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Multiplicity formulas for fundamental strings of representations of classical Lie algebras

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We call the *p-fundamental string* of a complex simple Lie algebra to the sequence of irreducible representations having highest weights of the form $k\omega_1 + \omega_p$ for $k \geq 0$, where ω_j denotes the jth fundamental weight of the associated root system. For a classical complex Lie algebra, we establish a closed explicit formula for the weight multiplicities of any representation in any p-fundamental string. *Published by AIP Publishing.* https://doi.org/10.1063/1.4993851

I. INTRODUCTION

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let (π, V_π) be a finite dimensional representation of $\mathfrak{g}$, that is, a homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V_\pi)$ with V_π a complex vector space. An element $\mu \in \mathfrak{h}^*$ is called a *weight* of π if

$$V_\pi(\mu) := \{v \in V_\pi : \pi(X)v = \mu(X)v \text{ for all } X \in \mathfrak{h}\} \neq 0.$$

The *multiplicity* of μ in the representation π, denoted by $m_\pi(\mu)$, is defined as $\dim V_\pi(\mu)$.

There are many formulas in the literature to compute $m_\pi(\mu)$ for arbitrary $\mathfrak{g}$, π, and μ. The ones by Freudenthal[12] and Kostant[19] are very classical. More recent formulas were given by Lusztig,[25] Littelmann,[24] and Sahi.[28] Although all of them are very elegant and powerful theoretical results, they may not be considered *closed explicit expressions*. Moreover, some of them are not adequate for computer implementation (cf. in Refs. 14 and 29).

Actually, it is not expected a closed formula in general. There should always be a sum over a symmetric group (whose cardinal grows quickly when the rank of $\mathfrak{g}$ does) or over partitions, or being recursive, or written in terms of combinatorial objects (e.g., Young diagrams like in Ref. 17), among other ways.

However, closed explicit expressions are possible for particular choices of $\mathfrak{g}$ and π. Obviously, this is the case for $\mathfrak{sl}(2,\mathbb{C})$ and π any of its irreducible representations (see Sec. I.9 in Ref. 16). Furthermore, for a classical Lie algebra $\mathfrak{g}$, it is not difficult to give expressions for the weight multiplicities of the representations $\mathrm{Sym}^k(V_{\mathrm{st}})$ and $\bigwedge^p(V_{\mathrm{st}})$ and also for their irreducible components (see, for instance, Lemmas III.2, IV.3, and V.3 and Theorem VI.1; these formulas are probably well known but they are included here for completeness). Here, V_{st} denotes the standard representation of $\mathfrak{g}$. A good example of a closed explicit formula in a non-trivial case was given by Cagliero and Tirao[5] for $\mathfrak{sp}(2,\mathbb{C}) \simeq \mathfrak{so}(5,\mathbb{C})$ and π arbitrary.

In order to end the description of previous results in this large area, we name a few recent related results, though the list is far from being complete: Refs. 1, 2, 6–11, 26, and 30.

The main goal of this article is to show, for each classical complex Lie algebra $\mathfrak{g}$ of rank n, a closed explicit formula for the weight multiplicities of any irreducible representation of $\mathfrak{g}$ having highest weight $k\omega_1 + \omega_p$, for any integers $k \geq 0$ and $1 \leq p \leq n$. Here, $\omega_1, \ldots, \omega_n$ denote the fundamental

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weights associated with the root system $\Sigma(\mathfrak{g},\mathfrak{h})$. We call the *p-fundamental string* to the sequence of irreducible representations of $\mathfrak{g}$ with highest weights $k\omega_1 + \omega_p$ for $k \geq 0$. We will write π_λ for the irreducible representation of $\mathfrak{g}$ with highest weight λ.

For types B_n, C_n, or D_n [i.e., $\mathfrak{so}(2n+1,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, or $\mathfrak{so}(2n,\mathbb{C})$, respectively] an accessory representation $\pi_{k,p}$ is introduced to unify the approach (see Definition II.2). We have that $\pi_{k,p}$ and $\pi_{k\omega_1+\omega_p}$ coincide except for $p = n$ in type B_n and $p = n-1, n$ in type D_n. The weight multiplicity formulas for $\pi_{k,p}$ are in Theorems III.1, IV.1, and V.1 for types C_n, D_n, and B_n, respectively. Their proofs follow the same strategy (see Sec. II). The formulas for the remaining cases, namely, the (spin) representations $\pi_{k\omega_1+\omega_n}$ in type B_n and $\pi_{k\omega_1+\omega_{n-1}}$, $\pi_{k\omega_1+\omega_n}$ in type D_n can be found in Theorems IV.2 and V.4, respectively.

Given a weight $\mu = \sum_{j=1}^n a_j\varepsilon_j$ (see Notation II.1) of a classical Lie algebra $\mathfrak{g}$ of types B_n, C_n, or D_n, we set

$$\|\mu\|_1 = \sum_{j=1}^n |a_j| \quad \text{and} \quad Z(\mu) = \#\{1 \leq j \leq n : a_j = 0\}. \tag{1}$$

We call $\|\mu\|_1$ the *one-norm* of μ. The function $Z(\mu)$ counts the number of zero coordinates of μ. It is not difficult to check that $m_{\pi_{k\omega_1}}(\mu)$ depends only on $\|\mu\|_1$ for a fixed $k \geq 0$. Moreover, it is known that $m_{\pi_{k,p}}(\mu)$ depends only on $\|\mu\|_1$ and $Z(\mu)$ for type D_n (see Lemma 3.3 in Ref. 23). This last property is extended to types B_n and C_n as a consequence of their multiplicity formulas.

Corollary I.1. For $\mathfrak{g}$, a classical Lie algebra of types B_n, C_n, or D_n and a weight $\mu = \sum_{i=1}^n a_i\varepsilon_i$, the multiplicity of μ in $\pi_{k,p}$ depends only on $\|\mu\|_1$ and $Z(\mu)$.

For $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ (type A_n), the multiplicity formula for a representation in a fundamental string is in Theorem VI.1. This case is simpler since it follows immediately from basic facts on Young diagrams. Although this formula should be well known, it is included for completeness.

Explicit expressions for the weight multiplicities of a representation in a fundamental string are required in several different areas. The interest of the authors on them comes from their application to spectral geometry. Actually, many multiplicity formulas have already been applied to determine the spectrum of Laplace and Dirac operators on certain locally homogeneous spaces. See Sec. VII for a detailed account of these applications.

It is important to note that all the weight multiplicity formulas obtained in this article have been checked with Sage[31] for many cases. This computer program uses the classical Freudenthal formula. Because of the simplicity of the expressions obtained in the main theorems, the computer takes usually a fraction of a second to calculate the result.

Throughout the article, we use the convention $\binom{b}{a} = 0$ if $a < 0$ or $b < a$.

The article is organized as follows. Section II explains the method to obtain $m_{\pi_{k,p}}(\mu)$ for types B_n, C_n, and D_n. These cases are considered in Secs. V, III, and IV, respectively, and type A_n is in Sec. VI. In Sec. VII, we include some conclusions.

II. STRATEGY

In this section, we introduce the abstract method used to find the weight multiplicity formulas for the cases B_n, C_n, and D_n. Throughout this section, $\mathfrak{g}$ denotes a classical complex Lie algebra of types B_n, C_n, and D_n, namely $\mathfrak{so}(2n+1,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{so}(2n,\mathbb{C})$, for some $n \geq 2$. We first introduce some standard notation.

Notation II.1. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathfrak{h}^*$. Thus, the sets of simple roots $\Pi(\mathfrak{g},\mathfrak{h})$ are given by $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$ for type B_n, $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$ for type C_n, and $\{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$ for type D_n. A precise choice for $\mathfrak{h}$ and ε_j will be indicated in each type.

We denote by $\Sigma(\mathfrak{g},\mathfrak{h})$ the set of roots, by $\Sigma^+(\mathfrak{g},\mathfrak{h})$ the set of positive roots, by $\omega_1, \ldots, \omega_n$ the fundamental weights, by $P(\mathfrak{g})$ the (integral) weight space of $\mathfrak{g}$, and by $P^{++}(\mathfrak{g})$ the set of dominant weights.

Let $\mathfrak{g}_0$ be the compact real form of $\mathfrak{g}$ associated with $\Sigma(\mathfrak{g}, \mathfrak{h})$, let G be the compact linear group with Lie algebra $\mathfrak{g}_0$ [e.g., $G = \mathrm{SO}(2n)$ for type D_n in place of spin$(2n)$], and let T be the maximal torus in G corresponding to $\mathfrak{h}$, that is, the Lie algebra $\mathfrak{t}$ of T is a real subalgebra of $\mathfrak{h}$. Write $P(G)$ for the set of G-integral weights and $P^{++}(G) = P(G) \cap P^{++}(\mathfrak{g})$.

By the highest weight theorem, the irreducible representations of $\mathfrak{g}$ and G are in correspondence with elements in $P^{++}(\mathfrak{g})$ and $P^{++}(G)$, respectively. For λ, an integral dominant weight, we denote by π_λ the associated irreducible representation of $\mathfrak{g}$.

We recall that, under Notation II.1, the fundamental weights are

$$\text{in type } \mathrm{B}_n, \qquad \omega_p = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_p & \text{if } 1 \leq p \leq n-1, \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n) & \text{if } p = n, \end{cases}$$

$$\text{in type } \mathrm{C}_n, \qquad \omega_p = \varepsilon_1 + \cdots + \varepsilon_p \quad \text{for every } 1 \leq p \leq n,$$

$$\text{in type } \mathrm{D}_n, \qquad \omega_p = \begin{cases} \varepsilon_1 + \cdots + \varepsilon_p & \text{if } 1 \leq p \leq n-2, \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n) & \text{if } p = n-1, \\ \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n) & \text{if } p = n. \end{cases}$$

We set $\widetilde{\omega}_p = \varepsilon_1 + \cdots + \varepsilon_p$ for any $1 \leq p \leq n$. Thus, $\widetilde{\omega}_p = \omega_p$ excepts for type B_n and $p = n$ when $\widetilde{\omega}_n = 2\omega_n$, and for type D_n and $p \in \{n-1, n\}$ when $\widetilde{\omega}_{n-1} = \omega_{n-1} + \omega_n$ and $\widetilde{\omega}_n = 2\omega_n$.

Definition II.2. Let $\mathfrak{g}$ be a classical Lie algebra of types B_n, C_n, or D_n. For $k \geq 0$ and $1 \leq p \leq n$ integers, let us denote by $\pi_{k,p}$ the irreducible representation of $\mathfrak{g}$ with highest weight $k\omega_1 + \widetilde{\omega}_p$, except for $p = n$ and type D_n when we set $\pi_{k,n} = \pi_{k\omega_1 + 2\omega_{n-1}} \oplus \pi_{k\omega_1 + 2\omega_n}$. By convention, we set $\pi_{k,0} = 0$ for $k \geq 0$.

Next we explain the procedure to determine the multiplicity formula for $\pi_{k,p}$.

Step 1: Obtain the decomposition in irreducible representations of

$$\sigma_{k,p} := \pi_{k\omega_1} \otimes \pi_{\widetilde{\omega}_p}, \tag{2}$$

and consequently, write $\pi_{k,p}$ in terms of representations of the form (2) in the virtual representation ring. Fortunately, this decomposition is already known and coincides for the types B_n, C_n, and D_n; thus, the second requirement has also a uniform statement (see Lemma II.3).

Step 2: Obtain a formula for the weight multiplicities of the extreme cases $\pi_{k\omega_1}$ and $\pi_{\widetilde{\omega}_p}$. It will be useful to realize these representations inside $\mathrm{Sym}^k(V_{\pi_{\omega_1}})$ and $\bigwedge^p(V_{\pi_{\omega_1}})$, respectively. Note that π_{ω_1} is the standard representation.

Step 3: Obtain a closed expression for the weight multiplicities on $\sigma_{k,p}$. This is the hardest step. One has that (see, for instance, Exercise V.14 in Ref. 16)

$$m_{\sigma_{k,p}}(\mu) = \sum_{\eta} m_{\pi_{k\omega_1}}(\mu - \eta)\, m_{\pi_{\widetilde{\omega}_p}}(\eta), \tag{3}$$

where the sum is over the weights of $\pi_{\widetilde{\omega}_p}$. Then, the multiplicity formulas obtained in step 2 can be applied.

Step 4: Obtain the weight multiplicity formula for $\pi_{k,p}$. We will replace the formula obtained in step 3 into the formula obtained in step 1.

The following result works out step 1.

Lemma II.3. Let $\mathfrak{g}$ be a classical Lie algebra of types B_n, C_n, or D_n *and let $k \geq 0$, $1 \leq p \leq n$ integers. Then*

$$\sigma_{k,p} = \pi_{k\omega_1} \otimes \pi_{\widetilde{\omega}_p} = \pi_{k-1,1} \otimes \pi_{0,p} \simeq \pi_{k,p} \oplus \pi_{k-1,p+1} \oplus \pi_{k-2,p} \oplus \pi_{k-1,p-1}. \tag{4}$$

Furthermore, in the virtual ring of representations, we have that

$$\pi_{k,p} = \sum_{j=1}^{p} (-1)^{j-1} \sum_{i=0}^{j-1} \sigma_{k+j-2i,p-j}. \tag{5}$$

Proof. The decomposition (4) is proved by Koike and Terada[18] [see Example (3) in p. 510], though their results are much more general and this particular case was probably already known.

We now show (5). The case $p = 1$ is trivial. Indeed, the right hand side equals $\sigma_{k+1,0} = \pi_{k,1}$ by definition. We assume that the formula is valid for values lower than or equal to p. By this assumption and (4), we have that

$$\begin{aligned}\pi_{k,p+1} &= \sigma_{k+1,p} - \pi_{k+1,p} - \pi_{k-1,p} - \pi_{k,p-1}\\ &= \sigma_{k+1,p} - \sum_{j=1}^{p}(-1)^{j-1}\sum_{i=0}^{j-1}\sigma_{k+1+j-2i,p-j} - \sum_{j=1}^{p}(-1)^{j-1}\sum_{i=0}^{j-1}\sigma_{k-1+j-2i,p-j} - \sum_{j=1}^{p-1}(-1)^{j-1}\sum_{i=0}^{j-1}\sigma_{k+j-2i,p-1-j}.\end{aligned}$$

By making the change of variables $h = j + 1$ in the last term, one gets

$$\pi_{k,p+1} = \sigma_{k+1,p} - \sum_{j=1}^{p}(-1)^{j-1}\sum_{i=0}^{j-1}\sigma_{k+1+j-2i,p-j} - \sigma_{k,p-1} - \sum_{j=2}^{p}(-1)^{j-1}\sigma_{k+1-j,p-j}.$$

The rest of the proof is straightforward. □

III. TYPE C

In this section, we consider the classical Lie algebra $\mathfrak{g}$ of type C_n, that is, $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$. In this case, according to Notation II.1, $\widetilde{\omega}_p = \omega_p$ for every p, thus $\pi_{k\omega_1+\omega_p} = \pi_{k,p}$. The next theorem gives the explicit expression of $m_{\pi_{k,p}}(\mu)$ for any weight μ. This expression depends on the terms $\|\mu\|_1$ and $Z(\mu)$, introduced in (1).

Theorem III.1. *Let $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$ for some $n \geq 2$ and let $k \geq 0$, $1 \leq p \leq n$ integers. For $\mu \in P(\mathfrak{g})$, if $r(\mu) := (k + p - \|\mu\|_1)/2$ is a non-negative integer, then*

$$\begin{aligned}m_{\pi_{k,p}}(\mu) = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j}{2}\rfloor}\frac{n-p+j+1}{n-p+j+t+1}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t}2^{p-j-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-j-2t-\beta}\\ \sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1}\binom{r(\mu)-i-p+\alpha+t+j+n-1}{n-1},\end{aligned}$$

and $m_{\pi_{k,p}}(\mu) = 0$ otherwise.

The rest of this section is devoted to prove this formula following the procedure described in Sec. II. We first set the notation for this case. Here $G = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n)$ where $\mathrm{Sp}(n,\mathbb{C}) = \{g \in \mathrm{SL}(2n,\mathbb{C}) : g^tJ_ng = J_n := \left(\begin{smallmatrix}0 & \mathrm{Id}_n\\ -\mathrm{Id}_n & 0\end{smallmatrix}\right)\}$, $\mathfrak{g}_0 = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(2n)$,

$$T = \left\{\mathrm{diag}\left(e^{\mathrm{i}\theta_1},\dots,e^{\mathrm{i}\theta_n},e^{-\mathrm{i}\theta_1},\dots,e^{-\mathrm{i}\theta_n}\right) : \theta_i \in \mathbb{R}\ \forall\, i\right\}, \tag{6}$$

$$\mathfrak{h} = \{\mathrm{diag}(\theta_1,\dots,\theta_n,-\theta_1,\dots,-\theta_n) : \theta_i \in \mathbb{C}\ \forall\, i\}, \tag{7}$$

$\varepsilon_i(\mathrm{diag}(\theta_1,\dots,\theta_n,-\theta_1,\dots,-\theta_n)) = \theta_i$ for each $1 \leq i \leq n$, $\Sigma^+(\mathfrak{g},\mathfrak{h}) = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{2\varepsilon_i : 1 \leq i \leq n\}$, and

$$\begin{aligned}P(\mathfrak{g}) &= P(G) = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n,\\ P^{++}(\mathfrak{g}) &= P^{++}(G) = \{\textstyle\sum_i a_i\varepsilon_i \in P(\mathfrak{g}) : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0\}.\end{aligned}$$

The following well-known identities (see, for instance, Sec. 17.2 in Ref. 13) will be useful to show step 2,

$$\pi_{k\omega_1} = \pi_{k\varepsilon_1} \simeq \mathrm{Sym}^k(\mathbb{C}^{2n}), \qquad \textstyle\bigwedge^p(\mathbb{C}^{2n}) \simeq \pi_{\omega_p} \oplus \bigwedge^{p-2}(\mathbb{C}^{2n}), \tag{8}$$

for any integers $k \geq 0$ and $1 \leq p \leq n$. Here, $\mathbb{C}^{2n}$ denotes the standard representation of $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$. Since $G = \mathrm{Sp}(n)$ is simply connected, π_λ descends to a representation of G for any $\lambda \in P^{++}(\mathfrak{g})$. In what follows we will work with representations of G for simplicity. Thus, $m_\pi(\mu) = \dim\{v \in V_\pi : \pi(\exp X)v = e^{\mu(X)}v \quad \forall\, X \in \mathfrak{t}\}$.

Lemma III.2. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, $k \geq 0$, $1 \leq p \leq n$, and $\mu = \sum_{j=1}^n a_j \varepsilon_j \in P(\mathfrak{g})$. Then,

$$m_{\pi_{k\omega_1}}(\mu) = m_{\pi_{k\varepsilon_1}}(\mu) = \begin{cases} \binom{r(\mu)+n-1}{n-1} & \textit{if } \ r(\mu) := \frac{k-\|\mu\|_1}{2} \in \mathbb{N}_0, \\ 0 & \textit{otherwise,} \end{cases} \tag{9}$$

$$m_{\pi_{\omega_p}}(\mu) = \begin{cases} \frac{n-p+1}{n-p+r(\mu)+1} \binom{n-p+2r(\mu)}{r(\mu)} & \textit{if } \ r(\mu) := \frac{p-\|\mu\|_1}{2} \in \mathbb{N}_0 \textit{ and } |a_j| \leq 1 \, \forall j, \\ 0 & \textit{otherwise.} \end{cases} \tag{10}$$

Proof. By (8), $\pi_{k\varepsilon_1}$ is realized in the space of homogeneous polynomials $\mathcal{P}_k \simeq \mathrm{Sym}^k(\mathbb{C}^{2n})$ of degree k in the variables $x_1, \ldots, x_{2n}$. The action of $g \in G$ on $f(x) \in \mathcal{P}_k$ is given by $(\pi_{k\varepsilon_1}(g) \cdot f)(x) = f(g^{-1}x)$, where x denotes the column vector $(x_1, \ldots, x_{2n})^t$.

The monomials $x_1^{k_1} \ldots x_n^{k_n} x_{n+1}^{l_1} \ldots x_{2n}^{l_n}$ with $k_1, \ldots, k_n, l_1, \ldots, l_n$ non-negative integers satisfying that $\sum_{j=1}^n k_j + l_j = k$ form a basis of $\mathcal{P}_k$ given by weight vectors. Indeed, one can check that the action of the element $h = \mathrm{diag}\left(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n}, e^{-\mathrm{i}\theta_1}, \ldots, e^{-\mathrm{i}\theta_n}\right) \in T$ on the monomial $x_1^{k_1} \ldots x_n^{k_n} x_{n+1}^{l_1} \ldots x_{2n}^{l_n}$ is given by the multiplication by $e^{\mathrm{i} \sum_{j=1}^n \theta_j (k_j - l_j)}$. Hence, the polynomial $x_1^{k_1} \ldots x_n^{k_n} x_{n+1}^{l_1} \ldots x_{2n}^{l_n}$ is a weight vector of weight $\mu = \sum_{j=1}^n (k_j - l_j)\varepsilon_j$.

Consequently, the multiplicity of a weight $\mu = \sum_{j=1}^n a_j \varepsilon_j \in \mathcal{P}(\mathfrak{g})$ in $\mathcal{P}_k$ is the number of different tuples $(k_1, \ldots, k_n, l_1, \ldots, l_n) \in \mathbb{N}_0^{2n}$ satisfying $\sum_{j=1}^n (k_j + l_j) = k$ and $a_j = k_j - l_j$ for all j. For such a tuple, we note that $k - \| \mu \|_1 = k - \sum_{i=1}^n |a_i| = 2 \sum_{i=1}^n \min(k_i, l_i)$. It follows that μ is a weight of $\mathcal{P}_k$ if and only if $k - \|\mu\|_1 = 2r$ with r a non-negative integer. Moreover, its multiplicity is the number of different ways one can write r as an ordered sum of n non-negative integers, which equals $\binom{r+n-1}{n-1}$. This implies (9).

For (10), we consider the representation $\bigwedge^p(\mathbb{C}^{2n})$. The action of G on $\bigwedge^p(\mathbb{C}^{2n})$ is given by $g \cdot v_1 \wedge \cdots \wedge v_p = (gv_1) \wedge \cdots \wedge (gv_p)$, where gv stands for the matrix multiplication between $g \in G \subset \mathrm{GL}(2n, \mathbb{C})$ and the column vector $v \in \mathbb{C}^{2n}$.

Let $\{e_1, \ldots, e_{2n}\}$ denote the canonical basis of $\mathbb{C}^{2n}$. For $I = \{i_1, \ldots, i_p\}$ with $1 \leq i_1 < \cdots < i_p \leq 2n$, we write $w_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$. Clearly, the set of w_I for all choices of I is a basis of $\bigwedge^p(\mathbb{C}^{2n})$. Since $h = \mathrm{diag}\left(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n}, e^{-\mathrm{i}\theta_1}, \ldots, e^{-\mathrm{i}\theta_n}\right) \in T$ satisfies $he_j = e^{\mathrm{i}\theta_j} e_j$ and $he_{j+n} = e^{-\mathrm{i}\theta_j} e_{j+n}$ for all $1 \leq j \leq n$, we see that w_I is a weight vector of weight $\mu = \sum_{j=1}^n a_j \varepsilon_j$ where

$$a_j = \begin{cases} 1 & \text{if } j \in I \text{ and } j+n \notin I, \\ -1 & \text{if } j \notin I \text{ and } j+n \in I, \\ 0 & \text{if } j, j+n \in I \text{ or } j, j+n \notin I. \end{cases} \tag{11}$$

Thus, an arbitrary element $\mu = \sum_j a_j \varepsilon_j \in P(\mathfrak{g})$ is a weight of $\bigwedge^p(\mathbb{C}^{2n})$ if and only if $|a_j| \leq 1$ for all j and $p - \|\mu\|_1 = 2r$ for some non-negative integer r.

It remains to determine the multiplicity in $\bigwedge^p(\mathbb{C}^{2n})$ of a weight $\mu = \sum_{j=1}^n a_j \varepsilon_j \in P(\mathfrak{g})$ satisfying $|a_j| \leq 1$ for all j and $r := \frac{p-\|\mu\|_1}{2} \in \mathbb{N}_0$. Let $I_\mu = \{i\colon 1 \leq i \leq n, a_i = 1\} \cup \{i\colon n+1 \leq i \leq 2n, a_{i-n} = -1\}$. The set I_μ has $p - 2r$ elements. For $I = \{i_1, \ldots, i_p\}$ with $1 \leq i_1 < \cdots < i_p \leq 2n$, it is a simple matter to check that w_I is a weight vector with weight μ if and only if I has p elements, $I_\mu \subset I$ and I has the property that $j \in I \setminus I_\mu \Leftrightarrow j+n \in I \setminus I\mu$ for $1 \leq j \leq n$. One can see that there are $\binom{n-p+2r}{r}$ choices for I. Hence $m_{\bigwedge^p(\mathbb{C}^{2n})}(\mu) = \binom{n-p+2r}{r}$. From (8), we conclude that $m_{\pi_{\omega_p}}(\mu) = m_{\bigwedge^p(\mathbb{C}^{2n})}(\mu) - m_{\bigwedge^{p-2}(\mathbb{C}^{2n})}(\mu) = \binom{n-p+2r}{r} - \binom{n-p+2+2r}{r}$ and (10) is proved. □

We next consider step 3, namely, a multiplicity formula for $\sigma_{k,p}$.

Lemma III.3. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, $k \geq 0$, $1 \leq p < n$, and $\mu \in P(\mathfrak{g})$. If $r(\mu) := (k + p - \|\mu\|_1)/2$ is a non-negative integer, then

$$m_{\sigma_{k,p}}(\mu) = \sum_{t=0}^{\lfloor p/2 \rfloor} \frac{n-p+1}{n-p+t+1}\binom{n-p+2t}{t} \sum_{\beta=0}^{p-2t} 2^{p-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-2t-\beta}$$
$$\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{r(\mu)-p+\alpha+t+n-1}{n-1},$$

and $m_{\sigma_{k,p}}(\mu) = 0$ *otherwise.*

Proof. Write $r = r(\mu)$ and $\ell = Z(\mu)$. We may assume that μ is dominant, thus $\mu = \sum_{j=1}^{n-\ell} a_j\varepsilon_j$ with $a_1 \geq \cdots \geq a_{n-\ell} > 0$ since it has ℓ zero-coordinates. In order to use (3), by Lemma III.2, we write the set of weights of π_{ω_p} as

$$\mathcal{P}(\pi_{\omega_p}) := \bigcup_{t=0}^{\lfloor p/2 \rfloor} \bigcup_{\beta=0}^{p-2t} \bigcup_{\alpha=0}^{\beta} \mathcal{P}^{(p)}_{t,\beta,\alpha},$$

where

$$\mathcal{P}^{(p)}_{t,\beta,\alpha} = \left\{ \sum_{h=1}^{p-2t} b_h \varepsilon_{i_h} : \begin{array}{l} i_1 < \ldots < i_\beta \leq n-\ell < i_{\beta+1} < \ldots < i_{p-2t} \\ b_j = \pm 1 \quad \forall j, \quad \#\{1 \leq j \leq \beta : b_j = 1\} = \alpha \end{array} \right\}. \tag{12}$$

A weight $\eta \in \mathcal{P}^{(p)}_{t,\beta,\alpha}$ has all entries in $\{0, \pm 1\}$ and satisfies $\|\eta\|_1 = p - 2t$, thus $m_{\pi_{\omega_p}}(\eta) = \frac{n-p+1}{n-p+t+1}\binom{n-p+2t}{t}$ by (10). It is a simple matter to check that

$$\#\mathcal{P}^{(p)}_{t,\beta,\alpha} = 2^{p-2t-\beta}\binom{n-\ell}{\beta}\binom{\beta}{\alpha}\binom{\ell}{p-2t-\beta}. \tag{13}$$

From (3), since the triple union above is disjoint, we obtain that

$$m_{\sigma_{k,p}}(\mu) = \sum_{t=0}^{\lfloor p/2 \rfloor} \sum_{\beta=0}^{p-2t} \sum_{\alpha=0}^{\beta} \sum_{\eta \in \mathcal{P}^{(p)}_{t,\beta,\alpha}} m_{\pi_{k\varepsilon_1}}(\mu-\eta)\, m_{\pi_{\omega_p}}(\eta).$$

One has that $\|\mu-\eta\|_1 = (k+p-2r)+(\beta-\alpha)-\alpha+(p-2t-\beta) = k-2(r+t+\alpha-p)$ for every $\eta \in \mathcal{P}^{(p)}_{t,\beta,\alpha}$. If $r \notin \mathbb{N}_0$, (9) forces $m_{\pi_{k\varepsilon_1}}(\mu-\eta) = 0$ for all $\eta \in \mathcal{P}^{(p)}_{t,\beta,\alpha}$, consequently $m_{\sigma_{k,p}}(\mu) = 0$. Otherwise,

$$m_{\sigma_{k,p}}(\mu) = \sum_{t=0}^{\lfloor p/2 \rfloor} \sum_{\beta=0}^{p-2t} \sum_{\alpha=0}^{\beta} \binom{r+t+\alpha-p+n-1}{n-1} \frac{n-p+1}{n-p+t+1}\binom{n-p+2t}{t} \#\mathcal{P}^{(p)}_{t,\beta,\alpha}$$

by Lemma III.2. The proof is complete by (13). □

Theorem III.1 follows by replacing the multiplicity formula given in Lemma III.3 into (5).

IV. TYPE D

We now consider type D_n, that is, $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$ and $G = \mathrm{SO}(2n)$. We assume that $n \geq 2$, so the non-simple case $\mathfrak{g} = \mathfrak{so}(4,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ is also considered.

Since G is not simply connected and has a fundamental group of order 2, the lattice of G-integral weights $P(G)$ is strictly included with index 2 in the weight space $P(\mathfrak{g})$. Consequently, a dominant weight λ in $P(\mathfrak{g}) \smallsetminus P(G)$ corresponds to a representation π_λ of spin$(2n)$, which does not descend to a representation of $G = \mathrm{SO}(2n)$.

In this case, for all $k \geq 0$ and $1 \leq p \leq n-2$, we have that

$$\pi_{k,p} = \pi_{k\omega_1+\omega_p}, \qquad \pi_{k,n-1} = \pi_{k\omega_1+\omega_{n-1}+\omega_n}, \qquad \pi_{k,n} = \pi_{k\omega_1+2\omega_{n-1}} \oplus \pi_{k\omega_1+2\omega_n}. \tag{14}$$

Each of them descends to a representation of G and its multiplicity formula is established in Theorem IV.1. The remaining cases $\pi_{k\omega_1+\omega_{n-1}}$ and $\pi_{k\omega_1+\omega_n}$ are spin representations. Their multiplicity formulas were obtained in Ref. 4 (Lemma 4.2) and are stated in Theorem IV.2.

Theorem IV.1. *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $G = \mathrm{SO}(2n)$ for some $n \geq 2$ and let $k \geq 0$, $1 \leq p \leq n$ integers. For $\mu \in P(G)$, if $r(\mu) := (k + p - \|\mu\|_1)/2$ is a non-negative integer, then*

$$m_{\pi_{k,p}}(\mu) = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t}2^{p-j-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-j-2t-\beta}$$
$$\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1}\binom{r(\mu)-i-p+\alpha+t+j+n-2}{n-2},$$

and $m_{\pi_{k,p}}(\mu) = 0$ otherwise. Furthermore, $m_{\pi_{k,p}}(\mu) = 0$ for every $\mu \in P(\mathfrak{g}) \smallsetminus P(G)$.

Theorem IV.2. *Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ and $G = \mathrm{SO}(2n)$ for some $n \geq 2$ and let $k \geq 0$ an integer. Let $\mu \in P(\mathfrak{g}) \smallsetminus P(G)$. Write $r(\mu) = k + \frac{n}{2} - \| \mu \|_1$, then*

$$m_{\pi_{k\omega_1+\omega_n}}(\mu) = \begin{cases}\binom{r(\mu)+n-2}{n-2} & \text{if } r(\mu) \geq 0 \text{ and } \operatorname{neg}(\mu) \equiv r(\mu) \pmod 2,\\ 0 & \text{otherwise,}\end{cases}$$
$$m_{\pi_{k\omega_1+\omega_{n-1}}}(\mu) = \begin{cases}\binom{r(\mu)+n-2}{n-2} & \text{if } r(\mu) \geq 0 \text{ and } \operatorname{neg}(\mu) \equiv r(\mu)+1 \pmod 2,\\ 0 & \text{otherwise,}\end{cases}$$

where $\operatorname{neg}(\mu)$ stands for the number of negative entries of μ. Furthermore, $m_{\pi_{k\omega_1+\omega_{n-1}}}(\mu) = m_{\pi_{k\omega_1+\omega_n}}(\mu)$ $= 0$ for every $\mu \in P(G)$.

The proof of Theorem IV.1 will follow the steps from Sec. II. Let us first set the elements introduced in Notation II.1. Define $\mathfrak{h} = \left\{\operatorname{diag}\left(\left[\begin{smallmatrix}0&\theta_1\\-\theta_1&0\end{smallmatrix}\right],\dots,\left[\begin{smallmatrix}0&\theta_n\\-\theta_n&0\end{smallmatrix}\right]\right) : \theta_i \in \mathbb{C}\,\forall i\right\}$ and $\varepsilon_i\left(\operatorname{diag}\left(\left[\begin{smallmatrix}0&\theta_1\\-\theta_1&0\end{smallmatrix}\right],\dots,\left[\begin{smallmatrix}0&\theta_n\\-\theta_n&0\end{smallmatrix}\right]\right)\right) = \theta_i$ for each $1 \leq i \leq n$. Thus $\Sigma^+(\mathfrak{g}, \mathfrak{h}) = \{\varepsilon_i \pm \varepsilon_j : i<j\}$,

$$P(\mathfrak{g}) = \{\textstyle\sum_i a_i\varepsilon_i : a_i \in \mathbb{Z}\,\forall i, \text{ or } a_i - 1/2 \in \mathbb{Z}\,\forall i\}, \qquad P(G) = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n,$$
$$P^{++}(\mathfrak{g}) = \{\textstyle\sum_i a_i\varepsilon_i \in P(\mathfrak{g}) : a_1 \geq \cdots \geq a_{n-1} \geq |a_n|\}, \qquad P^{++}(G) = P^{++}(\mathfrak{g}) \cap P(G).$$

It is now clear that $P(G)$ has index 2 in $P(\mathfrak{g})$.

The multiplicity formulas in type D_n for the extreme representations in step 2 are already determined. A proof can be found in Ref. 23 (Lemma 3.2).

Lemma IV.3. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $G = \mathrm{SO}(2n)$, $k \geq 0$, and $1 \leq p \leq n$. For $\mu = \sum_{j=1}^n a_j\varepsilon_j \in P(G)$, we have that

$$m_{\pi_{k\omega_1}}(\mu) = m_{\pi_{k\varepsilon_1}}(\mu) = \begin{cases}\binom{r(\mu)+n-2}{n-2} & \text{if } r(\mu) := \frac{k-\|\mu\|_1}{2} \in \mathbb{N}_0,\\ 0 & \text{otherwise,}\end{cases} \tag{15}$$
$$m_{\pi_{\widetilde{\omega}_p}}(\mu) = \begin{cases}\binom{n-p+2r(\mu)}{r(\mu)} & \text{if } r(\mu) := \frac{p-\|\mu\|_1}{2} \in \mathbb{N}_0 \text{ and } |a_j| \leq 1\,\forall j,\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

Lemma IV.4. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, $G = \mathrm{SO}(2n)$, $k \geq 0$, $1 \leq p \leq n-1$, and $\mu \in P(G)$. Write $r(\mu) = (k + p - \|\mu\|_1)/2$. If $r(\mu)$ is a non-negative integer, then

$$m_{\sigma_{k,p}}(\mu) = \sum_{t=0}^{\lfloor p/2\rfloor}\binom{n-p+2t}{t}\sum_{\beta=0}^{p-2t}2^{p-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-2t-\beta}\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{r(\mu)-p+\alpha+t+n-2}{n-2},$$

and $m_{\sigma_{k,p}}(\mu) = 0$ otherwise.

Proof. We will omit several details in the rest of the proof since it is very similar to the one of Lemma III.3. Write $r = (k + p - \|\mu\|_1)/2$ and $\ell = Z(\mu)$. We assume that μ is dominant. Lemma IV.3 implies that the set of weights of $\pi_{\widetilde{\omega}_p}$ is $\mathcal{P}(\pi_{\widetilde{\omega}_p}) := \bigcup_{t=0}^{\lfloor p/2\rfloor}\bigcup_{\beta=0}^{p-2t}\bigcup_{\alpha=0}^{\beta}\mathcal{P}_{t,\beta,\alpha}^{(p)}$, with $\mathcal{P}_{t,\beta,\alpha}^{(p)}$ as in (12).

One has that $\|\mu-\eta\|_1 = k-2(r+t+\alpha-p)$ for any $\eta\in\mathcal{P}^{(p)}_{t,\beta,\alpha}$. Hence, (3) and Lemma IV.3 imply $m_{\sigma_{k,p}}(\mu)=0$ if $r\notin\mathbb{N}_0$ and

$$m_{\sigma_{k,p}}(\mu)=\sum_{t=0}^{\lfloor p/2\rfloor}\sum_{\beta=0}^{p-2t}\sum_{\alpha=0}^{\beta}\binom{r+t+\alpha-p+n-2}{n-2}\binom{n-p+2t}{t}\#\mathcal{P}^{(p)}_{t,\beta,\alpha}$$

otherwise. The proof follows by (13). □

Theorem IV.1 then follows by substituting in (5) the multiplicity formula in Lemma IV.4.

Remark IV.5. By Definition II.2, $\pi_{k,n}$ in type D_n is the only case where $\pi_{k,p}$ is not irreducible. We have that $\pi_{k,n}=\pi_{k\omega_1+\widetilde{\omega}_n}\oplus\pi_{k\omega_1+\widetilde{\omega}_n-2\varepsilon_n}=\pi_{k\omega_1+2\omega_{n-1}}\oplus\pi_{k\omega_1+2\omega_n}$ for every $k\geq 0$. One can obtain the corresponding multiplicity formula for each of these irreducible constituents from Theorem IV.1 by proving the following facts. If $\mu\in P(G)$ satisfies $\|\mu\|_1=k+n$, then $m_{\pi_{k\omega_1+2\omega_n}}(\mu)=m_{\pi_{k,n}}(\mu)$ and $m_{\pi_{k\omega_1+2\omega_{n-1}}}(\mu)=0$ or $m_{\pi_{k\omega_1+2\omega_n}}(\mu)=0$ and $m_{\pi_{k\omega_1+2\omega_{n-1}}}(\mu)=m_{\pi_{k,n}}(\mu)$ and accordingly μ has an even or odd number of negative entries, respectively. Furthermore, if $\mu\in P(G)$ satisfies $\|\mu\|_1<k+n$, then $m_{\pi_{k\omega_1+2\omega_n}}(\mu)=m_{\pi_{k\omega_1+2\omega_{n-1}}}(\mu)=m_{\pi_{k,n}}(\mu)/2$.

V. TYPE B

We now consider $\mathfrak{g}=\mathfrak{so}(2n+1,\mathbb{C})$ and $G=\mathrm{SO}(2n+1)$, so $\mathfrak{g}$ is of type B_n. The same observation in the beginning of Sec. IV is valid in this case. Namely, a weight in $P^{++}(\mathfrak{g})\smallsetminus P^{++}(G)$ induces an irreducible representation of spin$(2n+1)$ which does not descend to G.

For any $k\geq 0$ and $1\leq p\leq n-1$, we have that

$$\pi_{k,p}=\pi_{k\omega_1+\omega_p},\qquad \pi_{k,n}=\pi_{k\omega_1+2\omega_n}. \tag{17}$$

All of them descend to representations of G. The corresponding multiplicity formula is in Theorem V.1 and the remaining case, $\pi_{k\omega_1+\omega_n}$ for $k\geq 0$, is considered in Theorem V.4.

Theorem V.1. *Let $\mathfrak{g}=\mathfrak{so}(2n+1)$, $G=\mathrm{SO}(2n+1)$ for some $n\geq 2$ and let $k\geq 0$, $1\leq p\leq n$ integers. For $\mu\in P(G)$, write $r(\mu)=k+p-\|\mu\|_1$, then*

$$\begin{aligned}
m_{\pi_{k,p}}(\mu)=&\sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t}2^{p-j-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-j-2t-\beta}\\
&\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1}\binom{\lfloor\frac{r(\mu)}{2}\rfloor-i-p+j+\alpha+t+n-1}{n-1}\\
&+\sum_{j=1}^{p-1}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j-1}{2}\rfloor}\binom{n-p+j+2t+1}{t}\sum_{\beta=0}^{p-j-2t-1}2^{p-j-2t-\beta-1}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-j-2t-\beta-1}\\
&\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1}\binom{\lfloor\frac{r(\mu)+1}{2}\rfloor-i-p+j+\alpha+t+n-1}{n-1}.
\end{aligned}$$

Furthermore, $m_{\pi_{k,p}}(\mu)=0$ for all $\mu\in P(\mathfrak{g})\smallsetminus P(G)$.

Remark V.2. Notice that, in Theorem V.1, $m_{\pi_{k,p}}(\mu)=0$ if $r(\mu)<0$ because of the convention $\binom{b}{a}=0$ if $b<a$.

We will omit most of the details since this case is very similar to the previous ones. According to Notation II.1, we set $\mathfrak{h}=\left\{\operatorname{diag}\left(\left[\begin{smallmatrix}0&\theta_1\\-\theta_1&0\end{smallmatrix}\right],\dots,\left[\begin{smallmatrix}0&\theta_n\\-\theta_n&0\end{smallmatrix}\right],0\right):\theta_i\in\mathbb{C}\;\forall i\right\}$, $\varepsilon_i\left(\operatorname{diag}\left(\left[\begin{smallmatrix}0&\theta_1\\-\theta_1&0\end{smallmatrix}\right],\dots,\left[\begin{smallmatrix}0&\theta_n\\-\theta_n&0\end{smallmatrix}\right],0\right)\right)=\theta_i$ for each $1\leq i\leq n$, $\Sigma^+(\mathfrak{g},\mathfrak{h})=\{\varepsilon_i\pm\varepsilon_j: i<j\}\cup\{\varepsilon_i\}$,

$$\begin{aligned}
P(\mathfrak{g})&=\{\textstyle\sum_i a_i\varepsilon_i: a_i\in\mathbb{Z}\,\forall i,\ \text{or } a_i-1/2\in\mathbb{Z}\,\forall i\}, && P(G)=\mathbb{Z}\varepsilon_1\oplus\cdots\oplus\mathbb{Z}\varepsilon_n,\\
P^{++}(\mathfrak{g})&=\{\textstyle\sum_i a_i\varepsilon_i\in P(\mathfrak{g}): a_1\geq a_2\geq\cdots\geq a_n\geq 0\}, && P^{++}(G)=P^{++}(\mathfrak{g})\cap P(G).
\end{aligned}$$

It is well known that (see Exercises IV.10 and V.8 in Ref. 16)

$$\mathrm{Sym}^k(\mathbb{C}^{2n+1}) \simeq \pi_{k\omega_1} \oplus \mathrm{Sym}^{k-2}(\mathbb{C}^{2n+1}), \qquad \pi_{\widetilde{\omega}_p} \simeq \textstyle\bigwedge^p(\mathbb{C}^{2n+1}), \tag{18}$$

where $\mathbb{C}^{2n+1}$ denotes the standard representation of $\mathfrak{g}$. Actually, $\pi_{k\omega_1}$ can be realized inside $\mathrm{Sym}^k(\mathbb{C}^{2n+1})$ as the subspace of harmonic homogeneous polynomials of degree k.

Lemma V.3. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$, $G = \mathrm{SO}(2n+1)$, $k \geq 0$, and $1 \leq p \leq n$. For $\mu = \sum_{j=1}^n a_j\varepsilon_j \in P(G)$, we have that

$$m_{\pi_{k\omega_1}}(\mu) = m_{\pi_{k\varepsilon_1}}(\mu) = \binom{r(\mu)+n-1}{n-1} \quad \textit{where}\ r(\mu) = \lfloor \tfrac{k-\|\mu\|_1}{2} \rfloor, \tag{19}$$

$$m_{\pi_{\widetilde{\omega}_p}}(\mu) = \begin{cases} \binom{n-p+r(\mu)}{\lfloor r(\mu)/2 \rfloor} & \textit{if}\ |a_j| \leq 1\, \forall j, \\ 0 & \textit{otherwise}, \end{cases} \quad \textit{where}\ r(\mu) = p - \|\,\mu\,\|_1. \tag{20}$$

Proof. Let $\mathcal{P}_k$ be the space of complex homogeneous polynomials of degree k in the variables $x_1, \ldots, x_{2n+1}$. Set $f_j = x_{2j-1} + ix_{2j}$ and $g_j = x_{2j-1} - ix_{2j}$ for $1 \leq j \leq n$. One can check that the polynomials $f_1^{k_1} \ldots f_n^{k_n} g_1^{l_1} \ldots g_n^{l_n} x_{2n+1}^{k_0}$ with $k_0, \ldots, k_n, l_1, \ldots, l_n$ non-negative integers satisfying that $\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = k$ form a basis of $\mathcal{P}_k$ given by weight vectors, each of them of weight $\mu = \sum_{j=1}^n (k_j - l_j)\varepsilon_j$. Notice that the number k_0 does not take part of μ.

Consequently, $m_{\pi_{\mathcal{P}_k}}(\mu)$ for $\mu = \sum_{j=1}^n a_j\varepsilon_j$ is the number of tuples $(k_0, \ldots, k_n, l_1, \ldots, l_n) \in \mathbb{N}_0^{2n+1}$ satisfying $a_j = k_j - l_j$ for all $1 \leq j \leq n$ and

$$\sum_{j=0}^n k_j + \sum_{j=1}^n l_j = k. \tag{21}$$

Note that (21) implies $k - \|\mu\|_1 - k_0 = 2s$ for some integer $s \geq 0$.

We fix an integer s satisfying $0 \leq s \leq r := \lfloor (k - \|\mu\|_1)/2 \rfloor$. Set $k_0 = k - \|\mu\|_1 - 2s \geq 0$. As in the proof of Lemma III.2, the number of $(k_1, \ldots, k_n, l_1, \ldots, l_n) \in \mathbb{N}_0^{2n}$ satisfying that $a_j = k_j - l_j$ for all $1 \leq j \leq n$ and (21) is equal to $\binom{s+n-1}{n-1}$. Hence,

$$m_{\mathcal{P}_k}(\mu) = \sum_{s=0}^r \binom{s+n-1}{n-1} = \binom{r+n}{n}.$$

The second equality is well known. It may be proven by showing that both sides are the r-term of the generating function $(1-z)^{-(n+1)}$. From (18), we conclude that $m_{\pi_{k\varepsilon_1}}(\mu) = m_{\mathcal{P}_k}(\mu) - m_{\mathcal{P}_{k-2}}(\mu) = \binom{r+n}{n} - \binom{r-1+n}{n} = \binom{r+n-1}{n-1}$.

We have that $\pi_{\widetilde{\omega}_p} \simeq \bigwedge^p(\mathbb{C}^{2n+1})$ by (18). By setting $v_j = e_{2j-1} - ie_{2j}$, $v_{j+n} = e_{2j-1} + ie_{2j}$ and $v_{2n+1} = e_{2n+1}$, one obtains that the vectors $w_I := v_{i_1} \wedge \cdots \wedge v_{i_p}$, for $I = \{i_1, \ldots, i_p\}$ satisfying $1 \leq i_1 < \cdots < i_p \leq 2n+1$, form a basis of $\bigwedge^p(\mathbb{C}^{2n+1})$. Furthermore, w_I is a weight vector of weight $\mu = \sum_{j=1}^n a_j\varepsilon_j$ given by (11). Note that the condition of $2n+1$ being or not in I does not influence on μ.

Hence, $\mu = \sum_j a_j\varepsilon_j$ is a weight of $\bigwedge^p(\mathbb{C}^{2n+1})$ if and only if $|a_j| \leq 1$ for all j and $p - \|\mu\|_1 \geq 0$. Proceeding as in Lemma III.2, by writing $s = \lfloor \frac{p-\|\mu\|_1}{2} \rfloor \geq 0$, the multiplicity of μ is $\binom{n-p+2s}{s}$ if $p - \|\mu\|_1$ is even and $\binom{n-p+2s+1}{s}$ if $p - \|\mu\|_1$ is odd. □

Theorem V.4. *Let $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$ and $G = \mathrm{SO}(2n+1)$ for some $n \geq 2$ and let $k \geq 0$ an integer. Let $\mu \in P(\mathfrak{g}) \smallsetminus P(G)$. Write $r(\mu) = k + \frac{n}{2} - \|\,\mu\,\|_1$, then*

$$m_{\pi_{k\omega_1+\omega_n}}(\mu) = \binom{r(\mu)+n-1}{n-1}. \tag{22}$$

Furthermore, $m_{k\omega_1+\omega_n}(\mu) = 0$ for all $\mu \in P(G)$.

Proof. This proof is very similar to Lemma 4.2 in Ref. 4. The assertion $m_{k\omega_1+\omega_n}(\mu) = 0$ for every $\mu \in P(G)$ is clear since any weight of $\pi_{k\omega_1+\omega_n}$ is equal to the highest weight $k\omega_1 + \omega_n$ minus a sum of positive roots, which clearly lies in $P(\mathfrak{g}) \smallsetminus P(G)$.

Let $\mu \in P(\mathfrak{g}) \setminus P(G)$. We may assume that μ is dominant, thus $\mu = \frac{1}{2}\sum_{i=1}^{n} a_i\varepsilon_i$ with $a_1 \geq \cdots \geq a_n \geq 1$ odd integers. One has that

$$\pi_{k\omega_1} \otimes \pi_{\omega_n} \simeq \pi_{k\omega_1+\omega_n} \oplus \pi_{(k-1)\omega_1+\omega_n}, \tag{23}$$

for any $k \geq 1$. Indeed, it follows immediately by Exercise V.19 in Ref. 16 since in its sum over the weights of π_{ω_n}, the only non-zero terms are attained at the weights ω_n and $\omega_n - \omega_1$.

It is well known that the set of weights of π_{ω_n} is $\mathcal{P}(\pi_{\omega_n}) := \{\frac{1}{2}\sum_{i=1}^{n} b_i\varepsilon_i : |b_i| = 1\}$ and $m_{\pi_{\omega_n}}(\nu) = 1$ for all $\nu \in \mathcal{P}(\pi_{\omega_n})$ (see, for instance, Exercise V.35 in Ref. 16).

We proceed now to prove (22) by induction on k. It is clear for $k = 0$ by the previous paragraph. Suppose that it holds for $k-1$. By this assumption and (23), we obtain that

$$m_{\pi_{k\omega_1+\omega_n}}(\mu) = m_{\pi_{k\omega_1}\otimes\pi_{\omega_n}}(\mu) - m_{\pi_{(k-1)\omega_1+\omega_n}}(\mu) = m_{\pi_{k\omega_1}\otimes\pi_{\omega_n}}(\mu) - \binom{r+n-2}{n-1}, \tag{24}$$

where $r = k + \frac{n}{2} - \|\mu\|_1$. It only remains to prove that $m_{\pi_{k\omega_1}\otimes\pi_{\omega_n}}(\mu) = \binom{r+n-1}{n-1} + \binom{r+n-2}{n-1}$.

Similarly to (3), we have that $m_{\pi_{k\omega_1}\otimes\pi_{\omega_n}}(\mu) = \sum_{\eta\in\mathcal{P}(\pi_{\omega_n})} m_{\pi_{k\omega_1}}(\mu-\eta)$. Since μ is dominant, for any $\eta = \frac{1}{2}\sum_{i=1}^{n} b_i\varepsilon_i \in \mathcal{P}(\pi_{\omega_n})$, it follows that

$$\|\mu - \eta\|_1 = \frac{1}{2}\sum_{i=1}^{n}(a_i - b_i) = \|\mu\|_1 + \frac{n}{2} - \ell_1(\eta) = k - r + n - \ell_1(\eta),$$

where $\ell_1(\eta) = \#\{1 \leq i \leq n \colon b_i = 1\}$. By Lemma V.3, $m_{\pi_{k\omega_1}}(\mu-\eta) \neq 0$ only if $r + \ell_1(\eta) - n \geq 0$. For each integer ℓ_1 satisfying $n - r \leq \ell_1 \leq n$, there are $\binom{n}{\ell_1}$ weights $\eta \in \mathcal{P}(\pi_{\omega_n})$ such that $\ell_1(\eta) = \ell_1$. On account of the above remarks,

$$m_{\pi_{k\omega_1}\otimes\pi_{\omega_n}}(\mu) = \sum_{\ell_1=n-r}^{n}\binom{\lfloor\frac{r+\ell_1-n}{2}\rfloor + n - 1}{n-1}\binom{n}{\ell_1} = \sum_{j=0}^{r}\binom{\lfloor\frac{r-j}{2}\rfloor + n - 1}{n-1}\binom{n}{j}. \tag{25}$$

We claim that the last term in (25) equals $\binom{r+n-1}{n-1} + \binom{r+n-2}{n-1}$. Indeed, a simple verification shows that both numbers are the rth term of the generating function $\frac{1+z}{(1-z)^n}$. From (24) and (25), we conclude that $m_{\pi_{k\omega_1+\omega_n}}(\mu) = \binom{r+n-1}{n-1}$ as asserted. □

Lemma V.5. Let $n \geq 2$, $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$, $G = \mathrm{SO}(2n+1)$, $k \geq 0$, $1 \leq p < n$, and $\mu \in P(G)$. Write $r(\mu) = k + p - \|\mu\|_1$. Then

$$\begin{aligned} m_{\sigma_{k,p}}(\mu) = &\sum_{t=0}^{\lfloor p/2\rfloor}\binom{n-p+2t}{t}\sum_{\beta=0}^{p-2t} 2^{p-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-2t-\beta}\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{\lfloor\frac{r(\mu)}{2}\rfloor - p + \alpha + t + n - 1}{n-1} \\ &+ \sum_{t=0}^{\lfloor (p-1)/2\rfloor}\binom{n-p+1+2t}{t}\sum_{\beta=0}^{p-1-2t} 2^{p-1-2t-\beta}\binom{n-Z(\mu)}{\beta}\binom{Z(\mu)}{p-1-2t-\beta} \\ &\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{\lfloor\frac{r(\mu)+1}{2}\rfloor - p + \alpha + t + n - 1}{n-1}. \end{aligned}$$

Proof. Write $r = k + p - \|\mu\|_1$ and $\ell = Z(\mu)$ and assume μ dominant. Define $\mathcal{P}_{t,\beta,\alpha}^{(p)}$ as in (12). From Lemma V.3, we deduce that the set of weights of $\pi_{\widetilde{\omega}_p}$ is

$$\mathcal{P}(\pi_{\widetilde{\omega}_p}) := \Big(\bigcup_{t=0}^{\lfloor p/2\rfloor}\bigcup_{\beta=0}^{p-2t}\bigcup_{\alpha=0}^{\beta}\mathcal{P}_{t,\beta,\alpha}^{(p)}\Big) \cup \Big(\bigcup_{t=0}^{\lfloor p-1/2\rfloor}\bigcup_{\beta=0}^{p-1-2t}\bigcup_{\alpha=0}^{\beta}\mathcal{P}_{t,\beta,\alpha}^{(p-1)}\Big).$$

This fact and (3) give

$$m_{\sigma_{k,p}}(\mu) = \sum_{t=0}^{\lfloor p/2\rfloor} \sum_{\beta=0}^{p-2t} \sum_{\alpha=0}^{\beta} \binom{\lfloor \frac{r}{2}\rfloor + t + \alpha - p + n - 1}{n-1} \binom{n-p+2t}{t} \#\mathcal{P}_{t,\beta,\alpha}^{(p)}$$
$$+ \sum_{t=0}^{\lfloor (p-1)/2\rfloor} \sum_{\beta=0}^{p-1-2t} \sum_{\alpha=0}^{\beta} \binom{\lfloor \frac{r-1}{2}\rfloor + t + \alpha - p + n}{n-1} \binom{n-p+1+2t}{t} \#\mathcal{P}_{t,\beta,\alpha}^{(p-1)},$$

since $\|\mu - \eta\|_1 = k - r - 2(t + \alpha - p)$ for all $\eta \in \mathcal{P}_{t,\beta,\alpha}^{(p)}$ and $\|\mu - \eta\|_1 = k - r - 2(t + \alpha - p) - 1$ for all $\eta \in \mathcal{P}_{t,\beta,\alpha}^{(p-1)}$. The proof follows by (13). □

Lemmas II.3 and V.5 complete the proof of Theorem V.1.

VI. TYPE A

Type A_n is the simplest case to compute the weight multiplicity formula of $\pi_{k,p}$. Actually, it follows immediately by standard calculations using Young diagrams. We include this formula to complete the list of all classical simple Lie algebras.

We consider in $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$, $\mathfrak{h} = \{\mathrm{diag}(\theta_1, \dots, \theta_{n+1}) : \theta_i \in \mathbb{C}\, \forall\, i,\ \sum_{i=1}^{n+1} \theta_i = 0\}$. We set $\varepsilon_i(\mathrm{diag}(\theta_1, \dots, \theta_{n+1})) = \theta_i$ for each $1 \le i \le n+1$. We will use the conventions of Ref. 13 in Lecture 15. Thus

$$\mathfrak{h}^* = \bigoplus_{i=1}^{n+1} \mathbb{C}\varepsilon_i / \langle \sum_{i=1}^{n+1} \varepsilon_i = 0\rangle,$$

the set of positive roots is $\Sigma^+(\mathfrak{g},\mathfrak{h}) = \{\varepsilon_i - \varepsilon_j : 1 \le i<j \le n+1\}$, and the weight lattice is $P(\mathfrak{g}) = \bigoplus_{i=1}^{n+1} \mathbb{Z}\varepsilon_i / \langle \sum_{i=1}^{n+1} \varepsilon_i = 0\rangle$. By abuse of notation, we use the same letter ε_i for the image of ε_i in $\mathfrak{h}^*$. A weight $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i$ is dominant if $a_1 \ge a_2 \ge \cdots \ge a_{n+1}$.

The representations having highest weights $\lambda = \sum_{i=1}^{n+1} a_i\varepsilon_i$ and $\mu = \sum_{i=1}^{n+1} b_i\varepsilon_i$ are isomorphic if and only if $a_i - b_i$ is constant, independent of i. Consequently, we can restrict to those $\lambda = \sum_{i=1}^{n+1} a_i\varepsilon_i$ with $a_{n+1} = 0$. Then,

$$P^{++}(\mathfrak{g}) = \left\{ \sum_{i=1}^{n} a_i\varepsilon_i \in P(\mathfrak{g}) : a_1 \ge a_2 \ge \cdots \ge a_n \ge 0 \right\}.$$

The corresponding fundamental weights are given by $\omega_p = \varepsilon_1 + \cdots + \varepsilon_p$ for each $1 \le p \le n$.

It is well known that for $\lambda \in P^{++}(\mathfrak{g})$ and μ a weight of π_λ, one can assume that $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i$ with $a_i \in \mathbb{N}_0$ for all i and $\sum_{i=1}^{n+1} a_i = \|\lambda\|_1$.

Theorem VI.1. *Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ for some $n \ge 1$ and let $k \ge 0$, $1 \le p \le n$ integers. Let $\mu = \sum_{i=1}^{n+1} a_i\varepsilon_i \in P(\mathfrak{g})$ with $a_i \in \mathbb{N}_0$ for all i and $\sum_{i=1}^{n+1} a_i = k + p$.* If $a_1 + a_2 + \cdots + a_j \le k + j$ *for all* $1 \le j \le p$, *then*

$$m_{\pi_{k\omega_1+\omega_p}}(\mu) = \binom{n - Z(\mu)}{p-1},$$

and $m_{\pi_{k,p}}(\mu) = 0$ otherwise.

Proof. The Young diagram corresponding to the representation $\pi_{k\omega_1+\omega_p}$ is the diagram with p rows, having all length 1, excepting the first one which has length $k+1$. It is well known that the multiplicity of the weight μ in this representation is equal to the number of ways one can fill its Young diagram with a_1 1's, a_2 2's, …, a_{n+1} $(n+1)$'s, in such a way that the entries in the first row are non-decreasing and those in the first column are strictly increasing (see, for instance, Sec. 15.3 in Ref. 13).

Consequently, the multiplicity of μ is equal to the number of ways of filling the first column. Since the first entry is uniquely determined, one has to choose $p-1$ different numbers for the rest of the entries. Hence, the theorem follows. □

VII. CONCLUDING REMARKS

For a classical complex Lie algebra $\mathfrak{g}$, it has been shown a closed explicit formula for the weight multiplicities of a representation in any p-fundamental string, namely, any irreducible representation of $\mathfrak{g}$ having highest weight $k\omega_1 + \omega_p$, for some integers $k \geq 0$ and $1 \leq p \leq n$. When $\mathfrak{g}$ is of type A_n, the proof was quite simple and the corresponding formula could be probably established from a more general result. To the best of the authors' knowledge, the obtained expressions of the weight multiplicities for types B_n, C_n, and D_n are new, except for small values of n, probably $n \leq 3$.

Although the formulas in Theorems III.1, IV.1, and V.1 (types C_n, D_n, and B_n, respectively) look complicated and long, they are easily handled in practice. It is important to note that all sums are over (integer) intervals, without including any sum over partitions or permutations. Furthermore, there are only combinatorial numbers in each term. Consequently, it is a simple matter to implement them in a computer program, obtaining a very fast algorithm even when the rank n of the Lie algebra is very large.

Moreover, for p and a weight μ fixed, the formulas become a quasi-polynomial on k. This fact was already predicted and follows by the Kostant multiplicity formula, such as M. Vergne pointed out to Kumar and Prasad[20] (see also Refs. 3 and 27).

For instance, when $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ (type D_n), Theorem IV.1 ensures that

$$m_{\pi_{k\omega_1}}(\mu) = \begin{cases} \binom{\frac{k-\|\mu\|_1}{2}+n-2}{n-2} & \text{if } k \geq \|\mu\|_1 \text{ and } k \equiv \|\mu\|_1 \pmod 2, \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

Consequently, the generating function encoding the numbers $\{m_{\pi_{k\omega_1}}(\mu) : k \geq 0\}$ is a rational function. Indeed,

$$\sum_{k\geq 0} m_{\pi_{k\omega_1}}(\mu) z^k = \sum_{k\geq 0} m_{\pi_{(2k+\|\mu\|_1)\omega_1}}(\mu) z^{2k+\|\mu\|_1} = \frac{z^{\|\mu\|_1}}{(1-z^2)^{n-1}}. \tag{27}$$

From a different point of view, for fixed integers k and p, the formulas are quasi-polynomials in the variables $\|\mu\|_1$ and $Z(\mu)$.

We end the article with a summary of past (and possible future) applications of multiplicity formulas in spectral geometry. We consider a locally homogeneous space $\Gamma\backslash G/K$ with the (induced) standard metric, where G is a compact semisimple Lie group, K is a closed subgroup of G, and Γ is a finite subgroup of the maximal torus T of G. When $G = \mathrm{SO}(2n)$, $K = \mathrm{SO}(2n-1)$, and Γ is cyclic acting freely on $G/K \simeq S^{2n-1}$, we obtain a *lens space*.

In order to determine explicitly the spectrum of a (natural) differential operator acting on smooth sections of a (natural) vector bundle on $\Gamma\backslash G/K$ (e.g., Laplace–Beltrami operator, Hodge–Laplace operator on p-form, Dirac operator), one has to calculate—among other things—numbers of the form $\dim V_\pi^\Gamma$ for π in a subset of the unitary dual $\widehat{G}$ depending on the differential operator. Since $\Gamma \subset T$, $\dim V_\pi^\Gamma$ can be computed by counting the Γ-invariant weights in π according to its multiplicity, so the problem is reduced to know $m_\pi(\mu)$.

At the moment, some weight multiplicity formulas have been successfully applied to the problem described above. The multiplicity formula for $\pi_{k\omega_1}$ in type D_n (Lemma IV.3) was used by Miatello, Rossetti, and the first named author[23] to determine the spectrum of the Laplace–Beltrami operator on a lens space. Furthermore, Corollary I.1 for type D_n was shown in the same article (Lemma 3.3) obtaining a characterization of lens spaces p-isospectral for all p (i.e., their Hodge–Laplace operators on p-forms have the same spectra). Later, Boldt and the first named author[4] considered in the Dirac operator on odd-dimensional spin lens spaces. In this work, it was obtained and used in Theorem IV.2, namely, the multiplicity formula for type D_n of the spin representations $\pi_{k\omega_1+\omega_{n-1}}$ and $\pi_{k\omega_1+\omega_n}$.

As a continuation of the study begun in Ref. 23, Theorem IV.1 was applied in the preprint[22] to determine explicitly every p-spectra of a lens space. Here, as usual, p-spectrum stands for the spectrum of the Hodge–Laplace operator acting on smooth p-forms. This article was the motivation to write the present paper.

The remaining formulas in the article may be used with the same goal. Actually, any application of the formulas for type D_n can be translated to an analogue application for type B_{n-1}, working in spaces covered by S^{2n-2} in place of S^{2n-1} (Sec. 4 in Ref. 15). This was partially done[21] by applying

Lemma V.3. The result extends Ref. 23 (for the Laplace–Beltrami operator) to even-dimensional lens orbifolds.

A different but feasible application can be done for type A_n. One may consider the complex projective space $P^n(\mathbb{C}) = \mathrm{SU}(n+1)/S(\mathrm{U}(n) \times \mathrm{U}(1))$. However, more general representations must be used. Indeed, in Ref. 21, it was considered the Laplace–Beltrami operator and the representations involved had highest weights $k(\omega_1 + \omega_n)$ for $k \geq 0$.

Theorem III.1 (type C_n) does not have an immediate application since the spherical representations of the symmetric space $\mathrm{Sp}(n)/(\mathrm{Sp}(n-1) \times \mathrm{Sp}(1))$ have highest weight of the form $k\omega_2$ for $k \geq 0$. Maddox[26] obtained a multiplicity formula for these representations. However, this expression is not explicit enough to be applied in this problem. An exception was the case $n = 2$, since in Ref. 21, it was applied the closed multiplicity formula in Ref. 5. It is not know by the authors if there is a closed subgroup K of $G = \mathrm{Sp}(n)$ such that the spherical representations of G/K are $\pi_{k\omega_1}$ for $k \geq 0$, that is,

$$\{\pi \in \widehat{G} : V_\pi^K \simeq \mathrm{Hom}_K(V_\pi, \mathbb{C}) \neq 0\} = \{\pi_{k\omega_1} : k \geq 0\}. \tag{28}$$

In such a case, Theorem III.1 could be used.

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