Disk one-point function for non-rational conformal theories

Juan Pablo Babaro [a] and Gaston Giribet [a,b]

[a] Departamento de Física, Universidad de Buenos Aires FCEN-UBA
Ciudad Universitaria, Pabellón 1, 1428, Buenos Aires, Argentina.

[b] Instituto de Física de Buenos Aires IFIBA-CONICET
Ciudad Universitaria, Pabellón 1, 1428, Buenos Aires, Argentina.

Abstract

We consider an infinite family of non-rational conformal field theories in the presence of a conformal boundary. These theories, which have been recently proposed in [1], are parameterized by two real numbers (b, m) in such a way that the corresponding central charges $c_{(b,m)}$ are given by $c_{(b,m)} = 1 + 6(b + b^{-1}(1 - m))^2$. For the disk geometry, we explicitly compute the expectation value of a bulk vertex operator in the generic case $m \in \mathbb{R}$, such that the result trivially reduces to the Liouville one-point function when $m = 0$. For the case $m = 1$ the observable we compute corresponds to the one-point function in the SL$(2,\mathbb{C})$/SU(2) Wess-Zumino-Novikov-Witten theory (WZNW). In fact, our calculation mimics the analysis recently performed in [2] for the case of the WZNW theory, showing how the boundary action for Euclidean AdS$_2$ D-branes in Euclidean AdS$_3$ space proposed therein admits a straightforward generalization to the whole family of m-parameterized non-rational models. A difference with respect to the WZNW theory arises in that, in contrast to the case $m = 1$, the CFT defined for generic m does not exhibit $sl(2)_k$ affine Kac-Moody symmetry but a Borel subalgebra of it, and this requires to implement conformal invariant boundary condition in a sligthly different manner. We perform the calculation of the disk one-point function in two different ways, obtaining results in perfect agreement, and giving the details of both the path integral and the free field derivations.

1 Introduction

Non-rational two-dimensional conformal field theories have important applications in physics. Systems of condensed matter, lower-dimensional quantum gravity, and string theory are examples of this. In particular, the problem of considering non-rational models on surfaces with boundaries has direct applications to the description of D-branes in string theory on non-compact backgrounds.

The study of conformal field theory (CFT) on two-dimensional manifolds with boundaries was initiated by J. Cardy in his early work on minimal models [3, 4, 5], and more recently it was extended to non-rational models by the experts, notably by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [6, 7]. The CFT description of branes in two-dimensional string theory, both in tachyonic and black hole backgrounds, and in three-dimensional string theory in Anti-de Sitter space (AdS), attracted the attention of the string theory community in the last ten years. The literature on string (brane) theory applications of boundary conformal field theory is certainly vast, and we cannot afford to give a complete list of references herein. Instead, let us address the reader's attention to the lectures [8, 9, 10], to the list of renowned works [11]-[41], and to the very interesting recent studies [42]-[46] on non-rational models on the disk geometry.

In this paper, we will consider the family of $c > 1$ non-rational two-dimensional conformal field theories recently proposed in [1]. This family of theories is parameterized by two real numbers (b, m) in such a way that the corresponding central charges $c_{(b,m)}$ are given by $c_{(b,m)} = 1 + 6(b + b^{-1}(1 - m))^2$.

The existence of this family of non-rational models was conjectured in Ref. [1] by S. Ribault, who presented convincing evidence by performing consistency checks. The way of introducing the new CFTs in [1] was constructive: These CFTs are defined in terms of their correlation functions which, on the other hand, admit to be expressed in terms of correlation functions in Liouville field theory. In addition, a Lagrangian representation of these new models was also proposed (see (1) below), and this was employed in [47] to compute two- and three-point functions on the topology of the sphere.

It was emphasized in [1] that, if these non-rational theories actually exist, then it would be possible to study issues like their solutions on Riemann surfaces with boundaries. And this is precisely the task we undertake in this paper: For the disk geometry, we explicitly compute the expectation value of a bulk vertex operator in the generic case $m \in \mathbb{R}$. In the case $m = 0$ the result trivially reduces to the Liouville disk one-point function. For the cases $m = 1$ and $m = b^2$ the observable we compute corresponds to the one-point function in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ Wess-Zumino-Novikov-Witten theory (WZNW), which was recently reconsidered in [2] in the context of AdS_3 string theory. In fact, our calculation generalizes the calculation of [2] to generic values of $m \in \mathbb{R}$, showing that the boundary action for Euclidean AdS_2 D-branes in Euclidean AdS_3 space proposed in [2] can be straightforwardly extended to the whole family of non-rational models of [1]. This is the case even when the theory for generic m does not exhibit $\widehat{sl}(2)_k \times \widehat{sl}(2)_k$ affine symmetry but just the Borel subalgebra of it. Consequently, the implementation of conformal invariant boundary conditions requires a condition on the stress-tensor explicitly. The boundary conditions are discussed in Section 3.

We perform the calculation of the bulk one-point function in two different ways: First, in Section 4 we give a path integral derivation. This consists of reducing the calculation of the expectation value of one bulk operator on the disk geometry to the analogous quantity in Liouville field theory, which, conveniently, is already known [6]. We closely follow the path integral techniques developed in [49], also reproduced in [2, 50]. With the intention to make the comparison with these reference easy, we aim to use a notation similar to the one used therein, up to minor differences. In Section 5, following the steps of [2], we perform a free-field calculations of the same one-point function, finding perfect agreement. As usual when dealing with the Coulomb gas representation in non-rational CFTs, an analytic extension will be necessary for the free field calculation to reproduce the result for generic $m \notin \mathbb{Z}$. Remarkably, the free field approach turns out to exactly agree with the general expression, what is ultimately verified by comparing with the path integral computation. In Section 6, we analyze some functional properties of the formula we obtain, of which the one-point function in Liouville theory and in the WZNW theory are particular cases.

2 Conformal field theory

The action of the non-rational conformal field theories discussed in [1] is given by

$$S_{\text{bulk}} \;=\; \frac{1}{2\pi}\int_{\Gamma} d^2z\, g^{1/2}\left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{Q_m}{4}R\phi + b^2(-\beta\bar{\beta})^m\, e^{2b\phi}\right) \tag{1}$$

where $Q_m = b + b^{-1}(1-m)$, and where we are using the standard notation $\partial = \frac{\partial}{\partial z} = \frac{1}{2}\partial_x - \frac{i}{2}\partial_y$ and $\overline{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\partial_x + \frac{i}{2}\partial_y$. Here, the notation S_{bulk} is chosen to emphasize that we are not including boundary terms yet (see (7) below).

Let us call $\mathcal{T}_{m,b}$ the theory defined by the action (1). Then, we notice that the case $\mathcal{T}_{0,b}$ corresponds to Liouville field theory coupled to a free β-γ ghost system. On the other hand, the case $\mathcal{T}_{1,1/\sqrt{k-2}}$ corresponds to the H_3^+=SL$(2,\mathbb{C})$/SU(2) WZNW theory with level $k = b^{-2}+2$ written in the Wakimoto free-field representation [48]. Besides, $\mathcal{T}_{k-2,\sqrt{k-2}}$ also yields the H_3^+ WZNW theory, with level $k = b^{+2}+2$; see [47]. In fact, the action above can be regarded as a generalization of these well-known non-rational conformal field theories.

The free stress-tensor associated to (1) is given by

$$T(z) \;=\; -\beta(z)\partial\gamma(z) - (\partial\phi(z))^2 + Q_m\,\partial^2\phi(z) \tag{2}$$

and by its anti-holomorphic counterpart $\overline{T}(\overline{z})$. This yields the central charge of the theory

$$c_{(b,m)} = 1 + 6Q_m^2. \tag{3}$$

The conformal dimension of the fields $e^{2\alpha\phi}$, β, and γ with respect to (2) are $(h_\alpha, \bar{h}_\alpha) = (\alpha(Q_m - \alpha), \alpha(Q_m-\alpha))$, $(h_\beta, \bar{h}_\beta) = (1,0)$ and $(h_\gamma, \bar{h}_\gamma) = (0,0)$, respectively. Therefore, the last term in the (1) is marginal with respect to the stress-tensor (2) if $h_b = \bar{h}_b = b\,(Q_m - b) = 1-m$, yielding the relation

$$Q_m \;=\; b + \frac{1-m}{b}. \tag{4}$$

The vertex operators we will consider are those of the form

$$\Phi^j(\mu|z) \;=\; e^{2h_j\sigma(z,\bar{z})}\,|\mu|^{2m(j+1)}\,e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\,e^{2b(j+1)\phi(z,\bar{z})} \tag{5}$$

whose holomorphic and anti-holomorphic conformal dimensions are given by

$$h_j \;=\; \bar{h}_j \;= (-b^2 j + 1 - m)(j+1), \tag{6}$$

and where μ is a complex variable. The spectrum of normalizable states of the theory, which is ultimately expressed by the values that j takes, is to be determined. The dependence on $\sigma(z,\bar{z})$ in (5) was introduced because throughout this paper we will work in the conformal gauge $ds^2 = e^{2\sigma(z,\bar{z})}dzd\bar{z}$, and such dependence is the one required for $\Phi^j(\mu|z)$ to transform as a primary $(h_j, \overline{h}_j)$-dimension operator under conformal transformations. We will later adopt the notation $|\rho(z)|^2 = e^{2\sigma(z,\bar{z})}$ for convenience.

3 Boundary conditions

The boundary term we add to the action (1) to define the theory in a surface with boundaries is the following

$$S_{\text{boundary}} = \frac{1}{2\pi}\int_{\partial\Gamma} dx\, g^{1/4}\left(Q_m K\phi + \frac{i}{2}\beta\left(\gamma + \bar{\gamma} - \xi\,\beta^{m-1}\,e^{b\phi}\right)\right), \tag{7}$$

where ξ is an arbitrary constant. This is the natural generalization of the boundary action proposed in [2].

We will consider the theory on the disk. In turn, conformal invariance permits to chose Γ as being the upper half plane $\text{Im}(z) = y \geq 0$, with $z = x + iy$, and, consequently, the boundary is given by the real line $\text{Re}(z) = x$.

Integrating by parts the γ-β terms, we have

$$\beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} = \bar{\partial}(\beta\gamma) - \gamma\bar{\partial}\beta + \partial(\bar{\beta}\bar{\gamma}) - \bar{\gamma}\partial\bar{\beta} \tag{8}$$

We see that the total derivatives change the boundary term; namely

$$\bar{\partial}(\beta\gamma) = \frac{1}{2}\partial_x(\beta\gamma) + \frac{i}{2}\partial_y(\beta\gamma) \tag{9}$$

$$\partial(\bar{\beta}\bar{\gamma}) = \frac{1}{2}\partial_x(\bar{\beta}\bar{\gamma}) - \frac{i}{2}\partial_y(\bar{\beta}\bar{\gamma}) \tag{10}$$

Integrating over the semi-infinite line $\text{Re}(y) > 0$, we see that the last terms in (9) and (10) contribute with a piece $-\frac{i}{2}\beta\gamma|_{y=0}$ and $+\frac{i}{2}\bar{\beta}\bar{\gamma}|_{y=0}$ respectively. This cancels the $\beta(\gamma + \bar{\gamma})$ contribution in (7), and the total action takes the form

$$\begin{aligned} S = {} & \frac{1}{2\pi}\int_{\Gamma} d^2z\, g^{1/2}\left(\partial\phi\bar{\partial}\phi - \gamma\bar{\partial}\beta - \bar{\gamma}\partial\bar{\beta} + \frac{Q_m}{4}R\phi + b^2(-\beta\bar{\beta})^m\,e^{2b\phi}\right) + \\ & + \frac{1}{2\pi}\int_{\partial\Gamma} dx\, g^{1/4}\left(Q_m K\phi - \frac{i\xi}{2}\beta^m\,e^{b\phi}\right) \end{aligned} \tag{11}$$

It is convenient to introduce bulk and boundary coupling constants (λ, λ_B) to control the strength of the interacting terms. This is achieved by shifting the zero-mode of ϕ as follows $\phi \to \phi + \frac{1}{2b}\ln(\lambda/b^2)$ and redefining $\xi = 2ib\lambda_B/\sqrt{\lambda}$. This makes the interaction term in the action to take the form $\lambda\int_\Gamma(-\beta\bar{\beta})^m e^{2b\phi} + \lambda_B\int_{\partial\Gamma}\beta^m e^{b\phi}$, which is useful to perform the free field calculation (see Section 5) as in this way one has access to a perturbative treatment of the screening effects.

As mentioned, the theory with $m = 1$ (and the theory with $m = b^2$) corresponds to the SL(2, $\mathbb{C}$)/SU(2) WZNW model, which exhibits $\widehat{sl}(2)_k \times \widehat{sl}(2)_k$ affine Kac-Moody symmetry. This symmetry is generated by the Kac-Moody current algebra

$$
\begin{aligned}
J^-(z) &= \beta(z) && (12)\\
J^3(z) &= \beta(z)\gamma(z) + b^{-1}\partial\phi(z) && (13)\\
J^+(z) &= \beta(z)\gamma^2(z) + 2b^{-1}\gamma(z)\partial\phi(z) - \left(b^{-2}+2\right)\partial\gamma(z) && (14)
\end{aligned}
$$

together with the anti-holomorphic counterparts $\overline{J}^{3,\pm}(\overline{z})$, where $b^{-2} = k-2$. In contrast, the theory (11) for generic m preserves only the Borel subalgebra of (12)-(14), which is generated only by the two currents $J^3(z)$ and $J^-(z)$,

$$J^-(z) = \beta(z), \qquad J^3(z) = \beta(z)\gamma(z) + \frac{m}{b}\partial\phi(z) \tag{15}$$

and by the pair of anti-holomorphic analogues. One can also consider the operator $J^+ = \beta\gamma^2 + 2mb^{-1}\gamma\partial\phi - (m^2b^{-2}+2)\,\partial\gamma$, which only represents a symmetry when $m = 0$ or $m = b^2$. Currents (15) obey the following operator product expansion (OPE)

$$J^-(z)J^3(w) \simeq \frac{J^-(w)}{(z-w)} + ... \qquad J^3(z)J^3(w) \simeq -\frac{(1+m^2b^{-2}/2)}{(z-w)^2} + ... \qquad J^-(z)J^-(w) \simeq ...$$

where the ellipses stand for regular terms that are omitted; and this realizes the Lie brackets for the modes $J^{3,-}_n = \frac{1}{2\pi i}\int dz J^{3,-}(z)z^{-n-1}$.

Varying the action in the boundary and imposing the condition $\delta(\beta+\bar{\beta})_{|z=\bar{z}} = 0$, one finds

$$\delta S|_{\partial\Gamma} = \frac{i}{4\pi}\int dx\Big[\left((-\partial+\bar{\partial})\phi - \xi\, b\,\beta^m\, e^{b\phi}\right)\delta\phi + \left(\gamma+\bar{\gamma} - \xi\, m\,\beta^{m-1}\, e^{b\phi}\right)\delta\beta\Big] \tag{16}$$

From this, one reads the gluing conditions in $z = \bar{z} = x$, which have to be

$$
\begin{aligned}
\beta(x) + \bar{\beta}(x)\ _{|z=\bar{z}} &= 0 && (17)\\
\gamma(x) + \bar{\gamma}(x)\ _{|z=\bar{z}} &= \xi\, m\, \beta^{m-1}(x)\, e^{b\phi(x)} && (18)\\
(-\partial + \bar{\partial})\, \phi(x)\ _{|z=\bar{z}} &= \xi\, b\, \beta^{m}\, (x) e^{b\phi(x)} && (19)
\end{aligned}
$$

And these conditions correspond to

$$
\begin{aligned}
J^{-}(x) + \bar{J}^{-}(x)\ _{|z=\bar{z}} &= 0, && (20)\\
J^{3}(x) - \bar{J}^{3}(x)\ _{|z=\bar{z}} &= 0. && (21)
\end{aligned}
$$

While (20) is evidently satisfied, condition (21) can be checked as follows

$$
\begin{aligned}
J^{3}(x) - \bar{J}^{3}(x) &= -\beta(x)\gamma(x) + \bar{\beta}(x)\bar{\gamma}(x) - \frac{m}{b}\left(\partial\phi(x) - \bar{\partial}\phi(x)\right) = \\
&= -\beta(x)\left(\gamma(x) + \bar{\gamma}(x) - \xi\, m\, \beta^{m-1}(x) e^{b\phi(x)}\right) = 0 && (22)
\end{aligned}
$$

where (17)-(19) were considered.

The other condition to be considered in the boundary is

$$
T\,(x) - \bar{T}(x)\ _{|z=\bar{z}} = 0 \tag{23}
$$

which guarantees boundary conformal symmetry. This classical analysis suggests that with these boundary conditions one obtains a theory that preserves the conformal symmetry generated by $J^{-}(z)$, $J^{3}(z)$, and $T(z)$. The free field computation ultimately helps to prove this explicitly.

Now we have discussed the boundary conditions and proposed the form of the boundary action, we are ready to undertake the calculation of the one-point function.

4 Path integral computation

The one-point function we are interested in is the vacuum expectation value $\langle \Phi^{j}(\mu|z)\rangle_{\mathcal{T}_{m,b}}$ of one bulk vertex operator (5) in the disk geometry. This is given by

$$\Omega_j^{(m,b)}(z) := \left\langle \Phi^j(\mu|z) \right\rangle_{\mathcal{T}_{m,b}} = \int \mathcal{D}\phi \mathcal{D}^2\beta \mathcal{D}^2\gamma \; e^{-S} \, |\rho(z)|^{2h_j} \, |\mu|^{2m(j+1)} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \, e^{2b(j+1)\phi(z,\bar{z})}, \tag{24}$$

with $\mathcal{D}^2\beta = \mathcal{D}\beta\mathcal{D}\bar{\beta}$, and $\mathcal{D}^2\gamma = \mathcal{D}\gamma\mathcal{D}\bar{\gamma}$, $|\rho(z)|^2 = e^{2\sigma(z,\bar{z})}$, and the action S is given by (11). Integrating over the fields γ and $\bar{\gamma}$ we get

$$\int \mathcal{D}\gamma \, e^{\frac{1}{2\pi}\int d^2w\gamma\bar{\partial}\beta} \, e^{\mu\gamma(z)} \;=\; \delta\left(\frac{1}{2\pi}\,\bar{\partial}\beta(w) + \mu\,\delta^{(2)}(w-z)\right) \tag{25}$$

and, respectively,

$$\int \mathcal{D}\bar{\gamma} \, e^{\frac{1}{2\pi}\int d^2\bar{w}\bar{\gamma}\partial\bar{\beta}} \, e^{\bar{\mu}\bar{\gamma}(\bar{z})} \;=\; \delta\left(\frac{1}{2\pi}\,\partial\bar{\beta}(\bar{w}) - \bar{\mu}\,\delta^{(2)}(\bar{w}-\bar{z})\right). \tag{26}$$

Using that $\bar{\partial}\left(\frac{1}{z}\right) = \partial\left(\frac{1}{\bar{z}}\right) = 2\pi\delta^{(2)}(z)$ we can write

$$\frac{1}{2\pi}\,\bar{\partial}\beta(w) + \frac{\mu}{2\pi}\,\bar{\partial}\left(\frac{1}{w-z}\right) \;=\; 0 \tag{27}$$

$$\frac{1}{2\pi}\,\partial\bar{\beta}(\bar{w}) - \frac{\bar{\mu}}{2\pi}\,\partial\left(\frac{1}{\bar{w}-\bar{z}}\right) \;=\; 0 \tag{28}$$

Then, in the boundary we impose

$$\beta + \bar{\beta}_{\;|z=\bar{z}} = 0. \tag{29}$$

Now, we integrate over β and $\bar{\beta}$. Considering the conditions above, we get a non-vanishing solution only if $\delta(\mu+\bar{\mu}) = 0$. The solution for β and $\bar{\beta}$ are thus given by

$$\beta_0(w) \;=\; -\frac{\mu}{w-z} - \frac{\bar{\mu}}{w-\bar{z}} \;=\; \frac{-\mu\,(z-\bar{z})}{(w-z)(w-\bar{z})}, \tag{30}$$

$$\bar{\beta}_0(\bar{w}) \;=\; \frac{\mu}{\bar{w}-z} + \frac{\bar{\mu}}{\bar{w}-\bar{z}} \;=\; \frac{-\bar{\mu}\,(z-\bar{z})}{(\bar{w}-z)(\bar{w}-\bar{z})}. \tag{31}$$

Now, we can follow the analysis of [49] closely. We consider the exact differential $\rho(w)\beta_0(w)$, whose expression we can give in terms of its poles, up to a global factor u. Then, write

$$\rho(w)\beta_0(z) \;=\; u\,\frac{1}{w-z}\,\frac{1}{w-\bar{z}}. \tag{32}$$

Finding the residues of $\rho(w)\beta_0(z)$, we obtain $\rho(z)\mu = -\rho(\bar{z})\bar{\mu} = u(z-\bar{z})^{-1}$, and therefore

$$u = \rho(z)\,\mu(z-\bar{z}) = \rho(z)\,\mu\,2iy. \tag{33}$$

Evaluating β according to (32), the function to be computed now takes the form

$$\Omega_j^{(m,b)}(z) = \int \mathcal{D}\phi \exp - \left(\frac{1}{2\pi} \int d^2w \left(\partial\phi\bar{\partial}\phi + b^2|u|^{2m}|w-z|^{-2m}|w-\bar{z}|^{-2m}\,|\rho(w)|^{-2m}\,e^{2b\phi} \right) \right) \times \\ \times \exp\left(-\frac{i\xi}{4\pi} \int d\tau\, u^m(\tau-z)^{-m}\,(\tau-\bar{z})^{-m}\,\rho(w)^{-m}\,e^{b\phi} \right) |\rho(z)|^{2h_j}\,|\mu|^{2m(j+1)}\,e^{2b(j+1)\phi(z,\bar{z})}\delta^{(2)}(\mu+\bar{\mu}) \tag{34}$$

Now, consider the following change of variables (i.e. field redefinition)

$$\phi(w) \to \phi(w) - \frac{m}{b}\ln|u|,$$

which consequently gives $e^{b\phi(w)} \to e^{b\phi(w)}\,|u|^{-m}$. This induces a change in the linear "dilaton" term $-\frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}\,Q_m R\phi - \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}KQ_m\phi$, which now takes the form

$$-\frac{1}{8\pi}\int d^2w\, g^{1/2}\,Q_m R\phi - \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}KQ_m\phi + \\ + \ln|u|m\left(1+\frac{1-m}{b^2}\right)\left[\frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}R + \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}K\right] \tag{35}$$

Using the Gauss-Bonnet theorem, which states that the Euler characteristic of the disk is

$$\chi(\Gamma) = \frac{1}{8\pi}\int_\Gamma d^2w\, g^{1/2}R + \frac{1}{2\pi}\int_{\partial\Gamma} d\tau\, g^{1/4}K = 1, \tag{36}$$

we find

$$\Omega_j^{(m,b)}(z) = \int \mathcal{D}\phi \exp - \left(\frac{1}{2\pi} \int d^2w \left(\partial\phi\bar{\partial}\phi + b^2|w-z|^{-2m}\,|w-\bar{z}|^{-2m}|\rho(z)|^{-2m}e^{2b\phi} \right) \right) \times \\ \times \exp\left(-\frac{i\xi}{4\pi}\left(\frac{u}{|u|}\right)^m \int d\tau(\tau-z)^{-m}\,(\tau-\bar{z})^{-m}\,\rho^{-m}(\tau)e^{b\phi} \right) \times \\ \times\, \delta^{(2)}(\mu+\bar{\mu})|\rho(z)|^{2h_j}\,|\mu|^{2m(j+1)}\,|u|^{-m(2j+1+b^{-2}(m-1))}e^{2b(j+1)\phi(z,\bar{z})} \tag{37}$$

Now, let us perform a second change of variables,

$$\phi(w,\bar{w}) \to \varphi(w,\bar{w}) + \frac{m}{2b}\left(\ln|w-z|^2 + \ln|w-\bar{z}|^2 + \ln|\rho(w)|^2\right), \tag{38}$$

which amounts to say

$$e^{b\phi} \to e^{b\varphi}\,|w-z|^m\,|w-\bar{z}|^m\,|\rho(w)|^m \tag{39}$$

and

$$\partial\bar{\partial}\phi(w,\bar{w}) \to \partial\bar{\partial}\varphi(w,\bar{w}) + \frac{m\pi}{b}\delta^{(2)}\left(|w-z|\right) + \frac{m\pi}{b}\delta^{(2)}(|w-\bar{z}|) + \frac{m}{2b}\partial\bar{\partial}\ln|\rho(w)|^2. \tag{40}$$

In the bulk action this change produces a transformation in the kinetic term $-\frac{1}{2\pi}\int_\Gamma d^2w\partial\phi\bar{\partial}\phi$, which becomes

$$\begin{aligned}
&\frac{1}{2\pi}\int_\Gamma d^2w\,\varphi\partial\bar{\partial}\varphi + \frac{m}{2b}\int_\Gamma d^2w\,\varphi\,\delta^{(2)}(|w-z|) + \frac{m}{2b}\int_\Gamma d^2w\,\varphi\,\delta^{(2)}(|w-\bar{z}|) + \\
&+\frac{m}{4\pi b}\int_\Gamma d^2w\,\varphi\,\partial\bar{\partial}\ln|\rho(w)|^2 + \frac{m^2}{4\pi b^2}\int_\Gamma d^2w\left(\ln|w-z|^2 + \ln|w-\bar{z}|^2 + \ln|\rho(w)|^2\right)\times \\
&\times\left(\frac{b}{m}\partial\bar{\partial}\varphi + \pi\delta^{(2)}(|w-z)| + \pi\delta^{(2)}(|w-\bar{z}|) + \frac{1}{2}\,\partial\bar{\partial}\ln|\rho(w)|^2\right).
\end{aligned} \tag{41}$$

Using the regularization [49]

$$\lim_{w\to z}\ln|w-z|^2 \equiv \ln|\rho(z)|^2, \tag{42}$$

with $2\sigma(z,\overline{z}) \equiv \ \ln|\rho(z)|^2$, and

$$ds^2 \;=\; |\rho(z)|^2\,dzd\bar{z}, \qquad \sqrt{g}R \;= -4\,\partial\bar{\partial}\ln|\rho(z)|^2, \tag{43}$$

one finds that the right hand side of (41) takes the form

$$-\frac{1}{2\pi}\int_\Gamma d^2w\,\partial\varphi\bar{\partial}\varphi \;-\; \frac{1}{2\pi}\int_\Gamma d^2w\,\sqrt{g}\frac{m}{4b}R\varphi \;+\; \frac{m}{b}\varphi(z) + \frac{m^2}{2b^2}\ln|z-\overline{z}| + \frac{m^2}{4b^2}\ln|\rho(z)|^2. \tag{44}$$

The boundary action also suffers a change. Recalling how to write the extrinsic curvature K in terms of $\partial_x^2\ln|\rho(\tau)|^2$, we find that, under the change (38), the boundary action changes as follows

$$\delta S_{\text{boundary}} = \;\frac{m}{2\pi b}\int_{\partial\Gamma} g^{1/4}K\phi. \tag{45}$$

In turn, the changes above induce a modification in the value of the background change Q_m, shifting it as follows

$$Q_m \;\rightarrow\; Q_m + \frac{m}{b} \;=\; b + \frac{1-m}{b} + \frac{m}{b} \;=\; b + \frac{1}{b} \;=\; Q_{m=0} \tag{46}$$

which gives the Liouville background charge $Q_{\text{Liouville}} = b + 1/b$ as the result.

With all this, the one-point function we are trying to compute takes the form

$$\begin{aligned}\Omega_j^{(m,b)}(z) = \;& \delta^2(\mu+\bar{\mu})|u|^{m(1+\frac{1-m}{b^2})}\,|z-\bar{z}|^{\frac{m^2}{2b^2}} \int \mathcal{D}\varphi \exp\left[-\left(\frac{1}{2\pi}\int_\Gamma d^2w\,\left(\partial\varphi\bar{\partial}\varphi + b^2 e^{2b\varphi}\right)\right)\right] \times \\ & \times \exp\left[-\frac{i\xi}{4\pi}\left(\frac{u}{|u|}\right)^m \int_{\partial\Gamma} d\tau\,\left(\frac{|\tau-z|}{\tau-z}\,\frac{|\tau-\bar{z}|}{\tau-\bar{z}}\,\frac{|\rho(z)|}{\rho(z)}\right)^m e^{b\varphi}\right]\, e^{2b(j+1)}\, e^{\frac{m}{b}\varphi(z)} \times \\ & \times |\mu|^{2m(j+1)}\,|z-\bar{z}|^{2m(j+1)}\,|u|^{-2m(j+1)}|\rho(z)|^{2h_j+\frac{m^2}{2b^2}}\end{aligned} \tag{47}$$

Then, taking into account (42),

$$e^{2b(j+1)\varphi(z,\bar{z})}\, e^{\frac{m}{b}\varphi(z,\bar{z})} \;=\; e^{\left(2b(j+1)+\frac{m}{b}\right)\varphi(z,\bar{z})}\,|\rho(z)|^{2m(j+1)}, \tag{48}$$

what amounts to extract the pole in the coincidence limit of the two operators $e^{2b(j+1)\varphi(z,\bar{z})}$ and $e^{\frac{m}{b}\varphi(w,\overline{w})}$. Then we find

$$\begin{aligned}\Omega_j^{(m,b)}(z) = \;& \delta^2(\mu+\bar{\mu})|\mu|^{mb^{-2}(b^2+1-m)}(2y)^{mb^{-2}(b^2+1-m/2)} \int \mathcal{D}\varphi \exp\left[-\frac{1}{2\pi}\int_\Gamma d^2w\,\left(\partial\varphi\bar{\partial}\varphi + b^2\,e^{2b\varphi}\right)\right] \times \\ & \times \exp\left[-\frac{i\xi}{4\pi}(\text{sgn}(\text{Im}\mu))^m \int_{\partial\Gamma} d\tau\, e^{b\varphi}\right]\,|\rho(z)|^{2h_j+m(b^2+1-m)/b^2}\, e^{(2b(j+1)+m/b)\varphi(z,\bar{z})}\end{aligned} \tag{49}$$

where sgn(Imμ) is the sign of the imaginary part of μ. Remarkably, the expression on the right hand side of (49) corresponds to the disk one-point function in Liouville field theory multiplied by a factor $\delta^2(\mu+\bar{\mu})|\mu|^{mb^{-2}(b^2+1-m)}(2y)^{mb^{-2}(b^2+1-m/2)}$, provided one agrees to identify the Liouville parameters α, b, μ_L and μ_B as follows

$$\mu_L = \frac{b^2}{2\pi} = \frac{\lambda}{2\pi}, \tag{50}$$

$$\mu_B = \frac{i\xi}{4\pi}(\mathrm{sgn}(\mathrm{Im}\mu))^m = \frac{\lambda_B}{4\pi}, \tag{51}$$

$$\alpha = b(j+1) + \frac{m}{2b}, \tag{52}$$

$$h_\alpha = \alpha(Q_L - \alpha) \ = h_j + \frac{m}{2b^2}(b^2 + 1 - \frac{m}{2}), \tag{53}$$

where we are using the standard notation; see for instance [11, 30]. This is to say

$$\left\langle \Phi_j(\mu|z) \right\rangle_{\mathcal{T}_{m,b}} = \delta^2(\mu+\bar{\mu})|\mu|^{m(1+\frac{1-m}{b^2})}\,(2y)^{m(1+\frac{1}{b^2}-\frac{m}{b^2})}\left\langle V_\alpha(z) \right\rangle_{\text{Liouville}} \tag{54}$$

where $V_\alpha(z) = e^{2\alpha\varphi(z,\bar{z})}$. This trick, which is exactly the one used in [2] to solve the case $m=1$, leads us to obtain the explicit expression for $\Omega_j^{(m,b)}$ in terms of the Liouville one-point function $\Omega_{j+m/2b^2}^{(m=0,b)}$. In fact, the expression for the disk one-point function of a bulk operator in Liouville theory is actually known [6, 11]; it reads

$$\left\langle V_\alpha(z) \right\rangle_{\text{Liouville}} = |z-\bar{z}|^{-2h_\alpha}\frac{2}{b}\Omega_0\left(\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{\frac{Q-2\alpha}{2b}}\cosh(2\pi s(2\alpha-Q))\,\Gamma(2b\alpha-b^2)\,\Gamma\left(\frac{2\alpha-Q}{b}\right) \tag{55}$$

where s is given by

$$\cosh(2\pi bs) = \frac{\mu_B}{\sqrt{\mu_L}}\sqrt{\sin(\pi b^2)} \tag{56}$$

and where Ω_0 is an irrelevant overall factor we determine below (see Section 5).

Thus, by following the trick in [2] (see also [37, 38, ?]), we managed to calculate the expectation value $\Omega_j^{(m,b)}$ of a bulk operator in the theory (1) on the disk by reducing such calculation to that of the observable $\Omega_{j=\frac{\alpha}{b}-\frac{m}{2b^2}-1}^{(m=0,b)}$ in Liouville field theory. The final result reads

$$\begin{aligned}\Omega_j^{(m,b)} = {} & \frac{2}{b}\Omega_0\,\delta(\mu+\bar{\mu})|\mu|^{m(1+\frac{1-m}{b^2})}\,|z-\bar{z}|^{-2h_j}\left(\pi\frac{\Gamma(1-b^2)}{\Gamma(b^2+1)}\right)^{j+\frac{1}{2}-\frac{1-m}{2b^2}}\times \\ & \times\Gamma\left(2j+1-\frac{1-m}{b^2}\right)\,\Gamma\left(b^2(2j+1)+m\right)\,\cosh\left(2\pi bs\left(2j+1-\frac{1-m}{b^2}\right)\right)\end{aligned} \tag{57}$$

with $h_j = -b^2 j(j+1) + (j+1)(1-m)$ and

$$\cosh(2\pi b s) \;=\; i\,\xi\,(\mathrm{sgn}(\mathrm{Im}\mu))^m \sqrt{\frac{\sin(\pi b^2)}{8\pi b^2}}. \tag{58}$$

In the next section we will reobtain this result using the free field calculation.

5 Free-field computation

In this section, we compute the bulk one-point function in a different way. We will consider the free field approach, which has proven to be a useful method to calculate correlation functions in this type of non-rational models on the sphere [51, 52, 53].

Considering Neumann boundary conditions for ϕ, $\beta = -\bar{\beta}$ and $\gamma = -\bar{\gamma}$, the non-vanishing correlators for the theory on the disk turn out to be

$$\big\langle \phi(z,\bar{z})\phi(w,\bar{w}) \big\rangle \;=\; -\ln|z-w||z-\bar{w}|\ , \tag{59}$$

$$\big\langle \beta(z)\gamma(w) \big\rangle \;=\; -\frac{1}{z-w}, \quad \big\langle \bar{\beta}(\bar{z})\bar{\gamma}(\bar{w}) \big\rangle = -\frac{1}{\bar{z}-\bar{w}}, \tag{60}$$

$$\big\langle \bar{\beta}(\bar{z})\gamma(w) \big\rangle \;=\; \frac{1}{\bar{z}-w}, \quad \big\langle \beta(z)\bar{\gamma}(\bar{w}) \big\rangle = \frac{1}{z-\bar{w}}. \tag{61}$$

where we see the mixing between holomorphic and anti-holomorphic modes due to the presence of the boundary conditions.

Now, we have to verify that the boundary term we added to the action actually corresponds to a theory that preserves the symmetry generated by $J^-(z)$, $J^3(z)$ and $T(z)$. Checking this at the first order in λ_B is actually simple, and it amounts to check that the following expectation values vanish,

$$\left\langle \left(J^-(z) + \bar{J}^-(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \dots \right\rangle_{|z=\bar{z}} \;=\; 0 \tag{62}$$

$$\left\langle \left(J^3(z) - \bar{J}^3(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \dots \right\rangle_{|z=\bar{z}} \;=\; 0 \tag{63}$$

Once again, this is completely analogous to the analysis done in [2] for $m=1$. However, in contrast to the case of the WZNW theory, where the condition $J^+(z) + \bar{J}^+(\bar{z}) = 0$ in known to hold as well [2], in the case of generic m we have to impose

$$\left\langle \left(T(z) - \bar{T}(\bar{z})\right) \int_{\partial\Gamma} d\tau \beta^m(\tau) e^{b\phi(\tau)} \ldots \right\rangle_{|z=\bar{z}} = 0 \tag{64}$$

explicitly. To verify that conditions (62)-(64) are obeyed, first we compute the following operator product expansions (OPEs)

$$J^-(z)\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{65}$$

$$\bar{J}^-(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim 0 \tag{66}$$

and

$$J^3(z)\beta^m(\tau)e^{b\phi(\tau)} \sim \frac{m\beta(z)\beta^{m-1}(\tau)e^{b\phi(\tau)}}{z-\tau} - \frac{m\beta^m(\tau)e^{b\phi(\tau)}}{z-\tau} \sim 0 \tag{67}$$

$$\bar{J}^3(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim -\frac{m\bar{\beta}(\bar{z})\beta^{m-1}(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau} - \frac{m\beta^m(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau} \sim 0 \tag{68}$$

where we considered that in the boundary $\bar{\beta}(x) = -\beta(x)$, and where the symbol ~ 0 means that the singular terms vanish when evaluating in the boundary.

Next, we have to impose condition (64). To do so, we have to compute the OPE

$$T(z)\beta^m(\tau)e^{b\phi(\tau)} \sim \partial_\tau \left(\frac{\beta^m(\tau)e^{b\phi(\tau)}}{z-\tau}\right) - \frac{ib\partial_\sigma\phi(\tau)\beta^m(\tau)e^{b\phi(\tau)}}{z-\tau} + \ldots \tag{69}$$

where, again, the symbol $\sim$ means that the equivalence holds up to regular terms and exact differentials when evaluating in the boundary. Analogously, we have the anti-holomorphic part

$$\bar{T}(\bar{z})\beta^m(\tau)e^{b\phi(\tau)} \sim \partial_\tau \left(\frac{\beta^m(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau}\right) - \frac{ib\partial_\sigma\phi(\tau)\beta^m(\tau)e^{b\phi(\tau)}}{\bar{z}-\tau} + \ldots \tag{70}$$

In the boundary we find that (69) and (70) contribute with the same piece and then we verify that (64) is actually satisfied.

Then, we can proceed and compute the bulk one-point function using the free field approach. This observable is given by

$$\Omega_j^{(m,b)} = \int \mathcal{D}\phi \mathcal{D}^2\beta \mathcal{D}^2\gamma \, \exp\Big[-\frac{1}{2\pi}\int_\Gamma \partial\phi\bar{\partial}\phi + \frac{1}{2\pi}\int_\Gamma \gamma\bar{\partial}\beta + \frac{1}{2\pi}\int_\Gamma \bar{\gamma}\partial\bar{\beta} - \frac{Q_m}{8\pi}\int_\Gamma g^{1/2}R\phi - $$
$$- \frac{b^2}{2\pi}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi} - \frac{Q_m}{2\pi}\int_{\partial\Gamma} g^{1/4}K\phi + \frac{i\xi}{4\pi}\int_{\partial\Gamma}\beta^m e^{b\phi}\Big] |\mu|^{2m(j+1)} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \, e^{2b(j+1)\phi(z,\bar{z})} \tag{71}$$

with $Q_m = b + \frac{1-m}{b}$. The notation we use is such that $z = x + iy$ and $w = \tau + i\sigma$.

Splitting the field ϕ in its zero-mode ϕ_0 and its fluctuations $\phi' = \phi - \phi_0$, and recalling that the Gauss-Bonnet contribution gives

$$\frac{1}{8\pi}\int_\Gamma g^{1/2}R \; Q_m\phi_0 + \frac{1}{2\pi}\int_{\partial\Gamma} g^{1/4}K \; Q_m\phi_0 \; = \; Q_m\phi_0, \tag{72}$$

we find

$$\Omega_j^{(m,b)} = \int \mathcal{D}\phi' d\phi_0 \mathcal{D}^2\beta \mathcal{D}^2\gamma \, e^{-S'_{|\lambda=0}} \exp\left[\frac{-b^2}{2\pi} e^{2b\phi_0}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi'}\right] \exp\left[\frac{i\xi}{4\pi} e^{b\phi_0}\int_{\partial\Gamma}\beta^m e^{b\phi'}\right] \times$$
$$\times \, e^{2b(j+1)\phi_0} \, e^{-Q_m\phi_0} \, |\mu|^{2m(j+1)} \, e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})} \, e^{2b(j+1)\phi'(z,\bar{z})} \tag{73}$$

where $S'_{|\lambda=0}$ means the free action, i.e. the action (11) with $\lambda = \lambda_B = 0$ evaluated on the field fluctuations ϕ'.

Then, we have to integrate over the zero-mode ϕ_0. The interaction terms in the actions give the following contribution to the integrand

$$\exp\left(\frac{-b^2}{2\pi} e^{2b\phi_0}\int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi'} + \frac{i\xi}{4\pi} e^{b\phi_0}\int_{\partial\Gamma}\beta^m e^{b\phi'}\right), \tag{74}$$

while the vertex operator itself contributes with an exponential $e^{(2b(j+1)-Q_m)\phi_0}$. There is also a contribution from the background charge. To handle the integration over the zero-mode it is useful to be reminded of what we do in the case of the topology of the sphere, $\xi = 0$, where one first rewrite the integrand in (74) as

$$\int_{\mathbb{R}_{>0}} dv \exp\left[-\frac{b^2}{2\pi}\, v \int_\Gamma (-\beta\bar{\beta})^m e^{2b\phi'}\right] \delta\left(v - e^{2b\phi_0}\right) \tag{75}$$

and then arrives to the integral

$$I[\phi'] \;=\; \int_{\mathbb{R}} d\phi_0 \int_{\mathbb{R}_{>0}} dv \;\; v^{j+1/2+(m-1)/2b^2}\, e^{-v\frac{b^2}{2\pi}\int_\Gamma(-\beta\bar{\beta})^m\, e^{2b\phi'}}\;\delta\left(v-e^{2b\phi_0}\right) \tag{76}$$

with $\delta\left(v-e^{2b\phi_0}\right) \;=\; \frac{1}{2bv}\delta\left(\phi_0-\frac{\ln v}{2b}\right)$. Then, one can integrate out the zero-mode contribution and use the integral expression of the Γ-function $\Gamma(x)=\int_{\mathbb{R}_{>0}} dt\; t^{x-1}e^{-t}$ to integrate over v. This gives rise to an overall factor $\Gamma(-n)$, with $n=-2j-1+(1-m)/b^2$, multiplied by n screening operators $\int_\Gamma(-\beta\bar{\beta})^m\, e^{2b\phi'}$. The factor $\Gamma(-n)$ develops single poles for $n\in\mathbb{Z}_{\geq 0}$, and the residues of such divergent contributions are from what we extract the form of the "resonant" correlators. For the disk geometry, where both bulk and boundary screening operators are present, it is possible to show that a similar expression is obtained. Standard techniques in the free field calculation yield the following expression for the residues of resonant correlators,

$$\begin{aligned}\underset{2j+1+\frac{m-1}{b^2}=-n}{\text{Res}}\;\Omega_j^{(m,b)} \;=\; & \frac{1}{2b}\,|\mu|^{2m(j+1)} \sum_{\substack{p,l=0\\2p+l=n}}^{\infty} \frac{1}{p!l!}\prod_{i=1}^{\infty}\int_\Gamma d^2w_i \prod_{k=1}^{\infty}\int_{\partial\Gamma} d\tau_k \Big\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\times \\ & \times e^{2b(j+1)\phi(z,\bar{z})}\prod_{i=1}^{p}\frac{b^2}{2\pi}(-\beta\bar{\beta})^m\, e^{2b\phi(w_i,\bar{w}_i)}\prod_{k=1}^{l}\frac{i\xi}{4\pi}\beta^m\, e^{b\phi(x_k)}\Big\rangle\end{aligned} \tag{77}$$

where, as mentioned, now both bulk and boundary screening operators, $\int_\Gamma(-\beta\bar{\beta})^m e^{2b\phi}$ and $\int_{\partial\Gamma}\beta^m e^{b\phi}$, appear. As shown above, the integration over the zero-mode ϕ_0 gives a charge conservation condition that demands to insert a precise amount of screening operators for the correlators not to vanish; p of these screening operators are to be inserted in the bulk, while l of them in the boundary, with $2p+l=n$. The precise relation is

$$2b(j+1)+2\,b\,p+b\,l \;=\; Q_m \;=\; b+\frac{1-m}{b}; \tag{78}$$

that is $n \;=\; 2\,p+l \;=-2j-1+(1-m)/b^2$.

The correlator then factorizes out in two parts: the part that depends on ϕ, and the contribution of the γ-β ghost system. The Coulomb gas calculation of the ϕ contribution yields

$$\left\langle e^{2(j+1)\phi(iy)}\prod_{i=1}^{p}e^{2b\phi(w_i,\bar{w}_i)}\prod_{k=1}^{l}e^{b\phi(\tau_k)}\right\rangle = \left[\prod_{k=1}^{l}(y^2+\tau_k^2)\prod_{i=1}^{p}|y^2+w_i^2|^2\right]^{-2b^2(j+1)} \times$$
$$\times\, |2y|^{-2b^2(j+1)^2}\left[\prod_{i,k}|w_i-x_k|^2\prod_{i<i'}|w_i-w_{i'}|^2\prod_{i,i'}|w_i-\bar{w}_{i'}|\prod_{k<k'}|x_k-x_{k'}|\right]^{-2b^2} \tag{79}$$

On the other hand, for working out the ghost contribution γ-β it is convenient first to consider the OPE

$$\left\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\beta(w)\right\rangle = \frac{\mu}{w-z}+\frac{\bar{\mu}}{\bar{w}-\bar{z}} = \frac{\mu\,(z-\bar{z})}{(w-z)(\bar{w}-\bar{z})}. \tag{80}$$

This implies that the γ-β correlator takes the form

$$\left\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\prod_{i=1}^{p}\left(\beta(w_i)\bar{\beta}(\bar{w}_i)\right)^m\right\rangle = \mu^{2pm}\,(2iy)^{2pm}\,(-1)^{pm}\prod_{i=1}^{p}\frac{1}{|y^2+w_i^2|^{2m}}$$

and

$$\left\langle e^{\mu\gamma(z)-\bar{\mu}\bar{\gamma}(\bar{z})}\prod_{k=1}^{l}\beta^m(\tau_k)\right\rangle = \mu^{lm}\,(2iy)^{lm}\prod_{k=1}^{l}\frac{1}{(y^2+\tau_k)^m}.$$

The full correlator is then given by

$$\left\langle e^{\mu\gamma(iy)-\bar{\mu}\bar{\gamma}(-iy)}\prod_{i=1}^{p}\frac{b^2}{2\pi}(\beta(w_i)\bar{\beta}(\bar{w}_i))^m\prod_{k=1}^{l}\frac{i\xi}{4\pi}\beta(\tau_k)^m\right\rangle =$$
$$= 2\pi\,\delta(\mu+\bar{\mu})\,(-1)^{pm}\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l|2u|^{nm}\,|\mu|^{nm}\prod_{i=1}^{p}\frac{1}{|y^2+w_i^2|^{2m}}\prod_{k=1}^{l}\frac{1}{(y^2+\tau_k^2)^m} \tag{81}$$

Notice that, in principle, integral expression (77) only makes sense for $p,l\in\mathbb{Z}_{\geq 0}$, what translates into a restriction on the values of j. However, one can try to define the generic case by means of an analytic continuation of the obtained expression. In addition to the analytic extension in the number of screening charges n (which is already present in the calculation performed for $m=0$ and $m=1$) the case of generic $m\in\mathbb{R}$ demands to deal with making sense of non-integer powers of β, i.e. a non-integer amount of fields β when computing the Wick contraction of the γ-β contribution. Free field calculations with non-integer m were already discussed in [51].

Returning to the form of the resonant correlators, for which p and l are integer numbers, the residues of these observables take the form

$$\begin{aligned}\operatorname*{Res}_{2j+1+\frac{m-1}{b^2}=-n}\Omega^{(m,b)} &= \frac{\pi}{b}\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{p!l!}\delta_{2p+l-n,0}(-1)^{pm}\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l\times\\ &\times\int\prod_{i=1}^{p}\frac{d^2w_i}{|y^2+w_i^2|^{4b^2(j+1)+2m}}\int\prod_{k=1}^{l}\frac{dx_k}{(y^2+x_k^2)^{b^2(j+1)+m}}\left[\prod_{i,k}|w_i-x_k|^2\prod_{i<i'}|w_i-w_{i'}|^2\times\right.\\ &\left.\times\prod_{i,i'}|w_i-\bar{w}_{i'}|\prod_{k<k'}|x_k-x_{k'}|\right]^{-2b^2}\delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}|2y|^{-2b^2(j+1)^2+nm}.\end{aligned}\tag{82}$$

To solve this we use the following integral formula (see Ref. [2] for the computation in the case $m=1$)

$$\begin{aligned}Y_{n,p}(a) &= \frac{1}{p!(n-2p)!}\int\prod_{i=1}^{p}\frac{d^2w_i}{|y^2+w_i^2|^{2a}}\int\prod_{k=1}^{l}\frac{dx_k}{(y^2+x_k^2)^a}\left[\prod_{i,k}|w_i-x_k|^2\prod_{i<i'}|w_i-w_{i'}|^2\times\right.\\ &\left.\times\prod_{i,i'}|w_i-\bar{w}_{i'}|\prod_{k<k'}|x_k-x_{k'}|\right]^{-2b^2}\end{aligned}\tag{83}$$

with $a=2b^2(j+1)+m=1+b^2-b^2n$. The solution of this multiple Selberg-type integral is given by

$$Y_{n,p}(a) = |z-\bar{z}|^{n\left(1-2a-(n-1)b^2\right)}\left(\frac{2\pi}{\Gamma(1-b^2)}\right)^n\frac{2^{-2p}}{n!(\sin(\pi b^2))^p}I_n(a)\,J_{n,p}(a),\tag{84}$$

where

$$I_n(a) = \prod_{i=0}^{n-1}\frac{\Gamma\left(1-(i-1)b^2\right)\Gamma\left(2a-1+(n-1+i)b^2\right)}{\Gamma^2\left(a+ib^2\right)}\tag{85}$$

and

$$J_{n,p}(a) = \sum_{i=0}^{p}(-1)^i \frac{\Gamma(n-p-i+1)}{\Gamma(p-i+1)\Gamma(n-2p+1)} \frac{\sin(\pi b^2(n-2i+1))}{\sin(\pi b^2(n-i+1))} \times$$
$$\times \prod_{r=0}^{i-1} \frac{\sin(\pi b^2(n-r))\sin(\pi a+\pi b^2(n-r))}{\sin(\pi b^2(r+1))\sin(\pi a+\pi b^2 r)}. \tag{86}$$

For the case of our interest, $a = 1 + b^2(1-n)$, and the function $I_n(a)$ simplifies substantially, taking the value $I_n(a) = \Gamma(1-b^2 n)$. The sum over $J_{n,p}(a)$ also simplifies notably; see below.

Then, we find the expression

$$\underset{2j+1+\frac{m-1}{b^2}=-n}{\mathrm{Res}}\, \Omega^{(m,b)} = \frac{\pi}{b}\,\delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}\,|z-\bar{z}|^{-2h_j}\left(\frac{2\pi}{\Gamma(1-b^2)}\right)^n \frac{\Gamma(1-b^2 n)}{n!}\times$$
$$\times \sum_{\substack{p,l=0\\ 2p+l=n}}^{\infty}(-1)^{pm}\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l \frac{2^{-2p}}{(\sin(\pi b^2))^p}\,J_{n,p}(a) \tag{87}$$

with $h_j = b^2 j(j+1) - (j+1)(1-m)$.

We may try to simplify the expression above further. Following [2], for m odd we find

$$\sum_{p=0}^{|n/2|}(-1)^p\,(2\cosh(2\pi bs))^{n-2p}\,J_{n,p}(1+b^2(1-n)) = \cosh(2\pi nbs), \tag{88}$$

where we replaced $a = 1 + b^2 - b^2 n$. According to the notation introduced in (58), we write

$$\xi = i(-\mathrm{sgn}(\mathrm{Im}\mu))^m\sqrt{\frac{8\pi b^2}{\sin(\pi b^2)}}\,\cosh(2\pi bs), \tag{89}$$

and this yields

$$\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^{n-2p} = \left(-\frac{1}{4\pi}\sqrt{\frac{8\pi b^2}{\sin(\pi b^2)}}\right)^{n-2p}(\cosh((2\pi bs)))^{n-2p}. \tag{90}$$

Replacing this into the sum in (87) we obtain

$$\sum_{p=0}^{\infty}\sum_{l=0}^{\infty}\delta_{2p+l-n,0}(-1)^{pm}\left(\frac{b^2}{2\pi}\right)^p\left(\frac{i\xi}{4\pi}(-\mathrm{sgn}(\mathrm{Im}\mu))^m\right)^l\frac{2^{-2p}}{(\sin(\pi b^2))^p}J_{n,p}(a) = \tag{91}$$
$$= (-1)^n\,2^{-\frac{3}{2}n}\,\pi^{-n}\left(\Gamma(1-b^2)\Gamma(1+b^2)\right)^{\frac{n}{2}}\cosh(2\pi nbs),$$

which can be proven using, in particular, the relation $\pi/\sin(\pi b^2) = \Gamma(1-b^2)\Gamma(b^2)$. It is worth noticing that the formula above holds for m odd, as we needed to use $(-1)^{pm} = (-1)^p$. Nevertheless, the final expression (after analytic continuation) turns out to be valid for generic $m \in \mathbb{R}$.

Putting all the pieces together, the final result for the resonant correlators reads

$$\begin{aligned}\underset{2j+1+\frac{m-1}{b^2}=-n}{\mathrm{Res}}\,\Omega^{(m,b)} &= 2^{-\frac{n}{2}}\,\frac{\pi}{b}\,\delta(\mu+\bar{\mu})|\mu|^{m\left(1+\frac{1-m}{b^2}\right)}\,|z-\bar{z}|^{-2h_j}\,\frac{(-1)^n}{n!}\left(\frac{\Gamma(1+b^2)}{\Gamma(1-b^2)}\right)^{\frac{n}{2}}\times\\ &\quad\times\Gamma(1-b^2n)\,\cosh(2\pi\,n\,b\,s)\end{aligned} \tag{92}$$

Analytic continuation in n is achieved by recalling that for $n \in \mathbb{Z}_{\geq 0}$ it holds

$$\lim_{\varepsilon\to 0}\varepsilon\Gamma(\varepsilon-n) = (-1)^n\frac{1}{\Gamma(n+1)},$$

which allows to replace the factor $(-1)^n/n!$ in (92) as follows

$$\frac{(-1)^n}{\Gamma(n+1)} \to \Gamma(2j+1+(m-1)b^{-2}).$$

were we used $(1-m)b^{-2}-2j-1 = n$. Written in terms of j, b, and m, the final result reads

$$\begin{aligned}\Omega_j^{(m,b)} &= \left(\frac{\pi}{2}\right)^{\frac{n}{2}}\frac{\pi}{b}\;\delta(\mu+\bar{\mu})|\mu|^{m(1+\frac{1-m}{b^2})}\,|z-\bar{z}|^{-2h_j}\left(\pi\,\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{(2j+1+(m-1)b^{-2})/2}\times\\ &\quad\times\Gamma\left(2j+1-\frac{1-m}{b^2}\right)\,\Gamma\left(b^2(2j+1)+m\right)\,\cosh\left(2\pi bs\left(2j+1-\frac{1-m}{b^2}\right)\right)\end{aligned} \tag{93}$$

with $h_j = -b^2j(j+1)+(j+1)(1-m)$ and $\cosh(2\pi bs) = i\xi(\mathrm{sgn}(\mathrm{Im}\mu))^m\sqrt{\sin(\pi b^2)/8\pi b^2}$. And we see that expression (93) coincides with the path integral result (57) if we chose the numerical global coefficient in (57) to be $\Omega_0 = (\pi/2)^{n/2}$. Therefore, we find exact agreement between the path integral approach and the free-field calculation. This manifestly shows that free field approach, if combined with the appropriate analytic continuation, turns out to be a useful method to find the general expression of correlation functions in non-rational CFTs. The path integral method actually reduced the problem to that of computing an observable of Liouville theory, which may be done by different methods, e.g. the bootstrap method. The relation existing between correlators of the conformal theories $\mathcal{T}_{m,b}$ defined by the action (11) and correlators of Liouville theory is a generalization of the so called H_3^+ WZNW-Liouville correspondence [56, 57, 58, 49]. The free field approach, on the other hand, gives a quite direct computation of the disk one-point function.

6 Analysis of the one-point function

Let us first analyze some special cases of the general result (93). The first particular example we may consider is clearly the theory for $m = 0$, namely $\mathcal{T}_{m=0,b}$. In this case, one-point function (93) trivially reduces to Liouville disk one-point function (55). Normalized states $|\alpha\rangle$ of the theory are created by the action of expònential vertex operators on the vacuum, $\lim_{z\to 0} e^{2\alpha\varphi(z)}\,|0\rangle = |\alpha\rangle$, having momentum $\alpha = Q + iP$, with $P \in \mathbb{R}$ and $Q = b + b^{-1}$; see for instance [54, 30, 55]. Then, Liouville one-point function reads

$$\Big\langle V_\alpha(z)\Big\rangle_{\text{Liouville}} \;=\; |z-\bar{z}|^{-2h_\alpha}\left(\pi\frac{\Gamma(b^2)}{\Gamma(1-b^2)}\right)^{\frac{-iP}{b}}\frac{\cos(4\pi sP)}{iP}\,\Gamma(1+2ibP)\,\Gamma\left(1+\frac{2iP}{b}\right) \qquad (94)$$

where s given by $\cosh(2\pi bs) = \sqrt{\sin(\pi b^2)\mu_B^2/\mu_L}$, and where we have fixed the Liouville "cosmological constant" μ_L to a specific value.

Next, we have the case $m = 1$, which corresponds to the H_+^3 WZNW theory with level k, $\mathcal{T}_{m=1,b=(k-2)^{-1/2}}$. In this case, the observable $\Omega_j^{(m=1,b=(k-2)^{-1/2})} = \langle\Phi_j(\mu|z)\rangle_{\text{WZNW}}$ represents AdS$_2$ branes in Euclidean AdS$_3$ space. This is

$$\Big\langle \Phi_j(\mu|z)\Big\rangle_{\text{WZNW}} = \pi\sqrt{k-2}\,\delta(\mu+\bar{\mu})\;|\mu|\;\;|z-\overline{z}|^{-2h_j}\left(\pi\frac{\Gamma(1-\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right)^{ip/2}\times$$
$$\times\Gamma\left(ip\right)\,\Gamma\left(1+\frac{ip}{k-2}\right)\cosh\left(2\pi ipbs\right). \qquad (95)$$

where $j = -1/2 + ip$ belongs to the continuous representation, with $p \in \mathbb{R}$, and where the string coupling cosntant $g_s^2 = e^{-2\langle\vartheta\rangle\chi} = e^{-\chi/\sqrt{2k-4}}$ was also fixed to a specific value to absorb Ω_0 (here, $\langle\vartheta\rangle$ represents the expectation value of the dilaton field, and χ is the Euler characteristic of the worldsheet manifold.) In terms of fields ϕ, γ and γ we used to describe the theory (1), AdS$_3$ metric is written in Poincaré coordinates as $ds^2 = l^2\left(d\phi^2 + e^{2\phi}d\gamma d\overline{\gamma}\right)$, where l is the "radius" of the space. When formulating string theory on this background, the level of the WZNW theory relates to the string tension as follows $k = l^2/\alpha'$. Branes in Lorentzian and Euclidean AdS$_3$

space were extensively studied in the literature; see for instance [16, 22, 20, 24, 17, 19, 23, 18, 21, 34, 2, 39] and references therein.

The case $m = b^2$ also yields the H_+^3 WZNW theory with level $k = b^{+2} + 2$. The fact that the WZNW theory is doubly represented in the family $\{\mathcal{T}_{mb}\}$ is associated to Langlands duality [47]. In the free field approach this is related to the existence of a "second" screening operator. In this context, free field calculations using the Wakimoto representation with non-integer m were already discussed in [51]. In fact, the computation we performed this paper can be regarded as a generalization of the one in [51] to the geometry of the disk and to the case where m is not necessarily equal to $b^{+2} + 2$. In general, the computation in the cases $m, n \notin \mathbb{Z}_{>0}$ requires a careful treatment of analytic continuation of the integral expressions in the Coulomb gas approach. This is seen, for instance, in equations (77) and (78), which makes perfect sense if the amount of screening operators is $n \in \mathbb{Z}_{\geq 0}$. The expressions for $n \notin \mathbb{Z}_{\geq 0}$ have to be understood just formally; see also equations (85) and (91).

Other case of interest is the theory for $m = 2$. In this case, correlation functions on the sphere were shown to obey third order differential equations that are associated to existence of singular vectors in the modulo [1]. This raises the question as to whether the existence of singular vectors, which is ultimately related to δ-function singularities in the correlation functions, can be used to compute observables in the theory with boundaries by means of the bootstrapt approach or some variation of it. It could be also interesting to attempt to compute observable in the presence of a boundary by using a free field representation similar to that proposed in [62] to describe the $m = 1$ theory. Such free field representation amounts to describe the H_+^3 WZNW theory as a $c < 1$ perturbed conformal field theory coupled to Liouville theory, resorting to the H_+^3 WZNW-Liouville correspondence.

Now, let us analyze here some properties of the one-point function for generic m we have computed in (93). Important information is obtained from studying how $\Omega_j^{(m,b)}$ transforms under certain changes in the set of quantum numbers (m, b, j) that leave the conformal dimension h_j unchanged. Looking at this gives important information about the symmetries of the theory. But, first, a few words on the scaling properties of the one-point function (93): The reason why we previously said that the precise value of the overall factor Ω_0 in (55) and (57) was "irrelevant" was that, by shifting the zero-mode of the field ϕ one easily introduces a KPZ scaling λ in the bulk expectation value, and this yields

$$\Omega_j^{(m,b)} \to \lambda^{(2j+1+(m-1)b^{-2})/2}\ \Omega_j^{(m,b)}.$$

Then, since Ω_0 also goes as a power $n/2 = -(2j+1+(m-1)b^{-2})/2$, its value can be absorbed and conventionally fixed to any (positive value). In turn, we prefer to write (93) by replacing

$$\Omega_0 \left(\pi \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{(2j+1+(m-1)b^{-2})/2} \to \left(\lambda\pi \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{(2j+1+(m-1)b^{-2})/2} \tag{96}$$

Now, we are ready to study the reflection properties of (93). Using properties of the Γ-function, it is easy to verify that the following relation holds

$$\Omega_j^{(m,b)}\Omega_{-1-j-\frac{m-1}{b^2}}^{(m,b)} = R_j^{(m,b)}, \quad \text{with} \quad R_j^{(m,b)} = \frac{1}{b^2}\left(\lambda\pi\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}\right)^{2j+1+\frac{m-1}{b^2}} \frac{\gamma(2j+1+(m-1)b^{-2})}{\gamma(-(2j+1)b^2-(m-1))},$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. It is remarkable that the reflection coefficient $R_j^{(m,b)}$, which is given by the two-point function on the sphere, arises in this expression. This generalizes what happens in Liouville field theory and in the H_3^+ WZNW model, and this is related to the fact that one eventually associates fields Φ_j and fields $R_j^{(m,b)}\Phi_{-1-j-b^{-2}(m-1)}$.

Other functional property of the one-point function that is interesting to analyze is how it behaves under duality transformation $b \to 1/b$. Actually, one can show that (93) obeys

$$\Omega_j^{(m,b)} = \Omega_{b^2(j+1-b^{-2})}^{(mb^{-2},b^{-1})},$$

provided the KPZ scaling parameter λ' associated to the function on the right hand side relates to that of the function on the left hand side through

$$\left(\lambda'\pi\frac{\Gamma(1-b^{-2})}{\Gamma(1+b^{-2})}\right)^{b^2} = \lambda\pi\frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}.$$

Last, let us mention another interesting problem that involves the theories $\mathcal{T}_{m,b}$ formulated on closed Riemann surfaces and that is in some sense related to the path integral techniques we discussed here. This is the problem of trying to use the path integral approach developed in [49] to define higher genus correlation functions for generic m, relating higher genus correlators in $\mathcal{T}_{m,b}$ to higher genus correlators in Liouville theory. Correlation functions on closed genus-g n-punctured Riemann surfaces in the theories $\mathcal{T}_{m,b}$ could be relevant to describe higher

m-monodromy operators in N=2 four-dimensional superconformal field theories, according to the recently proposed Alday-Gaiotto-Tachikawa conjecture [59, 60, 61]; this is because n-point functions in $\mathcal{T}_{m,b}$ are in correspondence with $(2n+2g-2)$-point functions in Liouville theory including $2g-2+n$ degenerate fields $V_{a=\frac{m}{2b}}$. Studying the relevance of the theories defined in [1] for the AGT construction is matter of future investigation.

This work was supported by UBA, ANPCyT, and CONICET through grants UBACyT X861, UBACyT X432, PICT-2007-00849 and PIP-2010-0396. The authors thank Lorena Nicolás for discussions.

References

[1] S. Ribault, *A Family of solvable non-rational conformal field theories*, JHEP **0805** (2008) 073, [arXiv:hep-th/08032099].

[2] V. Fateev and S. Ribault, *Boundary action of the H+(3) model*, JHEP **0802** (2008) 024, [arXiv:hep-th/07102093].

[3] J. Cardy, *Conformal Invariance and Surface Critical Behavior*, Nucl. Phys. **B240** [FS12] (1984) 514.

[4] J. Cardy, *Effect of Boundary Conditions on the Operator Content of Two-Dimensional Conformally Invariant Theories*, Nucl. Phys. **B275** (1986) 200.

[5] J. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, Nucl. Phys. **B 324** (1989) 581.

[6] V. Fateev, A.B. Zamolodchikov and Al. Zamolodchikov, *Boundary Liouville Field Theory I. Boundary State and Boundary Two-point Function*, [arXiv:hep-th/0001012]

[7] A.B. Zamolodchikov and Al. Zamolodchikov, *Liouville Field Theory on a Pseudosphere*, [arXiv:hep-th/0101152].

[8] C. Schweigert, J. Fuchs and J. Walcher, *Conformal field theory, boundary conditions and applications to string theory*, [arXiv:hep-th/0011109].

[9] V. Petkova and J-B. Zuber, *Conformal Boundary Conditions and what they teach us*, [arXiv:hep-th/0103007].

[10] V. Schomerus, *Non-compat string backgrounds and non-rational CFT*, Phys. Rept. **431** (2006) 39, [arXiv:hep-th/0509155].

[11] J. Teschner, *Remarks on Liouville theory with boundary*, Presented at the 4th Annual European TMR Conference on Integrability, Non-Perturbative Effects and Symmetry in Quantum Field Theory, France (2000), [arXiv:hep-th/0009138].

[12] J. McGreevy and H. Verlinde, *Strings from tachyons*, JHEP **0312** (2003) 054, [arXiv:hep-th/0304224].

[13] I. Klebanov, J. Maldacena and N. Seiberg, *D-brane Decay in Two-Dimensional String Theory*, JHEP **0307** (2003) 045, [arXiv:hep-th/0305159].

[14] J. McGreevy, J. Teschner and H. Verlinde, *Classical and Quantum D-branes in 2D String Theory*, JHEP **0401** (2004) 039, [arXiv:hep-th/0305194].

[15] S. Fredenhagen and V. Schomerus, *Boundary Liouville theory at $c = 1$*, JHEP **0505** (2005) pp. 025, [arXiv:hep-th/0409256].

[16] B. Ponsot, V. Schomerus and J. Teschner, *Branes in the Euclidean AdS(3)*, JHEP **0202** (2002) 016, [arXiv:hep-th/0112198].

[17] S. Stanciu, *D-branes in an AdS(3) background*, JHEP **9909** (1999) 028, [arXiv:hep-th/9901122].

[18] C. Bachas, M. Petropoulos, *Anti-de Sitter D-branes*, JHEP **0102** (2001) 025, [arXiv:hep-th/0012234].

[19] P. Lee, H. Ooguri, J. Park and J. Tannenhauser, *Open Strings on AdS(2) Branes*, Nucl. Phys. **B610** (2001) 03, [arXiv:hep-th/0106129].

[20] A. Giveon, D. Kutasov and A. Schwimmer, *Comments on D-branes in AdS(3)*, Nucl. Phys. **B615** (2001) 133, [arXiv:hep-th/0106005].

[21] A. Parnachev and D. Sahakyan, *Some remarks on D-branes in AdS(3)*, JHEP **0110** (2001) 022, [arXiv:hep-th/0109150].

[22] Y. Hikida and Y. Sugawara, *Boundary states of D-branes in AdS(3) Based on Discrete Series*, Prog. Theor. Phys. **107** (2002) 1245, [arXiv:hep-th/0107189].

[23] P. Lee, H. Ooguri and J. Park, *Boundary States for AdS(2) Branes in AdS(3)*, Nucl. Phys. **B632** (2002) 283, [arXiv:hep-th/0112188].

[24] B. Ponsot and S. Silva, *Are there really any AdS(2) branes in the euclidean (or not) AdS(3)?*, Phys. Lett. **B551** (2003) 173, [arXiv:hep-th/0209084].

[25] M. Gutperle and A. Strominger, *Timelike boundary Liouville theory*, Phys. Rev. **D67** (2003) 126002, [arXiv:hep-th/0301038].

[26] J. Teschner, *On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories*, JHEP **0404** (2004) 023, [arXiv:hep-th/0308140].

[27] S. Ribault and V. Schomerus, *Branes in the 2D black hole*, JHEP **0402** (2004) 019, [arXiv:hep-th/0310024].

[28] S. Ribault, *AdS2 D-branes in AdS3 spacetime*, Proceeding of the workshop "Across the present energy frontier: probing the origin of mass", Greece (2001), [arXiv:hep-th/0207094].

[29] S. Ribault, *Two AdS(2) branes in the Euclidean AdS(3)*, JHEP **0305** (2003) 003, [arXiv:hep-th/0210248].

[30] Yu Nakayama, *Liouville Field Theory - A decade after the revolution*, Int. J. Mod. Phys. **A19** (2004) 2771, [arXiv:hep-th/0402009].

[31] Yu Nakayama, *Tadpole Cancellation in Unoriented Liouville Theory*, JHEP **0311** (2003) 017, [arXiv:hep-th/0309063].

[32] Yu Nakayama, Soo-Jong Rey and Yuji Sugawara, *D-Brane Propagation in Two-Dimensional Black Hole Geometries*, JHEP **0509** (2005) 020, [arXiv:hep-th/0507040].

[33] Yu Nakayama, *Black Hole - String Transition and Rolling D-brane*, PhD Thesis, [arXiv:hep-th/0702221].

[34] D. Israel, *D-branes in Lorentzian AdS(3)*, JHEP **0506** (2005) 008, [arXiv:hep-th/0502159].

[35] D. Israel, A. Pakman and J. Troost, *D-branes in N=2 Liouville and its mirror*, Nucl. Phys. **B710** (2005) 529, [arXiv:hep-th/0405259].

[36] D. Israel, A. Pakman and J. Troost, *D-branes in little string theory*, Nucl. Phys. **B722** (2005) 3, [arXiv:hep-th/0502073].

[37] S. Ribault, *Discrete D-branes in AdS(3) and in the 2-D black hole*, JHEP **0608** (2006) 015, [arXiv:hep-th/0512238].

[38] K. Hosomichi and Ribault, *Solution of the H3+ model on a disc*, JHEP **0701** (2007) 057, [arXiv:hep-th/0610117].

[39] S. Ribault, *Boundary three-point function on AdS2 D-branes*, JHEP **0801** (2008) 004, [arXiv:hep-th/07083028].

[40] K. Hosomichi, *A correspondence between H+(3) WZW and Liouville theories in discs*, Nucl. Phys. Proc. Suppl. **171** (2007) 284. [arXiv:hep-th/0701260].

[41] K. Hosomichi, *Minimal String Theory*, JHEP **0806** (2008) 029, [arXiv:0804.4721].

[42] T. Creutzig and Y. Hikida, *Branes in the $OSP(1|2)$ WZNW model*, [arXiv:1004.1977].

[43] T. Creutzig, *Branes in Supergroups*, [arXiv:0908.1816].

[44] T. Creutzig, *Geometry of branes on supergroups*, Nucl. Phys. **B812** (2009) pp. 301, [arXiv:0809.0468].

[45] T. Creutzig and V. Schomerus, *Boundary Correlators in Supergroup WZNW Models*, Nucl. Phys. **B807** (2009) 471, [arXiv:0804.3469].

[46] T. Creutzig, T. Quella and V. Schomerus, *Branes in the $GL(1|1)$ WZNW-Model*, Nucl. Phys. **B792** (2008) 257, [arXiv:0708.0583].

[47] G. Giribet, Y. Nakayama and L. Nicolas, *The Langlands duality in Liouville-H(+)3 WZNW correspondence*, Int. J. Mod. Phys. **A24** (2009) 3137, [arXiv:hep-th/08051254].

[48] M. Wakimoto, *Fock representations of affine Lie algebra,* Comm. Math. Phys. **104** (1986) 605.

[49] Y. Hikida and V. Schomerus, *H+(3)-WZNW model from Liouville field theory*, JHEP **0710** (2007) 064, [arXiv:hep-th/07061030].

[50] Y. Hikida and V. Schomerus, *Structure constants of the OSP(1—2) WZNW model*, JHEP **0712** (2007) 100, [arXiv:hep-th/07110338].

[51] G. Giribet and C. Núñez, *Correlators in AdS3 string theory*, JHEP **0106** (2001) 010, [arXiv:hep-th/0105200].

[52] K. Becker and M. Becker, *Interactions in the SL(2,R)/U(1) black hole background*, Nucl. Phys. **B418** (1994) 206. [arXiv:hep-th/9310046].

[53] K. Hosomichi, K. Okuyama and Y. Satoh, *Free field approach to string theory on AdS3*, Nucl. Phys. **B598** (2001) 451, [arXiv:hep-th/0009107].

[54] J. Teschner, *Liouville theory revisited*, Class. Quant. Grav. **18** (2001) R153, [arXiv:hep-th/0104158].

[55] J. Teschner, *A lecture on the Liouville vertex operators*, Int. J. Mod. Phys. **A19S2** (2004) 436, [arXiv:hep-th/0303150].

[56] A. Stoyanovsky, *A relation between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov systems of partial differential equations*, [arXiv:math-ph/0012013].

[57] S. Ribault and J. Teschner, *H+(3)-WZNW correlators from Liouville theory*, JHEP **0506** (2005) 014, [arXiv:hep-th/0502048].

[58] S. Ribault, *Knizhnik-Zamolodchikov equations and spectral flow in AdS(3) string theory*, JHEP **0509** (2005) 045, [arXiv:hep-th/0507114].

[59] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, Lett. Math. Phys. **91** (2010) 167, [arXiv:0906.3219].

[60] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in N=2 gauge theory and Liouville modular geometry*, JHEP **1001** (2010) 113, [arXiv:0909.0945].

[61] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, JHEP **1002** (2010) 057, [arXiv:0909.1105].

[62] G. Giribet, *The string theory in AdS_3 as a marginal deformation of a linear dilaton background*, Nucl. Phys. **B737** (2006) 209, [arXiv:hep-th/0511252].

[63] Y. Hikida and V. Schomerus, *The FZZ-Duality Conjecture: A Proof*, JHEP **0903** (2009) 095, [arXiv:hep-th/08053931].