Counterterms in semiclassical Hořava-Lifshitz gravity

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We analyze the semiclassical Hořava-Lifshitz gravity for quantum scalar fields in $3+1$ dimensions. The renormalizability of the theory requires that the action of the scalar field contains terms with six spatial derivatives of the field, i.e. in the UV, the classical action of the scalar field should preserve the anisotropic scaling symmetry ($t \to L^{2z}t$, $\vec{x} \to L^2\vec{x}$, with $z = 3$) of the gravitational action. We discuss the renormalization procedure based on adiabatic subtraction and dimensional regularization in the weak field approximation. We verify that the divergent terms in the adiabatic expansion of the expectation value of the energy-momentum tensor of the scalar field contain up to six spatial derivatives, but do not contain more than two time derivatives. We compute explicitly the counterterms needed for the renormalization of the theory up to second adiabatic order and evaluate the associated β functions in the minimal subtraction scheme.

PACS numbers: 04.62.+v; 04.50.kd; 04.60.-m; 11.10.Gh

arXiv:1006.2870v1 [hep-th] 15 Jun 2010

One year and a half ago, Hořava proposed a new approach to formulate a quantum theory of gravity [1]. Hořava's theory, which has attracted enormous attention, consists of a non-diffeomorphism-invariant ultraviolet modification of Einstein's general relativity. The main idea in [1] is to extend Einstein-Hilbert action with higher spatial derivative terms, whose introduction, while manifestly breaking local Lorentz invariance, leads to heal the short distance divergences and ultimately yields a power counting renormalizable theory. The way this is achieved without introducing ghost instabilities is keeping the requirement of the theory to be of second-order in time derivatives. This introduces an asymmetry between the time coordinate t and the coordinates x^i associated to a preferable foliation that defines a three-dimensional space-like hypersurface of induced metric ${}^{(3)}g_{ij}$. In turn, four-dimensional diffeomorphism invariance results manifestly broken at short distances, and consequently the theory only exhibits diffeomorphism invariance in three-dimensions, in addition to the reparameterization invariance in time. According to this picture, the four-dimensional general covariance of gravity would emerge merely as an approximate symmetry at low energy.

Fragmentation of space-time diffeomorphism invariance in the form of a preferable three-dimensional space-like hypersurfaces defined at constant time, immediately suggests to consider the ADM decomposition for the metric as the convenient picture. Namely, consider

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{1}$$

where, as usual, Latin indices refer to the spatial coordinates, $i, j = 1, 2, 3$, and $g_{ij} = {}^{(3)}g_{ij}$. In the non-projectable theory, the lapse function N depends both on time and the spatial coordinates, in such a way general relativiy is captured within this formulation.

The action of Hořava's theory is given by

$$S = \frac{1}{16\pi G}\int dt\ dx^3\ N\sqrt{g}\left(K_{ij}K^{ij} - \lambda K^2 - 2\Lambda + \xi R - V\right) \tag{2}$$

where Λ is the bare cosmological constant and λ and ξ are arbitrary bare coupling constants; Einstein theory corresponds to the special choice $\lambda = 1$. The extrinsic curvature K_{ij} in the ADM variables takes the form $K_{ij} = (\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i)/(2N)$, whose trace is given by $K = K_{ij}g^{ij}$. Here, ∇_i denotes the spatial derivative, while the dot denotes the derivative with respect to time. The function V in (2) plays the rôle of a potential, as it only depends on spatial derivatives of the metric, which would include higher derivative contributions (see below). In (2), R represents the Ricci scalar curvature of the three-dimensional space-like hypersurface of induced metric g_{ij}.

The presence of terms in the action that involve higher spatial derivatives leads to different scaling dimensions for the time and the spatial coordinates. This is represented by the scaling symmetry

$$x^i \to L^2\ x^i, \qquad t \to L^{2z}\ t, \qquad N \to N, \qquad N_i \to L^{-4z}N_i \tag{3}$$

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which is characterized by the dynamical critical exponent z. A consistent choice is $z = 3$, which is the one we will consider throughout this paper. With this choice, we can consider the potential

$$V = \frac{1}{2}(a_1\Delta R + a_2 R_{ij}R^{ij} + ...) + 4\pi G(b_1\Delta^2 R + b_2 R_{ij}R^{jk}R^i_k + ...), \tag{4}$$

where we are using the notation $\Delta = \nabla_i\nabla^i$. The ellipses in (4) stand for other terms of the same dimension.

At low energy, the action turns out to be dominated by the term that involves the Ricci scalar R, with coefficient ξ. In turn, the theory would reproduce Einstein's general relativity in the infrared, provided λ flows to the value $\lambda_{\rm IR} = 1$. The consistency of the theory and the validity of this hypothesis were extensively discussed in the literature; see [2–5] and references therein. Of special interest is the discussion in [6, 7], where an improved version of Hořava gravity, which seems to be free of pathologies, was presented.

About renormalizability, of particular importance is the question on how the coupling of Hořava gravity to matter affects the properties of the theory in the UV. With the purpose of addressing this problem, we study the coupling of the theory to a quantum scalar field, representing the matter content. The gravitational field will be treated at a classical level, so we are considering a semiclassical Hořava-Lifshitz gravity. It is interesting to remark that, if the matter fields satisfy the usual dispersion relations (i.e if the classical action has four-dimensional general covariance), the theory is non renormalizable. Indeed, it is well known in the context of quantum field theory in curved spacetimes that in order to absorb the divergences associated to the matter fields it is necessary to include in the gravitational action terms proportional to $\mathcal{R}^2, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$, where $\mathcal{R}_{\mu\nu\rho\sigma}$ denotes the components of the four-dimensional Riemann tensor. These terms contain four time derivatives of the metric, and therefore are not included in Hořava gravity. As we will see, renormalizability of the field theory demands that the action for the matter sector contains terms with six spatial derivatives, implying that in the UV the coupling to the scalar field preserves the Lifshitz-type anisotropic scaling with critical exponent $z = 3$. We will verify that the divergent terms in the adiabatic expansion of the expectation value of the stress-tensor associated to the scalar field actually contains up to six spatial derivatives but it remains of second order in time derivatives. We will explicitly compute the counterterms needed for the renormalization of the theory up to second adiabatic order, and we will write down the corresponding β-functions in the minimal substraction scheme.

The computation techniques we will employ here have recently been employed with success to study renormalization in the so-called Einstein-aether theory and in other field theories with modified dispersion relations [8]. The idea for using the same techniques in Hořava-Lifshitz gravity comes from the observation that this theory is closely related to such Lorentz violating scenarios; see for instance [9] and [10].

Let us begin by considering the coupling of Hořava-Lifshitz gravity to a Lifshitz-type scalar field. In the ADM form, the components ${}^{(4)}g_{\mu\nu}$ of the four-dimensional metric (1) are given by

$${}^{(4)}g_{00} = -N^2 + g_{ij}N^iN^j, \qquad {}^{(4)}g_{0i} = g_{ij}N^j, \qquad {}^{(4)}g_{ij} = g_{ij},$$

where $i, j = 1, 2, 3$, and g_{ij} refers to the metric on the three-dimensional foliation of constant t. We consider small perturbations of the metric about flat space; namely, we write

$$N = 1 + \delta n, \qquad N^i = \delta N^i, \qquad g_{ij} = \delta_{ij} + h_{ij}. \tag{5}$$

We consider a matter Lagrangian giving by a scalar field φ that also exhibits anisotropic critical scaling; namely[11]

$$S_\varphi = \int dt dx^3 \sqrt{g} N \left(\frac{1}{2N^2}(\dot{\varphi} - N^i\partial_i\varphi)^2 + F(\varphi, \partial\varphi) - \frac{1}{2}m^2\varphi^2 \right)$$

where the potential $F(\varphi, \partial\varphi)$ is given by

$$F(\varphi, \partial\varphi) = -g_1\partial^i\varphi\partial_i\varphi - g_2\,(\Delta\varphi)^2 + g_3\Delta^2\varphi\Delta\varphi,$$

where we have to be reminded of the definition $\Delta\varphi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j\varphi)$, with $g^{ij} = {}^{(3)}g^{ij} = {}^{(4)}g^{ij} - ({}^{(4)}g^{i0}\,{}^{(4)}g^{j0})/{}^{(4)}g^{00}$, and ${}^{(4)}g^{00} = -N^{-2}$. The equation for the Green function reads

$$-\partial_t\left(\frac{\sqrt{g}}{N}(\partial_t - N^i\,\partial_i)G(x,x')\right) + \partial_j\left(\frac{N^i\sqrt{g}}{N}(\partial_t - N^i\,\partial_i)G(x,x')\right) - m^2\sqrt{g}N\,G(x,x') + 2g_1\partial_i\left(N\sqrt{g}\,\partial^i G(x,x')\right) -$$
$$2g_2\sqrt{g}\Delta(N\,\Delta G(x,x')) + g_3\sqrt{g}\Delta\left(N\,\Delta^2 G(x,x')\right) + g_3\sqrt{g}\Delta^2\left(N\,\Delta G(x,x')\right) = -\delta(x-x'). \tag{6}$$

At the linearized level we have (5), which yields $\sqrt{g} = 1 + h/2$, $g^{ij} = \delta_{ij} - h_{ij}$. This can be used to write the equation for the Green function (6) in the weak field approximation. The Feynman propagator of zero order in the metric perturbations reads

$$G_F^{(0)}(x,x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-x')}}{(-k_0^2+\omega_k^2-i\varepsilon)},$$

where $k = |\vec{k}|$ and

$$\omega_k^2 = m^2 + 2g_1k^2 + 2g_2k^4 + 2g_3k^6,$$

while the first order contribution can be written as follows

$$\begin{aligned} G_F^{(1)}(x,x') &= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}e^{ik\cdot(x-x')}\ f_k(p)}{(-k_0^2+\omega_k^2-i\varepsilon)\left(-(k_0+p_0)^2+\omega^2_{|\vec{k}+\vec{p}|}\right)} \\ &\equiv \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x}e^{ik\cdot(x-x')}\ f_k(p)\,(1+\epsilon_p)^{-1}}{(-k_0^2+\omega_k^2-i\varepsilon)\,(-k_0^2+\omega_k^2)}. \end{aligned} \tag{7}$$

Here, $f_k(p)$ is a function of k_0, k_i, p_0, and p_i that is linear in the metric perturbations

$$\begin{aligned} f_k(p) &= \left(\delta n - \frac{h}{2}\right) k_4^2 + 2ik_4k_i\delta N^i - \left(\delta n + \frac{h}{2}\right)\omega_k^2 + h_{ij}k^ik^j\frac{d\omega_k^2}{dk^2} + h_{ij}\delta_{rs}k^ik^jk^rp^s\frac{d^2\omega_k^2}{d(k^2)^2} \\ &+ik_4p_0\left(\delta n - \frac{h}{2}\right) - \delta N^ip_0k_i + ik_4\delta N^ip_i - \left(\left(\delta n + \frac{h}{2}\right)\delta_{ij}\ p^ik^j + h^{ij}p_ik_j\right)\frac{d\omega_k^2}{dk^2} \\ &-\frac{d^2\omega_k^2}{d(k^2)^2}\left(\frac{\delta n}{2}p^2k^2 + \frac{h}{2}\left(\delta_{ij}k^ip^j\right)^2 - \frac{1}{2}h_{ij}k^ik^jp^2 - h_{ij}\delta_{rs}p^ip^rk^jk^s\right) \\ &+\frac{d^3\omega_k^2}{d(k^2)^3}\left(\frac{\delta n}{4}p^2k^4 + \frac{2}{3}h_{ij}k^ik^j(\delta_{rs}k^rp^s)^2 - \frac{\delta n}{3}(\delta_{ij}k^ip^j)^2k^2\right) - \frac{d^2\omega_k^2}{d(k^2)^2}\left(\frac{h}{4}p^2\delta_{ij}k^ip^j - \frac{1}{2}h_{ij}k^ip^j\right) \\ &-\frac{d^3\omega_k^2}{d(k^2)^3}\left(\frac{\delta n}{3}p^2k^2\delta_{ij}k^ip^j + \frac{h}{3}\left(\delta_{ij}k^ip^j\right)^3 - \frac{2}{3}h_{ij}k^ik^jp^2\delta_{rs}k^rp^s - \frac{2}{3}h_{ij}p^ik^j(\delta_{rs}k^rp^s)^2\right) \\ &-\frac{d^3\omega_k^2}{d(k^2)^3}\left(\frac{\delta n}{12}p^4k^2 + \frac{h}{3}p^2\left(\delta_{ij}k^ip^j\right)^2 - \frac{1}{6}h_{ij}k^ik^jp^4 - \frac{2}{3}h_{ij}\delta_{rs}p^ip^rk^jk^sp^2\right) \\ &-\frac{d^3\omega^2}{d(k^2)^3}\left(\frac{h}{12}p^4\delta_{ij}k^ip^j - \frac{1}{6}h_{ij}k^ip^jp^4\right). \end{aligned} \tag{8}$$

with $p^2 = |\vec{p}|^2$, and ϵ_p is defined as

$$\epsilon_p = \frac{-2k_0p_0 - p_0^2 + \omega^2_{|\vec{k}+\vec{p}|} - \omega_k^2}{-k_0^2+\omega_k^2}. \tag{9}$$

In what follows, for the sake of convenience, we perform a Wick rotation in Eq.(7) and we call $k_4 = ik_0$.

To obtain the adiabatic expansion of the Feynman propagator we start by expanding the integrand of Eq.(7) in powers of p_0 and p_i. The different adiabatic orders of $f_k(p)$ are given by

$$f_k^{\mathrm{ad}(0)} = \left(\delta n - \frac{h}{2}\right)k_4^2 + 2ik_4k_i\delta N^i - \left(\delta n + \frac{h}{2}\right)\omega_k^2 + h_{ij}k^ik^j\frac{d\omega_k^2}{dk^2}, \tag{10a}$$

$$\begin{aligned} f_k^{\mathrm{ad}(1)} =& ik_4p_0\left(\delta n - \frac{h}{2}\right) - \delta N^ip_0k_i + ik_4\delta N^ip_i - \left(\left(\delta n + \frac{h}{2}\right)\delta_{ij}\ p^ik^j + h^{ij}p_ik_j\right)\frac{d\omega_k^2}{dk^2} \\ &+h_{ij}\delta_{rs}k^ik^jk^rp^s\frac{d^2\omega_k^2}{d(k^2)^2}, \end{aligned} \tag{10b}$$

$$\begin{aligned} f_k^{\mathrm{ad}(2)} =& -\frac{d^2\omega_k^2}{d(k^2)^2}\left(\frac{\delta n}{2}p^2k^2 + \frac{h}{2}\left(\delta_{ij}k^ip^j\right)^2 - \frac{1}{2}h_{ij}k^ik^jp^2 - h_{ij}\delta_{rs}p^ip^rk^jk^s\right) \\ &+\frac{d^3\omega_k^2}{d(k^2)^3}\left(\frac{\delta n}{4}p^2k^4 + \frac{2}{3}h_{ij}k^ik^j(\delta_{rs}k^rp^s)^2 - \frac{\delta n}{3}(\delta_{ij}k^ip^j)^2k^2\right). \end{aligned} \tag{10c}$$

From Eq.(8) one can see that no powers of p_0 appear in the adiabatic orders $f_k^{\mathrm{ad}(m)}$ with $m \geq 3$. Besides this property, these $m \geq 3$ adiabatic orders are not relevant for our discussion. Note also that the adiabatic orders $f_k^{\mathrm{ad}(m)}$ with $m \geq 6$ vanish.

We have also to expand ϵ_p in its adiabatic orders. That is,

$$\epsilon_p^{\mathrm{ad}(1)} = \frac{2ik_4p_0 + 2\frac{d\omega_k^2}{dk^2}\ \delta_{ij}k^ip^j}{\omega_k^2 + k_4^2}, \tag{11a}$$

$$\epsilon_p^{\mathrm{ad}(2)} = \frac{-p_0^2 + \frac{d\omega_k^2}{dk^2}\ p^2 + 2\frac{d^2\omega_k^2}{d(k^2)^2}\ (\delta_{ij}k^ip^j)^2}{\omega_k^2 + k_4^2}, \tag{11b}$$

and $\epsilon_p^{\mathrm{ad}(0)} = 0$. It is easy to see that $\epsilon_p^{\mathrm{ad}(m)}$ with $m \geq 3$ do not involve powers of p_0; explicit expressions of these adiabatic orders are not necessary for our present purposes.

The expression from which one obtains the components $\langle T_{\mu\nu}(x)\rangle$ of the expectation value of stress tensor after taking the coincidence limit $x \to x'$ is obtained by evaluating the derivatives of the propagator $G_F(x,x')$, as usual. Appropriate regularization turns out to be necessary. For the case of the component $T_{00}(x)$, we find

$$\begin{aligned}
\langle T_{00}(x)\rangle \;=\; & \lim_{x\to x'} \Big\{ \partial_t\partial_{t'} + \frac{1}{2}(\delta N^i\partial_i\partial_{t'} + \delta N^{i'}\partial_{i'}\partial_t) + \frac{m^2}{2}(1+2\delta n) \\
& - (1+2\delta n)\left(-g_1\delta^{ii'}\partial_i\partial_i' - g_2\partial^2\partial'^2 + \frac{1}{2}g_3(\partial^4\partial'^2 + \partial^2\partial'^4)\right) \\
& + \left(g_1(h^{ij'}\partial_i\partial_{j'} + h^{i'j}\partial_{i'}\partial_j) + g_2\left(h^{ij}\partial_i\partial_j\partial'^2 + h^{i'j'}\partial_{i'}\partial_{j'}\partial^2\right) - g_2(\partial_i\overline{h}_{ij}\partial_j\partial'^2 + \partial_i\overline{h}_{ij'}\partial^2\partial_{j'})\right) \\
& + \frac{1}{2}g_3\left(h_{ij}\partial_i\partial_j\partial'^4 + h_{i'j'}\partial_{i'}\partial_{j'}\partial^4 + \partial_i\overline{h}_{ij}\partial_j\partial'^4 + \partial_i\overline{h}_{ij'}\partial_{j'}\partial^4\right) \\
& - g_3\left(h^{i'j'}\partial^2\partial_{i'}\partial_{j'}\partial'^2 + h^{ij}\partial'^2\partial_i\partial_j\partial^2 - \partial_i\overline{h}_{ij}\partial_j\partial^2\partial'^2 - \partial_i\overline{h}_{ij'}\partial^2\partial_{j'}\partial'^2\right) \\
& - \frac{1}{2}g_3\left(\partial^2 h^{ij}\partial_i\partial_j\partial'^2 + \partial^2 h^{i'j'}\partial^2\partial_{i'}\partial_{j'} - \partial_i\partial^2\overline{h}_{ij}\partial_j\partial'^2 - \partial_i\partial^2\overline{h}_{ij'}\partial^2\partial_{j'}\right) \\
& - g_3\left(\partial_k h^{ij}\partial_i\partial_j\partial_k\partial'^2 + \partial_{k'}h^{i'j'}\partial^2\partial_{i'}\partial_{j'}\partial_{k'} - \partial_i\partial_k\overline{h}_{ij}\partial_k\partial_j\partial'^2 - \partial_i\partial_{k'}\overline{h}_{ij'}\partial^2\partial_{k'}\partial_{j'}\right) \Big\}\,\mathrm{Im}\ G_F(x,x')
\end{aligned} \tag{12}$$

where $\overline{h}_{ij} = h_{ij} - \frac{h}{2}\delta_{ij}$, $\partial^2 = \partial_i\partial_i$, and a primed index on a derivative indicates that the derivative is taken with respect to a primed coordinate.

For the sake of simplicity, and because it is enough for our present purposes, we partially fix the gauge by setting $\delta N^i = 0$. This greatly simplifies the expression of the $\langle T_{0i}(x)\rangle$ component, which reads

$$\langle T_{0i}(x)\rangle = \frac{1}{2}\lim_{x\to x'}(\partial_t\partial_{i'} + \partial_{t'}\partial_i)\ \mathrm{Im}\ G_F(x,x'). \tag{13}$$

To obtain the regularized expectation values of the stress tensor we use dimensional regularization. Therefore, after computing the derivatives of $\mathrm{Im}\ G_F(x,x')$ that appear in Eqns.(12) and (13), we can set $x = x'$. Then, it is straightforward to separate the different adiabatic orders of $\langle T_{0\mu}(x)\rangle$ ($\mu = 0,1,2,3$), before performing the integrations. In this way we obtain an integral expression for each adiabatic order. The next step is to use dimensional regularization to perform the integrals. We apply dimensional regularization to both temporal and spatial directions, but in a separated way. That is, we split the $d-$dimensional integrals into integrals in d_1 and d_2 dimensions, with $d = d_1 + d_2$, where $d_1 \to 1$ and $d_2 \to 3$. All integrals in k_4 are of the form

$$I_{d_1}(j,l) = \frac{\Omega_{d_1}}{(2\pi)^{d_1}}\int_0^{+\infty} dk_4 \frac{k_4^{d_1-1+j}}{(k_4^2+\omega_k^2)^l} = \frac{\Omega_{d_1}}{(2\pi)^{d_1}}\frac{\omega_k^{-2l+j+d_1}}{2\Gamma(l)}\Gamma\left(l - \frac{j+d_1}{2}\right)\Gamma\left(\frac{j+d_1}{2}\right)$$

where j is an even number ($I_{d_1}(j,l) = 0$ if j is odd) and l is an integer; $\Omega_{d_1} = 2\pi^{d_1/2}/\Gamma(d_1/2)$, with $\Gamma(z)$ being the Gamma function. Note that the right hand side is finite in the limit $d_1 \to 1$ (as is usual in dimensional regularization for odd dimensions), then, we set $d_1 = 1$.

Using the adiabatic expansion of the expectation value of the stress-tensor, by simple power counting one can study up to which adiabatic order it contains divergences and how many temporal derivatives do appear in the divergent terms. To illustrate this, let us consider as an example the following contribution to $\langle T_{00}\rangle$:

$$t_{00}(x) = \lim_{x\to x'}\partial_t\partial_{t'}\mathrm{Im}\ G_F(x,x').$$

It can be shown that this term, along with others which are similar, are the most divergent ones. The contribution that is linear in the metric perturbations can be written as

$$t_{00}(x) = -\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \int \frac{d^d k_E}{(2\pi)^d} \frac{k_4(k_4 + ip_0)}{(k_4^2 + \omega_k^2)^2} f_k(p) \sum_{r=0} (-\epsilon_p)^r, \tag{14}$$

Let us first analyze the terms that do not contain p_0. In a schematic way, it is simple to show that the ultraviolet behavior of a term in $f_k(p)$ and in ϵ_p, respectively, is given by $\epsilon_p \sim k^{6-n} p^n/(k_4^2 + \omega_k^2)$, with $1 \leq n \leq 6$, and $f_k(p) \sim \delta g\, k^{6-s} p^s$, where $0 \leq s \leq 5$ and δg represents a component of $\delta^{(4)} g_{\mu\nu}$. Then, for a term characterized by n, s and r, the integration in k_4 yields

$$\int d^{d_1} k_E \frac{k_4^2 f_k(p)}{(k_4^2 + \omega_k^2)^2} (-\epsilon_p)^r \sim \delta g \int d^{d_1} k_E \frac{k_4^2 k^{6(1+r)}}{(k_4^2 + \omega_k^2)^{2+r}} \left(\frac{p}{k}\right)^{s+nr} \sim \delta g\, k^3 \left(\frac{p}{k}\right)^{s+nr}.$$

Therefore, by power counting it can be shown that the integral in k is convergent only if $s + rn > d + 2$. That is, for $d = 4$, we have that $p^{d+2} = p^6$ is the maximum power of p that appears in a divergent contribution; i. e., terms of adiabatic order greater than six are finite.

Let us now analyze the terms that contain powers of p_0. In Eq.(14) p_0 appears explicitly and also implicitly through $f_k(p)$ and ϵ_p. Notice that of all the adiabatic orders only $f_k^{ad(1)}(p)$, $\epsilon_p^{ad(1)}$ and $\epsilon_p^{ad(2)}$ depend on p_0 (see Eqns.(10) and (11) and the paragraphs that follows each of them). Then, one can write all the terms of second adiabatic order that involve one or two powers of p_0 (i.e., terms with $p_0 p_i$ or with p_0^2 and no additional power of p_μ) and show that all of them are logarithmically divergent. Hence, as the convergence improves with the adiabatic order, we can conclude that the contribution of higher adiabatic orders with at least one power of p_0 will be finite. Below, we compute explicitly the terms of second adiabatic order of the 00 and $0i$ components of $\langle T_{\mu\nu}\rangle$ and we show that while the terms with one and two powers of p_0 are both logarithmically divergent, the ones with p_0^2 do not appear in the final result. We expect that terms with p_0^2 do appear in the ij components, but here we will not compute these explicitly.

In summary, in $\langle T_{\mu\nu}\rangle$ there appear divergences up to in the sixth *weighted adiabatic order*, where the weighted adiabatic order of a term is given by ($z = 3$)

$$W = z n_0 + n_i$$

where n_i and n_0 are, respectively, the number of spatial derivatives and time derivatives appearing in the term. This is analogous to the weighted power counting criterion introduced in [12] for field theories in Minkowski spacetime (see also [13]).

In order to illustrate the procedure by which we obtain the regularized adiabatic orders in terms of 3−tensorial quantities, let us consider as an example the zeroth adiabatic order of $\langle T_{00}(x)\rangle$. After performing the integrals in p_μ (which are straightforward) and the integral in k_4 (as described above), we obtain

$$\langle T_{00}(x)\rangle^{\mathrm{ad}(0)} = \frac{\mu^{4-d}}{2} \int \frac{d^{d_2}k}{(2\pi)^{d_2}} \left\{ \omega_k \left(1 + 2\delta n - \frac{h}{2}\right) - \frac{h_{ij}}{\omega_k} \left(g_1 k_i k_j + 2 g_2 k_i k_j k^2 + 3 g_3 k_i k_j k^4\right) \right\}, \tag{15}$$

where μ is an arbitrary parameter of dimensions of mass, introduced to ensure that φ has the correct dimensionality.

To carry out the angular integrations we use the following property [14]:

$$\int d^{d_2}k k^{i_1} ... k^{i_r} g(k^2) = \begin{cases} 0 & \text{if } r \text{ is odd,} \\ T^{i_1 \ldots i_r} A_r[g] & \text{if } r \text{ is even,} \end{cases}$$

where

$$T^{i_1 \ldots i_r} = \frac{1}{r!}[\delta^{i_1 i_2} \delta^{i_3 i_4} ... \delta^{i_{r-1} i_r} + \text{all permutations of the } i\text{'s}],$$
$$A_r[g] = \frac{2\pi^{d_2/2}\Gamma[(r+1)/2]}{\Gamma[1/2]\Gamma[(d_2 + r)/2]} \int_0^\infty dk k^{d_2+r-1} g(k^2).$$

The remaining integrals can be related by performing an integration by parts,

$$\begin{aligned} \langle T_{00}(x)\rangle^{\mathrm{ad}(0)} &= \frac{\mu^{4-d}\Omega_{d_2}}{4(2\pi)^{d_2}} \int_0^{+\infty} dk^2 k^{d_2-2} \left\{ \omega_k \left(1 + 2\delta n - \frac{h}{2}\right) - \frac{h}{d_2} k^2 \frac{d\omega_k}{dk^2} \right\} \\ &= \frac{\mu^{4-d}\Omega_{d_2}}{4(2\pi)^{d_2}} (1 + 2\delta n) \int_0^{+\infty} dk^2 k^{d_2-2} \omega_k = -{}^{(4)}g_{00} \frac{\mu^{4-d}\Omega_{d_2}}{4(2\pi)^{d_2}} I_0. \end{aligned} \tag{17}$$

where we have defined $I_0 = \int_0^{+\infty} dk^2 k^{d_2-2}\omega_k$. Moreover, one can easily show that, due to the gauge condition $\delta N^i = 0$, one has $\langle T_{0i}(x)\rangle^{\mathrm{ad}(0)} = 0$. Therefore, as expected, the lowest adiabatic order of the energy momentum tensor is proportional to the metric and can be absorbed into a redefinition of the cosmological constant (see below).

We follow the same procedure for the second adiabatic order of $\langle T_{0\mu}(x)\rangle$. After a long but straightforward calculation the results are

$$\begin{aligned}
\langle T_{00}(x)\rangle^{\mathrm{ad}(2)} &= -\frac{\mu^{4-d}\Omega_{d_2}}{48(2\pi)^{d_2}} I_1(\partial_i\partial_j h_{ij} - \partial^2 h) = -\frac{\mu^{4-d}\Omega_{d_2}}{24(2\pi)^{d_2}} I_1 G_{00}, \\
\langle T_{0i}(x)\rangle^{\mathrm{ad}(2)} &= -\frac{\mu^{4-d}\Omega_{d_2}}{48(2\pi)^{d_2}} \left\{ I_2(\partial_j \dot{h}_{ij} - \partial_i \dot{h}) + \frac{I_3}{d_2(d_2+2)}(2\partial_j \dot{h}_{ij} + \partial_i \dot{h}) \right\} \\
&= -\frac{\mu^{4-d}\Omega_{d_2}}{24(2\pi)^{d_2}} \left\{ I_2 G_{0i} + \frac{I_3}{d_2(d_2+2)}(2G_{0i} + 3\partial_i K^j_j) \right\}, \qquad (18)
\end{aligned}$$

where G_{00} and G_{0i} are components of the linearized Einstein tensor $G_{\mu\nu}$ and K^i_j is the linearized extrinsic curvature, and we have defined the following integrals:

$$I_1 = \int_0^{+\infty} dk^2 \frac{k^{d_2-2}}{\omega_k}\frac{d\omega_k^2}{dk^2}, \; I_2 = \int_0^{+\infty} dk^2 \frac{k^{d_2-2}}{\omega_k}, \; I_3 = \int_0^{+\infty} dk^2 \frac{k^{d_2+2}}{\omega_k^3}\frac{d^2\omega_k^2}{d(k^2)^2}.$$

Notice that while I_1 is quartically divergent, I_2 and I_3 are logarithmically divergent. The terms in Eq.(18) have first time derivatives of the metric and, as we have anticipated, result to be logarithmically divergent. We have repeated all the calculations without partially fixing the gauge $\delta N^i = 0$, and reobtained Eq. (18) as a cross-check.

With these results we can now analyze the renormalization of the bare constants associated to the terms of second adiabatic order that appear in the gravitational action (2). To do so, we start by writing the 00 and $0i$ parts of the semiclassical equations for the metric (in the weak field approximation), keeping only terms up to second adiabatic order; namely

$$\frac{1}{8\pi G}\left\{\Lambda^{(4)} g_{00} + \xi G_{00}\right\} = \langle T_{00}(x)\rangle = \langle T_{00}(x)\rangle_{ren} + \langle T_{00}(x)\rangle^{\mathrm{ad}(0)} + \langle T_{00}(x)\rangle^{\mathrm{ad}(2)}, \qquad (19)$$

$$\frac{1}{8\pi G}\left\{G_{0i} - (\lambda-1)\partial_i K^j_j\right\} = \langle T_{0i}(x)\rangle = \langle T_{0i}(x)\rangle_{ren} + \langle T_{0i}(x)\rangle^{\mathrm{ad}(2)}, \qquad (20)$$

where we have added and subtracted the adiabatic expansion of $\langle T_{0\mu}(x)\rangle$ in order to separate the renormalized part $\langle T_{0\mu}(x)\rangle_{ren}$ and the divergent contributions. The latter are to be absorbed into a redefinition of Λ, G, λ and ξ. Then, we introduce Eqns. (17) and (18) into (19) and (20), and we find that $\langle T_{00}(x)\rangle^{\mathrm{ad}(0)}$ and $\langle T_{0\mu}(x)\rangle^{\mathrm{ad}(2)}$ can be cancelled with the following choice of the bare constants:

$$\Lambda G^{-1} = (\Lambda G^{-1})_R - \frac{\mu^{4-d}\Omega_{d-1}}{(2\pi)^{d-2}} I_0, \qquad (21a)$$

$$\xi G^{-1} = (\xi G^{-1})_R - \frac{\mu^{4-d}\Omega_{d-1}}{6(2\pi)^{d-2}} I_1, \qquad (21b)$$

$$G^{-1} = (G^{-1})_R - \frac{\mu^{4-d}\Omega_{d-1}}{6(2\pi)^{d-2}}\left[I_2 + \frac{2I_3}{(d-1)(d+1)}\right], \qquad (21c)$$

$$G^{-1}(\lambda-1) = (G^{-1}(\lambda-1))_R + \frac{\mu^{4-d}\Omega_{d-1}}{2(2\pi)^{d-2}}\frac{I_3}{(d-1)(d+1)}, \qquad (21d)$$

where we denote the renormalized constants by a subscript R.

It is worth noting that from these equations we can recover the well-known results [15] corresponding to the usual ($z=1$) scalar field by setting $g_1 = 1/2$ and $g_2 = g_3 = 0$ before taking the limit $d \to 4$. In such a case, I_3 vanishes and in the limit $d \to 4$ we have $I_0 \sim m^4(d-4)^{-1}/4$ and $I_1 = I_2 \sim m^2/(d-4)$.

All the integrals on the right hand side in Eqs. (21) are divergent in the limit $d \to 4$. In the particular case of a massless field ($m = 0$) these integrals can be computed explicitly. We assume that $g_3 > 0$ and $g_2^2 - 4g_1 g_3 > 0$ in order to avoid zeros of ω_k. Thus, in the limit $d \to 4$ we have:

$$
\begin{aligned}
i_0 &\equiv \frac{\mu^{4-d}\Omega_{d-1}}{(2\pi)^{d-1}} I_0 = -\frac{g_2(g_2^2-4g_1g_3)}{8\sqrt{2}\pi^2 g_3^{5/2}}\left[\frac{1}{d-4} - \ln(\mu g_3^{1/4})\right] + FP, \\
i_1 &\equiv \frac{\mu^{4-d}\Omega_{d-1}}{(2\pi)^{d-1}} I_1 = -\frac{(g_2^2-4g_1g_3)}{4\sqrt{2}\pi^2 g_3^{3/2}}\left[\frac{1}{d-4} - \ln(\mu g_3^{1/4})\right] + FP, \\
i_2 &\equiv \frac{\mu^{4-d}\Omega_{d-1}}{(2\pi)^{d-1}} I_2 = -\frac{1}{\sqrt{2g_3}\pi^2}\left[\frac{1}{d-4} - \ln(\mu g_3^{1/4})\right] + FP, \\
i_3 &\equiv \frac{\mu^{4-d}\Omega_{d-1}}{(2\pi)^{d-1}} I_3 = -\frac{2}{5\sqrt{2g_3}\pi^2}\left[\frac{1}{d-4} - \ln(\mu g_3^{1/4})\right] + FP,
\end{aligned}
$$

where FP denotes the μ-independent finite part.

The renormalization group equations are obtained simply recalling that the bare constants are independent of μ, and are given by

$$\mu\frac{d}{d\mu}\left(\Lambda G^{-1}\right)_R = \frac{g_2}{\sqrt{2}}\frac{(g_2^2-4g_1g_3)}{4\pi g_3^{5/2}}, \tag{23a}$$

$$\mu\frac{d}{d\mu}\left(\xi G^{-1}\right)_R = \frac{1}{\sqrt{2}}\frac{(g_2^2-4g_1g_3)}{12\pi g_3^{3/2}}, \tag{23b}$$

$$\mu\frac{d}{d\mu}\left(G^{-1}\right)_R = \frac{3}{5\sqrt{2g_3}\pi}, \tag{23c}$$

$$\mu\frac{d}{d\mu}\left(G^{-1}(\lambda-1)\right)_R = -\frac{2}{5\sqrt{2g_3}\pi}. \tag{23d}$$

Note that, in contrast to what happens in the case of a usual ($z=1$) scalar field, here we have obtained that the renormalized constants depend on μ for a massless field. Note also that in these equations, as $g_3 > 0$ and $g_2^2 - 4g_1g_3 > 0$, the right hand side have a determined sign, except in the first equation which depends on the sign of g_2.

Assuming that the order of magnitude of the constants g_i are determined by a single mass scale M_C, i.e. $g_1 \sim g_2 M_C^2 \sim g_3 M_C^4 \sim \mathcal{O}(1)$, these equations give us information on the running of the renormalized constants in the UV [16], at scales larger than M_C. With the renormalization scheme we are considering (minimal subtraction), it is not possible to analyze the IR behaviour of the coupling constants. As pointed out in [14, 17, 18], this would require a mass dependent renormalization scheme. We hope to address this problem in a forthcoming publication.

Acknowledgments

This work was supported by Universidad de Buenos Aires UBA, CONICET and ANPCyT. The authors thank Diego Blas, Oriol Pujolàs and Jorge Russo for interesting discussions.

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