishes, therefore there are no δ-singularities contributing to it, e.g., $\partial_z\bar\tau'^{-1}$. Also note that the δ-singularities at the punctures cannot be seen since the latter are missing points on the Riemann surface.

Summarizing, we have the invariance

$$\mathcal{S}^{(m)}_{\varphi_{\bar z}} = \mathcal{S}^{(m)}_{\tau}, \tag{1.17}$$

which is equivalent to the invariance under $\tau' \to e^{\varphi}$ (the factor 2 can be adsorbed by a different transformation in (1.16)).

The above local invariance is very useful to write down the explicit form of the USO in terms of the classical Liouville field. In particular, as we will see, they depend only through the energy–momentum tensor and its derivatives.

The *anti-holomorphic* local Möbius transformations and the use of the inverse of the uniformizing map, have been first introduced in [5]. Such features distinguish our approach from the previous definitions of covariant operators generalizing the Schwarzian derivative (e.g., [6–8]).

Another feature of the above operators is that, once expressed in terms of the *trivializing* coordinate τ, seen as independent variable, they essentially reduces to ∂_τ^m. More precisely, we have

$$\mathcal{S}^{(m)}_{\tau} = \left(\frac{\partial z}{\partial \tau}\right)^{-\frac{m+1}{2}} \partial_\tau^m \left(\frac{\partial z}{\partial \tau}\right)^{\frac{1-m}{2}}. \tag{1.18}$$

This is equivalent to the fact that the classical energy–momentum tensor vanishes with such a coordinate choice. Conversely, we will show that the classical limit of the equations derived in [2] in $SL(2,\mathbb{R})_k$ can be rewritten in a Liouville-like manner and that the decoupling operators have such a simple form thanks to the vanishing of the associated energy–momentum tensor.

A crucial role in the analysis is played by the link between Liouville theory and the theory of uniformization of Riemann surfaces [5,9–16]. In particular, the $SL(2,\mathbb{R})$ symmetry which is manifest on the WZNW model side, is consistently mapped to the Liouville side where it acquires a geometrical meaning. It is in fact the isometry group of the hyperbolic upper half-plane. The Liouville decoupling operators are naturally $SL(2,\mathbb{R})$-invariant and the classical solutions of the equation $D_{2j+1}\psi = 0$, $j = 0, \frac{1}{2}, \ldots,$ span a spin j representation of $SL(2,\mathbb{R})$. It also follows that $e^{-j\varphi}$, $2j \in \mathbb{Z}^+$, can be decomposed in terms of these representations. The observation that this could be generalized at the quantum level using the representation theory of $U_q(sl(2))$ was at the basis of the algebraic approach to Liouville theory [19–21].

The paper is organized as follows. In Section 2, the basic aspects of uniformization of Riemann surfaces are introduced. The Liouville equation and its relation to the uniformizing equation are reviewed. In Section 3, the USO $\mathcal{S}^{(m)}_{\tau}$ of [5] are introduced. They are a generalization of the second-order linear differential operator associated to the uniformizing equation. In Section 3.2, we show that the decoupling operator D_m of LFT is the mth USO $\mathcal{S}^{(m)}_{\tau}$. This is consistent with the expressions given by Zamolodchikov for the first few values of m in [1]. The classical Zamolodchikov coefficients $(-1)^{m-1}2^{1-m}M^m m!(m-1)!$, which appear in Eq. (1.1), are also derived using two equivalent expressions of the USO, first in Section 3.3 and then in Appendix C.

Then we move on to the discussion on the $SL(2,\mathbb{R})$ side. In Section 4, we will rewrite the classical $SL(2,\mathbb{R})_k$ Zamolodchikov relations in a Liouville-like fashion, following the observation made in [2], Eq. (1.9). The vanishing of the associated projective connection provides an explanation of why the decoupling operators in the WZNW model have such a simple form. In this sense, the isotopic or boundary variable x plays the same role as the trivializing coordinate τ that belongs to the Poincaré upper half-plane $\mathbb{H}$ on the Liouville side. In Section 4.1, we point out that the covariance of the our discussion on the link between the Zamolodchikov relation of the $SL(2,\mathbb{R})$ model and Liouville theory can be understood from the AdS/CFT correspondence. In addition, we discuss how the Zamolodchikov relations in the $SL(2,\mathbb{R})$ will give Ward identities for the boundary CFT.

Section 5 is devoted to remarks suggesting the relation between our results and other known connections between Liouville theory and the WZNW model. Here we discuss how the $SL(2,\mathbb{R})$ symmetry which is manifest on the WZNW side is translated into the Liouville context, where it coincides with the isometry group of the hyperbolic upper half-plane. The solutions to $D_{2j+1}(\psi)=0$ span a spin j representation of $SL(2,\mathbb{R})$. Connection to the quantum group $U_q(sl(2))$ is also suggested. In Section 6, we present the conclusion and some future directions.

In Appendix A, we review how to express the classical Virasoro decoupling operator as a formal matrix determinant and show that the corresponding quantum decoupling operator reduces to the USO in the classical limit. In Appendix B, we show that the USO depend only on the energy–momentum tensor and its derivatives. In Appendix C, we provide another derivation of the Zamolodchikov relation from the USO.

2. Uniformization and Poincaré metric

Let us denote by D either the Riemann sphere $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$, the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}=\{\tau\in\mathbb{C}\mid \operatorname{Im}\tau>0\}$. The uniformization theorem states that every Riemann surface Σ is conformally equivalent to the quotient D/Γ, where Γ is a freely acting discontinuous group of fractional transformations preserving D, isomorphic to the fundamental group $\pi_1(\Sigma)$. In particular, for genus $g\geqslant 2$, the universal covering is given by $\mathbb{H}$. Let us consider this case and denote by $J_{\mathbb{H}}$ the complex analytic covering $J_{\mathbb{H}}:\mathbb{H}\to\Sigma$. Then, Γ is a finitely generated Fuchsian group belonging to $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{I,-I\}$. This acts on $\mathbb{H}$ by linear fractional transformations

$$\tau\to\gamma\cdot\tau=\frac{a\tau+b}{c\tau+d},\quad \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma,\qquad J_{\mathbb{H}}(\gamma\cdot\tau)=J_{\mathbb{H}}(\tau). \tag{2.1}$$

A Riemann surface isomorphic to the quotient $\mathbb{H}/\Gamma$ is endowed with a unique metric $\hat{g}$ with scalar curvature $R_{\hat{g}}=-1$ compatible with the complex structure. Consider the Poincaré metric on $\mathbb{H}$

$$ds^2=\frac{|d\tau|^2}{(\operatorname{Im}\tau)^2}. \tag{2.2}$$

Note that $PSL(2,\mathbb{R})$ transformations are isometries of $\mathbb{H}$ with the above metric. Then, the inverse of the uniformizing map $J_{\mathbb{H}}^{-1}:\Sigma\to\mathbb{H}$, $z\to\tau=J_{\mathbb{H}}^{-1}(z)$, induces the Poincaré metric on Σ

$$d\hat{s}^2=e^{\varphi(z,\bar{z})}|dz|^2,\quad e^{\varphi(z,\bar{z})}=\frac{|\tau'(z)|^2}{(\operatorname{Im}\tau(z))^2}, \tag{2.3}$$

which is invariant under $SL(2,\mathbb{R})$ transformations of $\tau(z)$. The condition

$$R_{\hat{g}} = \hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}}\ln\hat{g}_{z\bar{z}} = -1, \quad \hat{g}_{z\bar{z}} = \frac{1}{2}e^{\varphi(z,\bar{z})} \tag{2.4}$$

is equivalent to the Liouville equation

$$\partial_z\partial_{\bar{z}}\varphi(z,\bar{z}) = \frac{1}{2}e^{\varphi(z,\bar{z})}, \tag{2.5}$$

whereas the field $\tilde{\varphi} = \varphi + \ln M$, $M > 0$, defines a metric of constant negative curvature $-M$. The expression (2.3) is the unique solution to the Liouville equation on Σ.

2.1. The Liouville equation

Here we consider some aspects of the Liouville equation. First of all, note that, by the Gauss–Bonnet theorem, if $\int_\Sigma e^\varphi > 0$, then the equation

$$\partial_z\partial_{\bar{z}}\varphi(z,\bar{z}) = Me^{\varphi(z,\bar{z})} \tag{2.6}$$

has no solutions on surfaces with $\operatorname{sgn}\chi(\Sigma) = \operatorname{sgn} M$. In particular, on the Riemann sphere with $n \leqslant 2$ punctures[4] there are no solutions of the equation (let us set $M = \frac{1}{2}$)

$$\partial_z\partial_{\bar{z}}\varphi(z,\bar{z}) = \frac{1}{2}e^{\varphi(z,\bar{z})}, \quad \int\limits_\Sigma e^\varphi > 0. \tag{2.7}$$

The metric of curvature $+1$ on $\hat{\mathbb{C}}$

$$ds^2 = e^{\varphi_0}|dz|^2, \quad e^{\varphi_0} = \frac{4}{(1+|z|^2)^2} \tag{2.8}$$

satisfies the Liouville equation with the "wrong sign", that is

$$\partial_z\partial_{\bar{z}}\varphi_0(z,\bar{z}) = -\frac{1}{2}e^{\varphi_0(z,\bar{z})}. \tag{2.9}$$

If one insists on finding a solution of Eq. (2.7) on $\hat{\mathbb{C}}$, then inevitably one obtains at least three delta-singularities

$$\partial_z\partial_{\bar{z}}\varphi(z,\bar{z}) = \frac{1}{2}e^{\varphi(z,\bar{z})} - 2\pi\sum_{k=1}^{n}\delta^{(2)}(z-z_k), \quad n \geqslant 3. \tag{2.10}$$

However, the $(1,1)$-differential e^φ is not an admissible metric on $\hat{\mathbb{C}}$. In fact, since the unique solution of the equation $\partial_z\partial_{\bar{z}}\varphi = e^\varphi/2$ on the Riemann sphere is $\varphi = \varphi_0 + i\pi$ with $\varphi_0 \in \mathbb{R}$, to consider the Liouville equation on $\hat{\mathbb{C}}$ gives the unphysical metric $-e^{\varphi_0}|dz|^2$.

This discussion shows that in order to find a solution of Eq. (2.7) one needs at least three punctures, that is one must consider Eq. (2.7) on the surface $\Sigma = \hat{\mathbb{C}}\backslash\{z_1, z_2, z_3\}$ where the term $2\pi\sum_{k=1}^{3}\delta^{(2)}(z-z_k)$ does not appear simply because $z_k \notin \Sigma$, $k = 1,2,3$. In this case $\chi(\Sigma) = -1$, so that $\operatorname{sgn}\chi(\Sigma) = -\operatorname{sgn} M$ in agreement with the Gauss–Bonnet theorem.

[4] The 1-punctured Riemann sphere, i.e., $\mathbb{C}$, has itself as universal covering. For $n = 2$ we have $J_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}\backslash\{0\}$, $z \mapsto e^{2\pi i z}$. Furthermore, $\mathbb{C}\backslash\{0\} \cong \mathbb{C}/\langle T_1\rangle$, where $\langle T_1\rangle$ is the group generated by $T_1: z \mapsto z+1$.

2.2. The inverse map and the covariant Schwarzian operator

The Poincaré metric on Σ

$$e^{\varphi(z,\bar{z})} = -4\frac{|\tau'(z)|^2}{(\tau(z)-\bar{\tau}(\bar{z}))^2} \tag{2.11}$$

is invariant under $PSL(2,\mathbb{R})$ fractional transformations of τ. Eq. (2.11) makes it evident that from the explicit expression of the inverse map $\tau = J_{\mathbb{H}}^{-1}(z)$ we can find the dependence of e^{φ} on the moduli of Σ. Conversely, one can express the inverse map (up to a $PSL(2,\mathbb{C})$ fractional transformation) in terms of φ. This follows from the *Schwarzian equation*

$$\{\tau, z\} = T(z), \tag{2.12}$$

where

$$T(z) = -\frac{1}{2}(\partial_z\varphi)^2 + \partial_z^2\varphi \tag{2.13}$$

is the classical Liouville energy–momentum tensor (1.4), or *Fuchsian* projective connection, and

$$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2f'^{1/2}\left(f'^{-1/2}\right)'' \tag{2.14}$$

is the Schwarzian derivative of f. The Liouville equation implies that the classical energy–momentum tensor is holomorphic

$$\partial_{\bar{z}} T = 0. \tag{2.15}$$

Note also that $T(z)$ has the transformation properties of a projective connection under a change of coordinates, namely

$$\tilde{T}(\tilde{z}) = \left(\frac{\partial z}{\partial \tilde{z}}\right)^2 T(z) + \{z, \tilde{z}\} = \left(\frac{\partial z}{\partial \tilde{z}}\right)^2 T(z) - \left(\frac{\partial z}{\partial \tilde{z}}\right)^2 \{\tilde{z}, z\}. \tag{2.16}$$

Furthermore, $PSL(2,\mathbb{C})$ transformations of τ leave $T(z)$ invariant.

Let us define the *covariant Schwarzian operator*

$$\mathcal{S}_f^{(2)} = f'^{1/2}\partial_z f'^{-1}\partial_z f'^{1/2} \tag{2.17}$$

mapping $-\frac{1}{2}$ into $\frac{3}{2}$-differentials. In the above formula, it is understood that each ∂_z acts on each term on the right. Since

$$\mathcal{S}_f^{(2)}\psi = \left(\partial_z^2 + \frac{1}{2}\{f, z\}\right)\psi \tag{2.18}$$

the Schwarzian derivative can be written as

$$\{f, z\} = 2\mathcal{S}_f^{(2)} \cdot 1. \tag{2.19}$$

Like the Schwarzian derivative also $\mathcal{S}_f^{(2)}$ is invariant under $PSL(2,\mathbb{C})$ fractional transformations of f, that is

$$\mathcal{S}_{\gamma\cdot f}^{(2)} = \mathcal{S}_f^{(2)}, \quad \gamma \in PSL(2,\mathbb{C}). \tag{2.20}$$

Therefore, if the transition functions of Σ are linear fractional transformations, then $\{f, z\}$ transforms as a quadratic differential. However, except in the case of projective coordinates, the Schwarzian derivative does not transform covariantly on Σ. This is evident by (2.19) since in flat spaces only (e.g., the torus) a constant can be considered as a $-\frac{1}{2}$-differential.

2.3. The uniformizing equation

As we have seen, one of the important properties of the Schwarzian derivative is that the *Schwarzian equation* (2.12) can be linearized. Thus if ψ_1 and ψ_2 are linearly independent solutions of the *uniformizing equation*

$$\left(\partial_z^2 + \frac{1}{2}T(z)\right)\psi(z) = 0, \tag{2.21}$$

then ψ_2/ψ_1 is a solution of Eq. (2.12). That is, up to a $PSL(2,\mathbb{C})$ linear fractional transformation[5]

$$\tau = \psi_2/\psi_1. \tag{2.22}$$

Indeed by setting

$$\tau'^{1/2}\partial_z\tau'^{-1}\partial_z\tau'^{1/2}\psi_1 = \tau'^{1/2}\partial_z\tau'^{-1}\partial_z 1 = 0 \tag{2.23}$$

and

$$\tau'^{1/2}\partial_z\tau'^{-1}\partial_z\tau'^{1/2}\psi_2 = \tau'^{1/2}\partial_z 1 = 0 \tag{2.24}$$

it follows that

$$\psi_1 = \tau'^{-1/2}, \qquad \psi_2 = \tau'^{-1/2}\tau \tag{2.25}$$

are independent solutions of (2.21). Another way to prove (2.22) is to write Eq. (2.21) in the equivalent form

$$\tau'^{1/2}\partial_z\tau'^{-1}\partial_z\tau'^{1/2}\psi = 0 \tag{2.26}$$

and then to set $z = J_{\mathbb{H}}(\tau)$, where $J_{\mathbb{H}}: \mathbb{H} \to \Sigma$ is the uniformizing map.

The inverse map is locally univalent, that is if $z_1 \neq z_2$ then $\tau(z_1) \neq \tau(z_2)$. Furthermore, the solutions of the uniformizing equation have non-trivial monodromy properties. When z winds around non-trivial cycles of Σ

$$\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix} \to \begin{pmatrix}\tilde\psi_2\\ \tilde\psi_1\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\psi_2\\ \psi_1\end{pmatrix}, \tag{2.27}$$

[5] Note that the Poincaré metric is invariant under $PSL(2,\mathbb{R})$ fractional transformations of τ whereas the Schwarzian derivative $T(z) = \{\tau, z\}$ is invariant for $PSL(2,\mathbb{C})$ transformations of τ. Thus, with an arbitrary choice of ψ_1 and ψ_2 it may be that $\mathrm{Im}(\psi_2/\psi_1)$ is not positive definite, so that in general the identification $\tau = \psi_2/\psi_1$ is up to a $PSL(2,\mathbb{C})$ transformation.

which induces a linear fractional transformation of the inverse map $\tau(z)$

$$\tau \to \gamma \cdot \tau = \frac{A\tau + B}{C\tau + D}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \tag{2.28}$$

Thus, under a winding of z around non-trivial cycles of Σ, the point $\tau(z) \in \mathbb{H}$ moves from a representative $\mathcal{F}_i$ of the fundamental domain of Γ to an equivalent point of another representative $\mathcal{F}_j$. The monodromy group of the uniformizing equation is the automorphism group of the uniformizing map $J_{\mathbb{H}}(\tau)$ and is isomorphic to the fundamental group of Σ. However, note that (2.20) guarantees that, in spite of the polymorphicity (2.28), the classical Liouville energy–momentum tensor $T = 2\mathcal{S}_\tau^{(2)} \cdot 1$ is singlevalued.

Observe that, since ψ is a $-\frac{1}{2}$–differential, Eq. (2.21) on $\mathbb{H}$ reads $\tau'^{3/2}\partial_\tau^2\phi = 0$, that is we have the trivial equation

$$\partial_\tau^2\phi = 0. \tag{2.29}$$

In fact using

$$\psi(z)\, dz^{-1/2} = \phi(\tau)\, d\tau^{-1/2} \quad \Rightarrow \quad \tau'^{1/2}\psi(z) = \phi(\tau) \tag{2.30}$$

we find that Eq. (2.26) becomes

$$\tau'^{1/2}\partial_z\tau'^{-1}\partial_z\tau'^{1/2}\psi(z) = \tau'^{3/2}\partial_\tau^2\phi(\tau) = 0. \tag{2.31}$$

Note also that Eq. (2.29) is consistent with the fact that by (2.12) and (2.16)

$$\hat{T}(\tau) = \left(\frac{\partial z}{\partial \tau}\right)^2 T(z) - \left(\frac{\partial z}{\partial \tau}\right)^2 \{\tau, z\} = 0. \tag{2.32}$$

In this sense τ, or a general $PSL(2,\mathbb{C})$ transformation of it, is a *trivializing coordinate*. For any choice of the two linearly independent solutions we have $\phi_2/\phi_1 = \tau$ up to a $PSL(2,\mathbb{C})$ transformation. Going back to Σ we get $\tau = \psi_2/\psi_1$.

2.4. $PSL(2,R)$ symmetry

Note that any $GL(2,\mathbb{C})$ transformation

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2.33}$$

induces a linear fractional transformation of τ. It follows that the invariance of e^φ under $PSL(2,\mathbb{R})$ linear fractional transformations of τ corresponds to its invariance under $PSL(2,\mathbb{R})$ linear transformations of ψ_1, ψ_2. This leads us to the expression of $e^{-j\varphi}$ as

$$e^{-j\varphi} = (-4)^{-j}(\bar{\psi}_1\psi_2 - \bar{\psi}_2\psi_1)^{2j}. \tag{2.34}$$

In particular, when $2j$ is a non-negative integer, we get

$$e^{-j\varphi} = 4^{-j}\sum_{k=-j}^{j}(-1)^k \binom{2j}{j+k} \bar{\psi}_1^{j+k}\psi_1^{j-k}\psi_2^{j+k}\bar{\psi}_2^{j-k}, \quad 2j \in \mathbb{Z}^+. \tag{2.35}$$

On the other hand, since we can choose $\psi_2(z)=\psi_1(z)\int^z\psi_1^{-2}$, we have

$$e^{-j\varphi}=(-4)^{-j}\left|\psi(z)\right|^{4j}\left(\int^z\psi^{-2}-\int^{\bar z}\bar\psi^{-2}\right)^{2j},\quad\forall j,\tag{2.36}$$

with

$$\psi(z)=A\psi_1(z)\left(1+B\int^z\psi_1^{-2}\right),\quad A\in\mathbb{R}\backslash\{0\},\ B\in\mathbb{R}.\tag{2.37}$$

We note that the ambiguity in the definition of $\int^z\psi^{-2}$ reflects the polymorphicity of τ. This property of τ implies that, under a winding around non-trivial loops, a solution of (2.21) transforms in a linear combination involving itself and another (independent) solution. It is easy to check that

$$\left(\partial_z^2+\frac{1}{2}T(z)\right)e^{-\varphi/2}=0,\tag{2.38}$$

which shows that the uniformizing equation has the interesting property of admitting singlevalued solutions. The reason is that the $\bar z$-dependence of $e^{-\varphi/2}$ arises through the coefficients $\bar\psi_1$ and $\bar\psi_2$ in the linear combination of ψ_1 and ψ_2.

Since $[\partial_{\bar z},\mathcal{S}_\tau^{(2)}]=0$, the singlevalued solutions of the uniformizing equation are

$$\left(\partial_z^2+\frac{1}{2}T(z)\right)\partial_{\bar z}^{\ell}e^{-\varphi/2}=0,\quad\ell=0,1,\ldots.\tag{2.39}$$

Thus, since $e^{-\varphi}$ and $e^{-\varphi}\partial_{\bar z}\varphi$ are linearly independent solutions of Eq. (2.21), their ratio solves the Schwarzian equation

$$\{\partial_{\bar z}\varphi,z\}=T(z).\tag{2.40}$$

Higher order derivatives $\partial_{\bar z}^{\ell}e^{-\varphi/2}$, $\ell\geqslant 2$, are linear combinations of $e^{-\varphi/2}$ and $e^{-\varphi/2}\partial_{\bar z}\varphi$ with coefficients depending on $\bar T$ and its derivatives; for example

$$\partial_{\bar z}^2e^{-\varphi/2}=-\frac{\bar T}{2}e^{-\varphi/2}.\tag{2.41}$$

In particular if $\psi_2(z)=\bar T\psi_1(z)$ then, in spite of the fact that $\bar T$ is not a constant on Σ, ψ_1 and ψ_2 are linearly dependent solutions of Eq. (2.21).

Let us show what happens if one sets $\tau=\psi_1/\psi_2$ without considering the remark made in the previous footnote. As solutions of the uniformizing equation, we can consider $\psi_1=e^{-\varphi/2}$ and an arbitrary solution ψ_2 such that $\partial_z(\psi_2/\psi_1)=0$. Since $\partial_{\bar z}(e^{-\varphi/2}/\psi_2)\neq 0$, in spite of the fact that $\{e^{-\varphi/2}/\psi_2,z\}=T$, we have $\tau\neq\psi_1/\psi_2$.

We conclude the analysis of the uniformizing equation by summarizing some useful expressions for the Liouville energy–momentum tensor

$$\begin{aligned} T &= \{\tau, z\} = \{\partial_{\bar z}\varphi, z\} = 2\tau'^{1/2}\partial_z \frac{1}{\tau'}\partial_z \tau'^{1/2}\cdot 1 = 2e^{\varphi/2}\partial_z e^{-\varphi}\partial_z e^{\varphi/2}\cdot 1 \\ &= 2\big(e^{-\varphi/2}/\psi_2\big)'^{1/2}\partial_z\big(e^{-\varphi/2}/\psi_2\big)'^{-1}\partial_z\big(e^{-\varphi/2}/\psi_2\big)'^{1/2}\cdot 1 \\ &= -2e^{\varphi/2}\big(e^{-\varphi/2}\big)'' = -2\psi^{-1}\psi'', \end{aligned} \tag{2.42}$$

with ψ given in (2.37) and ψ_2 an arbitrary solution of Eq. (2.21) such that $\partial_z(e^{-\varphi/2}/\psi_2) \neq 0$.

3. USO and classical Zamolodchikov relations

Here we will consider a set of operators $\mathcal{S}_\tau^{(2j+1)}$, $j = \frac{1}{2}, 1, \ldots$, corresponding to ∂_z^{2j+1} covariantized by means of τ. These operators were first introduced in [5] and generalize the *Schwarzian operator* $\mathcal{S}_\tau^{(2)}$ that was studied above. In the next section, we will prove that they actually coincide with the classical decoupling operators in LFT D_m, $m = 2j+1$. Let us define

$$\mathcal{S}_f^{(n)} = f'^{(n-1)/2}\underbrace{\partial_z f'^{-1}\cdots\partial_z f'^{-1}\partial_z}_{n \text{ derivatives}} f'^{(n-1)/2}. \tag{3.1}$$

This is a linear operator mapping $(1-n)/2$ differentials to $(n+1)/2$ differentials. The Ker of $\mathcal{S}_f^{(n)}$ is generated by

$$s_k = \frac{f^{k-1}}{f'^{\frac{n-1}{2}}}, \quad k = 1, \ldots, n. \tag{3.2}$$

Under a $PSL(2,\mathbb{C})$ transformation of f we have

$$\gamma \cdot f = \frac{Af+B}{Cf+D}, \qquad (\gamma\cdot f)' = (Cf+D)^{-2}f', \tag{3.3}$$

the solutions s_k transform as

$$s_k \to \frac{1}{f'^{(n-1)/2}}(Cf+D)^{n-k}(Af+B)^{k-1}. \tag{3.4}$$

Since this is a linear combination of the s_k's, the Ker of $\mathcal{S}_f^{(n)}$ is $PSL(2,\mathbb{C})$-invariant. This means that the $\mathcal{S}_f^{(n)}$ themselves are $PSL(2,\mathbb{C})$-invariant

$$\mathcal{S}_{\gamma\cdot f}^{(n)} = \mathcal{S}_f^{(n)}. \tag{3.5}$$

From now on we will consider the operators with

$$f = \tau, \tag{3.6}$$

where $\tau = J_{\mathbb{H}}^{-1}(z)$ is the inverse of the uniformizing map. These operators have been introduced in [5]. Besides $PSL(2,\mathbb{C})$-invariance they satisfy some basic properties strictly related to Liouville and uniformization theories. We will call them uniformizing Schwarzian operators (USO).

3.1. Gauge invariance of the USO from the local univalence of τ

We want to show that

$$\mathcal{S}_\tau^{(n)} e^{\frac{(1-n)}{2}\varphi} = 0, \tag{3.7}$$

where φ is the classical Liouville field, that is

$$e^\varphi = \frac{|\tau'|^2}{(\operatorname{Im}\tau)^2} \tag{3.8}$$

is the Poincaré metric. A key observation in [5] is that e^φ can be seen as τ' after a Möbius transformation of τ with the coefficients depending on $\bar z$. More precisely, we see that under the $PSL(2,\mathbb{C})$ transformation $\tau \to (A\tau + B)/(C\tau + D)$, with

$$C = \frac{1}{2i\bar\tau'^{1/2}}, \qquad D = -\frac{\bar\tau}{2i\bar\tau'^{1/2}}, \tag{3.9}$$

that is

$$\tau \to 2i\bar\tau'^{1/2}A + \frac{4\bar\tau'}{\tau - \bar\tau} \tag{3.10}$$

the derivative of the inverse map is transformed into the Poincaré metric

$$\tau' \to e^\varphi = \partial_z(\gamma\cdot\tau) = \partial_{\bar z}(\bar\gamma\cdot\bar\tau). \tag{3.11}$$

A crucial step is to observe that nothing changes in the proof of (3.5) if the coefficients of γ are anti-holomorphic functions. In other words, the original global $PSL(2,\mathbb{C})$-invariance extends to a point dependent symmetry. This local symmetry is a rather particular one since it depends on $\bar z$ rather than on z. We can consider such a symmetry, related to the fact that anti-holomorphic univalent functions commute with ∂_z, as a "left gauge invariance" of the $\mathcal{S}_\tau^{(n)}$. In this respect we note that the dependence on $\bar z$ of the $PSL(2,\mathbb{C})$ transformation is through $\bar\tau$ and its derivatives. On the other hand, local univalence of τ implies that τ' never vanishes, so there are no δ-singularities contributing to, e.g., $\partial_z\bar\tau'^{-1}$. In other words, on the Riemann surface we always have[6]

$$[\partial_z, \bar\tau] = 0 = [\partial_z, \bar\tau']. \tag{3.12}$$

Apparently, Liouville theorem forbids non-trivial solutions of such equations. On the other hand, constancy of holomorphic functions on compact manifolds refers to true functions. In the present case we are treating with a polymorphic function, i.e., a function with a non-Abelian monodromy around non-trivial cycles. As such, the inverse of the uniformizing map can be seen as a polymorphic classical chiral boson.[7]

The above symmetry implies that

$$\mathcal{S}_\tau^{(n)} = e^{\frac{n-1}{2}\varphi}\partial_z e^{-\varphi}\cdots\partial_z e^{-\varphi}\partial_z e^{\frac{n-1}{2}\varphi}, \tag{3.13}$$

6 δ-singularities would appear for elliptic points or by filling-in possible punctures.

7 An issue which deserves to be investigated concerns the chiral boson defined in [17] whose properties suggest a relation with the inverse of the uniformizing map.

which makes (3.7) manifest. Observe that univalence of τ and thus the fact that $\tau'(z) \neq 0$ imply that the USO are holomorphic, so that

$$\left[\mathcal{S}_\tau^{(m)}, \bar{\mathcal{S}}_\tau^{(n)}\right] = 0. \tag{3.14}$$

Eq. (3.7) is manifestly covariant and singlevalued on Σ. Furthermore, we will show in Appendix B that the dependence of $\mathcal{S}_f^{(2j+1)}$ on f appears only through $\{f, z\}$ and its derivatives; for example

$$\mathcal{S}_\tau^{(3)} = \partial_z^3 + 2T\partial_z + T', \tag{3.15}$$

which is the second symplectic structure of the KdV equation. The operator $\mathcal{S}_f^{(3)}$ appears in the formulation of the covariant formulation of the KdV equation on Riemann surfaces [6] where single-valued vector fields, explicitly constructed in [18] and also admitting essential singularities of Baker–Akhiezer type, were used instead of the polymorphic vector field $1/\tau'$.

An important property of the equation $\mathcal{S}_\tau^{(2j+1)}\psi = 0$ is that its projection on $\mathbb{H}$ is the trivial equation

$$w'^{\,j+1}\partial_w^{2j+1}\tilde{\psi} = 0, \quad w \in \mathbb{H}, \tag{3.16}$$

where $w = \tau(z)$ and

$$\psi(z)\,dz^{-j} = \tilde{\psi}(w)\,dw^{-j}. \tag{3.17}$$

As we saw previously in (2.32) this is consistent with the fact that $\hat{T}(w) = 0$. This also explains why only for $j > 0$ it is possible to have finite expansions of $e^{-j\varphi}$ such as in Eq. (2.35). The reason is that the solutions of Eq. (3.16) are $\{w^k \mid k = 0, \ldots, 2j\}$ so that the best thing we can do is to consider linear combinations of positive powers of the non-chiral solution $\operatorname{Im} w$ which is just the square root of inverse of the Poincaré metric on $\mathbb{H}$.

Note that Eq. (3.13) can also be derived in the following way. By the $PSL(2, \mathbb{C})$ invariance of the Schwarzian derivative, in particular the fact that $\{\tau, z\} = \{\partial_{\bar{z}}\varphi, z\}$, we find

$$\mathcal{S}_\tau^{(2j+1)} = \mathcal{S}_{\partial_{\bar{z}}\varphi}^{(2j+1)}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots. \tag{3.18}$$

On the other hand, by Liouville equation

$$\mathcal{S}_{\partial_{\bar{z}}\varphi}^{(2j+1)} = e^{j\varphi}\partial_z e^{-\varphi}\partial_z e^{-\varphi}\cdots\partial_z e^{-\varphi}\partial_z e^{j\varphi}, \quad j = 0, \frac{1}{2}, 1, \ldots. \tag{3.19}$$

The above expression will be crucial to prove that the USO and the classical Liouville decoupling operators are the same.

3.2. Classical decoupling operators in Liouville theory

In [1] Zamolodchikov considered the fields

$$V_m = e^{(1-m)\varphi/2}, \quad m \in \mathbb{Z}^+, \tag{3.20}$$

and showed that the first few representatives satisfy the ODEs

$$\begin{aligned}
&\partial_z \cdot 1 = 0,\\
&\left(\partial_z^2 + \frac{1}{2}T\right)e^{-\varphi/2} = 0,\\
&\left(\partial_z^3 + 2T\partial_z + T'\right)e^{-\varphi} = 0,\\
&\left(\partial_z^4 + 5T\partial_z^2 + 5T'\partial_z + \left(\frac{9}{4}T^2 + \frac{3}{2}T''\right)\right)e^{-3\varphi/2} = 0,\\
&\left(\partial_z^5 + 10T\partial_z^3 + 15T'\partial_z^2 + \left(16T^2 + 9T''\right)\partial_z + 16TT' + 2T'''\right)e^{-2\varphi} = 0,
\end{aligned} \tag{3.21}$$

together with the complex conjugates ($\partial_z \to \partial_{\bar z}$, $T \to \bar T$). By using the classical Liouville equation

$$\partial_z \partial_{\bar z}\varphi = Me^{\varphi} \tag{3.22}$$

it is then possible to show that the fields φV_m satisfy the relations

$$\begin{aligned}
&\bar D_1 D_1 \varphi = Me^{\varphi},\\
&\bar D_2 D_2\left(\varphi e^{-\varphi/2}\right) = -M^2 e^{3\varphi/2},\\
&\bar D_3 D_3\left(\varphi e^{-\varphi}\right) = 3M^3 e^{2\varphi},\\
&\bar D_4 D_4\left(\varphi e^{-3\varphi/2}\right) = -18M^4 e^{5\varphi/2},\\
&\bar D_5 D_5\left(\varphi e^{-2\varphi}\right) = 180M^5 e^{3\varphi},
\end{aligned} \tag{3.23}$$

which are some particular cases of (1.1). The main result of [1] is the proof that the above relations hold for general m at the quantum level. However, the general form of the classical decoupling operators D_m was considered unknown there.

In the following, we will prove that the operators D_m coincide with the USO $\mathcal{S}_\tau^{(m)}$ introduced in[8] [5] (another independent proof is given in Appendix A). First of all, the operators $\mathcal{S}_\tau^{(m)}$, like the D_m, have $e^{(1-m)\varphi/2}$ as solution. Secondly, also the $\mathcal{S}_\tau^{(m)}$ depend on $\partial_z\varphi$ only through the classical energy–momentum tensor T and its derivatives. This is shown in Appendix B. Furthermore, both $\mathcal{S}_\tau^{(m)}$ and D_m are covariant operators mapping $(1-m)/2$-differentials to $(m+1)/2$-differentials. In this respect note that covariance of the D_m is understood a priori: a possible inhomogeneous term in changing coordinates in the intersection of two patches would imply that (3.21) are not covariantly satisfied. Next, since

$$\left[\partial_{\bar z}, \mathcal{S}_\tau^{(m)}\right] = 0 \tag{3.24}$$

it follows that, besides $e^{(1-m)\varphi/2}$, other solutions of $\mathcal{S}_\tau^{(m)}\psi = 0$ have the form $\partial_{\bar z}^{\ell} e^{(1-m)\varphi/2}$. Furthermore, a basis of solutions of $\mathcal{S}_\tau^{(m)}\psi = 0$ is given by [5]

$$\psi_j = (\partial_{\bar z}\varphi)^j e^{(1-m)\varphi/2}, \qquad j = 0, \ldots, m-1. \tag{3.25}$$

8 The coincidence of such operators was pointed out to us by Giulio Bonelli.

To see this it is sufficient to insert ψ_j on the RHS of (3.13) and systematically use the Liouville equation

$$e^{-\varphi}\partial_z(\partial_{\bar z}\varphi)^j = jM(\partial_{\bar z}\varphi)^{j-1}. \tag{3.26}$$

On the other hand, since

$$[\partial_{\bar z}, D_m] = 0 \tag{3.27}$$

and $\partial_z\bar T = 0$, the functions (3.25) are also a basis of solutions of $D_m\psi = 0$. Therefore, we proved that $D_m = \mathcal{S}_\tau^{(m)}$, which turns out to yield the classical Zamolodchikov relations (1.1) as we will see in the next subsection. Appendix C presents another equivalent derivation of these relations starting from the expression (3.13) of the decoupling operators.

3.3. Classical Zamolodchikov relations

By means of the above results we can now investigate the classical Zamolodchikov relation in Liouville theory. Let us evaluate

$$\bar{\mathcal{S}}_\tau^{(m)}\mathcal{S}_\tau^{(m)}\left(\varphi e^{\frac{(1-m)}{2}\varphi}\right). \tag{3.28}$$

Since

$$e^{\tilde\varphi} = \left|\frac{dz}{d\tilde z}\right|^2 e^{\varphi} \tag{3.29}$$

it follows that[9]

$$\tilde\varphi(\tilde z, \bar{\tilde z}) = \varphi(z,\bar z) + \ln\left(\frac{dz}{d\tilde z}\right) + \ln\left(\frac{d\bar z}{d\bar{\tilde z}}\right), \tag{3.30}$$

implying that $\varphi e^{\frac{(1-m)}{2}\varphi}$ is not a covariant quantity. However, (3.28) is still a differential of weight $(\frac{m+1}{2}, \frac{m+1}{2})$. In fact, the inhomogeneous term appearing under a holomorphic coordinate transformation $z \to \tilde z$

$$\tilde{\bar{\mathcal{S}}}_\tau^{(m)}\tilde{\mathcal{S}}_\tau^{(m)}\left(\tilde\varphi e^{\frac{(1-m)}{2}\tilde\varphi}\right) = \left|\frac{dz}{d\tilde z}\right|^{m+1}\bar{\mathcal{S}}_\tau^{(m)}\mathcal{S}_\tau^{(m)}\left(\varphi e^{\frac{(1-m)}{2}\varphi} + \ln\left|\frac{dz}{d\tilde z}\right|^2 e^{\frac{(1-m)}{2}\varphi}\right) \tag{3.31}$$

cancels. Actually, by (3.14) and (3.7) we have

$$\begin{aligned}
&\bar{\mathcal{S}}_\tau^{(m)}\mathcal{S}_\tau^{(m)}\left(\ln\left|\frac{dz}{d\tilde z}\right|^2 e^{\frac{1-m}{2}\varphi}\right)\\
&\quad= \mathcal{S}_\tau^{(m)}\ln\left(\frac{dz}{d\tilde z}\right)\bar{\mathcal{S}}_\tau^{(m)}e^{\frac{1-m}{2}\varphi} + \bar{\mathcal{S}}_\tau^{(m)}\ln\left(\frac{d\bar z}{d\bar{\tilde z}}\right)\mathcal{S}_\tau^{(m)}e^{\frac{1-m}{2}\varphi} = 0.
\end{aligned} \tag{3.32}$$

[9] In the quantum theory the geometrical nature of Liouville field is different; in that case the transformation is given by $\tilde\varphi = \varphi + Q\ln(\frac{dz}{d\tilde z}) + Q\ln(\frac{d\bar z}{d\bar{\tilde z}})$ where Q is the background charge which, after the appropriate rescaling in the classical limit turns out to be $Q \to 1$.

We can express the operators in a form that considerably simplifies the calculations. Actually, note the identity

$$\mathcal{S}_\tau^{(m)} = \left(\frac{\partial z}{\partial \tau}\right)^{-\frac{m+1}{2}} \partial_\tau^m \left(\frac{\partial z}{\partial \tau}\right)^{\frac{1-m}{2}}. \tag{3.33}$$

It is therefore convenient to consider z as function of τ rather than vice versa. Thus we have

$$\varphi e^{\frac{1-m}{2}\varphi} = -\left(\ln\left|\frac{\partial z}{\partial \tau}\right|^2 + 2\ln y\right) y^{m-1} \left|\frac{\partial z}{\partial \tau}\right|^{m-1}, \tag{3.34}$$

where $y = \operatorname{Im}\tau$ and (3.28) becomes

$$\begin{aligned} &-\left|\frac{\partial z}{\partial \tau}\right|^{-(m+1)} \partial_{\bar\tau}^m \partial_\tau^m \left[y^{m-1}\left(\ln\left|\frac{\partial z}{\partial \tau}\right|^2 + 2\ln y\right)\right] \\ &\quad = -2\left|\frac{\partial z}{\partial \tau}\right|^{-(m+1)} \partial_{\bar\tau}^m \partial_\tau^m \left(y^{m-1}\ln y\right). \end{aligned} \tag{3.35}$$

Noticing that $4^{-m}\partial_y^{2m}$ is the only term in $\partial_{\bar\tau}^m \partial_\tau^m$ that does not contain $\partial_{\mathrm{Re}\,\tau}$, we see that our problem reduces to compute the numerical coefficient b_m in

$$\partial_y^{2m} y^{m-1} \ln y = b_m y^{-m-1}. \tag{3.36}$$

Rather than evaluating b_m using the Leibniz formula, we easily obtain it by induction. By (3.36) we have

$$\begin{aligned} \partial_y^{2m+2} y^m \ln y &= \partial_y \partial_y^{2m}\left(m y^{m-1}\ln y + y^{m-1}\right) = m b_m \partial_y y^{-m-1} \\ &= -m(m+1) b_m y^{-m-2}, \end{aligned} \tag{3.37}$$

which gives $b_{m+1} = (-1)^m m!(m+1)! b_1$ and since $\partial_y^2 \ln y = -y^{-2}$, we have $b_1 = -1$, that is

$$b_m = (-1)^m m!(m-1)!. \tag{3.38}$$

The final result is

$$\bar{\mathcal{S}}_\tau^{(m)} \mathcal{S}_\tau^{(m)} \left(\varphi e^{\frac{1-m}{2}\varphi}\right) = 2(-1)^{m+1} 4^{-m} m!(m-1)! e^{\frac{m+1}{2}\varphi}, \tag{3.39}$$

which, for $M = \frac{1}{2}$, coincides with the expression Eq. (1.1) argued in [1] by inspection of the first few cases.

4. Liouville-like equations in $SL(2,R)_k$ WZNW model

Now, we move to the $SL(2,\mathbb{R})$ structure of Liouville hierarchy. In particular, in this section, we will discuss the Zamolodchikov hierarchy of differential equations in the context of the finite-dimensional representations of $sl(2,\mathbb{R})_k$ algebra.

Let us first recall some basic facts about differentiable functions on $SL(2,\mathbb{C})/SU(2) = \mathbb{H}_3^+$. These are associated to vertex operators of string theory on Euclidean AdS_3 and in particular certain non-normalizable states in $\mathbb{H}_3^+$ describe hermitian representations of $SL(2,\mathbb{R})$.

Among the representations of the $sl(2)_k$ affine algebra, there is a set of reducible finite-dimensional representations that are similar to those of the $SU(2)$ group and are classified by an index j as usual. These representations precisely correspond to the classical branch of the Kac–Kazhdan degenerate states considered in [2]. They are labelled by $2j+1 = m \in \mathbb{Z}^+$ and are associated to the following functions

$$\Phi_m(z|x) = \frac{m}{\pi}\left(\left|x - x_0(z)\right|^2 e^{\phi(z)} + e^{-\phi(z)}\right)^{m-1}, \tag{4.1}$$

which correspond to the Gauss parametrization of the homogeneous space $SL(2,\mathbb{C})/SU(2)$. They can be related to vertex operators in AdS_3 with Poincaré metric whose sigma model action is given by

$$\begin{aligned} S &= \frac{k}{2\pi}\int d^2z \left(\frac{\partial_z t_0 \partial_{\bar z} t_0 + \partial_z x_0 \partial_{\bar z}\bar x_0}{t_0^2}\right) \\ &= \frac{k}{2\pi}\int d^2z \left(\partial_z\phi\partial_{\bar z}\phi + \partial_z x_0 \partial_{\bar z}\bar x_0 e^{2\phi}\right), \end{aligned} \tag{4.2}$$

where the spacetime coordinates and their dependence on the worldsheet variable $(z,\bar z)$ are given by $\{t_0(z,\bar z) = e^{-\phi(z,\bar z)}, x_0(z,\bar z), \bar x_0(z,\bar z)\}$. We will also need the auxiliary functions

$$\begin{aligned} \tilde\Phi_m(z|x) = \frac{m}{\pi}&\left(\left|x - x_0(z)\right|^2 e^{\phi(z)} + e^{-\phi(z)}\right)^{m-1} \\ &\times \ln\left(\left|x - x_0(z)\right|^2 e^{\phi(z)} + e^{-\phi(z)}\right). \end{aligned} \tag{4.3}$$

For $m=1$ we simply have the identity, $\Phi_1(z|x) = \frac{1}{\pi}$. The most important case is

$$\Phi_2(z|x) = \frac{2}{\pi}\left(\left|x - x_0(z)\right|^2 e^{\phi(z)} + e^{-\phi(z)}\right), \tag{4.4}$$

which can be thought of as the basic block in the construction of any other function. In fact

$$\begin{aligned} \Phi_m(z|x) &= \frac{m}{\pi}\left(\frac{\pi}{2}\Phi_2\right)^{m-1}, \\ \tilde\Phi_m(z|x) &= \frac{m}{\pi}\left(\frac{\pi}{2}\Phi_2\right)^{m-1}\ln\left(\frac{\pi}{2}\Phi_2\right). \end{aligned} \tag{4.5}$$

Then, if one defines the field

$$\varphi(z|x) = -2\ln\left(\frac{\pi}{2}\Phi_2\right), \tag{4.6}$$

the first ($m=1$) of Eqs. (1.5) becomes simply

$$\partial_x\partial_{\bar x}\varphi = -2e^{\varphi}, \tag{4.7}$$

which is the Liouville equation with the “wrong sign”, since $M=-2$ (cf. Eq. (2.9)). This is not a coincidence since e^{φ} is actually a metric of constant positive curvature on the sphere parametrized by x_0 (cf. Eq. (2.8)). In this respect we note that it is possible to Weyl rescale the metric in such a way that the curvature be -1 everywhere except that at $n \geqslant 3$ singular points. Then, removing the singularities will lead to the standard Liouville equation on the punctured sphere.

In general, the classical Zamolodchikov relations in $SL(2,\mathbb{R})_k$ WZNW model, Eq. (1.5), are equivalent to

$$\partial_{\bar{x}}^m \partial_x^m \big[\varphi e^{\frac{1-m}{2}\varphi}\big] = -2m!(m-1)!\big[e^{\frac{1+m}{2}\varphi}\big]. \tag{4.8}$$

Notice also that the coefficient on the RHS of the above equation matches the corresponding coefficient in (1.1) upon setting $M=-2$. The decoupling equations (1.8) are

$$\partial_{\bar{x}}^m \big[e^{\frac{1-m}{2}\varphi}\big] = \partial_x^m \big[e^{\frac{1-m}{2}\varphi}\big] = 0, \tag{4.9}$$

which are completely analogous to Eqs. (1.1), (1.2) in LFT, except for the fact that the differential operator D_m is now ∂_x^m. In other words, whereas in the case of LFT one considers the Riemann surface rather than the upper half-plane, in the present case, that is the Riemann sphere, the surface corresponds to its universal covering. Thus, in the case of the Riemann sphere the operators simplify to ∂_x^m, just as in the case of the operators D_m on the negatively curved Riemann surfaces that essentially reduce to[10] ∂_τ^m. In other words, the simplification of the covariant operators on the Riemann sphere corresponds to the simplification of the D_m once seen on $\mathbb{H}$, which in turn has the same origin of the simplification of the Poincaré metric from the Riemann surface to $\mathbb{H}$ as it reduces to $1/(\operatorname{Im}\tau)^2$, and solves the Liouville equation in the $\mathbb{H}$ variable τ, losing its Jacobian $|\tau'|^2$.

As in the case of the upper half-plane, where the Fuchsian projective connection vanishes,[11] the analogous quantity on the Riemann sphere

$$T = -\frac{1}{2}(\partial_x\varphi)^2 + \partial_x^2\varphi \tag{4.10}$$

is identically zero. Actually, one sees that (4.4) and (4.6) give $(\partial_x\varphi)^2 = 2\partial_x^2\varphi$. Besides, we can directly observe the vanishing of the $T(x)$ by noticing that $T(x) = 2e^{\varphi/2}\partial_x^2 e^{-\varphi/2}$ and, thus, since the decoupling operators are simply derivatives (i.e., $\partial^2 e^{-\varphi/2} = 0$), we eventually find $T(x) = 0$ as a direct consequence.

Therefore, one can immediately conclude that the analogues of the Liouville decoupling operators D_m reduce to ∂_x^m. In this sense, the variable x is a *trivializing* coordinate. In order to better understand this statement, notice that upon a change of coordinates, $x \to y$, the projective connection $T(x)$ transforms almost like a quadratic differential but not in a homogeneous way

$$\tilde{T}(y) = \left(\frac{\partial x}{\partial y}\right)^2 T\big(x(y)\big) + \{x, y\}. \tag{4.11}$$

[10] More precisely, note that by (3.33) the solutions of $\mathcal{S}_\tau^{(m)} \cdot \psi = 0$, have the common global term $(\frac{\partial z}{\partial \tau})^{\frac{m-1}{2}}$, so that $\mathcal{S}_\tau^{(m)} \cdot \psi = (\frac{\partial z}{\partial \tau})^{-\frac{m+1}{2}} \partial_\tau^m \phi = 0$, where $\psi = (\frac{\partial z}{\partial \tau})^{\frac{m-1}{2}} \phi$. Therefore, finding the inverse of the uniformizing map reduces $\mathcal{S}_\tau^{(m)} \cdot \psi = 0$ to the trivial equation $\partial_\tau^m \phi = 0$.

[11] Since the Fuchsian projective connection is given by $\{\tau, z\}$, on $\mathbb{H}$ we simply have $\{\tau, \tau\} = 0$.

In particular, the presence of the Schwarzian derivative implies that $T(y)$ will be in general non-vanishing, unless y is a linear fractional transformation of x.

Furthermore, recall that the variable τ introduced in LFT plays the role of a *trivializing* coordinate. Correspondingly, in terms of τ, the LFT decoupling operators $D_m = \mathcal{S}_\tau^{(m)}$ are $(\partial_\tau z)^{-\frac{m+1}{2}} \partial_\tau^m (\partial_\tau z)^{\frac{1-m}{2}}$.[12]

4.1. $SL(2, R)$ covariant hierarchy and AdS_3/CFT_2 correspondence

In the previous subsection, we showed that the classical $SL(2,\mathbb{R})_k$ Zamolodchikov relations derived in [2], Eqs. (1.5), (4.8), are in one-to-one correspondence with the classical relations derived by Zamolodchikov in LFT [1], Eq. (1.1). Furthermore, the isotopic coordinate x, which is interpreted as the boundary variable in the AdS_3/CFT_2 correspondence [3], plays the role of a *trivializing* coordinate. This means that the decoupling operators reduce to simple partial derivatives, ∂_x^m and $\partial_{\bar{x}}^m$. Then, the $SL(2,\mathbb{R})_k$ decoupling operators are in one-to-one correspondence with the USO $\mathcal{S}_\tau^{(m)}$ [5], with τ the *trivializing* coordinate in LFT.

Because of the physical meaning of variables $(x, \bar{x})$ as the coordinates on the manifold where the boundary conformal field theory is formulated, we find a motivation to pay particular attention to these. For instance, let us note that the first relation

$$\partial_x \partial_{\bar{x}} \varphi(z|x) = -2e^{\varphi(z|x)}, \tag{4.12}$$

is *covariant* if and only if the RHS transforms as a $(1,1)$-differential under a holomorphic change of coordinates $x \to \tilde{x}$. This, in the context of the AdS_3/CFT_2 correspondence, amounts to saying that the above operator has conformal weight $(h, \bar{h}) = (1,1)$ in the boundary CFT. In fact, the boundary conformal dimension h_{boundary} is determined by the highest weight j of the vertex operator Φ_{2j+1} via [3]

$$h_{\text{boundary}} = -j, \tag{4.13}$$

which encodes the fact that the generators of the global conformal symmetry of the boundary correspond to the generators of the global $SL(2,\mathbb{C})$ symmetry of AdS_3 [22–24]. Besides, observe that $e^{\varphi(z|x)}$ is the classical limit of a Kac–Moody primary operator Φ_{2j+1}, whose highest weight is $j = \tilde{j}_{1,1}^{+} = -1$ [2]. Thus, e^{φ} is a $(1,1)$-differential in the boundary variables, as is suggested from the fact that it satisfies the Liouville equation. A similar analysis shows that the higher order relations (1.5), (4.8) are also naturally covariant in the boundary variables.

Furthermore, observe also that the relations found in [2] turn out to be a set of Ward identities for the boundary CFT. This is due to the fact that the AdS_3/CFT_2 correspondence

[12] One may realize an important subtlety here. While the Liouville geometry considered in the last section has a negative curvature, e^{φ} for $SL(2,\mathbb{R})$ has a sphere geometry and hence a positive curvature. However, even though we have fully utilized the Liouville geometry of the negative curvature (Poincaré upper half-plane) to derive the explicit form of the Zamolodchikov relations, once we have written them down in an algebraic manner as differential equations, the analytic continuation of the cosmological constant obviously works (see also the discussion in Section 2.1). Thus the parallelism we propose does not break here.

states that correlation functions in the BCFT are directly related to correlation functions of appropriate bulk-boundary operators in the worldsheet [3]

$$\left\langle \prod_i \Phi_{j_i}(x_i, \bar{x}_i) \right\rangle_{\text{BCFT}} = \left\langle \prod_i \int d^2 z_i \, \Phi_{j_i}(z_i, \bar{z}_i | x_i, \bar{x}_i) \right\rangle_{\text{worldsheet}}. \tag{4.14}$$

Therefore, via Eq. (4.14), operator-valued relations on the RHS yield non-trivial Ward identities for the boundary CFT on the LHS. In particular, Eqs. (1.5), (4.8) and their quantum counterpart play a special role since the corresponding decoupling operators naturally involve only the isotopic or boundary coordinates $(x, \bar{x})$.[13]

5. $SL(2, R)$ finite representations in WZNW and LFT

Certainly, it is impossible to avoid the question about whether the connection between Liouville theory and $SL(2, \mathbb{R})$ symmetry we have studied so far is or is not connected with the other known relations existing between this pair of CFTs. In fact, since the relation between the WZNW model formulated on $SL(2, \mathbb{R})$ and the LFT (or deformations of these models) frequently appears in different contexts, we find it convenient here to shortly discuss how the specific connection we point out in this paper relates itself to those other works linking both conformal theories.

Let us first briefly comment on the geometrical nature of the $SL(2, \mathbb{R})$ symmetry, as being the isometry group of the hyperbolic upper half-plane involved in the uniformization problem where Liouville theory plays a central role.

The meaning of the decoupling equations (1.8) within the context of $sl(2)_k$ algebra is that the Kac–Moody primary fields Φ_m generate a finite-dimensional spin $j = \frac{m-1}{2}$ representation of $SL(2, \mathbb{R})$. This fact is reflected in the polynomial form of Φ_m in terms of x. Thus, the solutions of the decoupling equation that span the representation are the monomials

$$\psi_{j,m} = x^{j+m}, \quad m = -j, \ldots, j. \tag{5.15}$$

In terms of this realization, the generators of the $sl(2)$ algebra are ($a = \{+, -, 3\}$)

$$D^3 = x\partial_x - j, \qquad D^- = -\partial_x, \qquad D^+ = -x^2\partial_x + 2jx \tag{5.16}$$

and then, the action of the currents on the vertex operators $\Phi_j(z|x)$ is given by

$$\left[J_n^a, \Phi_j(z|x)\right] = z^n D_j^a \Phi_j(z|x). \tag{5.17}$$

[13] In general, the worldsheet correlation functions on the RHS will contain other vertex operators that come from the CFTs that are combined with the AdS_3 WZNW factor. For instance, the full (super)string background may be $AdS_3 \times S^3 \times M$. The presence of these vertex operators that multiply each $\Phi_{j_i}(z_i, \bar{z}_i | x_i, \bar{x}_i)$ may actually be necessary for the above formula to be covariant. Namely, covariance requires that the full vertex operators $\Phi_{j_i}(z_i|x_i) \times V_N$ have worldsheet conformal weight $(1, 1)$, where V_N refers to the vertex operator in the manifold $S^3 \times M$.

The monomials $\psi_{j,m}$ are eigenfunctions of D_j^3 and correspond to the usual basis $|j,m\rangle$ of the spin j representation

$$D^3\psi_{j,m} = m\psi_{j,m}, \qquad D^{\pm}\psi_{j,m} = (\pm j - m)\psi_{j,m\pm 1}. \tag{5.18}$$

Similarly, the solutions of the classical Liouville decoupling equation $S_\tau^{(2j+1)}\chi = 0$ span a spin j representation of $SL(2,\mathbb{R})$

$$\chi_{j,m}(z) = \frac{\tau(z)^{j+m}}{\tau'(z)^j}, \quad m = -j, \ldots, j. \tag{5.19}$$

The generators are given by

$$\begin{aligned} &\mathcal{D}^3 = \tau'(z)^{-j}(\tau\partial_\tau - j)\tau'(z)^j, \qquad \mathcal{D}^- = -\tau'(z)^{-j}\partial_\tau\tau'(z)^j, \\ &\mathcal{D}^+ = \tau'(z)^{-j}\big(-\tau^2\partial_\tau + 2j\tau\big)\tau'(z)^j. \end{aligned} \tag{5.20}$$

As we discussed in Section 2, the $SL(2,\mathbb{R})$ symmetry has a geometrical origin. It corresponds to the isometry group of the hyperbolic upper half-plane. The exponential of the Liouville field

$$e^{\varphi(z)} = \frac{\tau'(z)\bar\tau'(\bar z)}{(\operatorname{Im}\tau(z))^2} \tag{5.21}$$

is invariant under $SL(2,\mathbb{R})$ transformations. It also follows that for $2j \in \mathbb{Z}^+$

$$e^{-j\varphi} = 4^{-j}\sum_{m=-j}^{j}(-1)^m\binom{2j}{j+m}\chi_{j,m}(z)\bar\chi_{j,-m}(\bar z). \tag{5.22}$$

The above expression shows that at the classical level negative powers of the metric are decomposed into irreducible representations of $SL(2,\mathbb{R})$. Likewise, Gervais and Neveu have shown that at the quantum level there exists a decomposition of these negative powers of the metric into operators that transform under irreducible representations of the quantum group $U_q(sl(2))$ [19]. This observation is the basis of the algebraic approach to Liouville theory [20,21]. Representations of $U_q(sl(2))$ were also studied by Ponsot and Teschner [25] who expressed the fusion coefficients of Liouville theory in terms of the appropriate Racah–Wigner coefficients. Certainly, it would be interesting to completely understand the connection existing between the mentioned algebraic approach and the relation we presented here. This requires further study.

6. Conclusion

We proved that the classical Liouville decoupling operators are given by USO [5], which once again shows the close relationship between Liouville theory and the theory of uniformization of Riemann surfaces. This result enables us to define a *trivializing* coordinate τ, such that the decoupling operators become simple partial derivatives in τ and the classical energy–momentum tensor vanishes. Conversely, we showed that the classical $SL(2,\mathbb{R})_k$ Zamolodchikov relations derived in [2], Eqs. (1.5), (4.8), can be written in a manifestly Liouville-like fashion and are in one-to-one correspondence with the classical relations derived by Zamolodchikov in LFT [1], Eq. (1.1). In particular, the isotopic coordinate x, which is a boundary variable in the AdS_3/CFT_2 correspondence, plays the same role of a *trivializing* coordinate as τ does. The manifest $SL(2,\mathbb{R})$ symmetry on the WZNW side is mirrored by the $SL(2,\mathbb{R})$ isometry of the hyperbolic upper half-plane, where τ lives.

There are some future directions worthwhile pursuing. First of all, it is important to extend our results beyond the classical limit (i.e., finite b or k). As we can easily see, the Zamolodchikov coefficients for the both theories have a similar structure even quantum mechanically. As was signaled in [26], the decoupling operator in the $SL(2,\mathbb{R})_k$ model has an explicitly factorized form, while it does not in the Liouville (Virasoro) case. Therefore, the connection at the quantum level may suggest an elegant way to derive Virasoro decoupling operators explicitly. In this context, the Hamiltonian reduction may be also useful. Furthermore, the geometrical meaning of the Zamolodchikov coefficients will shed a new light on the quantum LFT itself.

Secondly, our results have an obvious application to the AdS_3/CFT_2 duality. As we have discussed in Section 4, the Zamolodchikov relations in the $SL(2,\mathbb{R})$ model provide a set of Ward identities for the boundary CFT after the integration over the worldsheet coordinate z. In the case of the Liouville theory, it is believed that constraints from the Zamolodchikov relations will give an important clue to the integrability of the minimal string theory (even in higher genus). Therefore, it is very plausible, in view of the correspondence between the two theories discussed in this paper, that further study of the Zamolodchikov relations in the $SL(2,\mathbb{R})_k$ model will yield a hint towards the complete solution of the AdS_3/CFT_2 correspondence.

Finally, the fact that the inverse uniformizing map $\tau(z)$ becomes a trivializing coordinate of the Liouville decoupling equation in the classical limit suggests the introduction of a quantum $\tau(z)$ as a fundamental block of the quantization of the LFT. In the literature (e.g., [19,21]) some attempts of the quantization based on the Bäcklund transformation were discussed in the case of the simple geometry. On more complicated Riemann surfaces, the non-trivial global transformation of $\tau(z)$ besides the univalence should play an important role. The quantization of such a "scalar" with non-trivial global transformation properties will be an interesting problem and also useful for the quantization of the Liouville theory on the higher genus Riemann surfaces.

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Appendix A. Matrix formulation and explicit form of Virasoro null vectors

Here we give another independent proof of the fact that the classical decoupling operators D_n coincide with the USO $\mathcal{S}_\tau^{(n)}$ defined in Section 3. The strategy is the following. First we introduce a nice and compact way to express our operators as "formal determinants" of matrices. In particular, as found in [8], this leads to an explicit formulation of the quantum decoupling operator $D_{n,1}$. Taking the classical limit, we recover the USO $\mathcal{S}_\tau^{(n)}$.

A.1. Operators as matrices

Now we explain how to write a general operator as a formal determinant of a matrix. First we need a realization of the $sl(2)$ algebra in the $n \times n$ space of matrices (we use also $2j + 1 = n$). We take

$$[J_-]_{p,q} = \delta_{p,q+1}, \qquad [J_0]_{p,q} = (j - p + 1)\delta_{p,q},$$
$$[J_+]_{p,q} = p(n - p)\delta_{p+1,q}. \tag{A.1}$$

These matrices satisfy the commutation relations: $[J_0, J_\pm] = \pm J_\pm$, $[J_+, J_-] = 2J_0$. We call a generic operator O_n, and the corresponding matrix operator $\hat{O}_n = -J_- + A$, where A is an upper triangular matrix. The operator O_n is the formal determinant of the matrix operator $\hat{O}_n$. The formal determinant is defined as follows. We set $\vec{f} = (f_1, f_2, \ldots, f_n)$ and $\vec{F} = (F_0, 0, \ldots, 0)$. If the matrix satisfies the relation $\vec{F} = \hat{O}_n \vec{f}$, then its formal determinant satisfies $F_0 = O_n f_n$. There are a lot of matrices that correspond to the same operator, so we can make a sort of gauge transformation for matrices. We take N as an upper triangular matrix with one on the diagonal. Two matrix operators related by the gauge transformation

$$\hat{O}'_n = N^{-1}\hat{O}_n N \tag{A.2}$$

define the same differential operator.[14]

A.2. The Virasoro null vector $(n, 1)$

The operator $D_{n,1}$, which annihilates the level n null Virasoro primary, is obtained as the formal determinant of the following matrix operator [8]:

$$\hat{D}_{n,1} = -J_- + \sum_{k=0}^{\infty} \left(b^2 J_+\right)^k L_{-k-1}. \tag{A.3}$$

To take the classical limit [1] we send $b \to 0$, $L_{-1} \to d_z$ and $2(k-2)!b^2 L_{-k} \to {d_z}^{k-2} T$ at $k > 1$. By using (2.12), the classical decoupling operator is

$$\hat{D}_n = -J_- + d_z \mathbf{1} + \frac{1}{2}\{\tau, z\} J_+. \tag{A.4}$$

[14] Note that also N^{-1} is an upper triangular matrix with 1 on the diagonal. The gauge transformation on the vector $(f_1, \ldots, f_n)$ leaves f_n invariant, and on the vector $(f_0, 0, \ldots, 0)$ leaves f_0 invariant. At the end the formal determinant takes into account only the dependence of f_0 from f_n.

This is indeed the matrix formulation of the USO. In fact, defining the logarithmic derivative $l = \tau''/\tau'$ and making the following gauge transformation

$$\hat{D}_n = e^{-lJ_+/2}\hat{\mathcal{S}}_\tau^{(n)} e^{lJ_+/2} \tag{A.5}$$

we obtain

$$\hat{\mathcal{S}}_\tau^{(n)} = -J_- + d\mathbf{1} - lJ_0. \tag{A.6}$$

The formal determinant is equivalent to (1.10).

Appendix B. Schwarzian operators and Schwarzian derivative

The operator $\mathcal{S}_\tau^{(m)}$ depends on $\varphi_z \equiv \partial_z\varphi$ only through the classical energy–momentum tensor

$$T = \{\tau, z\} = \{\partial_z\varphi, z\} = -\frac{1}{2}(\partial_z\varphi)^2 + \partial_z^2\varphi. \tag{B.1}$$

In order to prove this, we are going to show that the variation of the operator under a deformation of φ_z that keeps T invariant is vanishing [27].

First of all, for δT to be vanishing we need

$$\delta T = \partial_z\delta\varphi_z - \varphi_z\delta\varphi_z = (\partial_z - \varphi_z)\delta\varphi_z = 0. \tag{B.2}$$

The crucial observation is that this is equivalent to

$$(\partial_z - j\varphi_z)\delta\varphi_z = \delta\varphi_z\big(\partial_z - (j-1)\varphi_z\big) \quad \leftrightarrow \quad A_j\delta\varphi_z = \delta\varphi_z A_{j-1}, \tag{B.3}$$

where $A_j \equiv \partial_z - j\varphi_z$. On the other hand, since

$$\mathcal{S}_\tau^{(m)} = \mathcal{S}_\tau^{(2j+1)} = e^{j\varphi}\partial_z e^{-\varphi}\partial_z \cdots \partial_z e^{-\varphi}\partial_z e^{j\varphi} \tag{B.4}$$

can be rewritten as

$$\begin{aligned}\mathcal{S}_\tau^{(m)} = \mathcal{S}_\tau^{(2j+1)} &= \big(e^{j\varphi}\partial_z e^{-j\varphi}\big)\big(e^{(j-1)\varphi}\partial_z e^{-(j-1)\varphi}\big)\cdots\big(e^{-j\varphi}\partial_z e^{j\varphi}\big)\\ &\equiv A_j A_{j-1}\cdots A_{-(j-1)}A_{-j}\end{aligned} \tag{B.5}$$

one finds

$$\begin{aligned}\delta S_\tau^{(2j+1)} &= \delta A_j A_{j-1}\cdots A_{-j} + A_j\delta A_{j-1}\cdots A_{-j} + \cdots + A_j A_{j-1}\cdots\delta A_{-j}\\ &= -j\delta\varphi_z A_{j-1}\cdots A_{-j} - (j-1)A_j\delta\varphi_z\cdots A_{-j} + \cdots\\ &= -j\delta\varphi_z A_{j-1}\cdots A_{-j} - (j-1)\delta\varphi_z A_{j-1}\cdots A_{-j} + \cdots\\ &= \left(\sum_{k=-j}^{j} k\right)\delta\varphi_z A_{j-1}\cdots A_{-j} = 0.\end{aligned} \tag{B.6}$$

Appendix C. Classical Zamolodchikov relations, II

Here we show another way to derive the classical Zamolodchikov relations. First of all note that

$$\bar{S}_\tau^{(2j+1)} S_\tau^{(2j+1)}\big(\varphi e^{-j\varphi}\big) = \bar{S}_\tau^{(2j+1)} \sum_{k=0}^{2j+1} \beta_{k,2j+1}\varphi_z^k e^{-j\varphi}, \tag{C.1}$$

where we used

$$S_\tau^{(2j+1)}\big(\varphi e^{-j\varphi}\big) = e^{(j+1)\varphi}\big(e^{-\varphi}\partial_z\big)^{2j+1}\varphi \tag{C.2}$$

and

$$\big(e^{-\varphi}\partial_z\big)^\ell \varphi = e^{-\ell\varphi}\sum_{m=0}^{\ell}\beta_{m,\ell}\varphi_z^m. \tag{C.3}$$

The latter formula is valid because one can express a higher order derivative of φ in terms of φ_z and the energy–momentum tensor together with its derivatives. Then, since

$$\begin{aligned} e^{-\varphi}\partial_{\bar z}\big(\varphi_z^{2j+1}\big) &= e^{-\varphi}(2j+1)\varphi_z^{2j}\partial_{\bar z}\partial_z\varphi = e^{-\varphi}(2j+1)\varphi_z^{2j}Me^{\varphi} \\ &= (2j+1)M\varphi_z^{2j} \end{aligned} \tag{C.4}$$

one finds that

$$\begin{aligned} \bar{S}_\tau^{(2j+1)}\big(\varphi_z^{2j+1}e^{-j\varphi}\big) &= e^{(j+1)\varphi}\big(e^{-\varphi}\partial_{\bar z}\big)^{2j+1}\big(\varphi_z^{2j+1}\big) \\ &= (2j+1)!M^{2j+1}e^{(j+1)\varphi} \end{aligned} \tag{C.5}$$

and likewise

$$\bar{S}_\tau^{(2j+1)}\big(\varphi_z^k e^{-j\varphi}\big) = 0 \tag{C.6}$$

for $k < 2j+1$. Hence

$$\bar{S}_\tau^{(2j+1)} S_\tau^{(2j+1)}\big(\varphi e^{-j\varphi}\big) = \beta_{2j+1,2j+1}(2j+1)!M^{2j+1}e^{(j+1)\varphi}. \tag{C.7}$$

One can evaluate $\beta_{2j+1,2j+1}$ by induction. In fact

$$\begin{aligned} &\big(e^{-\varphi}\partial_z\big)^{\ell+1}\varphi \\ &\quad = e^{-\varphi}\partial_z e^{-\ell\varphi}\sum_{m=0}^{\ell}\beta_{m,\ell}\varphi_z^m \\ &\quad = e^{-(\ell+1)\varphi}\sum_{m=0}^{\ell}\left[\partial_z\beta_{m,\ell}\varphi_z^m + m\beta_{m,\ell}\varphi_z^{m-1}\left(T+\frac{1}{2}\varphi_z^2\right) - \ell\beta_{m,\ell}\varphi_z^{m+1}\right], \end{aligned} \tag{C.8}$$

which implies that

$$\beta_{\ell+1,\ell+1} = -\frac{\ell}{2}\beta_{\ell,\ell} \quad\Rightarrow\quad \beta_{n,n} = (-1)^{n+1}\frac{(n-1)!}{2^{n-1}}. \tag{C.9}$$

Finally

$$\bar{S}_{\tau}^{(2j+1)} S_{\tau}^{(2j+1)}\big(\varphi e^{-j\varphi}\big) = (-1)^{2j} \frac{(2j)!}{2^{2j}} (2j+1)! M^{2j+1} e^{(j+1)\varphi}, \tag{C.10}$$

which is exactly Zamolodchikov's result (1.1) once we use $j = \frac{m-1}{2}$.

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