General aspects of Gauss-Bonnet models without potential in dimension four

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Abstract

In the present work, the isotropic and homogenous solutions with spatial curvature $k = 0$ of four dimensional Gauss-Bonnet models are characterized. The main assumption is that the scalar field ϕ which is coupled to the Gauss-Bonnet term has no potential [51]-[52]. Some singular and some eternal solutions are described. The evolution of the universe is given in terms of a curve $\gamma = (H(\phi), \phi)$ which is the solution of a polynomial equation $P(H^2, \phi) = 0$ with ϕ dependent coefficients. In addition, it is shown that the initial conditions in these models put several restrictions on the evolution. For instance, an universe initially contracting will be contracting always for future times and an universe that is expanding was always expanding at past times. Thus, there are no cyclic cosmological solutions for this model. These results are universal, that is, independent on the form of the coupling $f(\phi)$ between the scalar field and the Gauss-Bonnet term. In addition, a proof that at a turning point $\dot{\phi} \to 0$ a singularity necessarily emerges is presented, except for some specific choices of the coupling. This is valid unless the Hubble constant $H \to 0$ at this point. This proof is based on the Raychaudhuri equation for the model. The description presented here is in part inspired in the works [33]-[35]. However, the mathematical methods that are implemented are complementary of those in these references, and they may be helpful for study more complicated situations in a future.

1. Introduction

One of the main interests in higher derivative gravity theories is that they can describe inflation by a higher order curvature in the Einstein-Hilbert action [1]-[2] without the addition of dark energy or scalar fields. An important role in this context is played by the Gauss-Bonnet invariant, since it appears in QFT renormalization in curved space times [3]. In addition, the Gauss-Bonnet term arises in low-energy effective actions of some string theories. For instance, the tree-level string effective action has been calculated up to several orders in the α' expansion in [4]-[5]. The result is that there is no moduli dependence of the tree-level couplings. However, one loop corrections to the gravitational couplings have been considered in the context of orbifold compactifications of the heterotic superstring [6]. It has been shown in that reference that there are no moduli dependent corrections to the Einstein term while there are nontrivial curvature contributions. They appear as the Gauss-Bonnet combination multiplied by a function of the modulus field.

The results described above partially motivated the study of cosmological consequences of the Gauss-Bonnet term. In four dimensions, this term is topological and does not have any dynamical effect. However, when this term is non-minimally coupled with any other field such as a scalar field ϕ, the resulting dynamics is non trivial. Several cosmological consequences has been exploited in recent literature, and we refer the reader to [7]-[29] and references therein. But the aim of the present letter is not focused in inflationary aspects of the theory, instead in the characterization of singular and eternal solutions of the theory. It is important to mention that there exist preliminary works on this

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subject, examples are given in [33]-[50]. In particular, the results of [33]-[35] suggest the existence of singular solutions as well as regular solutions. The singular solutions are confined to an small portion of the phase space, while the non singular fill the rest. This situation is different than in GR, where the Gauss-Bonnet term is absent, and the powerful Hawking-Penrose theorems apply [53].

Despite the existence of the results [33]-[50], in the present paper an independent characterization of the possible solutions of the theory will be presented. The main assumption is that the universe is isotropic and homogeneous, that a potential is absent [51]-[52], and that the spatial curvature is flat. In the authors opinion, the mathematical characterization achieved below has not been employed yet, and it may be helpful in order to understand more complicated situations later on.

The present paper is organized as follows. In section 2 the equations of motion for Gauss-Bonnet vacuums in four dimensions coupled to an scalar field ϕ will be presented. In particular, the full system for an isotropic and homogeneous solution with flat spatial curvature will be derived. This section do not contain any new material, all its content is standard. In section 3 it will be shown that the possible vacuums of the theory are given by a curve $\gamma=(H(\phi),\phi)$ in the space (H,ϕ) characterized by an equation $P(H^2,\phi)=0$ which is polynomial on H^2. The relation is purely algebraic. The time dependence of these fields is also derived in this section. In section 4, some numerical aspects related to these solutions are analyzed. In particular, eternal and singular solutions are characterized. The algebraic relation found in section 3 plays an important role in this characterization. Section 5 contains a discussion of the obtained results.

2. Gauss-Bonnet equation

The model that will be considered here is a generalization of the Gauss-Bonnet one. Recall that a pure Gauss-Bonnet gravity model is described in D dimensions by the following action

$$S=\int d^Dx\sqrt{-g}G. \tag{2.1}$$

The Gauss-Bonnet invariant G is given by

$$G\equiv R^2-4R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}. \tag{2.2}$$

The signature to be employed in the following is $(-,+,...,+)$. The equations of motions $\delta S=0$ that arise by considering variations $\delta g_{\mu\nu}$ are the following

$$-\frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\alpha\beta}-\nabla_\alpha\nabla_\beta R-2R_{\rho\beta\alpha\sigma}R^{\sigma\rho}+\frac{1}{2}g_{\alpha\beta}\Box R+\Box R_{\alpha\beta}+\frac{1}{2}R^2g_{\alpha\beta}-2RR_{\alpha\beta}$$
$$-2\nabla_\beta\nabla_\alpha R+2g_{\alpha\beta}\Box R-\frac{1}{2}R+R_{\alpha\beta}=0. \tag{2.3}$$

The resulting model is non trivial for $D\neq 4$. In four dimensions instead, the term $\sqrt{-g}G$ can be expressed as a total derivative

$$\sqrt{-g}G=\partial_\alpha K^\alpha,\qquad K^\alpha=\sqrt{-g}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu}_{\rho\sigma}\Gamma^\rho_{\mu\beta}\bigg[\frac{R^\sigma_{\nu\gamma\delta}}{2}+\frac{\Gamma^\sigma_{\lambda\gamma}\Gamma^\lambda_{\nu\sigma}}{3}\bigg]. \tag{2.4}$$

The references [32] contains further details about this term. But we conclude that, in $D=4$ for a manifold without boundary, this model is irrelevant. However, the following modified action

$$S=\int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+V(\phi)+f(\phi)G\right\}, \tag{2.5}$$

is physically non trivial, as the Gauss-Bonnet is coupled to the real scalar field ϕ by the coupling $f(\phi)$. Due to this coupling, the lagrangian it is not a total derivative anymore and contributes to the equations of motion.

In the following, a Gauss-Bonnet model with $V(\phi) = 0$ will be considered, as in references [51]-[52]. The equation for the scalar field ϕ under this circumstance is given by

$$\nabla^2\phi + f'(\phi)G = 0. \tag{2.6}$$

The equations of motion for the metric $g_{\mu\nu}$ are more involved. The variation of the action throws the following result

$$0 = \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}g^{\mu\nu}f(\phi)G - 2f(\phi)RR^{\mu\nu}$$

$$+2\nabla^\mu\nabla^\nu\left(f(\phi)R\right) - 2g^{\mu\nu}\nabla^2\left(f(\phi)R\right) + 8f(\phi)R^\mu{}_\rho R^{\nu\rho} - 4\nabla_\rho\nabla^\mu\left(f(\phi)R^{\nu\rho}\right) - 4\nabla_\rho\nabla^\nu\left(f(\phi)R^{\mu\rho}\right)$$

$$+4\nabla^2\left(f(\phi)R^{\mu\nu}\right) + 4g^{\mu\nu}\nabla_\rho\nabla_\sigma\left(f(\phi)R^{\rho\sigma}\right) - 2f(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4\nabla_\rho\nabla_\sigma\left(f(\phi)R^{\mu\rho\sigma\nu}\right).$$

However, by taking into account the following identities

$$\nabla^\rho R_{\rho\tau\mu\nu} = \nabla_\mu R_{\nu\tau} - \nabla_\nu R_{\mu\tau},$$

$$\nabla^\rho R_{\rho\mu} = \frac{1}{2}\nabla_\mu R,$$

$$\nabla_\rho\nabla_\sigma R^{\mu\rho\nu\sigma} = \nabla^2 R^{\mu\nu} - \frac{1}{2}\nabla^\mu\nabla^\nu R + R^{\mu\rho\nu\sigma}R_{\rho\sigma} - R^\mu{}_\rho R^{\nu\rho},$$

$$\nabla_\rho\nabla^\mu R^{\rho\nu} + \nabla_\rho\nabla^\nu R^{\rho\mu} = \frac{1}{2}\left(\nabla^\mu\nabla^\nu R + \nabla^\nu\nabla^\mu R\right) - 2R^{\mu\rho\nu\sigma}R_{\rho\sigma} + 2R^\mu{}_\rho R^{\nu\rho},$$

$$\nabla_\rho\nabla_\sigma R^{\rho\sigma} = \frac{1}{2}\Box R,$$

which are consequences of the Bianchi identities, the last expression can be written as [30]

$$0 = \frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \left(\frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi\right) + \frac{1}{2}g^{\mu\nu}f(\phi)G$$

$$-2f(\phi)RR^{\mu\nu} + 4f(\phi)R^\mu{}_\rho R^{\nu\rho} - 2f(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4f(\phi)R^{\mu\rho\sigma\nu}R_{\rho\sigma}$$

$$+2\left(\nabla^\mu\nabla^\nu f(\phi)\right)R - 2g^{\mu\nu}\left(\nabla^2 f(\phi)\right)R - 4\left(\nabla_\rho\nabla^\mu f(\phi)\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f(\phi)\right)R^{\mu\rho}$$

$$+4\left(\nabla^2 f(\phi)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f(\phi)\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f(\phi)\right)R^{\mu\rho\nu\sigma}. \tag{2.7}$$

The equations (2.6) and (2.7) are the full system of equations describing the theory.

The following discussion is focused on the isotropic and homogeneous vacuums of the model with zero spatial curvature. The corresponding distance element for these vacuums is given by

$$g_4 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2.$$

The formulas for the Levi-Civita connection and the curvature of this background are well known, they are explicitly

$$\Gamma^t_{ij} = a^2 H\delta_{ij}\ , \qquad \Gamma^i_{jt} = \Gamma^i_{tj} = H\delta^i{}_j\ , \qquad R_{itjt} = -\left(\dot{H} + H^2\right)\delta_{ij},$$

$$R_{ijkl} = a^4 H^2\left(\delta_{ik}\delta_{lj} - \delta_{il}\delta_{kj}\right), \qquad R_{tt} = -3\left(\dot{H} + H^2\right)\ ,$$

$$R_{ij} = a^2\left(\dot{H} + 3H^2\right)\delta_{ij}, \qquad R = 6\dot{H} + 12H^2. \tag{2.8}$$

The other components are all zero. By use of these formulas, equation (2.6) becomes

$$\ddot{\phi}+3H\dot{\phi}-24H^2f'(\phi)(H^2+\dot{H})=0. \tag{2.9}$$

The two other independent equations that follows from (2.7) are

$$H^2=\frac{\kappa^2}{3}\rho_{eff},\qquad 2\dot{H}+3H^2=-\kappa^2 p_{eff}, \tag{2.10}$$

where the energy density and the pressure of the scalar field ϕ are given by

$$\rho_{eff}=\frac{\dot{\phi}^2}{2}-24H^3\dot{f},\qquad p_{eff}=\frac{\dot{\phi}^2}{2}+8H^2f''(\phi)\dot{\phi}^2+8H^2f'(\phi)\ddot{\phi}+16H\dot{H}f'(\phi)\dot{\phi}+16H^3f'(\phi)\dot{\phi},$$

respectively. The equations (2.9) and (2.10) characterize the isotropic and homogeneous vacuums of the theory, with zero spatial curvature.

3. Formal aspects

The next task is to characterize the possible solutions of the system (2.9)-(2.10). By a redefinition of the scalar field ϕ and the function $f(\phi)$ the Newton constant κ may be set to one. In addition the redefinition $8f(\phi)\to f(\phi)$ is convenient, in order to simplify some numerical factors. After these redefinitions, the equations of motion (2.9) and (2.10) are given by

$$\frac{\dot{\phi}^2}{2}=3H^2(1+\dot{f}(\phi)H), \tag{3.11}$$

$$\frac{\dot{\phi}^2}{2}=-2(H^2+\dot{H})(1+\dot{f}(\phi)H)-H^2(1+\ddot{f}(\phi)), \tag{3.12}$$

$$\ddot{\phi}=-3H\dot{\phi}+3H^2f'(\phi)(H^2+\dot{H}). \tag{3.13}$$

This is a highly non linear system for ϕ and H. However, several conclusions may be drawn from (3.11)-(3.13) without reference to a particular choice of the coupling $f(\phi)$. First, by taking into account that $\dot{f}(\phi)=f'(\phi)\dot{\phi}$, the equation (3.11) becomes a quadratic algebraic relation for $\dot{\phi}$. Its solution is

$$\dot{\phi}=\frac{H}{2}\bigg[6H^2f'(\phi)\pm\sqrt{36H^4f'(\phi)^2+24}\bigg]. \tag{3.14}$$

This equation, when inserted into (3.13), gives the following expression for $\ddot{\phi}$

$$\ddot{\phi}=-\frac{3}{2}H^2\bigg[6H^2f'(\phi)\pm\sqrt{36H^4f'(\phi)^2+24}\bigg]+3H^2f'(\phi)(H^2+\dot{H}). \tag{3.15}$$

On the other hand, a different expression for $\ddot{\phi}$ can be obtained by directly taking the time derivative of (3.14), the result is

$$\ddot{\phi}=\frac{\dot{H}}{2}\bigg[18H^2f'(\phi)\pm\frac{216H^5f'(\phi)^2+48H}{\sqrt{36H^6f'(\phi)^2+24H^2}}\bigg]$$

$$+\frac{H}{2}\bigg[6H^2f'(\phi)\pm\sqrt{36H^4f'(\phi)^2+24}\bigg]\bigg[6H^3f''(\phi)\pm\frac{72H^6f'(\phi)f''(\phi)}{\sqrt{36H^6f'(\phi)^2+24H^2}}\bigg]. \tag{3.16}$$

Both expressions (3.16) and (3.15) should be equal, and this equality implies that

$$\bigg[6H^2f'(\phi)\pm\frac{216H^5f'(\phi)^2+48H}{2\sqrt{36H^6f'(\phi)^2+24H^2}}\bigg]\dot{H}=-\frac{3}{2}H^2\bigg[6H^2f'(\phi)\pm\sqrt{36H^4f'(\phi)^2+24}\bigg]$$

$$+3H^4 f'(\phi) - \left[3H^2 f'(\phi) \pm \sqrt{9H^4 f'(\phi)^2 + 6}\right]\left[6H^4 f''(\phi) \pm \frac{72H^7 f'(\phi) f''(\phi)}{\sqrt{36H^4 f'(\phi)^2 + 24}}\right]. \tag{3.17}$$

This is an equation which expresses the time derivative $\dot{H}$ of the Hubble parameter as a function $\dot{H} = \dot{H}(\phi, H)$. However, the formula (3.12) was not used to get this relation. When this formula is taken into account, a non equivalent expression for $\dot{H} = H(\phi, H)$ is obtained as follows. By use of the elementary formula $\ddot{f} = f''(\phi)\dot{\phi}^2 + f'(\phi)\ddot{\phi}$ together with (3.11) and (3.14), the formula (3.12) can be expressed as follows

$$-H^2 f'(\phi)\ddot{\phi} = H^2 + \left[1 + \frac{H^2}{2} f'(\phi)\left[6H^2 f'(\phi) \pm \sqrt{36H^4 f'(\phi)^2 + 24}\right]\right](5H^2 + 2\dot{H} + 6H^4 f''(\phi)). \tag{3.18}$$

By replacing $\ddot{\phi}$ by (3.15) it follows after some simple rearrangement of terms that

$$\left\{2 + H^2 f'(\phi)\left[3H^2 f'(\phi) \pm \sqrt{36H^4 f'(\phi)^2 + 24}\right]\right\}\dot{H} = -H^2$$

$$-\frac{3H^4 f'(\phi)}{2}\left[6H^2 f'(\phi) \pm \sqrt{36H^4 f'(\phi)^2 + 24}\right] + 3H^6 f'(\phi)^2 - H^2$$

$$-\frac{1}{2}\left[2 + H^2 f'(\phi)\left[6H^2 f'(\phi) \pm \sqrt{36H^4 f'(\phi)^2 + 24}\right]\right](5H^2 + 6H^4 f''(\phi)), \tag{3.19}$$

which is an expression for $\dot{H} = \dot{H}(\phi, H)$ non equivalent to (3.17).

Now, the crucial point is that the non equivalent expressions (3.17) and (3.19) are of the form

$$A\dot{H} = B, \qquad C\dot{H} = D,$$

respectively, where $A, .., D$ are functions of H and ϕ but not functions of $\dot{H}$. The compatibility conditions for both equations to be true is $AD = BC$. By reading the coefficients $A, ..., D$ directly from (3.17) and (3.19) it follows that the relation $AD = BC$ translates into an equation of the form

$$(aH^{10} f'(\phi)^4 f''(\phi) + bH^8 f'(\phi)^4 + cH^6 f'(\phi)^2 f''(\phi) + dH^4 f'(\phi)^4 + eH^2 f'(\phi) + f)H^\alpha$$

$$= \left[gH^8 f'(\phi)^4 + hH^6 f'(\phi)^2 f''(\phi) + mH^4 f'(\phi)^4 + nH^2 f'(\phi) + q\right]H^\alpha \sqrt{36H^4 f'(\phi)^2 + 24}. \tag{3.20}$$

Here $a, .., q$ are some real numbers and $\alpha > 0$ is an integer, whose explicit value is not of importance in the following discussion. The factor H^α is important, since it shows that $H = 0$ may be a solution of this equation. In fact, when $H \to 0$ both equations (3.17) and (3.19) show that, if $f(\phi)$ and its derivatives are non divergent, then $\dot{H} \to 0$. Thus, the flat space is included in the model.

The solutions of (3.20) give a relation between H and ϕ, which is of course non univoque. By taking the square of this expression it follows that these branches $H(\phi)$ satisfy

$$H^{2\alpha}\left[aH^{10} f'(\phi)^4 f''(\phi) + bH^8 f'(\phi)^4 + cH^6 f'(\phi)^2 f''(\phi) + dH^4 f'(\phi)^4 + eH^2 f'(\phi) + f\right]^2$$

$$-H^{2\alpha}\left[gH^8 f'(\phi)^4 + hH^6 f'(\phi)^2 f''(\phi) + mH^4 f'(\phi)^4 + nH^2 f'(\phi) + q\right]^2 (36H^4 f'(\phi)^2 + 24) = 0. \tag{3.21}$$

This is an algebraic relation of the form $P(H^2, f'(\phi), f''(\phi)) = 0$, which is polynomial of degree ten for H^2 (or an expression for H of order twenty), up to the factor $H^{2\alpha}$. The equation $P(H^2, f'(\phi), f''(\phi)) = 0$ describe a curve $\gamma = (H(\phi), \phi)$ in the space (H, ϕ) of possible evolutions of the vacuum, without reference to the proper time t. This should be supplemented with the time dependence of some of the fields ϕ or H^2. This dependence follows directly from (3.14), the result is

$$t - t_0 = \int_{\phi_0}^{\phi} \frac{d\phi}{H\left[6H^2 f'(\phi) \pm \sqrt{36H^4 f'(\phi)^2 + 24}\right]}, \tag{3.22}$$

and is obtained by simple integration.

Both expressions (3.21)-(3.22) represent the full solution of the Einstein system in the isotropic and homogeneous case. The first gives implicitly $H = H(\phi)$ and this, combined with the second gives that $\phi = \phi(t)$ and consequently $H = H(t)$. In the general case, an explicit expression for the solution is hopeless by the Galois theorem. Nevertheless, as it will be explained below, this algebraic relation will be useful for partially describing the solutions of the theory.

4. Numerical aspects

4.1 Bounds for the evolution

In addition to the formal characterization made above, some numerical consequences of the model may found as follows. First of all, from equation (3.11) it is immediately deduced that

$$\frac{\dot{\phi}^2}{6H^2} = 1 + \dot{f}H \geq 0. \tag{4.23}$$

This implies that the inequality

$$Hf'(\phi)\dot{\phi} \geq -1,$$

is satisfied during the whole evolution of the universe. On the other hand, the equation (3.12) can be expressed by use of (3.11) as follows

$$\frac{5\dot{\phi}^2}{6} + H^2 = -2\frac{dH}{dt} - \frac{d(\dot{f}H^2)}{dt} \geq 0.$$

Therefore, as the proper time t grows, the following bound is obtained

$$H(2 + \dot{f}H) \leq C_0, \qquad t \geq 0, \tag{4.24}$$

with C_0 being the value of the quantity $H(2 + \dot{f}H)$ at $t = 0$.

The inequality (4.24) derived above has important consequences. Assume for a moment that the initial condition is $C_0 < 0$. The inequality (4.23) shows that the factor $2 + \dot{f}H \geq 1$, therefore this case corresponds to $H < 0$. Thus the universe is contracting at $t = 0$. By taking into account (4.24) and that $2 + \dot{f}H \geq 1$ it follows that

$$H \leq \frac{C_0}{2 + \dot{f}H} \leq 0, \qquad t \geq 0. \tag{4.25}$$

Thus the universe is always contracting in the future if it is contracting at $t = 0$. This is an universal conclusion, no matter the form of the coupling $f(\phi)$. Furthermore, for the past, the equation (4.24) is converted into

$$H \geq \frac{C_0}{2 + \dot{f}H} \geq C_0, \qquad t \leq 0, \tag{4.26}$$

Again, in the last step the inequality $2 + \dot{f}H \geq 1$ was taken into account.

Suppose now that the initial condition is $C_0 > 0$. Then

$$H \leq \frac{C_0}{2 + \dot{f}H} \leq C_0, \qquad t \geq 0. \tag{4.27}$$

For the past, the equation (4.24) is converted into

$$H \geq \frac{C_0}{2 + \dot{f}H} \geq 0, \qquad t \leq 0, \tag{4.28}$$

Thus, if the universe is expanding at $t = 0$, it was expanding always in the past. Again, this is an universal conclusion, without reference to the particular form of $f(\phi)$. These results show in particular that there are no cyclic cosmologies for these models.

Consider now the behavior of ϕ and $\dot{\phi}$. If the negative branch is chosen in (3.14) then the time derivative is given by

$$2\dot{\phi} = H\left[6H^2 f'(\phi) - \sqrt{36H^4 f'(\phi)^2 + 24}\right]. \tag{4.29}$$

It is convenient to express this formula as

$$2\dot{\phi} = 6H^3 f'(\phi)\left[1 - \sqrt{1 + \frac{24}{36H^4 f'(\phi)^2}}\right]. \tag{4.30}$$

In these terms it follows from (4.30) that if $f'(\phi) \to \pm\infty$ then $\dot{\phi} \to 0$. Also, if $f'(\phi) \neq 0$ then $\dot{\phi} \to 0$ when $H \to \pm\infty$. Instead, if $f'(\phi) = 0$ then $\dot{\phi} \to \pm\infty$ when $H \to \mp\infty$, as follows from (4.29). For this reason, a coupling $f'(\phi)$ which never reaches a zero will be chosen. An example of this may be a coupling $f'(\phi) > 0$ with a minimum $f'_m > 0$. If this condition is satisfied, $\dot{\phi}$ is never divergent. In addition $\dot{\phi} \to 0$ when $H \to 0$.

It is important to remark that the restrictions just imposed are not due to a fundamental reason, but it is due simply due to the author technical limitations. To give a counter example, the solutions of [33] and [51]-[52] and [55] are outside of these conditions. Still these authors are able to find regular solutions. But these assumptions will allow us to characterize other type of solutions that these authors have not considered.

Now, if $\dot{\phi}$ is interpreted as a function of the two variables $(f'(\phi), H)$ given by (4.29), then the properties shown above show that it vanishes at the point $(f'_m, 0)$ and at any point of the infinite. The fact that $\dot{\phi}$ is a continuous function shows that it must have a minimum and a maximum somewhere. In fact, the restriction of this function to any straight line in the space $(f'(\phi), H)$ connecting the point $(f'_m, 0)$ with some point of at the infinite interpolates between the two zeros continuously. The Bolzano theorem implies the presence of a minimum or a maximum in any of these directions. These directions are parameterized an "angular" coordinate ϑ and since the function $\dot{\phi}$ is well behaved, these extrema varies continuously with the angle. As the angular coordinate $0 \leq \vartheta < 2\pi$ is compact, it follows that there should exist a global minimum $\dot{\phi}_1$ and a global maximum $\dot{\phi}_2$. Therefore

$$\dot{\phi}_2 \leq \dot{\phi} \leq \dot{\phi}_1,$$

and, by simple integration of the last expression, it follows that

$$\begin{aligned} \phi_0 + \dot{\phi}_1 t \leq \phi \leq \phi_0 + \dot{\phi}_2 t, \qquad t \geq 0, \\ \phi_0 + \dot{\phi}_2 t \leq \phi \leq \phi_0 + \dot{\phi}_1 t, \qquad t \leq 0. \end{aligned} \tag{4.31}$$

This means that the values of ϕ are bounded by two linear time functions, and therefore are bounded for any finite time t.

4.2 Characterization of the singular and regular solutions

In order to characterize the regular and singular solutions of the theory, it should be remarked that the curvature invariants that can be constructed by (2.8) are all dependent on H and its derivative $\dot{H}$. For instance, the following invariants

$$R = 6\dot{H} + 12H^2, \qquad R_{ij}R^{ij} = 12\left(\dot{H} + 3H^2\right)^2,$$

are solely dependent on these two quantities. The following discussion is focused in the possible singularities of H and $\dot{H}$.

The bounds described in the previous subsection are useful in order to understand these singularities. First, note that the equation (3.21) can be considered as an algebraic one with coefficients determined the first and second derivatives $f'(\phi)$ and $f''(\phi)$. If these functions exist and are continuous for any finite value of ϕ and are never zero, then it is obtained from (3.21) and (4.31) that these coefficients $f'(\phi(t))$ and $f''(\phi(t))$ are well behaved for every finite value of the proper time t. Of course, this assumption implies that these functions have no vertical asymptotes. The assumption that the coefficients are never zero is for simplicity, otherwise the behavior of the roots of a polynomial may be singular when some of the coefficients vanish.[1] Under these conditions, by taking into account that the coefficients are finite and are symmetric functions of the roots, it follows that the roots $H(\phi)$ of (3.21) are finite for every finite time t.

The next step is to understand the possible singularities of $\dot{H}$. It is convenient to consider a simple example first. Suppose for a moment that there is a (fictitious) theory whose vacuum is described by a circle $H^2+\phi^2=1$. Then it is clear that, at the point $(H,\phi)=(1,0)$ the value of $H'(\phi)$ is divergent, since the tangent to the curve is vertical to the H axis. If the dynamic is such that this point $(H,\phi)=(1,0)$ is attained at finite time, then $\dot{H}=H'(\phi)\dot{\phi}$ appears to be divergent due to the fact that $H'(\phi)\to\infty$ at this point. However, it is also clear that $\dot{\phi}$ goes to zero at $(H,\phi)=(1,0)$, since it is a turning point for ϕ. Thus a $0.\infty$ type of indetermination appears. But one may think a plenty of situations such that the fields are evolving around the circle with finite velocity, for instance, $H=\sin(t)$ and $\phi=\cos(t)$. In those cases, the $0.\infty$ indetermination gives a finite answer and $\dot{H}$ is finite at the turning point.

For the present model, the situation is more complicated than in the example just described, as the curve $\gamma=(H^2(\phi),\phi)$ is described by (3.21) which is much harder than a circle equation. Still, some conclusions may be obtained. The equation (3.21) for H^2 is of generic form

$$P_n(H^2)=\sum_{n=0}^{10}a_nH^{2n}=0,$$

with ϕ dependent coefficients a_n, up to a factor $H^{2\alpha}$. This last factor describes a flat space solution. Now, the coefficients of this equation are well behaved functions of ϕ and therefore an infinitesimal change $d\phi$ induces an smooth change in the coefficients da_i. This induces a change dH^2 in any of the roots of the polynomial, which is related to the variations da_i by the following formula

$$P_n'(H^2)dH^2=-\sum_{n=0}^{9}H^{2n}da_n. \tag{4.32}$$

If this formula is not satisfied, then H^2+dH^2 would cease to be a root. From the last relation, it is deduced that

$$\frac{\partial H^2}{\partial a_i}=-\frac{H^{2n}}{P_n'(H^2)}. \tag{4.33}$$

Thus the derivative (4.33) is well behaved when $P_n(H^2)=0$ but $P_n'(H^2)\neq 0$. For a polynomial function, this condition is the statement that the roots H^2 are simple roots. In other words, if the evolution of ϕ is such that two roots H_1^2 and H_2^2 merge at a finite time t_0, then at this point the derivative (4.33) will be divergent. Now, the time derivative of H^2 is

$$\frac{dH^2}{dt}=\frac{\partial H^2}{\partial a_i}\frac{da_i}{d\phi}\dot{\phi}. \tag{4.34}$$

As was argued above, the Hubble constant H has finite values at any finite time t. Thus the possible divergent behavior of the derivative would not be due to a vertical asymptote. Instead, it will be due to the presence of a turning point in the curve $\gamma=(H^2(\phi),\phi)$, which is characterized by $\dot{\phi}\to 0$.

[1]To see an example, consider a quadratic whose principal coefficient goes to zero. It is easy to see that one of the roots is divergent in this limit.

It is tempting to conclude, in view of the circle example given above, that the $0.\infty$ indetermination that appears in (4.34) at a turning point is generically solved to give a finite answer. Therefore the resulting value of $\dot{H}$ and consequently, the value of R, will be finite at such point. However, this is absolutely not the case, and a rigorous proof can be given as follows. Consider the Raychaudhuri identity [53]

$$\frac{d\theta}{dt} = -R_{tt} - \frac{\theta^2}{3}, \tag{4.35}$$

which is an useful tool for studying singularities. Here θ is the expansion parameter of the universe at a given time t, which in the isotropic and homogenous case reduces $\theta = 3H$. For the space times in consideration one has that

$$R_{tt} = -3\left(\dot{H} + H^2\right).$$

The insertion of this expression into (4.35) gives a trivial identity. However, a non trivial relation may be found by use of (2.10). By taking (2.10) into account, together with the redefinition $\kappa^2 = 1$ and $8f(\phi) \to f(\phi)$ the last formula becomes

$$R_{tt} = \frac{1}{2}(\rho_{eff} + 3p_{eff}) = \dot{\phi}^2 + \frac{3}{2}H^2 f''(\phi)\dot{\phi}^2 + \frac{3}{2}H^2 f'(\phi)\ddot{\phi} + 3H\dot{H}f'(\phi)\dot{\phi} + \frac{3}{2}H^3 f'(\phi)\dot{\phi}.$$

This, together with identification $\theta = 3H$, converts the Raychaudhuri equation (4.35) into

$$3(1 + H\dot{f})\frac{dH}{dt} = -\dot{\phi}^2 - \frac{3}{2}H^2(\ddot{f} + H\dot{f}) - 3H^2. \tag{4.36}$$

By further use of (4.23) the last equation can be expressed as

$$\dot{\phi}^2\frac{dH}{dt} = -2H^2\dot{\phi}^2 - 3H^4(\ddot{f} + H\dot{f}) - 6H^4. \tag{4.37}$$

This equation is completely equivalent to (3.12). But some indirect conclusions may be drawn from this precise expression. First, if $H \neq 0$ then, for a turning point characterized by $\dot{\phi} \to 0$ it follows $\dot{H} \to \pm\infty$. This is unless there is a precise cancellation between all the terms in the right hand. There is no reason to expect such an ending. Thus, generically for the present model, a singular value for $\dot{H}$ and for the curvature R are present at a turning point. This is an important conclusion.

In brief, the analysis given above shows that the curvature R of the universe will be regular if there are no turning points for ϕ. If this is combined with (4.34) it follows that there is no turning point when the derivative $\partial_{a_i}H^2$ is finite. This implies that the curvature R will be regular if the roots of the polynomial equation (3.21) do not merge at any finite proper time t. Therefore, the task to construct universes with regular curvature R is reduced to find a function $f(\phi)$ is C^3, such that $f'(\phi) \neq 0$ and is such that the roots of (3.21) never merge, no matter which values ϕ takes during the evolution. There exist plenty of coupling functions of this type. Consider a coupling such that $f'(\phi)$ has a minimum $f'_1 > 0$ and a maximum f'_2, and such that these extrema are reached asymptotically at $\phi \to \pm\infty$. Furthermore assume that the second derivative $f''(\phi)$ is never zero except at $\phi \to \pm\infty$. Then both $f'(\phi)$ and $f''(\phi)$ take bounded values, and so the coefficients of the polynomial (3.21). By adjusting the range of values of $f'(\phi)$ and $f''(\phi)$ conveniently, it should be possible to construct a model for which the roots never merge due to the limited values of the polynomial coefficients.[2]

A further possibility is that the evolution of ϕ, which is already constrained by (4.31), is in fact constrained further to a compact set $\phi_1 \leq \phi(t) \leq \phi_2$ for which the image of the polynomial coefficients of (3.21) determined by $f'(\phi(t))$ and $f''(\phi(t))$ are such that the roots never merge. However, without knowing the precise form of the coupling and the evolution of $\phi(t)$ we can not make further comments about this possibility.

[2]Note that the condition that the roots never merge is too restrictive. In fact, the condition that is really needed is that *some* of the roots do not merge.

5. Discussion

The present work is complementary with some models presented in the literature such as [33] and [51]-[52]. In [33] certain Gauss-Bonnet models without potential were considered, such as the ones presented here. These authors already noticed that for $k = 0$ there are a large set of non singular solutions. These works are related to couplings of the form $f(\phi) = a\phi^n$. These authors also studied the behavior of the scale factor $a(t)$ near a singularity. The authors suggest the existence of a large set of non singular solutions. This situation seem to be different than in GR, where the powerful singularity theorems of Hawking insure that the universe is going to fall into a curvature singularity in the past or future if it is everywhere expanding or contracting [53]. Generalizations of these theorems in presence of scalar fields in GR where obtained in [54].

Similar models were studied in [51], where the authors are able to find explicit cosmological solutions for $n = 1$ and $n = 2$ in the case of spatial curvature $k = 0$. The authors study the early stages of the universe by neglecting the effect of the Ricci scalar R and are able suggest in this approximation the existence of solution which interpolate between a de Sitter and a Milne universe. This approximation is confirmed for $n = 1$ in [52], and more general exponents are considered as well. These authors are able to find for $n = 2$ a de Sitter phase with an exit at large times and also singularity free cosmologies with no big bang. But in the strong curvature regime, the non singular solutions are found for $n = 2$, for other exponents $n \neq 2$ the resulting universes begin or end in a singularity.

The results of the present work do not contradict these results. In fact, the mathematical characterization presented here is not related to a particular coupling of the form $f(\phi) = a\phi^n$. We are not restricted to polynomial couplings, Instead, the main idea is to obtain the most possible general results on the dynamics, as independent as possible of the form of the coupling. One of these universal results is that the Hubble constant H is bounded by the initial condition for any future time, independently of the form of the coupling $f(\phi)$. In particular, an initially contracting universe is contracting forever in the future. Also, an expanding universe at the present was always expanding in the past. This implies that no cyclic cosmologies are allowed in this model, no matter the functional form of $f(\phi)$. Another universal result is that, at a turning point $\dot{\phi} \to 0$ the universe is necessarily singular, unless it is flat. This result was obtained by use of the Raychaudhuri equation adapted to these models, and it is true up to some very specific choice of the couplings.

Another general result obtained in the present work is that, for an spatially flat universe, the relation between the Hubble constant H and the value of the scalar field ϕ is described in terms of a polynomial equation (3.21) for H with ϕ dependent coefficients. This is a polynomial of degree larger than five, thus its explicit roots can not be found analytically. This of course may complicate the analysis of the resulting theories. Nevertheless, we were able put some conditions on the domain of the coupling $f(\phi)$ which give rise to singularity free cosmologies. These are bounds of $f(\phi)$ and its first and second derivative. If the domain of these functions is restricted enough, the resulting solution is regular at any finite proper time. It is important to emphasize however that these restrictions do not consist in a no-go theorem. In other words, there can exist couplings which violates these conditions and still giving rise to singularity free cosmologies. For instance, it has been shown that when the derivative of the coupling $f'(\phi)$ between the inflaton and the Gauss Bonnet term is bounded from above and below, and when the domain is restricted conveniently, there exist non trivial cosmological solutions whose curvature is regular for any past and future times. However, the reference [55] present some singularity free solutions with $f(\phi) = a\phi^2$. Clearly, $f'(\phi)$ does not have a minimum value, and the resulting solution is still regular. The use of the conditions we have derived is just due to our mathematical limitations, not due to any fundamental reason.

It should be remarked that, even taking into account that the conditions presented here are not satisfied by the couplings $f(\phi) = a\phi^n$ of references [33] and [51]-[52], there is an uncanny resemblance between our results and those of these references. In our case, we are requiring a bounded domain for $f'(\phi)$ and $f''(\phi)$. In the case in [33] and [51]-[52], the singularity free cosmologies are found only for $n=2$, while for $n>2$ they are all singular. These and our results suggest that singularity free cosmologies take place when the coupling is bounded conveniently or posses a sufficiently slow growing. Otherwise a singularity will be unavoidable.

Another interesting detail to remark is that, for the small power case $f(\phi)=a\phi$ one has that $f''(\phi)=0$, which is a condition that was avoided in the present work. This condition was also imposed not for any fundamental reason, instead due to our limitation on analytical methods. But it is interesting to note that $f(\phi)=a\phi$ violates this condition and give rise to singular cosmologies, even though it is slowly growing. We have the following heuristic interpretation of this apparent coincidence. For $n=0$ the Gauss-Bonnet term is just topological, and the resulting model is GR with an scalar field without potential. In this context the Hawking theorems [53] apply and a singularity takes place. For $n=1$, the model is not deviated enough from GR, and still a singularity appears. For $n=2$ is deviated enough such that a singularity is avoided. For $n>2$ the coupling is very explosive and again a singularity takes place. Our restrictions also are not close to GR neither close to an explosive behavior, and for this reason the singularity is also avoided.The difference is that we are not limited to polynomial couplings. It may be interesting to develop this non rigorous argument further in a future.

Finally, we would like to remark that, even though the absence of a potential for the inflaton may be non realistic from the cosmological point of view [31], the mathematical methods implemented here may be useful to characterize solutions with the potential term $V(\phi)$ or spatial curvature $k=\pm 1$ turned on. We will come to this point in a near future.

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