

Reentrant Phase Transitions in Rotating AdS Black Holes

Natacha Altamirano,^{1,2,*} David Kubizňák,^{1,†} and Robert B. Mann^{1,2,‡}

¹*Perimeter Institute, 31 Caroline St. N. Waterloo Ontario, N2L 2Y5, Canada*

²*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1*
(Dated: June 24, 2013)

We study the thermodynamics of higher-dimensional singly spinning asymptotically AdS black holes in the canonical (fixed J) ensemble of extended phase space, where the cosmological constant is treated as pressure and the corresponding conjugate quantity is interpreted as thermodynamic volume. Along with the usual small/large black hole phase transition, we find a new phenomenon of reentrant phase transitions for all $d \geq 6$ dimensions, in which a monotonic variation of the temperature yields two phase transitions from large to small and back to large black holes. This situation is similar to that seen in multicomponent liquids.

PACS numbers: 04.50.Gh, 04.70.-s, 05.70.Ce

Introduction. In view of the AdS/CFT correspondence, phase transitions in asymptotically AdS black holes allow for a dual interpretation in the thermal conformal field theory (CFT) living on the AdS boundary—the principal example being the well known radiation/Schwarzschild-AdS black hole Hawking–Page transition [1] which can be interpreted as a confinement/deconfinement phase transition in the dual quark gluon plasma [2]. Charged [3–6] and rotating [7, 8] asymptotically AdS black holes possess an interesting feature—they allow for a first order small-black-hole/large-black-hole phase (SBH/LBH) transition which is in many ways reminiscent of the liquid/gas transition of the Van der Waals fluid. This superficial analogy was recently found more intriguing [9] by considering a thermodynamic analysis in an extended phase space where the cosmological constant is identified with thermodynamic pressure and its variations are included in the first law of black hole thermodynamics. This notion emerges from geometric derivations of the Smarr formula [10] that i) imply the mass of an AdS black hole should be interpreted as the enthalpy of the spacetime and ii) allow for a computation of the conjugate thermodynamic volume. Intensive and extensive quantities are now properly identified [9] and the SBH/LBH transition can be understood as a liquid/gas phase transition by employing Maxwell’s equal area law to the $P - V$ diagram. Coexistence lines and critical exponents are then seen to match those of a Van der Waals fluid.

In this paper we report the finding of an interesting phenomena, observed previously in multicomponent fluids, e.g., [11], of *black hole reentrant phase transitions* (RPTs). A system undergoes an RPT if a monotonic variation of any thermodynamic quantity results in two (or more) phase transitions such that the final state is macroscopically similar to the initial state. We find for a certain range of pressures (and a given angular momentum) that a monotonic lowering of the temperature yields a large-small-large black hole transition, where we refer to the latter ‘large’ state as an intermediate black hole

(IBH). This situation is accompanied by a discontinuity in the global minimum of the Gibbs free energy, referred to as a *zeroth-order phase transition*, a phenomenon seen in superfluidity and superconductivity [12], and recently for Born–Infeld black holes [13]. We find the RPT to be generic for all rotating AdS black holes in $d \geq 6$ dimensions.

Extended phase space thermodynamics. Rotating AdS black holes were constructed in $d = 4$ by Carter [14] and later generalized to all higher dimensions [15–17]. In what follows we limit ourselves to the case of singly spinning black holes for which only one of the rotation parameters is non-trivial. In d spacetime dimensions the metric reads

$$ds^2 = -\frac{\Delta}{\rho^2} \left(dt - \frac{a}{\Xi} \sin^2 \theta d\varphi \right)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Sigma} d\theta^2 + \frac{\Sigma \sin^2 \theta}{\rho^2} \left[a dt - \frac{(r^2 + a^2)}{\Xi} d\varphi \right]^2 + r^2 \cos^2 \theta d\Omega_{d-2}^2, \quad (1)$$

where $d\Omega_{d-2}^2$ is the metric for the $(d - 2)$ -sphere and

$$\begin{aligned} \Delta &= (r^2 + a^2) \left(1 + \frac{r^2}{l^2} \right) - 2mr^{5-d}, & \Sigma &= 1 - \frac{a^2}{l^2} \cos^2 \theta, \\ \Xi &= 1 - \frac{a^2}{l^2}, & \rho^2 &= r^2 + a^2 \cos^2 \theta, \end{aligned} \quad (2)$$

with l the AdS radius. The associated thermodynamic quantities read (in Planck units) [17]

$$M = \frac{\omega_{d-2}}{4\pi} \frac{m}{\Xi^2} \left(1 + \frac{(d-4)\Xi}{2} \right), \quad (3)$$

$$J = \frac{\omega_{d-2}}{4\pi} \frac{ma}{\Xi^2}, \quad \Omega_H = \frac{a}{l^2} \frac{r_+^2 + l^2}{r_+^2 + a^2}, \quad (4)$$

$$T = \frac{1}{2\pi} \left[r_+ \left(\frac{r_+^2}{l^2} + 1 \right) \left(\frac{1}{a^2 + r_+^2} + \frac{d-3}{2r_+^2} \right) - \frac{1}{r_+} \right], \quad (5)$$

$$S = \frac{\omega_{d-2}}{4} \frac{(a^2 + r_+^2) r_+^{d-4}}{\Xi} = \frac{A}{4}, \quad (6)$$

where r_+ is the black hole horizon radius (the largest positive real root of $\Delta = 0$) and $\omega_d = 2\pi^{\frac{d+1}{2}} / \Gamma[(d+1)/2]$ is the volume of the unit d -sphere.

We interpret the negative cosmological constant Λ as a positive thermodynamic pressure P [7, 10, 18, 19]

$$P = -\frac{1}{8\pi}\Lambda = \frac{(d-1)(d-2)}{16\pi l^2}, \quad (7)$$

in which case the first law of black hole thermodynamics and Smarr formula [10] are

$$\delta M = T\delta S + \Omega_H\delta J + V\delta P, \quad (8)$$

$$\frac{d-3}{d-2}M = TS + \Omega_H J - \frac{2}{d-2}VP, \quad (9)$$

where the thermodynamic volume conjugate to P is [18]

$$V = \frac{r_+ A}{d-1} \left[1 + \frac{a^2}{\Xi} \frac{1 + r_+^2/l^2}{(d-2)r_+^2} \right]. \quad (10)$$

It obeys the so called reverse isoperimetric inequality [18] and in the non-rotating case becomes $V = \frac{\omega_{d-2} r_+^{d-1}}{d-1}$, which is the spatial volume of a round sphere of radius r_+ in the Euclidean space.

Reentrant phase transition. The thermodynamic behavior of the system is governed by the Gibbs free energy $G = G(T, P, \dots)$, which reads (I being the Euclidean action [17])

$$G = M - TS = \frac{I}{\beta} + \Omega_H J \quad (11)$$

$$= \frac{\omega_{d-2} r_+^{d-5}}{16\pi \Xi^2} \left(3a^2 + r_+^2 - \frac{(r_+^2 - a^2)^2}{l^2} + \frac{3a^2 r_+^4 + a^4 r_+^2}{l^4} \right)$$

and depends on an external parameter J . We can plot it for fixed J parametrically, first expressing $a = a(J, r_+, P)$ using (4) and then inserting the (well-behaved) solution into the expressions for G and T , consequently expressed as functions of P and r_+ .

A simple criterion used for investigating the thermodynamic stability is the positivity of the specific heat. For a canonical ensemble in the extended phase space, it is natural to consider the specific heat at constant pressure

$$C_P = T \left(\frac{dS}{dT} \right)_P \quad (12)$$

which is different from the specific heat at constant thermodynamic volume, C_V . We take negativity of C_P as a sign of local thermodynamic instability. Note that, as always, we calculate this quantity for fixed J . That is, our specific heat at constant P is in fact a specific heat at constant (P, J) and coincides with C_J considered in previous studies, e.g. [20]. When plotting the Gibbs free energy we plot branches with $C_P > 0$ in red solid lines and branches with $C_P < 0$ in dashed blue lines.

The behavior of G depends crucially on the dimension d . For $d = 4$ and $d = 5$ the situation is illustrated in fig. 1. For $P > P_c$ and any temperature there is only one branch of locally thermodynamically stable black holes

(with positive C_P) whereas for $P < P_c$ the characteristic swallowtail behaviour indicating the SBH/LBH phase transition emerges, with the global minimum of G having $C_P > 0$. The corresponding $P - T$ diagram (not shown) is reminiscent of what was observed for charged black holes in [9] and is analogous to the Van der Waals $P - T$ diagram.

In $d \geq 6$ the situation is more subtle and markedly different as shown in Figs. 2 and 3. For $P > P_c$, G resembles the curve characteristic for the Schwarzschild-AdS black hole, known from the Hawking–Page transition [1]: the upper branch corresponds to small unstable black holes with $C_P < 0$ whereas the lower branch describes stable large black holes with $C_P > 0$. There is a critical point at $P = P_c$, and for the range of pressures $P \in (P_t, P_c)$ and temperatures $T \in (T_t, T_c)$ there is a standard first order SBH/LBH phase transition, reminiscent of the Van der Waals phase transition [9]. However for $P_c > P_z = P > P_t$ three separate phases of black holes emerge (see fig. 4): intermediate black holes (IBH) (on the left), small (middle), and large (on the right). This holds for $T \in (T_t, T_z)$, $P \in (P_t, P_z)$ and terminates at $T = T_t$. Small and large black holes are separated by a standard first order phase transition, but the intermediate and small are separated by a finite jump in G , which in this range has a discontinuous global minimum (fig. 3). This is the RPT, first observed in a nicotine/water mixture [21], and since seen in multicomponent fluid systems, gels, ferroelectrics, liquid crystals, and binary gases [11]. Finally, for $T < T_t$ only one LBH phase exists.

Consider fig. 3. If we start decreasing the temperature from, say $T = 0.24$, the system follows the lower solid (red) curve until it joins the upper solid (red) curve—this corresponds to a first order SBH/LBH phase transition. As T continues to decrease the system follows this upper curve until $T = T_0 \in (T_t, T_z)$, where G has a discontinuity at its global minimum. Further decreasing T , the system jumps to the uppermost red line—this corresponds to the zeroth order phase transition between small and intermediate black holes.

This novel situation is clearly illustrated in the $P - T$ diagrams in fig. 4. There is the expected SBH/LBH line of coexistence corresponding to the liquid/gas Van der Waals case, ending in a critical point (T_c, P_c) . This line terminates at (T_t, P_t) , where there is a “triple point” between the small, intermediate, and large black holes. For smaller values of T there is an unstable line of coexistence (not shown) between the IBHs and LBHs. For $T \in (T_t, T_z)$ there is a new IBH/SBH line of coexistence (see inset of fig. 4) that terminates in another “critical point” (T_z, P_z) . The range for the RPT is quite narrow and must be determined numerically. For example for $J = 1$ and $d = 6$ we obtain $(T_t, T_z, T_c) \approx (0.2332, 0.2349, 0.3004)$ and $(P_t, P_z, P_c) \approx (0.0553, 0.0579, 0.0958)$.

Equation of state. The equation of state $P = P(V, T, J)$ in the canonical (fixed J) ensemble can

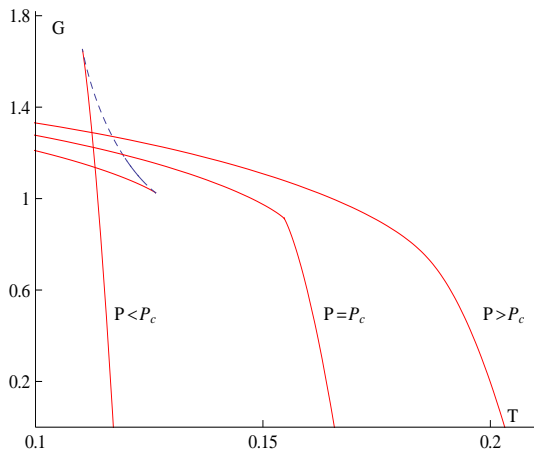


FIG. 1. **Gibbs free energy in $d = 5$ for various values of P and $J = 1$.** As with RN-AdS black holes in any dimension, we see characteristic swallowtail behaviour indicating an SBH/LBH transition. Solid-red/dashed-blue lines correspond to C_P positive/negative respectively; the $C_P < 0$ line indicates a local thermodynamic instability where the Gibbs energy is not a local minimum; at the joins C_P diverges. The behaviour of G in $d = 4$ is similar.

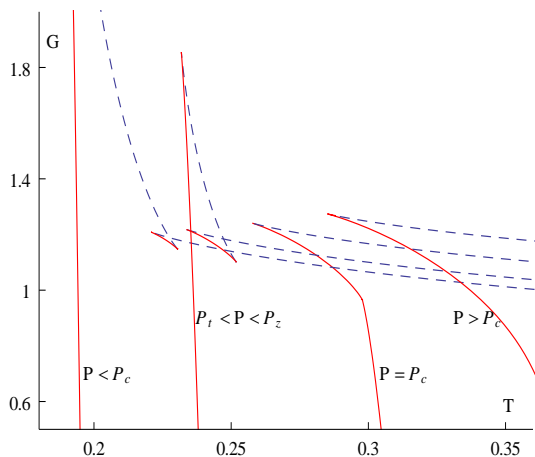


FIG. 2. **Gibbs free energy in $d = 6$ for various values of P and $J = 1$.** Solid-red/dashed-blue lines correspond to C_P positive/negative respectively. As with Schwarzschild-AdS black holes, for $P \geq P_c$, the (lower) LBH branch is thermodynamically stable whereas the upper branch is unstable. For $P = P_c$ we observe critical behaviour. At the joins of dashed (blue) and solid (red) lines C_P diverges. For $P \in (P_t, P_z)$ we observe a “zeroth-order phase transition” signifying the onset of an RPT.

be computed by solving Eqs. (3)–(6) to eliminate (m, a, r_+) in terms of the basic thermodynamic variables. The result can be obtained numerically (see fig. 5), but there are two cases of physical interest that can be approximated analytically: the slowly rotating case ($a \rightarrow 0$) and the ultraspinning ($a \rightarrow l$) regime.

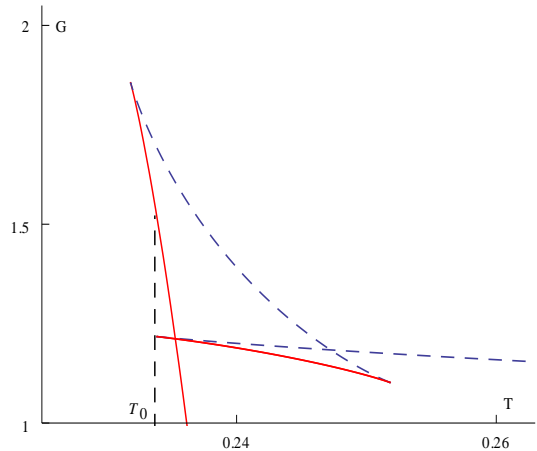


FIG. 3. **Zeroth order phase transition in $d = 6$.** A close-up of fig. 2 illustrating the discontinuity in the global minimum of G at $T = T_0 \approx 0.2339 \in (T_t, T_z)$ (denoted by the vertical line). We have set $P = 0.0564 \in (P_t, P_z)$ and $J = 1$. Solid-red/dashed-blue lines correspond to C_P positive/negative respectively.

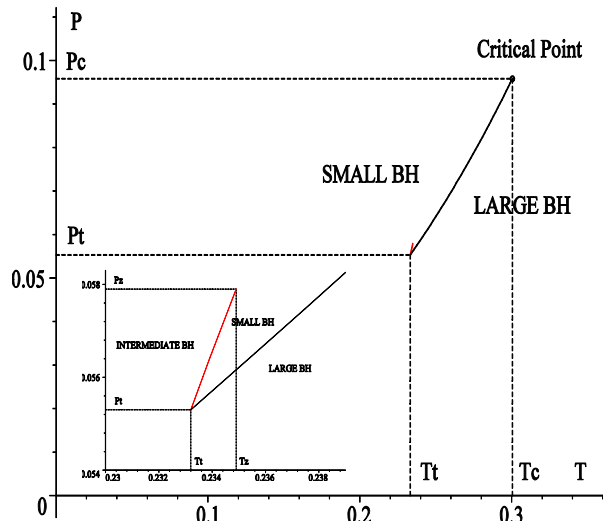


FIG. 4. **$P - T$ diagram in $d = 6$.** The coexistence line of the first order phase transition between small and large black holes is depicted by a thick black solid line for $J = 1$. It initiates from the critical point (P_c, T_c) and terminates at (P_t, T_t) . The red solid line in the inset indicates the ‘coexistence line’ of small and intermediate black holes, separated by a finite gap in G , indicating the RPT. It commences from (T_z, P_z) and terminates at (P_t, T_t) . A similar figure is valid for any $d \geq 6$.

In the slowly rotating case, we expand in the parameter $\epsilon = a/l$, to obtain

$$P = \frac{T}{v} - \frac{d-3}{\pi(d-2)v^2} + \frac{\pi(d-1)16^d J^2}{4\omega_{d-2}^2 [(d-2)v]^{2(d-1)}} + O(\epsilon^4) \quad (13)$$

and it is straightforward to show that Van der Waals

behaviour occurs as in $d = 4$ [13]. The specific volume of the fluid v is defined by $V = \frac{(\kappa v)^{d-1} \omega_{d-2}}{d-1}$, $\kappa = \frac{1}{4}(d-2)$. Critical points (P_c, v_c, T_c) can be computed from $\frac{\partial P}{\partial v} = \frac{\partial^2 P}{\partial v^2} = 0$. We find that the critical exponents $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ match those of a Van der Waals fluid, though the critical ratio $\rho_c = \frac{P_c v_c}{T_c} = \frac{2d-3}{4(d-1)}$ differs; note that it reduces to $\rho_c = 5/12$ for $d = 4$ [13].

In the ultraspinning limit $a \rightarrow l$, $r_+ \rightarrow 0$. For all $d \geq 6$ the geometry of a black hole approaches that of a black membrane [22, 23]; setting $f = 1 - \mu R^{5-d}$ the metric is

$$ds_M^2 = -f d\tau^2 + \frac{dR^2}{f} + d\sigma^2 + \sigma^2 d\varphi^2 + R^2 d\Omega_{d-4}^2,$$

known to be unstable due to the Gregory–Laflamme instability [24]. The entropy vanishes and the temperature diverges but both the angular momentum and (specific) volume $V = \frac{8\pi M l^2}{(d-1)(d-2)}$ remain finite, while the equation of state is $P = \frac{4\pi}{(d-1)(d-2)} \frac{J^2}{V^2}$. The same equation of state is valid for ultraspinning black rings [25]. Expanding about the ultraspinning limit, $\Xi \rightarrow 0$, we find

$$P_6 = \frac{\pi J^2}{5V^2} - \frac{\sqrt{5\omega_4}}{16\sqrt{\pi^3 V T}} + \frac{9\sqrt{5\omega_4} J^2}{3200\sqrt{\pi^3 V^5 T^5}} + O(\Xi^9),$$

$$G_6 = \frac{2\sqrt{5\pi} P J}{5} + \frac{\sqrt{25\omega_4} J}{8\sqrt{T}(\pi P)^{\frac{1}{4}}} - \frac{5\omega_4}{1024\pi^3 T P} + O(\Xi^3)$$

for $d = 6$; higher-dimensional expansions can likewise be computed.

In between these two limiting cases a phase transition to a novel family of black objects branching off the spherical black holes (e.g., black rings) is expected (see, e.g., [26, 27] and references therein). The exact critical point is shifted; we find numerically that the critical ratio ρ_c is slightly smaller and that all critical exponents remain the same. The two expansions for the equation of state are, together with the exact numerical solution, displayed for the critical temperature in fig. 6. Obviously, the slow rotation approximation is effectively accurate for large values of r_+ (rightmost curve) whereas the ultraspinning one (leftmost curve) is accurate as r_+ approaches zero.

Discussion. Although the cosmological constant is generally regarded as fixed in the action, it is possible to dynamically generate it using a $(d-1)$ -form gauge potential and incorporate it into a generalized first-law [28]. We see that identifying it as pressure and incorporating its conjugate volume yields not only a consistent Smarr formula, but also a qualitatively new phase structure in the thermodynamics of rotating black holes similar to binary fluids. In binary fluids at low temperatures, directional bonding between unlike species can lead to a miscible state, which is restored at high temperatures since entropy of mixing dominates. At intermediate temperatures the two fluids become immiscible. The corresponding physics for $d \geq 6$ rotating black holes remains an interesting subject for further study.

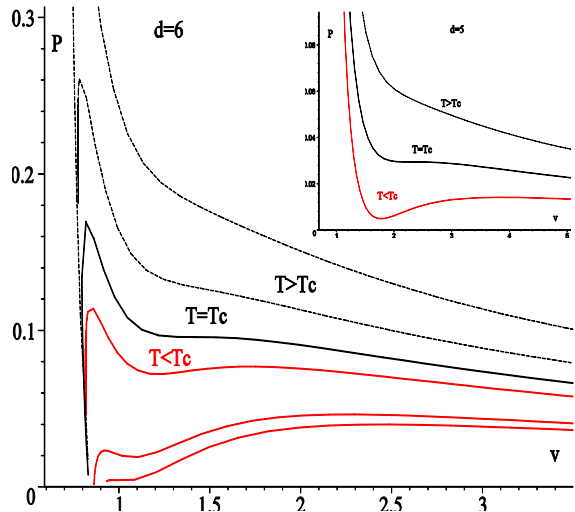


FIG. 5. $P - v$ diagram in $d = 5$ (inset) and $d = 6$. Obviously, in $d \geq 6$ the $P - v$ diagram is more complex than that of the standard Van der Waals, and reflects the interesting behavior of the Gibbs free energy and a possible RPT.

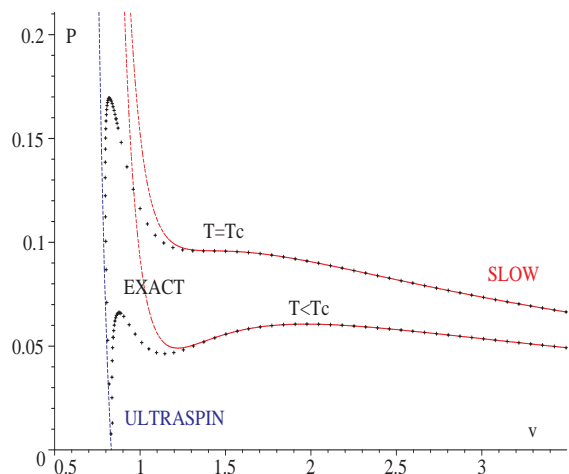


FIG. 6. **The two approximations.** Exact critical and subcritical isotherms, depicted by black crosses, are compared to the slow spinning expansion (13) denoted by red curves and the ultraspinning expansion (14) denoted by a blue curve. Note that the ultraspinning black holes correspond to the upper branch in the Gibbs free energy which is unstable. We have considered $d = 6$ and set $J = 1$.

Acknowledgments. This work was supported in part by the Natural Sciences and Engineering Research Council of Canada. We are grateful to the Perimeter Institute where part of this work was carried out.

* naltamirano@perimeterinstitute.ca

† dkubiznak@perimeterinstitute.ca

‡ rbmann@uwaterloo.ca

- [1] S. Hawking and D. N. Page, *Thermodynamics of Black Holes in anti-De Sitter Space*, *Commun.Math.Phys.* **87** (1983) 577.
- [2] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, *Adv.Theor.Math.Phys.* **2** (1998) 505–532, [[hep-th/9803131](#)].
- [3] A. Chamblin, R. Emparan, C. Johnson, and R. Myers, *Charged AdS black holes and catastrophic holography*, *Phys.Rev.* **D60** (1999) 064018, [[hep-th/9902170](#)].
- [4] A. Chamblin, R. Emparan, C. Johnson, and R. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys.Rev.* **D60** (1999) 104026, [[hep-th/9904197](#)].
- [5] M. Cvetič and S. Gubser, *Phases of R charged black holes, spinning branes and strongly coupled gauge theories*, *JHEP* **9904** (1999) 024, [[hep-th/9902195](#)].
- [6] C. V. Johnson, *Large N Phase Transitions, Finite Volume, and Entanglement Entropy*, [arXiv:1306.4955](#).
- [7] M. M. Caldarelli, G. Cognola, and D. Klemm, *Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories*, *Class.Quant.Grav.* **17** (2000) 399–420, [[hep-th/9908022](#)].
- [8] Y.-D. Tsai, X. Wu, and Y. Yang, *Phase Structure of Kerr-AdS Black Hole*, *Phys.Rev.* **D85** (2012) 044005, [[arXiv:1104.0502](#)].
- [9] D. Kubiznak and R. B. Mann, *P-V criticality of charged AdS black holes*, *JHEP* **1207** (2012) 033, [[arXiv:1205.0559](#)].
- [10] D. Kastor, S. Ray, and J. Traschen, *Enthalpy and the Mechanics of AdS Black Holes*, *Class.Quant.Grav.* **26** (2009) 195011, [[arXiv:0904.2765](#)].
- [11] T. Narayanan and A. Kumar, *Reentrant phase transitions in multicomponent liquid mixtures*, *Physics Reports* **249** (1994) 135–218.
- [12] V. P. Maslov, *Zeroth-order phase transitions*, *Mathematical Notes* **76** (2004), no. 5-6 697–710.
- [13] S. Gunasekaran, D. Kubiznak, and R. Mann, *Extended phase space thermodynamics for charged and rotating black holes and Born-Infeld vacuum polarization*, *JHEP* **1211** (2012) 110, [[arXiv:1208.6251](#)].
- [14] B. Carter, *Hamilton-Jacobi and Schrödinger separable solutions of Einstein's equations*, *Commun. Math. Phys.* **10** (1968) 280–310.
- [15] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, *Rotation and the AdS/CFT correspondence*, *Phys. Rev.* **D59** (1999) 064005, [[hep-th/9811056](#)].
- [16] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, *Rotating black holes in higher dimensions with a cosmological constant*, *Phys. Rev. Lett.* **93** (2004) 171102, [[hep-th/0409155](#)].
- [17] G. W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, *The general Kerr-de Sitter metrics in all dimensions*, *J. Geom. Phys.* **53** (2005) 49–73, [[hep-th/0404008](#)].
- [18] M. Cvetič, G. Gibbons, D. Kubiznak, and C. Pope, *Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume*, *Phys.Rev.* **D84** (2011) 024037, [[arXiv:1012.2888](#)].
- [19] B. P. Dolan, *Where is the PdV term in the first law of black hole thermodynamics?*, [arXiv:1209.1272](#).
- [20] R. Monteiro, M. J. Perry, and J. E. Santos, *Thermodynamic instability of rotating black holes*, *Phys.Rev.* **D80** (2009) 024041, [[arXiv:0903.3256](#)].
- [21] C. Hudson *Z. Phys. Chem.* **47** (1904) 113.
- [22] R. Emparan and R. C. Myers, *Instability of ultra-spinning black holes*, *JHEP* **0309** (2003) 025, [[hep-th/0308056](#)].
- [23] M. Caldarelli, R. Emparan, and M. J. Rodriguez, *Black Rings in (Anti)-deSitter space*, *JHEP* **0811** (2008) 011, [[arXiv:0806.1954](#)].
- [24] R. Gregory and R. Laflamme, *Black strings and p-branes are unstable*, *Phys.Rev.Lett.* **70** (1993) 2837–2840, [[hep-th/9301052](#)].
- [25] N. Altamirano, D., D. Kubiznak, and R. Mann, *Thermodynamic volume, criticality, and the ultraspinning limit of higher-dimensional black holes, in preparation.*
- [26] O. Dias, P. Figueras, R. Monteiro, and J. Santos, *Ultraspinning instability of anti-de Sitter black holes*, *JHEP* **1012** (2010) 067, [[1011.0996](#)].
- [27] G. S. Hartnett and J. E. Santos, *Non-Axisymmetric Instability of Rotating Black Holes in Higher Dimensions*, [arXiv:1306.4318](#).
- [28] J. Creighton and R. B. Mann, *Quasilocal thermodynamics of dilaton gravity coupled to gauge fields*, *Phys.Rev.* **D52** (1995) 4569–4587, [[gr-qc/9505007](#)].